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Forced Solutions of Disturbed Pendulum-like Lur'e Systems

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***Abstract*— The mathematical model of a viscously damped pendulum is an example of a Lur'e system with periodic nonlinearity. Systems of this type arise in many applications, describing e.g. phase-locked loops and other synchronization systems arising in communication engineering, networks of oscillators and power generators. Periodic nonlinearities usually imply multistability of the system and the existence of multiple stable and unstable equilibria. This makes inapplicable many tools of classical nonlinear control, developed for systems with globally stable equilibria. To study asymptotic properties of such systems, special techniques have been developed stemming from Popov's method of "a priori integral indices", or integral quadratic constraints. These tools lead to efficient frequency-domain criteria, providing convergence of any solution to one of equilibria. In this paper, we further develop Popov's method, addressing the problem of *robustness* of the convergence property against external disturbances that do not oscillate at infinity (more precisely, decomposable into the sum of a constant excitation and decaying L_1 or L_2 signal). Will the forced solutions of the disturbed system also converge to one of the equilibria points (in general, the set of equilibria depends on the disturbance)? In this paper, we find a sufficient frequency-domain condition ensuring such a robust convergence, showing also that a relaxed form of this condition guarantees absence of high-frequency periodic oscillations in the system.**

***Index Terms*— Stability, integral equation, periodic nonlinearity, robustness, phase-locked loop**

I. INTRODUCTION

A number of systems, arising in natural sciences and engineering, can be represented by a *feedback interconnection* of an asymptotically stable linear stationary system and *periodic nonlinearity*. The simplest example of such a system is a pendulum with viscous damping; for the sake of brevity all system of the aforementioned type are said to be *pendulum-like*. Examples of such systems include vibrational units, electric motors, power generators and various synchronization circuits such as phase,frequency and delay-locked loops (PLL/FLL/DLL) [1]–[5] broadly used in electrical, communication and computer engineering.

Pendulum-like systems are featured by complex multistable dynamics with infinite sequences of stable and unstable equilibria points (in fact, their natural phase space is a cylindric or toric manifold [6], [7]). Many effects in such systems, e.g. oscillations, hidden attractors and "cycle slipping" [8]–[12] cannot be examined by tools of classical nonlinear control and require special techniques. One of the central problems, concerned with dynamics of synchronization systems, is the convergence of all solutions to equilibria points. This counterpart of global asymptotic stability in systems with unique equilibria is referred to as the *gradient-like* behavior [13], [14]; the gradient-like behavior excludes, in particular, limit cycles and other hidden attractors.

To study the challenging dynamics of pendulum-like system, two groups of methods have been developed. Analysis of finite-dimensional system relies on special non-quadratic Lyapunov functions in [6]–[8], [11], [13]–[15]. Much less studied are their *infinite-dimensional* counterparts, describing e.g. synchronization systems with delays [16], [17] and non-rational low-pass filters [18]. The central method proposed to study them [19]–[25] stems from the seminal V.M. Popov's technique [26], [27], that is referred to as the method of "a priory integral indices" and has grown into the elegant *integral quadratic constraints* (IQC) approach [28]–[30]. Using this approach, conditions for the solutions' convergence, as well as estimates for their transient behavior, have been obtained [19]–[25]; these criteria can be called "frequency-algebraic" as they consist of parameter-dependent *frequency-domain* conditions (similar to those arising in classical absolute stability) and algebraic restrictions on the parameters.

In this paper, we address the following question: assuming that a pendulum-like system has convergent solutions, how robust is this property against external disturbances? Obviously, if such a disturbance persistently excites the solution (being e.g. harmonic or other periodic oscillatory signal), the solution no longer converges to an equilibrium point but rather oscillates. In synchronization systems, such disturbances are typically modeled as combinations of stationary random signals and polyharmonic signals [31]–[33] to be rejected or damped. In this paper, we deal with other type of disturbances that have finite limit at infinity (being thus combinations of constant and decaying signals), enabling thus the disturbed system to have equilibria. Such disturbances are often considered in regard to the phenomenon of *cycle slipping* [34], [35], that is, the solution's "migration" between the basins of attraction of different equilibria points under random external perturbations. In spite of the

Theorems 1, 4, 5 supported solely by Russian Foundation for Basic Research (RFBR) grant 17-08-00715 held by St. Petersburg State University. Theorems 2, 3 supported solely by Russian Science Foundation (RSF) grant 16-19-00057 held by IPME RAS

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existence of equilibria, it is unclear whether each solution of the disturbed system converges to one of them, or this convergence is destroyed by the disturbance. In this paper, we take an important step in analysis of pendulum–like systems and establish "frequency-algebraic" criteria that ensure the convergence of forced solutions under the disturbances of the aforementioned type. In this paper, we deal with scalar nonlinearities; a more complicated case of vector-valued nonlinearity will be studied in our next work [36]. Similar to stability conditions for undisturbed systems [19]–[25], the relaxation of the frequency-domain inequality in these criteria leads to the conditions, providing the absence of high-frequency periodic solutions.

II. Problem setup.

Consider a control system with distributed parameters described by an integro–differential Volterra equation

$$\begin{aligned}\dot\sigma(t) &= b(t) + \rho(\psi(\sigma(t-h)) + f(t-h)) - \\ &- \int_0^t \gamma(t-\tau)(\psi(\sigma(\tau)) + f(\tau))\,d\tau, \quad t>0.\end{aligned} \tag{1}$$

Here $h \geq 0$; $\psi : \mathbb{R} \to \mathbb{R}$; $f : [-h,+\infty) \to \mathbb{R}$, $\gamma, b : [0,+\infty) \to \mathbb{R}$, The solution to the system (1) is uniquely determined by the initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t). \tag{2}$$

Throughout the paper, the following conditions are adopted.

Assumption 1: The system (1) and its initial condition (2) satisfy the following conditions

1) the function $b(\cdot)$ is continuous and $b(t) \to 0$ as $t \to +\infty$; the function $\gamma(\cdot)$ is piece–wise continuous;
2) $b(t)$ and $\gamma(t)$ exponentially decay as $t \to \infty$, in particular, for some $r > 0$ one has
$$b(t)e^{rt} \in L_2[0,+\infty), \tag{3}$$
$$\gamma(t)e^{rt} \in L_2[0,+\infty); \tag{4}$$
3) the function $f(t)$ is continuous and
$$\lim_{t\to+\infty} f(t) = L; \tag{5}$$
4) $\sigma^0(\cdot)$ is continuous and $\sigma(0+0) = \sigma^0(0)$;
5) $\psi(\sigma)$ is Δ-periodic ($\psi(\sigma+\Delta) = \psi(\sigma)$); it is C^1-smooth with
$$\begin{aligned}\alpha_1 &\triangleq \inf_{\zeta\in[0,\Delta)} \psi'(\zeta);\\ \alpha_2 &\triangleq \sup_{\zeta\in[0,\Delta)} \psi'(\zeta)\end{aligned} \tag{6}$$
(this implies that $\alpha_1 < 0 < \alpha_2$);
6) the following function has simple isolated roots
$$\varphi(\zeta) \triangleq \psi(\zeta) + L. \tag{7}$$

Under Assumption 1 the problem (1), (2) has a unique $C^1(\mathbb{R}^+)$–smooth solution [20](Theorem 2.19.1).

Notice that $f(t)$ can be interpreted as an external *disturbance*, and we are interested in the behavior of *forced* solutions in presence of this disturbance. Assuming that any solution of the system (1) with $f \equiv 0$ converges, we are interested in the conditions which guarantee the same property for the forced solutions ($f \not\equiv 0$). Obviously, this is possible not for any type of disturbance, e.g. a periodic signal, persistently disturbing the system, typically leads to its oscillatory behavior. This motivates us to introduce the assumption (5), enabling the forced solutions to converge.

Introducing the shifted disturbance

$$g(t) \triangleq f(t) - L, \tag{8}$$

one may easily notice that (1) shapes into

$$\begin{aligned}\dot\sigma(t) &= b(t) + (\varphi(\sigma(t-h)) + g(t-h)) - \\ &- \int_0^t \gamma(t-\tau)(\varphi(\sigma(\tau)) + g(\tau))\,d\tau \quad (t>0),\end{aligned} \tag{9}$$

where φ is defined in (7) and the new disturbance $g(t)$ vanishes at infinity. Additionally, we will assume that

$$\textbf{either} \quad g(t) \in L_1[0,+\infty), \tag{10}$$
$$\textbf{or} \quad g(t) \in L_2[0,+\infty). \tag{11}$$

The cases (10) and (11) are examined in different ways, using Popov's method of "a priori integral indices" [27] and, similar to this method, leading to *frequency-domain* convergence criteria. We first introduce the transfer function of linear part of (1) from $\psi(\sigma) + f$ to $(-\dot\sigma)$ as follows

$$K(p) \triangleq -\rho e^{-ph} + \int_0^\infty \gamma(t)e^{-pt}\,dt \quad (p \in \mathbb{C}). \tag{12}$$

In next sections we establish novel frequency–algebraic conditions, ensuring the convergence of each solution of (9) (and hence each solution of (1)) to one of the equilibria, provided that at least one of the conditions (10) and (11) holds. We also show that relaxation of the frequency-domain conditions in these criteria entails the absence of fast-oscillating periodic solutions. All criteria obtained in this paper will use the following frequency–domain inequality

$$\begin{aligned}\Pi(\omega) &\triangleq \Re e\{-\tau(K(\imath\omega) + \imath m_1^{-1}\omega)^*(K(\imath\omega) + \imath m_2^{-1}\omega) - \\ &- \varepsilon|K(\imath\omega)|^2 + \varkappa K(\imath\omega)\} - \delta \geq 0.\end{aligned} \tag{13}$$

Here $\imath^2 = -1$, $\varkappa, \varepsilon > 0, \delta > 0, \tau > 0$ and $m_1 \in [-\infty, \alpha_1]$, $m_2 \in [\alpha_2, +\infty]$ are variable parameters (if $m_i = \pm\infty$, $m_i^{-1} = 0$), and the symbol * stands for the complex conjugation.

III. Stability and oscillations of systems with L_2-summable external disturbance

In this section we assume that (11) is valid and, additionally, $g(t)$ is differentiable and

$$\dot g(t) \in L_2[0,+\infty). \tag{14}$$

Theorem 1. *If for a certain set of parameters $\varkappa, \varepsilon > 0, \delta > 0, \tau > 0, m_1 \leq \alpha_1, m_2 \geq \alpha_2$ the frequency-domain inequality* (13) *holds for all $\omega \geq 0$, then any bounded solution of* (1) *converges in the sense that*

$$\sigma(t) \xrightarrow[t\to\infty]{} \sigma_{eq} \tag{15}$$

$$\dot\sigma(t) \xrightarrow[t\to\infty]{} 0, \tag{16}$$

where the limit point σ_{eq} is a root of the equation

$$\psi(\sigma_{eq}) = -L.$$

Proof: Let $\eta(t) \triangleq \varphi(\sigma(t))$ and $\xi(t) = \eta(t) + g(t)$,

$$\mu(t) \triangleq \begin{cases} 0, & t<0,\\ t, & 0\leq t\leq 1,\\ 1. & t>1. \end{cases} \tag{17}$$

For arbitrary $T \geq 1$ and a given solution of (1) we introduce the functions

$$\xi_T(t) \triangleq \begin{cases} \mu(t)\xi(t), & t<T,\\ \xi(T)e^{c(T-t)}, & t\geq T \quad (c>0); \end{cases} \tag{18}$$

$$\sigma_0(t) = b(t) + \rho\xi(t-h)(1-\mu(t-h)) - \\ -\int_0^t (1-\mu(\tau))\gamma(t-\tau)\xi(\tau)\,d\tau; \tag{19}$$

$$\sigma_T(t) = \rho\xi_T(t-h) - \int_0^t \gamma(t-\tau)\xi_T(\tau)\,d\tau, \tag{20}$$

so that

$$\dot\sigma(t) = \sigma_0(t) + \sigma_T(t) \quad \text{for } t\in[0,T]. \tag{21}$$

Consider a set of functionals

$$J_T \triangleq \int_0^\infty \{\varkappa\sigma_T\xi_T + \delta\xi_T^2 + \\ +\varepsilon\sigma_T^2 + \tau(\sigma_T - m_1^{-1}\dot\xi_T)(\sigma_T - m_2^{-1}\dot\xi_T)\}\,dt. \tag{22}$$

By virtue of (20) and (18)

$$\mathfrak{F}(\sigma_T)(\imath\omega) = -K(\imath\omega)\mathfrak{F}(\xi_T)(\imath\omega) \tag{23}$$

and

$$\mathfrak{F}(\dot\xi_T)(\imath\omega) = \imath\omega\mathfrak{F}(\xi_T)(\imath\omega), \tag{24}$$

where $\mathfrak{F}$ stands for the Fourier transform. Due to Plancherel theorem and (23), (24) we have

$$J_T = -\tfrac{1}{2\pi}\int_{-\infty}^{+\infty} \Pi(\omega)|\mathfrak{F}(\xi_T)|^2(\imath\omega)\,d\omega. \tag{25}$$

It follows from (13) that for $T\geq 1$

$$J_T \leq 0. \tag{26}$$

On the other hand we have

$$J_T = \rho_T + J_0 + J_{1T} + J_{2T} + \int_T^\infty (\varepsilon+\tau)\sigma_T^2\,dt, \tag{27}$$

where

$$\rho_T \triangleq \int_0^T \{\varkappa\dot\sigma\xi + \delta\xi^2 + \varepsilon\dot\sigma^2 + \\ +\tau(\dot\sigma - m_1^{-1}\dot\xi)(\dot\sigma - m_2^{-1}\dot\xi)\}\,dt; \tag{28}$$

$$J_{1T} \triangleq -\int_0^T \{\varkappa\sigma_0\xi_T + 2(\varepsilon+\tau)\sigma_0\dot\sigma - \\ -(\varepsilon+\tau)\sigma_0^2 - \tau\sigma_0(m_1^{-1}+m_2^{-1})\dot\xi_T\}\,dt; \tag{29}$$

$$J_{2T} \triangleq \int_T^\infty \{\varkappa\sigma_T\xi_T + \delta\xi_T^2 + \tau m_1^{-1}m_2^{-1}(\dot\xi_T)^2 - \\ -\tau(m_1^{-1}+m_2^{-1})\sigma_T\dot\xi_T\}\,dt \tag{30}$$

and J_0 is an integral from 0 to 1 of a bounded function. It is clear that

$$\sigma_0(t)e^{rt} \in L_2[0,+\infty) \tag{31}$$

and functions $\dot\sigma(t), \xi(t), \dot\xi(t)$ are bounded on $\mathbb{R}_+$. So

$$|J_{1T}| + |J_{2T}| + |J_0| < C_0, \tag{32}$$

where C_0 does not depend on T. Then it follows from (26) and (27) that

$$\rho_T < C_0, \tag{33}$$

On the other hand

$$\rho_T \geq \int_0^T G_1(\dot\sigma,\eta,\dot\eta)\,dt - \int_0^T |G_2(\dot\sigma,\eta,\dot\eta,g,\dot g)|\,dt, \tag{34}$$

where

$$G_1(\dot\sigma,\eta,\dot\eta) \triangleq \varkappa\dot\sigma\eta + \delta\eta^2 + \varepsilon\dot\sigma^2 + \\ +\tau(\dot\sigma - m_1^{-1}\dot\eta)(\dot\sigma - m_2^{-1}\dot\eta); \tag{35}$$

$$G_2(\dot\sigma,\eta,\dot\eta,g,\dot g) \triangleq \varkappa\dot\sigma g + 2\delta g\eta - \tau m_1^{-1}(\dot\sigma - m_2^{-1}\dot\eta)\dot g - \\ -\tau m_2^{-1}\dot g(\dot\sigma - m_1^{-1}\dot\eta) + \tau m_1^{-1}m_2^{-1}\dot g^2. \tag{36}$$

It is clear that

$$(\dot\sigma - m_i^{-1}\dot\eta)^2 \leq C_1\dot\sigma^2 \quad (i=1,2), \tag{37}$$

where a constant C_1 depends on m_1^{-1} and m_2^{-1}.

Then by means of the obvious inequality

$$ab \leq \lambda a^2 + \frac{1}{4\lambda}b^2 \qquad (a,b,\lambda\in\mathbb{R}, \lambda>0) \tag{38}$$

one can easily establish that for any $\varepsilon_1 > 0$ the estimate

$$|G_2| \leq \varepsilon_1 C_2\dot\sigma^2 + \varepsilon_1\delta\eta^2 + \tfrac{C_3}{\varepsilon_1}g^2 + \tfrac{C_4}{\varepsilon_1}\dot g^2 + \\ +\tau|m_1^{-1}m_2^{-1}|\dot g^2 \tag{39}$$

with some constants C_2, C_3, C_4 depending on $\varkappa, \delta, \tau, m_1^{-1}, m_2^{-1}$ is true.

Let us choose $\varepsilon_1 < 1$ so small that $\varepsilon_1 C_2 < \varepsilon$. Then it follows from (11), (14), (33) and (34) that

$$\int_0^T \{\varkappa\dot\sigma\eta + \delta(1-\varepsilon_1)\eta^2 + \\ +(\varepsilon - \varepsilon_1 C_2)\dot\sigma^2 + \tau(\dot\sigma - m_1^{-1}\dot\eta)(\dot\sigma - m_2^{-1}\dot\eta)\}\,dt < C_5, \tag{40}$$

where C_5 does not depend on T. Notice that

$$(\dot{\sigma} - m_1^{-1}\dot{\eta})(\dot{\sigma} - m_2^{-1}\dot{\eta}) \geq 0 \tag{41}$$

and

$$\int_0^T \dot{\sigma}\eta\, dt = \int_{\sigma(0)}^{\sigma(T)} \varphi(\sigma)\, d\sigma. \tag{42}$$

Since $\sigma(t)$ is bounded on $[0, \infty)$ the integral $\int_{\sigma(0)}^{\sigma(t)} \varphi(\zeta)\, d\zeta$ is bounded on $[0, \infty)$ as well. Then it follows from (40), (41), (42) that

$$\dot{\sigma}(t), \varphi(\sigma(t)) \in L_2[0, +\infty) \tag{43}$$

The functions $\varphi(\sigma(t))$ and $\dot{\sigma}(t)$ are uniformly continuous on $[0, +\infty)$ [20](pp.262-263). According to Barbalat lemma [26]

$$\dot{\sigma}(t) \to 0 \text{ as } t \to +\infty, \tag{44}$$

$$\varphi(\sigma(t)) \to 0 \text{ as } t \to +\infty. \tag{45}$$

Then $\sigma(t)$ tends to a zero of $\varphi(\sigma)$as $t \to +\infty$. Theorem 1 is proved. ■

Remark. *The value of σ_{eq} varies. It depends on $\sigma^0(t)$.*

Notice that Theorem 1 gives conditions for convergence of *bounded* solutions, being thus a criterion of *dichotomy*: any solution is either convergent or unbounded. This result may serve to prove convergence of *any* solution in the case where all solutions are bounded (Lagrange stability of (1)). One sufficient condition for this has been found in [37], two other conditions, being nonlinear algebraic constraints on the parameters, are given in the following criteria, established in [38]. To formulate them, we introduce the functions

$$\Phi(\zeta) \triangleq \sqrt{\left(1 - m_1^{-1}\psi'(\zeta)\right)\left(1 - m_2^{-1}\psi'(\zeta)\right)}, \tag{46}$$

$$P(\zeta; \varepsilon, \tau) \triangleq \sqrt{\varepsilon + \tau\Phi^2(\zeta)}, \tag{47}$$

and the constants

$$\nu := \frac{\int_0^\Delta (\psi(\zeta) + L)\, d\zeta}{\int_0^\Delta |\psi(\zeta) + L| d\zeta}, \quad \nu_0 := \frac{\int_0^\Delta (\psi(\zeta) + L)\, d\zeta}{\int_0^\Delta \Phi(\zeta)|(\psi(\zeta) + L)|\, d\zeta}, \tag{48}$$

$$\nu_1(\varepsilon, \tau) := \frac{\int_0^\Delta (\psi(\zeta) + L)\, d\zeta}{\int_0^\Delta |\psi(\zeta) + L| P(\zeta; \varepsilon, \tau)\, d\zeta}. \tag{49}$$

Theorem 2. [38] *Suppose that the assumptions of Theorem 1 hold and, additionally, for some $a \in [0, 1]$ the following quadratic form is positive definite*

$$\begin{gathered} Q(x, y, z) \triangleq \varepsilon x^2 + \delta y^2 + \tau z^2 + \\ + a\varkappa\nu xy + (1 - a)\varkappa\nu_0 yz > 0 \\ \forall x, y, z : |x| + |y| + |z| \neq 0. \end{gathered} \tag{50}$$

Then any *solution is featured by the convergence property from Theorem 1 (in particular, all solutions are bounded).*

Theorem 3. [38] *Theorem 2 remains valid, replacing* (50) *by the following inequality*

$$4\delta > \varkappa^2\nu_1^2(\varepsilon, \tau). \tag{51}$$

If the dichotomy condition from Theorem 1 does not hold, in general the system may have non-convergent bounded solutions. We are going to show that a "relaxed" criterion from Theorem 1 implies absence of *periodic* solutions of high frequencies. By "periodic solutions" we mean solutions for which the function $\varphi(\sigma(t))$ is periodic, or, equivalently, the following condition holds.

Definition 1. *We say that a solution $\sigma(t)$ of* (1) *is periodic if there exist a number $T > 0$ and an integer I such that*

$$\sigma(t + T) = \sigma(t) + I\Delta, \quad \forall t \geq 0 \tag{52}$$

The minimal number T is called the period *and the number $\omega = \frac{2\pi}{T}$ is said to be the* frequency *of the periodic solution.*

Next two theorems show that frequency-algebraic conditions of gradient-like behavior can serve to evaluate the frequency of periodic solutions.

Theorem 4. *Let $\omega_0 > 0$, $\varkappa$, $\varepsilon > 0$, $\delta > 0$, $\tau > 0$, and $a \in [0, 1]$, exist such that for $\omega = 0$ and $\omega \geq \omega_0$ the inequality* (13) *holds and the condition* (50) *is valid. Then* (1) *has no periodic solutions of frequency $\omega \geq \omega_0$.*

Theorem 5. *Suppose there exist an $\omega_0 > 0$, $\varkappa$, $\varepsilon > 0$, $\delta > 0$, $\tau > 0$, such that for $\omega = 0$ and all $\omega \geq \omega_0$ the inequality* (13) *holds and condition* (51) *is valid. Then the equation* (1) *has no periodic solutions of frequency $\omega \geq \omega_0$.*

To prove Theorems 4 and 5, we rewrite (9) as follows

$$\begin{gathered} \dot{\sigma}(t) = \beta(t) + \rho\varphi(\sigma(t - h)) - \\ - \int_0^t \gamma(t - \tau)\varphi(\sigma(\tau))\, d\tau \quad (t > 0) \end{gathered} \tag{53}$$

where we denote

$$\beta(t) \triangleq b(t) + \rho g(t - h) - \int_0^t \gamma(t - \tau) g(\tau)\, d\tau. \tag{54}$$

Since the convolution of two quadratically summable functions tends to zero as $t \to +\infty$ [39] it follows from the assumptions 1) and 2) and the assumption (11) that

$$\beta(t) \to 0 \text{ as } t \to +\infty. \tag{55}$$

One can repeat now the argument of theorems from [40].

IV. Stability and oscillations of systems with summable external disturbance.

In this section we suppose that the function $g(t)$ is summable on $[0,+\infty)$, i.e. the inclusion (10) is valid.

Proposition 1. *The following relations are true:*

$$z(t) \triangleq \int_0^t \gamma(t-\tau)g(\tau)\,d\tau \to 0 \quad \textit{as } t \to +\infty, \tag{56}$$

$$z(t) \in L_1[0,+\infty) \cap L_2[0,+\infty). \tag{57}$$

Proof: It follows from (4) that

$$\gamma(t) \in L_1[0,+\infty) \cap L_2[0,+\infty). \tag{58}$$

The function $z(t)$ is a convolution of a summable $\gamma(t)$ and a vanishing $g(t)$. Then in virtue of [39] the relation (56) is true. On the other hand

$$\int_0^\infty |z(t)|\,dt \le \int_0^\infty \int_0^t |\gamma(t-\tau)g(\tau)|\,d\tau\,dt =$$

$$= \int_0^\infty |g(\tau)| \int_\tau^\infty |\gamma(t-\tau)|\,dt\,d\tau \le \int_0^\infty |g(\tau)| \int_0^\infty |\gamma(\lambda)|\,d\lambda\,d\tau, \tag{59}$$

which implies that $z(t)$ is summable on $[0,+\infty)$. This fact together with (56) implies that $z(t) \in L_2[0,+\infty)$. ■

Rewriting (9) in the form (53), Proposition 1 implies, in view of conditions 1),2),3) in Assumption 1, that (55) holds and, moreover,

$$\beta(t) \in L_1[0,+\infty) \cap L_2[0,+\infty). \tag{60}$$

Asymptotic behavior of equation (53) with β satisfying (55) (60), γ satisfying (4) and periodic $\varphi(\sigma)$ with simple isolated roots has been investigated in a number of published works, which allows to extend Theorems 1-5 to the case of L_1-summable disturbance. Theorem 1 follows from dichotomy theorem proved in [20] (Theorem 2.20.2). Theorem 2 and Theorem 3 for (53) are proved in [22]. Theorem 4 and Theorem 5 follow from [40].

V. Stability and oscillations of phase–locked loops

In this section we consider the phase–locked loops (PLL) with sine-shaped characteristic of phase detector, reducible [41] to the equation (1) with

$$\psi(\zeta) = \sin\zeta - \beta \quad (\beta \in (0,1)). \tag{61}$$

We consider two cases of the transfer function $K(p)$, characterizing the low-pass filter of a PLL.

Example 1. Consider first a PLL with the integrating filter

$$K(p) = \frac{T}{Tp+1} \quad (T > 0). \tag{62}$$

The mathematical equation of the corresponding system (1) coincides with that of the disturbed pendulum

$$\ddot{\sigma} + \frac{1}{T}\dot{\sigma} + (\sin\sigma - \beta) + f(t) = 0. \tag{63}$$

Using Theorem 2, it is possible to estimate the set of the coefficients $\{(T,\beta)\}$, for which all solutions of the system (63) converge, provided that either $f, \dot{f} \in L_2[0,\infty)$ or $f \in L_1[0,\infty)$. In Fig. 1, the estimate for the frontier of the stability domain in the parameter space $\{T^2,\beta\}$ is shown by the lower curve. The upper curve is the frontier of the *exact* stability domain, obtained in [42] for the *undisturbed* system by using qualitative-numerical methods. Theorem 2 thus gives a good approximation of the stability domain.

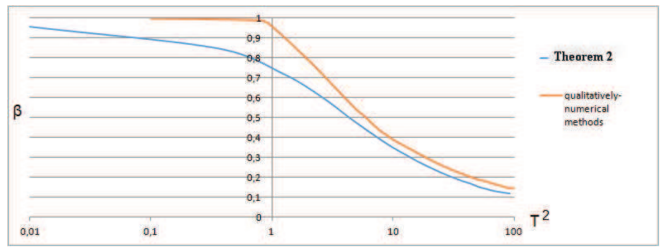

Fig. 1. The exact stability domain for the system (63) vs. its estimate.

Example 2. Our next example deals with a PLL with the proportional integrating filter, whose transfer function is

$$K(p) = T\frac{Tmp+1}{Tp+1} \quad (m \in (0,1)), \tag{64}$$

In Fig. 2, we compare the stability domain for $m = 0.4$ with the domains, where periodic solutions of the frequency $\omega \ge \omega_0$ are absent. The leftmost curve is the frontier of the stability domain (conditions of Theorem 2 hold). In particular, under this curve periodic trajectories are absent ($\omega_0 = 0$). Up to this curve we cannot guarantee convergence of solutions. But we can guarantee that between the leftmost curve and orange (respectively, grey and magenta) dashed curves periodic solutions, if they exist, cannot have frequency greater than $\omega_0 = 3$ (respectively, $\omega_0 = 7$ and $\omega_0 = 10$) thanks to Theorem 4. As $\omega_0 \to \infty$, such curves converge to the line L, on which the conditions of Theorem 4 fail.

VI. Conclusion

In this paper we study forced solutions of a pendulum-like system in presence of uncertain disturbance, assumed to be nonoscillatory at infinity and thus enabling the disturbed system to have the infinite set of equilibria. We offer novel frequency–domain criteria for convergence of any forced solution (which may be considered as the criterion of *robust* convergence in the corresponding class of disturbances). These critera are based on techniques, stemming from V.M. Popov's method of integral quadratic constraints.

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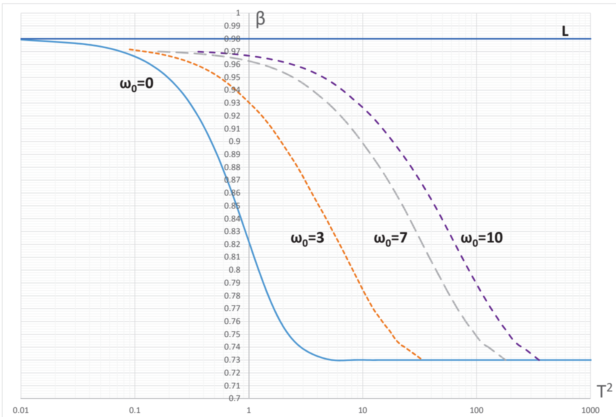

Fig. 2. Domains of stability and slow oscillation for the system (64),(61).

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