ON THE p-ADIC VALUATION OF STIRLING NUMBERS OF THE FIRST KIND

PAOLO LEONETTI

Università "Luigi Bocconi", Department of Statistics, Milan, Italy
email: `leonetti.paolo@gmail.com`

CARLO SANNA

Università degli Studi di Torino, Department of Mathematics, Turin, Italy
email: `carlo.sanna.dev@gmail.com`

Abstract. For all integers $n \geq k \geq 1$, define $H(n,k) := \sum 1/(i_1 \cdots i_k)$, where the sum is extended over all positive integers $i_1 < \cdots < i_k \leq n$. These quantities are closely related to the Stirling numbers of the first kind by the identity $H(n,k) = s(n+1,k+1)/n!$. Motivated by the works of Erdős–Niven and Chen–Tang, we study the p-adic valuation of $H(n,k)$. Lengyel proved that $\nu_p(H(n,k)) > -k \log_p n + O_k(1)$ and we conjecture that there exists a positive constant $c = c(p,k)$ such that $\nu_P(H(n,k)) < -c \log n$ for all large n. In this respect, we prove the conjecture in the affirmative for all $n \leq x$ whose base p representations start with the base p representation of $k-1$, but at most $3x^{0.835}$ exceptions. We also generalize a result of Lengyel by giving a description of $\nu_2(H(n,2))$ in terms of an infinite binary sequence.

1. Introduction

It is well known that the n-th harmonic number $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is not an integer whenever $n \geq 2$. Indeed, this result has been generalized in several ways (see, e.g., [2, 7, 13]). In particular, given integers $n \geq k \geq 1$, Erdős and Niven [8] proved that

$$H(n,k) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{1}{i_1 \cdots i_k}$$

is an integer only for finitely many n and k. Precisely, Chen and Tang [4] showed that $H(1,1)$ and $H(3,2)$ are the only integral values. (See also [11] for a generalization to arithmetic progressions.)

A crucial step in both the proofs of Erdős–Niven and Chen–Tang's results consists in showing that, when n and k are in an appropriate range, for some prime number p the p-adic valuation of $H(n,k)$ is negative, so that $H(n,k)$ cannot be an integer.

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Moreover, a study of the p-adic valuation of the harmonic numbers was initiated by Eswarathasan and Levine [9]. They conjectured that for any prime number p the set $\mathcal{J}_p$ of all positive integers n such that $\nu_p(H_n) > 0$ is finite. Although Boyd [3] gave a probabilistic model predicting that $\#\mathcal{J}_p = O(p^2(\log\log p)^{2+\varepsilon})$, for any $\varepsilon > 0$, and Sanna [21] proved that $\mathcal{J}_p$ has asymptotic density zero, the conjecture is still open. Another result of Sanna [21] is that $\nu_p(H_n) = -\lfloor \log_p n \rfloor$ for any n in a subset $\mathcal{S}_p$ of the positive integers with logarithmic density greater than 0.273.

In this paper, we study the p-adic valuation of $H(n,k)$. Let $s(n,k)$ denotes an unsigned Stirling number of the first kind [10, §6.1], i.e., $s(n,k)$ is the number of permutations of $\{1,\dots,n\}$ with exactly k disjoint cycles. Then $H(n,k)$ and $s(n,k)$ are related by the following easy identity.

Lemma 1.1. *For all integers $n \geq k \geq 1$, we have $H(n,k) = s(n+1,k+1)/n!$.*

In light of Lemma 1.1, and since the p-adic valuation of the factorial is given by the formula [10, p. 517, 4.24]

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1},$$

where $s_p(n)$ is the sum of digits of the base p representation of n, it follows that

$$\nu_p(H(n,k)) = \nu_p(s(n+1,k+1)) - \frac{n - s_p(n)}{p-1}, \tag{1}$$

hence the study of $\nu_p(H(n,k))$ is equivalent to the study of $\nu_p(s(n+1,k+1))$. That explains the title of this paper.

In this regard, p-adic valuations of sequences with combinatorial meanings have been studied by several authors (see, e.g., [5, 15, 17, 19, 20, 22]). In particular, the p-adic valuation of Stirling numbers of the second kind have been extensively studied [1, 6, 12, 14, 16]. On the other hand, very few seems to be known about the p-adic valuation of Stirling numbers of the first kind. Indeed, up to our knowledge, the only systematic work on this topic is due to Lengyel [18]. Among several results, he showed (see the proof of [18, Theorem 1.2]) that, for all primes p and positive integers k, it holds

$$\nu_p(H(n,k)) > -k\log_p n + O_k(1). \tag{2}$$

The main aim of this article is to provide an upper bound for $\nu_p(H(n,k))$. In this respect, we believe that inequality (2) is *nearly* optimal, and our Theorem 2.3 confirms this in the special case when the base p representation of n starts with the base p representation of $k-1$. We think that our method could be improved to remove this condition on the base p representation of n, however we restrict ourselves to this special case since the proofs are already quite involved.

Lastly, we also formulate the following:

Conjecture 1.1. *For any prime number p and any integer $k \geq 1$, there exists a constant $c = c(p,k) > 0$ such that $\nu_p(H(n,k)) < -c\log n$ for all sufficiently large integers n.*

2. Main results

Before stating our results, we need to introduce some notation and definition. For any prime number p, we write

$$\langle a_0, \ldots, a_v \rangle_p := \sum_{i=0}^{v} a_i p^{v-i}, \text{ where } a_0, \ldots, a_v \in \{0, \ldots, p-1\},\ a_0 \neq 0, \tag{3}$$

to denote a base p representation. In particular, hereafter, the restrictions of (3) on $a_0, \ldots, a_v$ will be implicitly assumed any time we will write something like $\langle a_0, \ldots, a_v \rangle_p$.

For any positive integer $a = \langle a_0, \ldots, a_v \rangle_p$, let $\mathcal{S}_p(a)$ be the set of all positive integers whose base p representations start with the base p representation of a, that is,

$$\mathcal{S}_p(a) := \big\{ \langle b_0, \ldots, b_u \rangle_p : u \geq v \text{ and } b_i = a_i \text{ for } i = 0, \ldots, v \big\}.$$

We call *p-tree of root* $a = \langle a_0, \ldots, a_v \rangle_p$ a set of positive integers $\mathcal{T}$ such that:

(T1) $\langle a_0, \ldots, a_v \rangle_p \in \mathcal{T}$;
(T2) If $\langle b_0, \ldots, b_u \rangle_p \in \mathcal{T}$ then $u \geq v$ and $b_i = a_i$ for $i = 0, \ldots, v$;
(T3) If $\langle b_0, \ldots, b_u \rangle_p \in \mathcal{T}$ and $u > v$ then $\langle b_0, \ldots, b_{u-1} \rangle_p \in \mathcal{T}$.

Hence, it is clear that $\mathcal{T} \subseteq \mathcal{S}_p(a)$. Moreover, for any $n = \langle d_0, \ldots, d_s \rangle_p \in \mathcal{S}_p(a) \setminus \mathcal{T}$ we denote by $\mu_p(\mathcal{T}, n)$ the least positive integer r such that $\langle d_0, \ldots, d_r \rangle_p \notin \mathcal{T}$. Note that $\mu_p(\mathcal{T}, n)$ is indeed well defined and that obviously $\mu_p(\mathcal{T}, n) \leq s$. Finally, the *girth* of $\mathcal{T}$ is the least integer g such that for all $\langle b_0, \ldots, b_u \rangle_p \in \mathcal{T}$ we have $\langle b_0, \ldots, b_u, c \rangle_p \in \mathcal{T}$ for at most g values of $c \in \{0, \ldots, p-1\}$.

We are ready to state our results about the p-adic valuation of $H(n, k)$.

Theorem 2.1. *Let p be a prime number and let $k \geq 2$ be an integer. Then there exist a p-tree $\mathcal{T}_p(k)$ of root $k-1$ and a nonnegative integer $W_p(k)$ such that for all integers $n = \langle d_0, \ldots, d_s \rangle_p \in \mathcal{S}_p(k-1)$ we have:*

(i) *If $n \notin \mathcal{T}_p(k)$ then $\nu_p(H(n,k)) = W_p(k) + \mu_p(\mathcal{T}_p(k), n) - ks$;*
(ii) *If $n \in \mathcal{T}_p(k)$ then $\nu_p(H(n,k)) > W_p(k) - (k-1)s$.*

Moreover, the girth of $\mathcal{T}_p(k)$ is less than $p^{0.835}$. In particular, $\mathcal{T}_2(k)$ is infinite and its girth is equal to 1.

Note that the case $k = 1$ has been excluded from the statement. (As mentioned in the introduction, see [3, 9, 21] for results on the p-adic valuation of $H(n, 1) = H_n$.)

For given p and k, the proof of Theorem 2.1 shows how to compute $W_p(k)$, while in Section 5 we explain a method to effectively compute the elements of $\mathcal{T}_p(k)$. Therefore, Theorem 2.1(i) gives an effective formula for $\nu_p(H(n,k))$ for any $n \in \mathcal{S}_p(k-1) \setminus \mathcal{T}_p(k)$. Note also that the bound on the girth of $\mathcal{T}_p(k)$ implies that $\mathcal{S}_p(k-1) \setminus \mathcal{T}_p(k)$ has infinitely many elements. Furthermore, for some p and k we have that $\mathcal{T}_p(k)$ is finite (see Section 5), hence in such cases computing $\nu_p(H(n,k))$ for the finitely many $n \in \mathcal{T}_p(k)$ and using Theorem 2.1(i) for $n \in \mathcal{S}_p(k-1) \setminus \mathcal{T}_p(k)$, we obtain a complete description of $\nu_p(H(n,k))$ for all $n \in \mathcal{S}_p(k-1)$.

Since the statement of Theorem 2.1 is a bit complicated, for the sake of clarity we give a numerical example: Take $p = 3$ and $k = 2$. Then $\mathcal{T}_p(k)$ is the finite set of 8 integers drawn in Figure 1, while $W_p(k) = 0$. If we choose $n = 1257 = \langle 1, 2, 0, 1, 1, 2, 0 \rangle_3$, then it follows easily that $n \in \mathcal{S}_p(k-1) \setminus \mathcal{T}_p(k)$ and $\mu_p(\mathcal{T}_p(k), n) = 3$ thus Theorem 2.1 gives $\nu_p(H(n,k)) = 0 + 3 - 2 \cdot 6 = -9$.

Lengyel [18, Theorem 2.5] proved that for each integer $m \geq 2$ it holds

$$\nu_2(s(2^m, 3)) = 2^m - 3m + 3$$

which, in light of identity (1), is in turn equivalent to

$$\nu_2(H(2^m - 1, 2)) = 4 - 2m. \tag{4}$$

As an application of Theorem 2.1, we give a corollary that generalizes (4) and provides a quite precise description of $\nu_2(H(n, 2))$.

Corollary 2.2. *There exists a sequence $f_0, f_1, \ldots \in \{0, 1\}$ such that for any integer $n = \langle d_0, \ldots, d_s \rangle_2 \geq 2$ we have:*

- (i) *If $d_0 = f_0, \ldots, d_{r-1} = f_{r-1}$, and $d_r \neq f_r$, for some positive integer $r \leq s$, then $\nu_2(H(n, 2)) = r - 2s$;*
- (ii) *If $d_0 = f_0, \ldots, d_s = f_s$, then $\nu_2(H(n, 2)) > -s$.*

Precisely, the sequence $f_0, f_1, \ldots$ can be computed recursively by $f_0 = 1$ and

$$f_s = \begin{cases} 1 & \text{if } \nu_2(H(\langle f_0, \ldots, f_{s-1}, 1 \rangle_2, 2)) > -s, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

for any positive integer s. In particular, $f_0 = 1$, $f_1 = 1$, $f_2 = 0$.

Note that (4) is indeed a consequence of Corollary 2.2. In fact, on the one hand, for $m = 2$ the identity (4) can be checked quickly. On the other hand, for any integer $m \geq 3$ we have $2^m - 1 = \langle d_0, \ldots, d_{m-1} \rangle_2$ with $d_0 = \cdots = d_{m-1} = 1$, so that $d_0 = f_0$, $d_1 = f_1$, and $d_2 \neq f_2$, hence (4) follows from Corollary 2.2(i), with $s = m - 1$ and $r = 2$.

Finally, we obtain the following upper bound for $\nu_p(H(n, k))$.

Theorem 2.3. *Fix a prime number p, and integer $k \geq 2$, and $x \geq (k-1)p$. Then*

$$\nu_p(H(n, k)) < -(k - 1)(\log_p n - \log_p(k - 1) - 1)$$

holds for all $n \in \mathcal{S}_p(k - 1) \cap [(k - 1)p, x]$, but at most $3x^{0.835}$ exceptions.

Note that $\#(\mathcal{S}_p(k - 1) \cap [(k - 1)p, x]) \gg_{p,k} x$. Hence Theorem 2.3 gives an upper bound for $\nu_p(H(n, k))$ for almost all $n \in \mathcal{S}_p(k - 1)$, with respect to the its asymptotic relative density. In particular, there exists a positive constant $c = c(p, k)$ such that

$$\nu_p(H(n, k)) < -c \log(n)$$

for almost all $n \in \mathcal{S}_p(k - 1)$, which provides, in turn, a sort of evidence in support of Conjecture 1.1.

3. Preliminaries

Let us start by proving the identity claimed in Lemma 1.1.

Proof of Lemma 1.1. By [10, Eq. 6.11] and $s(n + 1, 0) = 0$, we have the polynomial identity

$$\prod_{i=1}^{n} (X + i) = \sum_{k=0}^{n} s(n + 1, k + 1) X^k,$$

hence

$$1 + \sum_{k=1}^{n} H(n,k)X^k = \prod_{i=1}^{n} \left(\frac{X}{i} + 1\right) = \frac{1}{n!}\prod_{i=1}^{n}(X+i) = \sum_{k=0}^{n} \frac{s(n+1,k+1)}{n!} X^k$$

and the claim follows. □

From here later, let us fix a prime number p and let $k = \langle e_0, \dots, e_t \rangle_p + 1 \geq 2$ and $n = \langle d_0, \dots, d_s \rangle_p$ be positive integers with $s \geq t+1$ and $d_i = e_i$ for $i = 0, \dots, t$. For any $a_0, \dots, a_v \in \{0, \dots, p-1\}$, define

$$B_p(a_0, \dots, a_v) := \langle a_0, \dots, a_v \rangle_p - \langle a_0, \dots, a_{v-1} \rangle_p,$$

where by convention $\langle a_0, \dots, a_{v-1} \rangle_p = 0$ if $v = 0$, and also

$$\mathcal{B}_p(a_0, \dots, a_v) := \big\{c_p(i) : i = 1, \dots, B_p(a_0, \dots, a_v)\big\}$$

where $c_p(1) < c_p(2) < \cdots$ denotes the sequence of all positive integers not divisible by p. Lastly, put

$$\mathcal{A}_p(n, v) := \big\{m \in \{1, \dots, n\} : \nu_p(m) = s - v\big\},$$

for each integer $v \geq 0$. The next lemma relates $\mathcal{A}_p(n,v)$ and $\mathcal{B}_p(d_0, \dots, d_v)$.

Lemma 3.1. *For each nonnegative integer $v \leq s$, we have*

$$\mathcal{A}_p(n, v) = \big\{jp^{s-v} : j \in \mathcal{B}_p(d_0, \dots, d_v)\big\}.$$

In particular, $\#\mathcal{A}_p(n,v) = B_p(d_0, \dots, d_v)$ and $\mathcal{A}_p(n,v)$ depends only on $p, s, d_0, \dots, d_v$.

Proof. For $m \in \{1, \dots, n\}$, we have $m \in \mathcal{A}_p(n,v)$ if and only if $p^{s-v} \mid n$ but $p^{s-v+1} \nmid n$. Therefore,

$$\begin{aligned}
\#\mathcal{A}_p(n,v) &= \left\lfloor \frac{n}{p^{s-v}} \right\rfloor - \left\lfloor \frac{n}{p^{s-v+1}} \right\rfloor = \left\lfloor \sum_{i=0}^{s} d_i p^{v-i} \right\rfloor - \left\lfloor \sum_{i=0}^{s} d_i p^{v-i-1} \right\rfloor \\
&= \sum_{i=0}^{v} d_i p^{v-i} - \sum_{i=0}^{v-1} d_i p^{v-i-1} = \langle d_0, \dots, d_v \rangle_p - \langle d_0, \dots, d_{v-1} \rangle_p \\
&= B_p(d_0, \dots, d_v),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_p(n,v) &= \big\{c_p(i)p^{s-v} : i = 1, \dots, \#\mathcal{A}_p(n,v)\big\} \\
&= \big\{c_p(i)p^{s-v} : i = 1, \dots, B_p(d_0, \dots, d_v)\big\} \\
&= \big\{jp^{s-v} : j \in \mathcal{B}_p(d_0, \dots, d_v)\big\},
\end{aligned}$$

as claimed. □

Before stating the next lemma, we need to introduce some additional notation. First, we define

$$\mathcal{C}_p(n,k) := \bigcup_{v=0}^{t} \mathcal{A}_p(n,v) \quad \text{and} \quad \Pi_p(k) := \prod_{j \in \mathcal{C}_p(n,k)} \frac{1}{\mathrm{free}_p(j)},$$

where $\operatorname{free}_p(m) := m/p^{\nu_p(m)}$ for any positive integer m. Note that, since $d_i = e_i$ for $i = 0, \ldots, t$, from Lemma 3.1 it follows easily that indeed $\Pi_p(k)$ depends only on p and k, and not on n. Then we put

$$U_p(k) := \sum_{v=0}^{t} B_p(e_0, \ldots, e_v)v + t + 1,$$

while, for $a_0, \ldots, a_{t+v+1} \in \{0, \ldots, p-1\}$, with $v \geq 0$ and $a_i = e_i$ for $i = 0, \ldots, t$, we set

$$H_p'(a_0, \ldots, a_{t+v}) := \sum_{\substack{0 \leq v_1, \ldots, v_k \leq t+v \\ v_1 + \cdots + v_k = U_p(k) + v}} \sum_{\substack{j_1/p^{v_1} < \cdots < j_k/p^{v_k} \\ j_1 \in \mathcal{B}_p(a_0, \ldots, a_{v_1}), \ldots, j_k \in \mathcal{B}_p(a_0, \ldots, a_{v_k})}} \frac{1}{j_1 \cdots j_k}$$

and

$$H_p(a_0, \ldots, a_{t+v+1}) := H_p'(a_0, \ldots, a_{t+v}) + \Pi_p(k) \sum_{j \in \mathcal{B}_p(a_0, \ldots, a_{t+v+1})} \frac{1}{j}.$$

Note that $\nu_p(H_p(a_0, \ldots, a_{t+v+1})) \geq 0$, this fact will be fundamental later.

The following lemma gives a kind of p-adic expansion for $H(n,k)$. We use $O(p^v)$ to denote a rational number with p-adic valuation greater than or equal to v.

Lemma 3.2. *We have*

$$H(n,k) = \sum_{v=0}^{s-t-1} H_p(d_0, \ldots, d_{t+v+1}) \cdot p^{v-ks+U_p(k)} + O\big(p^{s-t-ks+U_p(k)}\big).$$

Proof. Clearly, we can write

$$H(n,k) = \sum_{v=0}^{V_p(n,k)} J_p(n,k,v) \cdot p^{v-V_p(n,k)},$$

where

$$V_p(n,k) := \max\big\{\nu_p(i_1 \cdots i_k) : 1 \leq i_1 < \cdots < i_k \leq n\big\},$$

and

$$J_p(n,k,v) := \sum_{\substack{1 \leq i_1 < \cdots < i_k \leq n \\ \nu_p(i_1 \cdots i_k) = V_p(n,k) - v}} \frac{1}{\operatorname{free}_p(i_1 \cdots i_k)},$$

for each nonnegative integer $v \leq V_p(n,k)$.

We shall prove that $V_p(n,k) = ks - U_p(k)$. On the one hand, we have

$$\begin{aligned}\sum_{v=0}^{t} B_p(e_0, \ldots, e_v) &= \sum_{v=0}^{t} \big(\langle e_0, \ldots, e_v\rangle_p - \langle e_0, \ldots, e_{v-1}\rangle_p\big) \\ &= \langle e_0, \ldots, e_t\rangle_p = k - 1.\end{aligned} \tag{6}$$

On the other hand, by (6) and thanks to Lemma 3.1, we obtain

$$\#\mathcal{C}_p(n,k) = \sum_{v=0}^{t} \#\mathcal{A}_p(n,v) = \sum_{v=0}^{t} B_p(e_0, \ldots, e_v) = k - 1. \tag{7}$$

Hence, in order to maximize $\nu_p(i_1 \cdots i_k)$ for positive integers $i_1 < \cdots < i_k \le n$, we have to choose $i_1, \dots, i_k$ by picking all the $k-1$ elements of $\mathcal{C}_p(n,k)$ and exactly one element from $\mathcal{A}_p(n, t+1)$. Therefore, using again (6) and Lemma 3.1, we get

$$\begin{aligned} V_p(n,k) &= \sum_{v=0}^{t} \#\mathcal{A}_p(n,v)(s-v) + (s-t-1) \qquad (8) \\ &= \sum_{v=0}^{t} B_p(e_0,\dots,e_v)(s-v) + (s-t-1) \\ &= \left(\sum_{v=0}^{t} B_p(e_0,\dots,e_v) + 1\right) s - U_p(k) \\ &= ks - U_p(k), \end{aligned}$$

as desired.

Similarly, if $\nu_p(i_1 \cdots i_k) = V_p(n,k) - v$, for some positive integers $i_1 < \cdots < i_k \le n$ and some nonnegative integer $v \le s-t-1$, then only two cases are possible: $\nu_p(i_1), \dots, \nu_p(i_k) \ge s-t-v$; or $i_1, \dots, i_k$ consist of all the $k-1$ elements of $\mathcal{C}_p(n,k)$ and one element of $\mathcal{A}_p(n, t+v+1)$. As a consequence,

$$J_p(n,k,v) = \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ \nu_p(i_1 \cdots i_k) = V_p(n,k) - v \\ \nu_p(i_1), \dots, \nu_p(i_k) \ge s-t-v}} \frac{1}{\mathrm{free}_p(i_1 \cdots i_k)} + \Pi_p(k) \sum_{i \in \mathcal{A}_p(n,t+v+1)} \frac{1}{\mathrm{free}_p(i)}, \qquad (9)$$

for all nonnegative integers $v \le s-t-1$.

By putting $v_\ell := s - \nu_p(i_\ell)$ and $j_\ell := \mathrm{free}_p(i_\ell)$ for $\ell = 1, \dots, k$, the first sum of (9) can be rewritten as

$$\begin{aligned} &\sum_{\substack{0 \le v_1, \dots, v_k \le t+v \\ (s-v_1) + \cdots + (s-v_k) = V_p(n,k) - v}} \; \sum_{\substack{i_1 < \cdots < i_k \\ i_1 \in \mathcal{A}_p(n,v_1), \dots, i_k \in \mathcal{A}_p(n,v_k)}} \frac{1}{\mathrm{free}_p(i_1 \cdots i_k)} \\ &= \sum_{\substack{0 \le v_1, \dots, v_k \le t+v \\ v_1 + \cdots + v_k = U_p(k) + v}} \; \sum_{\substack{j_1/p^{v_1} < \cdots < j_k/p^{v_k} \\ j_1 \in \mathcal{B}_p(d_0,\dots,d_{v_1}), \dots, j_k \in \mathcal{B}_p(d_0,\dots,d_{v_k})}} \frac{1}{j_1 \cdots j_k} = H_p'(d_0, \dots, d_{t+v}), \end{aligned}$$

where we have also made use of (8) and Lemma 3.1, hence

$$J_p(n,k,v) = H_p(d_0, \dots, d_{t+v+1}), \qquad (10)$$

for any nonnegative integer $v \le s-t-1$.

At this point, being $s > t$, by (8) it follows that $V_p(n,k) > s-t-1$, hence

$$H(n,k) = \sum_{v=0}^{s-t-1} J_p(n,k,v) \cdot p^{v-ks+U_p(k)} + O\big(p^{s-t-ks+U_p(k)}\big), \qquad (11)$$

since clearly $\nu_p(J_p(n,k,v)) \ge 0$ for any nonnegative integer $v \le V_p(n,k)$.

In conclusion, the claim follows from (10) and (11). □

Finally, we need two lemmas about the number of solutions of some congruences. For rational numbers a and b, we write $a \equiv b \bmod p$ to mean that $\nu_p(a-b) > 0$.

Lemma 3.3. *Let r be a rational number and let x, y be positive integers with $y < p$. Then the number of integers $v \in [x, x+y]$ such that $H_v \equiv r \bmod p$ is less than $\frac{3}{2}y^{2/3}+1$.*

Proof. The case $r = 0$ is proved in [21, Lemma 2.2] and the proof works exactly in the same way even for $r \neq 0$. ☐

Lemma 3.4. *Let q be a rational number and let a be a positive integer. Then the number of $d \in \{0, \ldots, p-1\}$ such that*

$$\sum_{i=a}^{a+d} \frac{1}{c_p(i)} \equiv q \bmod p \tag{12}$$

is less than $p^{0.835}$.

Proof. It is easy to see that there exists some $h \in \{0, \ldots, p-2\}$ such that

$$c_p(i) = \begin{cases} c_p(a) + i - a & \text{for } i = a, \ldots, a+h, \\ c_p(a) + i - a + 1 & \text{for } i = a+h+1, \ldots, a+p-1. \end{cases}$$

Therefore, by putting $x := c_p(a)$, $y := h$, and $r := q + H_{x-1}$ in Lemma 3.3, we get that the number of $d \leq h$ satisfying (12) is less than $\frac{3}{2}h^{2/3}+1$. Similarly, by putting $x := c_p(a) + h + 2$, $y := p - h - 2$, and

$$r := q + H_{x-1} - \sum_{i=a}^{a+h} \frac{1}{c_p(i)}$$

in Lemma 3.3, we get that the number of $d \in [h+1, p-1]$ satisfying (12) is less than $\frac{3}{2}(p-h-2)^{2/3}+1$. Thus, letting N be the number of $d \in \{0, \ldots, p-1\}$ that satisfy (12), we have

$$N \leq \frac{3}{2}h^{2/3} + 1 + \frac{3}{2}(p-h-2)^{2/3} + 1 \leq 3\left(\frac{p-2}{2}\right)^{2/3} + 2.$$

Furthermore, it is clear the d and $d+1$ cannot both satisfy (12), hence $N \leq \lceil p/2 \rceil$. Finally, a little computation shows that the maximum of

$$\log_p\left(\min\left(3\left(\frac{p-2}{2}\right)^{2/3} + 2, \left\lceil\frac{p}{2}\right\rceil\right)\right)$$

is obtained for $p = 59$ and is less than 0.835, hence the claim follows. ☐

4. PROOF OF THEOREM 2.1

Now we are ready to prove Theorem 2.1. For any $a_0, \ldots, a_{t+u+1} \in \{0, \ldots, p-1\}$, with $u \geq 0$ and $a_i = e_i$ for $i = 0, \ldots, t$, let

$$\Sigma_p(a_0, \ldots, a_{t+u+1}) := \sum_{v=0}^{u} H_p(a_0, \ldots, a_{t+v+1}) \cdot p^v.$$

Furthermore, define the sequence of sets $\mathcal{T}_p^{(0)}(k), \mathcal{T}_p^{(1)}(k), \ldots$ as follows: $\mathcal{T}_p^{(0)}(k) := \{\langle e_0, \ldots, e_t \rangle_p\}$, and for any integer $u \geq 0$ put $\langle a_0, \ldots, a_{t+u+1} \rangle_p \in \mathcal{T}_p^{(u+1)}(k)$ if and only if $\langle a_0, \dots, a_{t+u} \rangle_p \in \mathcal{T}_p^{(u)}(k)$ and $\nu_p(\Sigma_p(a_0, \dots, a_{t+u+1})) \geq u + 1$. At this point, setting

$$\mathcal{T}_p(k) := \bigcup_{u=0}^{\infty} \mathcal{T}_p^{(u)}(k),$$

it is straightforward to see that $\mathcal{T}_p(k)$ is a p-tree of root $\langle e_0, \dots, e_t \rangle_p$. Put $W_p(k) := U_p(k) - t - 1$.

If $n \notin \mathcal{T}_p(k)$ then, for the sake of convenience, set $r := \mu_p(\mathcal{T}_p(k), n)$. Thus $r > t$, $\langle d_0, \dots, d_{r-1} \rangle_p \in \mathcal{T}_p(k)$ but $\langle d_0, \dots, d_r \rangle \notin \mathcal{T}_p(k)$, so that

$$\nu_p(\Sigma_p(d_0, \dots, d_r)) \leq r - t - 1. \tag{13}$$

Now we distinguish between two cases. If $r = t + 1$, then $\nu_p(\Sigma_p(d_0, \dots, d_{t+1})) = 0$ and by Lemma 3.2 we obtain $\nu_p(H(n, k)) = W_p(k) + r - ks$. If $r > t + 1$ then by $\langle d_0, \dots, d_{r-1} \rangle \in \mathcal{T}_p(k)$ we get that $\nu_p(\Sigma_p(d_0, \dots, d_{r-1})) \geq r - t - 1$, which together with (13) and

$$\Sigma_p(d_0, \dots, d_r) = \Sigma_p(d_0, \dots, d_{r-1}) + H_p(d_0, \dots, d_r) \cdot p^{r-t-1}$$

implies that $\nu_p(\Sigma_p(d_0, \dots, d_r)) = r - t - 1$, hence by Lemma 3.2 we get $\nu_p(H(n, k)) = W_p(k) + r - ks$, and (i) is proved.

If $n \in \mathcal{T}_p(k)$ then, by the definition of $\mathcal{T}_p(k)$, we have $\nu_p(\Sigma_p(d_0, \dots, d_s)) \geq s - t$. Therefore, by Lemma 3.2 it follows that $\nu_p(H(n, k)) > W_p(k) - (k-1)s$, and this proves (ii).

It remains only to bound the girth of $\mathcal{T}_p(k)$. Let u be a nonnegative integer and pick $\langle a_0, \dots, a_{t+u} \rangle_p \in \mathcal{T}_p^{(u)}(k)$. By the definition of $\mathcal{T}_p^{(u+1)}(k)$, we have $\langle a_0, \dots, a_{t+u+1} \rangle_p \in \mathcal{T}_p^{(u+1)}(k)$ if and only if $\nu_p(\Sigma_p(a_0, \dots, a_{t+u+1})) \geq u + 1$, which in turn is equivalent to

$$\sum_{v=0}^{u-1} H_p(a_0, \dots, a_{t+v+1}) \cdot p^{v-u} + H'_p(a_0, \dots, a_{t+u}) + \Pi_p(k) \sum_{j \in \mathcal{B}_p(a_0, \dots, a_{t+u+1})} \frac{1}{j} \tag{14}$$

$$\equiv \sum_{v=0}^{u} H_p(a_0, \dots, a_{t+v+1}) \cdot p^{v-u} \equiv 0 \bmod p.$$

Using the definition of $\mathcal{B}_p(a_0, \dots, a_{t+u+1})$ and the facts that

$$B_p(a_0, \dots, a_{t+u+1}) = a_{t+u+1} + (p-1) \sum_{v=0}^{u} a_v p^{u-v},$$

and $\nu_p(\Pi_p(k)) = 0$, we get that (14) is equivalent to

$$\sum_{i=a}^{a+a_{t+u+1}} \frac{1}{c_p(i)} \equiv -\sum_{i=1}^{a-1} \frac{1}{c_p(i)}$$

$$- \frac{1}{\Pi_p(k)} \left(\sum_{v=0}^{u-1} H_p(a_0, \dots, a_{t+v+1}) \cdot p^{v-u} + H'_p(a_0, \dots, a_{t+u}) \right) \bmod p, \tag{15}$$

where

$$a := (p-1) \sum_{v=0}^{u} a_v p^{u-v}.$$

Note that both a and the right-hand side of (15) do not depend on a_{t+u+1}. As a consequence, by Lemma 3.4 we get that $\langle a_0, \dots, a_{t+u+1} \rangle_p \in \mathcal{T}_p^{(u+1)}(k)$ for less than $p^{0.835}$ values of $a_{t+u+1} \in \{0, \dots, p-1\}$. Thus the girth of $\mathcal{T}_p(k)$ is less than $p^{0.835}$.

Finally, consider the case $p = 2$. Obviously, $1/c_2(i) \equiv 1 \bmod 2$ for any positive integer i, while the right-hand side of (15) is equal to 0 or 1 (mod 2). Therefore, there exists one and only one choice of $a_{t+u+1} \in \{0, 1\}$ such that (15) is satisfied. This means that $\mathcal{T}_2(k)$ is infinite and its girth is equal to 1.

The proof is complete.

5. The computation of $\mathcal{T}_p(k)$

Given p and k, it might be interesting to effectively compute the elements of $\mathcal{T}_p(k)$. Clearly, $\mathcal{T}_p(k)$ could be infinite — by Theorem 2.1 this is indeed the case when $p = 2$ — hence the computation should proceed by first enumerating all the elements of $\mathcal{T}_p^{(0)}(k)$, then all the elements of $\mathcal{T}_p^{(1)}(k)$, and so on. An obvious way to do this is using the recursive definition of the $\mathcal{T}_p^{(u)}(k)$'s. However, it is easy to see how this method is quite complicated and impractical. A better idea is noting that from Theorem 2.1 we have

$$\begin{aligned}\mathcal{T}_p^{(u+1)}(k) = \big\{ \langle a_0, \dots, a_{t+u}, b \rangle_p : \langle a_0, \dots, a_{t+u} \rangle_p \in \mathcal{T}_p^{(u)}(k), \\ \nu_p(H(\langle a_0, \dots, a_{t+u}, b \rangle_p, k)) > W_p(k) - (k-1)s \big\},\end{aligned} \tag{16}$$

for all integers $u \geq 0$. Therefore, starting from $\mathcal{T}_p^{(0)}(k) = \{\langle e_0, \dots, e_t \rangle_p\}$, formula (16) gives a way to compute recursively all the elements of $\mathcal{T}_p(k)$. In particular, if $\mathcal{T}_p(k)$ is finite, then after sufficient computation one will get $\mathcal{T}_p^{(u)}(k) = \varnothing$ for some positive integer u, so the method actually proves that $\mathcal{T}_p(k)$ is finite.

The authors implemented this algorithm in SageMath, since it allows computations with arbitrary-precision p-adic numbers. In particular, they found that $\mathcal{T}_3(2), \dots, \mathcal{T}_3(6)$ are all finite sets, with respectively $8, 24, 16, 7, 23$ elements, while the cardinality of $\mathcal{T}_3(7)$ is at least 43. Through these numerical experiments, it seems that, in general, $\mathcal{T}_p(k)$ does not exhibit any trivial structure (see Figures 1, 2, 3), hence the question of the finiteness of $\mathcal{T}_p(k)$ is probably a difficult one.

6. Proof of Corollary 2.2

Only for this section, let us focus on the case $p = 2$ and $k = 2$, so that $t = 0$, $e_0 = 1$, and $W_2(2) = 0$. Thanks to Theorem 2.1 we know that $\mathcal{T}_2(2)$ is infinite and its girth is equal to 1. Hence, it follows easily that there exists a sequence $f_0, f_1, \dots \in \{0, 1\}$ such that $\mathcal{T}_2^{(u)}(2) = \{\langle f_0, \dots, f_u \rangle_2\}$ for all integers $u \geq 0$. In particular, $f_0 = e_0 = 1$. At this point, (i) and (ii) are direct consequences of Theorem 2.1, while the recursive formula (5) is just a special case of (16).

7. Proof of Theorem 2.3

On the one hand, if $n = \langle d_0, \dots, d_s \rangle_p \in (\mathcal{S}_p(k-1) \setminus \mathcal{T}_p(k)) \cap [(k-1)p, x]$ then by Theorem 2.1 we get that

$$\begin{aligned}\nu_p(H(n,k)) &= W_p(k) + \mu_p(\mathcal{T}_p(k), n) - ks \\ &\leq W_p(k) - (k-1)s \\ &= \sum_{v=0}^{t} B_p(e_0, \dots, e_v) v - (k-1)s \\ &\leq \sum_{v=0}^{t} B_p(e_0, \dots, e_v) t - (k-1)s < -(k-1)(\log_p n - \log_p(k-1) - 1),\end{aligned}$$

where we have made use of (6) and the inequalities $\mu_p(\mathcal{T}_p(k), n) \leq s$, $s > \log_p n - 1$, and $t \leq \log_p(k-1)$.

On the other hand, by Theorem 2.1, the girth of $\mathcal{T}_p(k)$ is less than $p^{0.835}$, hence it follows easily that $\#\mathcal{T}_p^{(u)}(k) < p^{0.835u}$, for any positive integer u. As a consequence,

$$\#(\mathcal{T}_p(k) \cap [(k-1)p, x]) \leq \sum_{u=1}^{\lfloor \log_p x \rfloor - t} \#\mathcal{T}_p^{(u)}(k) < \sum_{u=1}^{\lfloor \log_p x \rfloor} p^{0.835u} < 3x^{0.835},$$

and the claim follows.

8. Figures

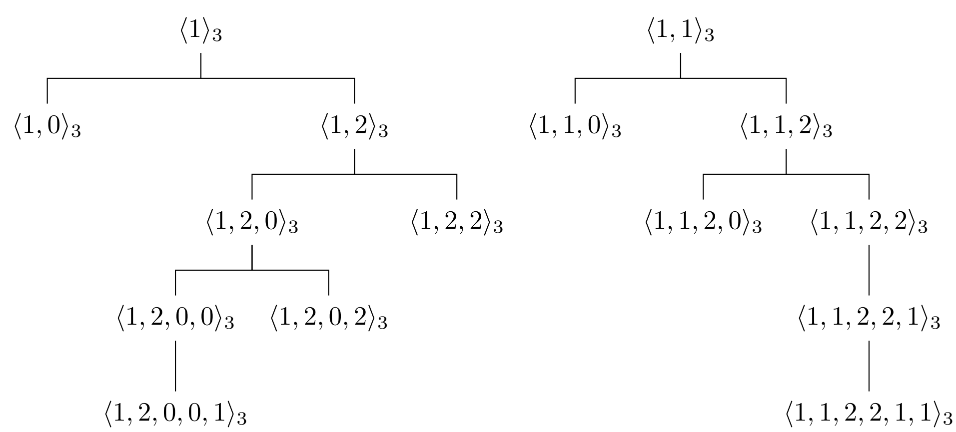

Figure 1. The 8 elements of $\mathcal{T}_3(2)$ (left tree), and the 7 elements of $\mathcal{T}_3(5)$ (right tree).

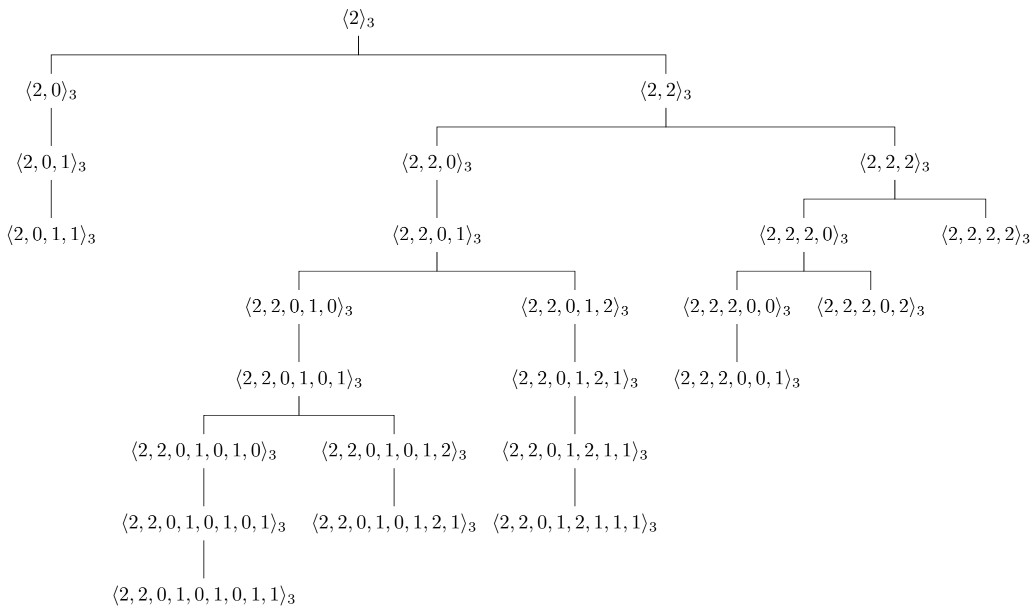

FIGURE 2. The 24 elements of $\mathcal{T}_3(3)$.

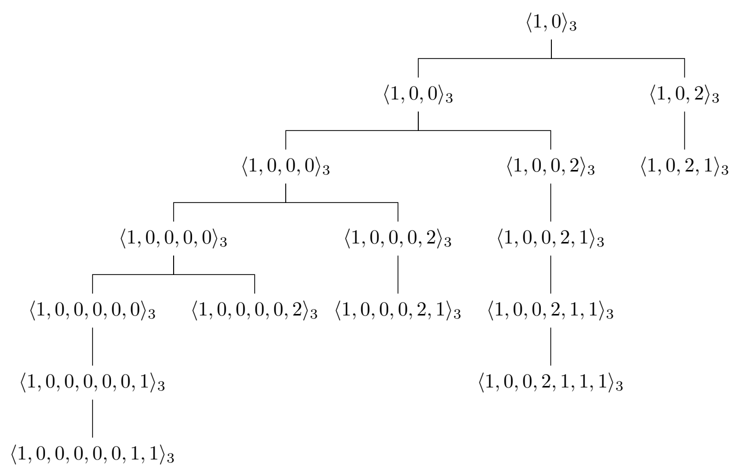

FIGURE 3. The 16 elements of $\mathcal{T}_3(4)$.

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