Fusion of partial orderings for decision problems in Quality Management

ICQEM 18

"A better world with Quality!"

3rd International Conference on Quality Engineering and Management

July 11-13, 2018

Universitat Internacional de Catalunya

Barcelona, Spain

Proceedings Book

Technical record

Title

Proceedings book of the 3rd International Conference on Quality Engineering and Management, 2018

Authors/Editors

Berbegal-Mirabent, Jasmina; Marimon, Frederic; Casadesús, Martí; Sampaio, Paulo

Publisher

International Conference on Quality Engineering and Management

Date

July 2018

Cover Design

Luís Coutinho

ISBN

978-989-20-8521-0

Papers index

Circular economy: Challenges for the supply chain Dagnino, B.V.	1
Assessing the financial effects of adopting multiple certifications Hernandez-Vivanco, A., Domingues, P., Sampaio, P., Bernardo, M. and Cruz-Cázares, C.	20
Amine foaming modelling and reduction using binary logistic regression Abdulla, A.F. and Awad, M.I.	40
Delight in excellent tourism experiences: The role of the exceptional value and the appraisal Escobar Rivera, D., Casadesús Fa, M. and Simon Villar, A.	56
Human error prevention in manufacturing Deshpande, M.	75
QM and innovation in the Greek agri-food sector Kafetzopoulos, D., Sakali, P.C. and Skalkos, D.	93
A maturity model to assess supply chain quality management integration Cubo, C., Oliveira, R., Estrada, R., Fernandes, A.C., Carvalho, M.S., Afonso, P., Roque, J. and Sampaio, P.	115
Fusion of partial orderings for decision problems in quality management Franceschini, F. and Maisano, D.A.	126
Coaching and organizational improvement: A balanced scorecard approach Fernández, C.L., Bernardo, M. and Karsperkaya, Y.	148
Capability index analysis in short-terms production at the machine supplier and initial manufacturing facility Oprime, P.C., Lizarelli, F.L., Pimenta, M., Alliprandini, D.H. and Chakraborti, S.	171
Control chart and capability indices: Systematic literature review Lizarelli, F.L., Oprime, P.C., Pimenta, M., Alliprandini, D.H. and Chakraborti, S.	184
Collaborative networking of researchers in quality: A Portuguese case Pires, A.R., Saraiva, M. and Rolo, A.	201
Barriers to the development of continuous improvement maturity in the manufacturing company Kucińska-Landwójtowicz, A.	219
A QFD-based approach to identify product-service systems functional requirements Barravecchia, F., Mastrogiacomo, L. and Franceschini, F.	235
Robustness analysis of inspection design parameters for assembly of short-run manufacturing processes Galetto, M., Verna, E., Genta, G. and Franceschini, F.	255
University absenteeism: Students' and lecturers' point of view Bernardo, M., Aparicio-Chueca, P., Elasri-Ejjaberi, A., Triado-Ivern, X.M., Guàrdia-Olmos, J., Presas, P., Maestro-Yarza, I., Turull-Rubinat, M. and Roca-Acedo, B.	275
Practices adopted to the execution and quality control of foundations in building construction Costa de Jesus, R. and dos Santos, J.	290
Quality and satisfaction – The case of nautical tourist services in the region of Alto Douro Vinhateiro – Portugal Saraiva, M., Soares, M., Jorge, F. and Pires, A.R.	309
The influence of a lean six sigma green belt course on European students. A practical case Dirkwinkel, T., Panieri, C., Preś, I. and Rosi, L.	329
Economic production lot-sizing model in an imperfect production system with maintenance Park, M.	345

Multi-objective mathematical model for a sustainable supply chain considering customer satisfaction and SR concepts: NSGAII approach Zavvar Sabegh, M.H., Moslemi, SH., Zrymiak, D. and Leonard, D.	360
Hassan's approach: Novel 5-step model transforming 'performance' to 'high performance' Hassan, H.	384
Theoretical comparison of universitary quality assurance maturity models Sanchez-Puchol, F., Pastor-Collado, J.A. and Casanovas, J.	401
Quality and innovation in the hospitality industry: A literature review Gkana V. and Gotzamani K.	420
Potentials of blockchain technology for the quality management of interconnected processes Kuhn, M., Paul, M., Schaefer, F. and Otten, H.	438
Quality management potentials – A future trend based analysis and strategy Schäfer, F., Kuhn, M., Nguyen, H.G. and Otten, H.	458
World state of quality 2017: Results of a worldwide approach to measure macroquality Cubo, C., Saraiva, P., Sampaio, P. and Reis, M.	473
Service quality for retail banks in the digital age Nguyen, H.G., Schäfer, F. and Otten, H.	484
The EFQM excellence model in a research centre Mendes, A.J.C. and Sampaio, P.	502
Measurement systems improvement in a tire industry process Maia, H. and Nóvoa, H.	518
Identification of inefficiencies in a complaints handling process Santos, M., Alvelos, H.A., Xambre, A.R. and Teixeira, L.	537
Contributions towards the improvement of the IMS-MM©: A Delphi method study Saraiva, C., Domingues, J.P.D., Sampaio, P. and Arezes, P.M.	550
Strategic quality management for airports: The case of the Libyan airports Abugeddida, R., Donnellan, P. and Fallon, E.	569
Public works in Brazil: Contracting, execution and inspection Menaget de Andrade, V.H. and Santos, J.	593
Digital transformation of quality management Cots, S.	605
The market concentration on education and the effects on quality Spolavori, R. and Curth, M.	614
Development of a methodical approach for requirements management in cross-company networks (ReMAiN) Bielefeld, O., Schlueter, N. and Winzer, P.	632

Fusion of partial orderings for decision problems in Quality Management

Franceschini, F. and Maisano, D.A.

Dept. of Management and Production Engineering, Politecnico di Torino, Turin, Italy

ABSTRACT

Purpose – In a rather common problem for the Quality Management field, (i) a set of *judges* express their individual (*subjective*) judgments about a specific *attribute*, which is related to some *objects* of interest, and (ii) these judgments have to be fused into a *collective* one. This paper develops a new technique where individual judgments – which are expressed in the form of *partial* preference orderings, including the more/less preferred objects only – are fused into a collective judgment, which is expressed in the form of a *ratio* scaling of the objects of interest. An application example concerning the design of a civilian aircraft seat is presented.

Design/methodology/approach – The proposed technique borrows the architecture and the underlying postulates from the Thurstone's *Law of Comparative Judgment* (LCJ), adopting a more user-friendly response mode, which is based on (partial) preference orderings instead of paired-comparison relationships. By aggregating and processing these orderings, an overdefined system of equations can be constructed and solved through the *Generalized Least Squares* method. Apart from a ratio scaling of the objects of interest, this approach makes it possible to estimate the relevant uncertainty, by propagating the uncertainty of input data.

Findings – Preliminary results show the effectiveness of the proposed technique, even when preference orderings are rather "incomplete", i.e., they include a relatively limited number of objects, with respect to those available.

Research limitations/implications – Thanks to the relatively simple and practical response mode, the proposed technique is applicable to a variety of practical contexts, such as telephone and street interviews. Although preliminary results are promising, the technique will be tested in a more organic way, considering several factors (e.g., number of judges, number of objects, degree of completeness of preference orderings, degree of agreement of judges, etc.).

Originality/value – Even though the scientific literature includes many techniques that are inspired by the LCJ, the proposed one is characterized by two important novelties: (i) it is based on a more user-friendly response mode and (ii) it allows to obtain a ratio scaling of objects with a relevant uncertainty estimation.

Paper type: Research paper

Keywords: Group decision making, Law of comparative judgment, Partial preference ordering, Generalized least squares

INTRODUCTION

A problem that is rather common to several scientific field is articulated as follows (Keeney and Raiffa, 1993; Franceschini et al., 2007; Coaley, 2014):

- a set of *objects* (o_1, o_2, …) should be compared on the basis of the degree of a specific *attribute*;
- a set of *judges* (j_1, j_2, …) individually express their *subjective* judgments on these objects;
- these judgments should be fused into a single *collective judgment*, which is usually expressed in the form of a *scaling*, i.e., assignment of numbers to the objects, according to a conventional rule/method (De Vellis, 2016).

With reference to the Quality Management field, possible examples of this problem are: (i) fusing judgments related to the customer satisfaction of set of competing products, or (ii) fusing judgments by reliability/maintenance engineers on the severity of potential process failures, etc..

Figure 1 shows a pedagogical representation of the problem of interest, in which four final consumers (i.e., judges) have to express their judgments on the taste (i.e., attribute) of three types of candies (i.e., objects). It general, judges may refrain from judging part of the objects, when lacking adequate knowledge of them (e.g., see judgments with "???").

The scientific literature encompasses a plurality of fusion techniques, which differ from each other for at least three features: (i) the *response mode* for collecting subjective judgments; (ii) the underlying *rationale* of the fusion technique, and (iii) the *form* of the resulting collective judgment. For an exhaustive discussion of the existing techniques in various fields (e.g., Quality Management, Multi-Criteria Decision Making, etc.), we refer the reader to the vast literature and reviews (Coaley, 2014; De Vellis, 2016).

Regardless of the peculiarities of the individual fusion techniques, a key element for their success is the simplicity of response mode (Franceschini et al., 2007; Harzing et al., 2009). For example, various studies show that comparative judgments of objects (e.g., "o_i is more/less preferred than o_j") are simpler and more reliable than judgments in absolute terms (e.g., "the degree of the attribute of o_i is low/intermediate/high") (Harzing et al., 2009; Edwards, 1957).

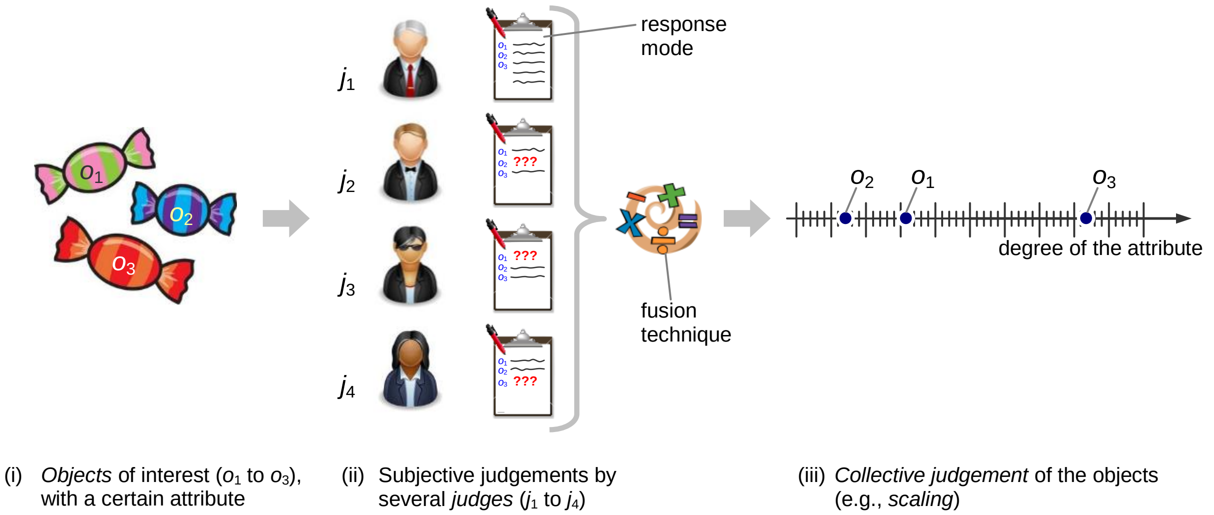


Figure 1 – Pedagogical representation of the problem of interest.

As to the typology of collective judgments, we note that they are often treated as if they were defined on a *ratio* scale – i.e., a scale with non-arbitrary zero and meaningful distance – even when they actually are not; e.g., rankings or ordinal-scale values of the objects are often improperly "promoted" to ratio-scale values, in the moment in which they are combined with other indicators through weighted sums, geometric averages, or – more in general – statistics that are permissible to ratio-scale values only (Roberts, 1979).

In a recent paper, the authors have developed a technique, denominated "*ZM*-technique", that combines the Thurstone's *Law of Comparative Judgment* (LCJ) (Thurstone, 1927; Edwards, 1957) with a response mode based on preference orderings (Franceschini and Maisano, 2018). The resulting collective judgment is expressed in the form of a *ratio* scaling, which can be constructed without any conceptually prohibited "promotion". An important requirement of the *ZM*-technique is that, apart from "regular" objects (i.e., o_1, o_2, …, o_n), preference orderings also include two "dummy" or "anchor" objects: i.e., o_Z, which corresponds to the *absence* of the attribute of interest, and o_M, which corresponds to the *maximum-imaginable* degree of the attribute (Franceschini and Maisano, 2018).

The *ZM*-technique requires judges to formulate *linear* preference orderings, i.e., orderings including all (regular and dummy) objects, according to a hierarchical sequence with relationships of *strict preference* (“>”) and/or *indifference* (“~”) (Nederpelt and Kamareddine, 2004). This is certainly a limitation, as it makes the response mode unsuitable for some practical contexts where ranking a number of objects can be problematic. It has also been observed that, when formulating preference orderings, judges tend to focus on the more/less preferred objects, providing more reliable judgments about them, to the detriment of the remaining objects (Lagerspetz, 2016; Harzing et al., 2009). Another limitation of the *ZM*-technique – and the traditional LCJ too (Montag, 2006) – is the impossibility to estimate the uncertainty related to the resulting scaling of objects.

The above considerations raise the following research question: “How could the *ZM*-technique be modified so as to (1) make the response more user-friendly and reliable and (2) determine a (statistically sound) estimate of the uncertainty related to the solution?”.

The aim of this paper is to address the previous research question, proposing a new technique that overcomes the limitations of the *ZM*-technique while preserving the basic principles. The new technique replaces linear preference orderings with “incomplete” orderings, which are focussed exclusively on the more/less preferred objects. Borrowing the language from Mathematics’ Order Theory, these other orderings can be classified as *partial*, i.e., apart from strict preference and indifference relationships, they may also contain *incomparability* relationships among (some of) the objects (Nederpelt and Kamareddine, 2004).

The rest of the paper is organized into five sections. Section “Background information” briefly recalls the LCJ and *ZM*-technique. Section “Methodology” illustrates the new technique, which includes the construction of an overdetermined system of equations and its solution through the *Generalized Least Squares* (GLS) method (Kariya and Kurata, 2004). This technique also allows to estimate the uncertainty related to the solution, “propagating” the uncertainty of input data. Section “Application example” applies the new technique to a real-life example, concerning the design of a civilian aircraft seat. Section “Conclusions” summarizes the original contributions of this paper and its practical implications, limitations and suggestions for future research. Further information on the GLS method is contained in the appendix.

BACKGROUND INFORMATION

This section "prepares the field" to better understand the proposed technique and is organized in two subsections, which respectively recall the LCJ and the *ZM*-technique.

Thurstone's LCJ

Thurstone (1927) postulated the existence of a *psychological continuum*, i.e., an abstract and unknown unidimensional scale, in which objects are positioned depending on the degree of a certain *attribute* – i.e., a specific feature of the objects, which evokes a subjective response in each judge. The position of a generic *i*-th object (o_i) is postulated to be distributed normally, in order to reflect the intrinsic judge-to-judge variability: $o_i \sim N(x_i, \sigma_i^2)$, where x_i and σ_i^2 are the unknown mean value and variance related to the degree of the attribute of that object. Considering two generic objects, o_i and o_j, it can be asserted that:

$$o_i - o_j \sim N(x_i - x_j, \sigma_i^2 + \sigma_j^2 - 2 \cdot \rho_{ij} \cdot \sigma_i \cdot \sigma_j), \tag{1}$$

where ρ_{ij} is the Pearson coefficient, denoting the correlation between the positioning of objects o_i and o_j. The probability that the position of o_i in the psychological continuum is higher than that of o_j can be expressed as:

$$p_{ij} = P(o_i - o_j > 0) = 1 - \Phi\left[\frac{0-(x_i - x_j)}{\sqrt{\sigma_i^2 + \sigma_j^2 - 2 \cdot \rho_{ij} \cdot \sigma_i \cdot \sigma_j}}\right], \tag{2}$$

Φ being the cumulative distribution function of the standard normal distribution $z \sim N(0, 1)$.

The LCJ (*case V*) includes the following additional simplifying assumptions (Thurstone, 1927; Edwards, 1957): $\sigma_i^2 = \sigma^2 \ \forall i$, $\rho_{ij} = \rho, \forall i, j$, and $2 \cdot \sigma^2 \cdot (1-\rho) = 1$. Eq. 2 can therefore be expressed as:

$$p_{ij} = P(o_i - o_j > 0) = 1 - \Phi[-(x_i - x_j)]. \tag{3}$$

Although p_{ij} is unknown, it can be estimated using the information contained in a set of (subjective) judgments by a number (m) of judges (Thurstone, 1927). Precisely, each judge expresses his/her judgment for each paired comparison (i.e., $\forall i, j$), through relationships of *strict preference* (e.g., "$o_i > o_j$" or "$o_i < o_j$") or *indifference* (e.g., "$o_1 \sim o_2$"). Then, for each judge who prefers o_i to o_j, a frequency indicator f_{ij} is incremented by one unit. In the case the two objects are considered indifferent, f_{ij} is conventionally incremented by 0.5, so that:

$$f_{ij} = m_{ij} - f_{ji}, \tag{4}$$

m_{ij} being the total number of judges who express their judgment for the *i*,*j*-th paired comparison. In general, $m_{ij} \leq m$ since judges may sometimes refrain from expressing their judgments on some of the possible paired comparisons. We remark that the condition in Eq. 4 is a *sine qua non* for the application of the LCJ (Thurstone, 1927).

The observed proportion of judges that prefer o_i to o_j can be used to estimate the unknown probability p_{ij}:

$$\hat{p}_{ij} = \frac{f_{ij}}{m_{ij}}. \tag{5}$$

Of course, the relationship of complementarity $\hat{p}_{ij} = 1 - \hat{p}_{ji}$ holds.

Returning to Eq. 3, it can be expressed as:

$$\hat{p}_{ij} = 1 - \Phi[-(x_i - x_j)], \tag{6}$$

from which:

$$x_i - x_j = -\Phi^{-1}(1 - \hat{p}_{ij}). \tag{7}$$

In general, objects are judged differently by judges; however, if all judges express the same judgment, the model is no more viable ($\hat{p}_{ij}$ values of 1.00 and 0.00 would correspond to $-\Phi^{-1}(1-\hat{p}_{ij})$ values of $\pm\infty$). A simplified approach for tackling this problem is associating values of $\hat{p}_{ij} \geq 0.977$ with $-\Phi^{-1}(1 - 0.977) = 1.995$ and values of $\hat{p}_{ij} \leq 0.023$ with $-\Phi^{-1}(1 - 0.023) = -1.995$. More sophisticated solutions to deal with this issue have been proposed (Edwards, 1957).

Extending the reasoning to all possible paired comparisons for which $m_{ij} \geq 1$ (i.e., at least one judge expresses his/her own judgment), the relevant $\hat{p}_{ij}$ values can be determined and the following system of equations can be constructed:

$$\begin{cases} \vdots \\ x_i - x_j + \Phi^{-1}\left(1 - \hat{p}_{ij}\right) = 0 \qquad \forall i, j : m_{ij} \geq 1. \\ \vdots \end{cases} \tag{8}$$

Since, the rank of the system is lower than the number (n) of unknowns of the problem (i.e., x_1, x_2, …, x_n) – and the system itself would be indeterminate (Thurstone, 1927) – the following conventional condition is introduced:

$$\sum_{i=1}^{n} x_i = 0 \, . \tag{9}$$

Eqs. 8 and 9 are then aggregated into a new system, which is *over-determined* (i.e., it has rank *n* while the total number of equations (*q*) is higher than *n*) and *linear* with respect to the unknowns:

$$\begin{cases} \begin{bmatrix} \vdots & \\ x_i - x_j + \Phi^{-1}\left(1 - \hat{p}_{ij}\right) = 0 & \forall i, j : m_{ij} \geq 1 \\ \vdots & \end{bmatrix} \\ \sum_{i=1}^{n} x_i = 0 \end{cases} . \tag{10}$$

This system can be expressed in matrix form as:

$$\begin{cases} \vdots \\ \sum_{k=1}^{n} \left(a_{hk} \cdot x_k\right) - b_h = 0 \quad \forall h \in [0, q] \\ \vdots \end{cases} \Rightarrow \quad \boldsymbol{A} \cdot \boldsymbol{X} - \boldsymbol{B} = \boldsymbol{0} \, , \tag{11}$$

$\boldsymbol{X} = [x_1, x_2, \ldots, x_n]^T \in R^{n \times 1}$ being the column vector containing the unknowns of the problem, a_{hk} being a generic element of matrix $\boldsymbol{A} \in R^{q \times n}$, and b_h being a generic element of vector $\boldsymbol{B} \in R^{n \times 1}$. For details on the construction of $\boldsymbol{A}$ and $\boldsymbol{B}$, see (Gulliksen, 1956).

In the case each judge expresses his/her judgment on the totality of the $C_2^n = n \cdot (n-2)/2$ paired comparisons, the system in Eq. 10 is "complete" – i.e., with $q = C_2^n + 1$ equations – and can be solved in a closed form as (Thurstone, 1927):

$$\hat{x}_j = \sum_{i=1}^{m} \Phi^{-1}\left(1 - \hat{p}_{ij}\right) \qquad \forall j \, . \tag{12}$$

The LCJ unfortunately has some limitations, including the following ones:

1. The response mode is relatively tedious for judges;
2. The LCJ results into an *interval* scaling, i.e., objects are defined on a scale with meaningful distance but arbitrary zero point (Thurstone, 1927; Roberts, 1979);
3. The solution can be determined only when the system of equations is "complete";
4. No uncertainty estimation is provided.

ZM-method

This technique has been proposed to overcome some of the limitations of the LCJ (Franceschini and Maisano, 2018). A significant drawback of the LCJ response mode is that paired comparisons can be tedious and complex to manage, due to the fact that much repetitious information is required from judges. This problem can be overcome asking each judge to formulate a *preference ordering*, i.e., a sequence of objects in order of preference (more preferred ones in the top positions and less preferred ones in the bottom ones).

Apart from regular objects (o_1, o_2, …, o_n), judges should include two *dummy* objects in their orderings: one (o_Z) corresponding to the *absence* of the attribute of interest and one (o_M) corresponding to the *maximum-imaginable* degree of the attribute, consistently with the current technological and socio-economic context (Franceschini and Maisano, 2018). When dealing with these special objects, two important requirements should be considered by judges:

1. o_Z should be positioned at the bottom of a preference ordering, i.e., there should not be any other object with preference lower than o_Z. In the case the attribute of another object is judged to be absent, that object will be considered indifferent to o_Z and positioned at the same hierarchical level.
2. o_M should be positioned at the top of a preference orderings, i.e., there should not be any other object with preference higher than o_M. In the case the attribute of another object is judged to be the maximum-imaginable, that object will be considered indifferent to o_M and positioned at the same hierarchical level.

Next, the preference orderings of judges can be turned into paired-comparison data (e.g., the four-object ordering $(o_3 \sim o_1) > o_2 > o_4$ is turned into the $C_2^4 = 6$ paired-comparison relationships: "$o_1 > o_2$", "$o_1 \sim o_3$", "$o_1 > o_4$", "$o_2 < o_3$", "$o_2 > o_4$", and "$o_3 > o_4$"; it can be noticed that this response mode forces judges to be *transitive* (e.g., if "$o_1 > o_2$" and "$o_2 > o_4$", then "$o_1 > o_4$").

Next, the traditional LCJ can be applied to the resulting paired-comparison data and a scaling (x) of the objects can be determined (Eq. 12). Through the following transformation, the resulting scaling (x) is transformed into a new one (y), which is defined in the conventional range [0, 100]:

$$\hat{y}_i = \hat{y}_i\left(\hat{\boldsymbol{X}}\right) = 100 \cdot \frac{\hat{x}_i - \hat{x}_Z}{\hat{x}_M - \hat{x}_Z} \quad \forall i\,, \tag{13}$$

where: $\hat{x}_Z$ and $\hat{x}_M$ are the scale values of o_Z and o_M, resulting from the LCJ; $\hat{x}_i$ is the scale value of a generic *i*-th object, resulting from the LCJ; $\hat{y}_i$ is the scale value of a generic *i*-th object in the new scale *y*. This transformation can also be expressed in vector form as:

$$\hat{\boldsymbol{Y}} = \hat{\boldsymbol{Y}}(\hat{\boldsymbol{X}}) = [\hat{y}_1(\hat{\boldsymbol{X}}), \hat{y}_2(\hat{\boldsymbol{X}}), \ldots]^T, \tag{14}$$

being $\hat{\boldsymbol{Y}}$ a column vector whose components result from a system of *n* decoupled equations. Since scale *y* "inherits" the *interval* property from scale *x* and has a conventional zero point that corresponds to the absence of the attribute (i.e., $\hat{y}_Z$), it can be reasonably considered as a *ratio* scale, without any conceptually prohibited "promotion". We note that the two dummy objects, o_Z and o_M, are used to "anchor" the *x* scale to the *y* scale (Paruolo et al., 2013).

Although the *ZM*-technique simplifies the response mode and allows to obtain a ratio scaling, it still does not solve other relevant limitations of the traditional LCJ:

- The procedure is not applicable to the system in Eq. 10 when it is not "complete" (i.e., there is at least one (*i*, *j*) paired comparison for which $m_{ij} = 0$).
- It does not contemplate neither the variability of $\hat{p}_{ij}$ values, which are actually treated as deterministic parameters (not probabilistic ones), nor the "propagation" of this variability on the $\hat{\boldsymbol{X}}$ solution (and therefore on the "transformed" solution, $\hat{\boldsymbol{Y}}$).

In fact, since f_{ij} is determined considering a sample of m_{ij} paired comparisons (as illustrated in section "Thurstone's LCJ"), it will be distributed binomially; $\hat{p}_{ij}$ is the best estimator of p_{ij}, according to the information available. In formal terms:

$$f_{ij} \sim B[\mu_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij}, \sigma^2_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij} \cdot (1 - \hat{p}_{ij})], \tag{15}$$

In the hypothesis that $m_{ij} \cdot \hat{p}_{ij} \geq 5$, when $0 \leq \hat{p}_{ij} \leq 0.5$, or $m_{ij} \cdot (1 - \hat{p}_{ij}) \geq 5$, when $0.5 < \hat{p}_{ij} \leq 1$, the following approximations can be reasonably introduced (Ross, 2014):

$$\begin{aligned} &f_{ij} \sim N[\mu_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij}, \sigma^2_{f_{ij}} \approx m_{ij} \cdot \hat{p}_{ij} \cdot (1 - \hat{p}_{ij})] \\ &p_{ij} \sim N\left[\mu_{p_{ij}} \approx \hat{p}_{ij}, \sigma^2_{p_{ij}} \approx \frac{\hat{p}_{ij} \cdot (1 - \hat{p}_{ij})}{m_{ij}}\right] \end{aligned}. \tag{16}$$

It is worth remarking that, even when all judges express their judgments for all the possible paired comparisons (i.e., $m_{ij} = m \ \forall i, j$), the variance of p_{ij} may change from one paired comparison to one other, as it also depends on the relevant $\hat{p}_{ij}$ value.

METHODOLOGY

Response-mode simplification

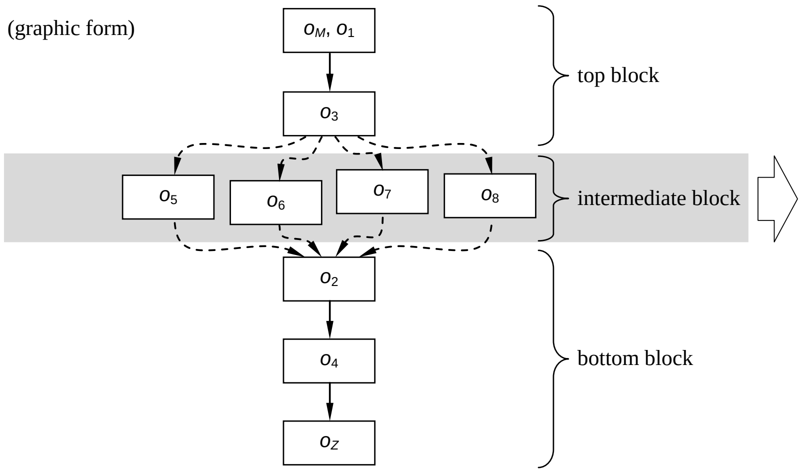
Although the formulation of preference orderings is less tedious and complex to manage than the direct formulation of paired-comparison relationships, it still may be problematic for some practical situations, e.g., asking judges to rank more than a handful of objects during a telephone or street interview may put a very high demand on their cognitive abilities (Harzing et al., 2009; Lenartowicz and Roth, 2001).

To further simplify the response mode, judges could formulate “incomplete” orderings of the more and/or less preferred objects only, neglecting the remaining ones. These orderings can be decomposed into three blocks: (i) a block including the *top* objects (i.e., the more preferred ones, plus o_M), (ii) a block including the *bottom* objects (i.e., the less preferred ones, plus o_Z), and (iii) a block including the *intermediate* objects. Surely the objects in the intermediate block will not be comparable to each other (i.e., it cannot be asserted that one object is more/less/equally preferred to one other), but their hierarchical level will be (1) lower than that of the objects in the top block and (2) higher than that of the objects in the bottom block. The resulting preference orderings can be classified as *partial* since – apart from the relationships of *strict preference* and *indifference* – they may also contain *incomparability* relationships among pairs of objects (Nederpelt and Kamareddine, 2004). Figure 2(a) contains a fictitious partial ordering of n = 10 (regular and dummy) objects, which is divided into the three afore-described blocks.

Any generic partial ordering can be translated into paired-comparison relationships. Among the $C_2^{n=10} = 45$ possible paired-comparison relationships in the example in Figure 2(b), thirty-nine are of *strict preference* (“>” or “<”) or *indifference* (“~”), while the remaining six are of incomparability (“||”) and concern exclusively the elements in the intermediate block (o_5, o_6, o_7, o_8). We note that the objects in the top and bottom blocks are mutually ordered and translated into paired-comparison relationships of strict preference and indifference. An even more simplified response mode could be that one in which each judge merely identifies the more or less preferred objects, without ordering them; this other form of judgment can be translated into a partial ordering too, which also includes mutual relationships of incomparability among the objects in the top and bottom blocks.

(a) Partial preference ordering

(graphic form)



(analytic form) $(o_M \sim o_1) > o_3 > \{o_5 \parallel o_6 \parallel o_7 \parallel o_8\} > o_2 > o_4 > o_Z$

(b) Paired-comparison relationships

	o_Z	o_M	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8
o_Z	-	<	<	<	<	<	<	<	<	<
o_M		-	~	>	>	>	>	>	>	>
o_1			-	>	>	>	>	>	>	>
o_2				-	<	>	<	<	<	<
o_3					-	>	>	>	>	>
o_4						-	<	<	<	<
o_5							-	$\parallel$	$\parallel$	$\parallel$
o_6								-	$\parallel$	$\parallel$
o_7									-	$\parallel$
o_8										-

Possible relationships:
"$>$" and "$<$" $\rightarrow$ strict preference;
"$\sim$" $\rightarrow$ indifference;
"$\parallel$" $\rightarrow$ incomparibility.

Figure 2 – (a) Example of partial preference ordering with corresponding *top*, *intermediate* and *bottom* blocks; (b) the partial ordering is turned into paired-comparison relationships.

Returning to the general problem, all the (partial) preference orderings that have been formulated by m judges can be translated into a number of paired-comparison relationships. For each paired comparison, there will therefore be m_{ij} "usable" relationships for determining $\hat{p}_{ij}$. In fact, the (m_{ij}) relationships that contribute to the estimate of p_{ij} values are those of strict preference and indifference, while the remaining (m – m_{ij}) relationships of incomparability do not contribute to this estimate; regarding the example in Figure 2, m_{ij} = 45 – 6 = 39. In the case m_{ij} = 0, p_{ij} cannot be estimated.

GLS solution

In general, the system in Eq. 10 will not necessarily be "complete", as the number of equations (q) could be lower than $C_2^n + 1$ (i.e., for any paired comparison with m_{ij} = 0, no equation can be formulated) and therefore cannot be solved through the LCJ.

The literature dealt with the problem of solving such "incomplete" systems through the *Ordinary Least Squares* (OLS) method. For example, Gulliksen (1956) discusses some approximate numerical methods for the OLS solution formula to Eq. 11 (Kariya and Kurata, 2004; Ross, 2014):

$$\hat{\boldsymbol{X}} = \left(\boldsymbol{A}^T \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{B}. \tag{17}$$

Also, it can be demonstrated that, in the case in which the equation system is "complete", the LCJ solution coincides with the OLS one – i.e., that one minimizing the sum of the squared residuals related to the equations in Eq. 11 (Gulliksen, 1956):

$$\sum_{h=1}^{q}\left[\sum_{k=1}^{n}\left(a_{hk}\cdot x_k\right)-b_h\right]^2, \tag{18}$$

n being the number of elements in $\hat{\boldsymbol{X}}$, and q being the total number of equations available. In general, the OLS solution is possible even for "incomplete" systems, as long as $q \geq n$; this condition is easily met in practice (Gulliksen, 1956).

Even though the OLS method provides an effective solution to the problem of interest, it does not provide any practical estimate of the uncertainty associated with the elements of $\hat{\boldsymbol{X}}$. In fact, although it is possible to calculate the covariance matrix of $\hat{\boldsymbol{X}}$ as:

$$\Sigma_{\boldsymbol{X}} = \left(\boldsymbol{A}^T \cdot \boldsymbol{A}\right)^{-1}, \tag{19}$$

it is of no practical use for this specific problem, as the uncertainties of the $\hat{\boldsymbol{X}}$ elements are identical and not affected by the real uncertainty of input data (i.e., $\hat{p}_{ij}$ values, see section "*ZM*-method") (Gulliksen, 1956). This limitation can be overcome using the *Generalized Least Squares* (GLS) method, which is more articulated than the OLS method as it includes several additional steps (see the qualitative representation in Figure 3).

The idea of applying the GLS to the problem of interest in the "incomplete" case had already been advanced several decades ago by Arbuckle and Nugent (1973), who contemplated this and other goodness-of-fit criteria, such as *maximum likelihood*. These techniques, however, have not been applied extensively, probably due to some computational constraints that are nowadays overcome. Additionally, the GLS solution proposed by Arbuckle and Nugent (1973) was combined with a "classic" response mode, based on the direct formulation of paired-comparison relationships.

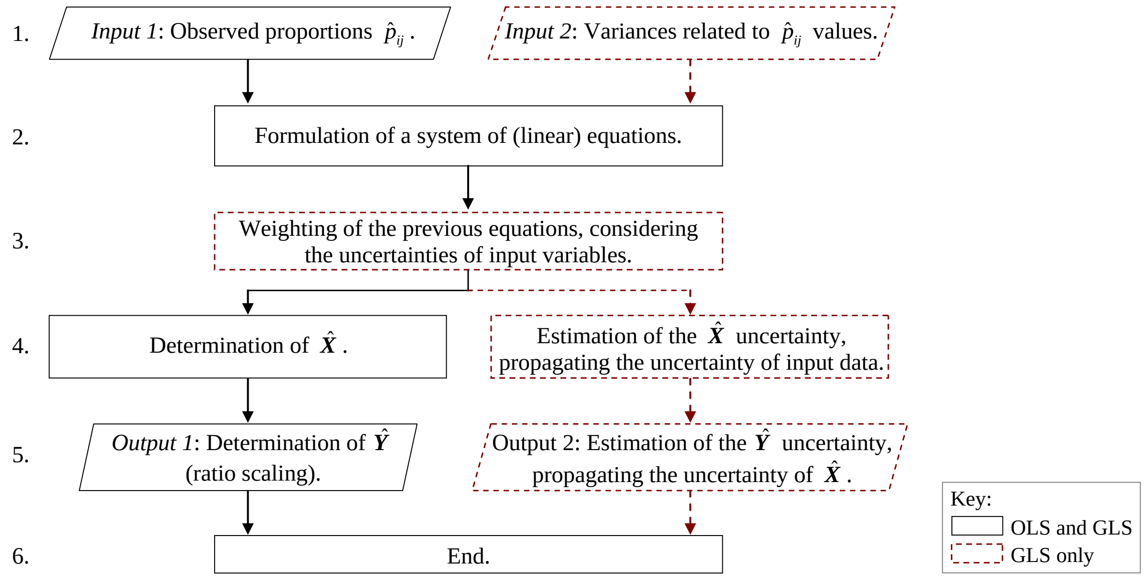


Figure 3 – Flow chart representing the main steps of the OLS and GLS solution to the problem of interest; it can be noticed that the GLS solution includes several additional steps (see dashed blocks) with respect to the OLS one.

From a technical point of view, the GLS method allows obtaining a solution that minimizes the weighted sum of the squared residuals related to the equations in Eq. 11, i.e.:

$$\sum_{h=1}^{q} w_h \cdot \left[\sum_{k=1}^{n} (a_{hk} \cdot x_k) - b_h \right]^2, \tag{20}$$

in which weights (w_h) take into account the uncertainty in the $\hat{p}_{ij}$ values. It can be demonstrated that, for a generic equation related to a generic *ij*-th paired comparison: $w_h = \left[\frac{\partial \Phi^{-1}(1-\hat{p}_{ij})}{\partial \hat{p}_{ij}} \right]^2 \Big/ \sigma^2_{\hat{p}_{ij}}$ (Arbuckle and Nugent, 1973).

Next, weights are aggregated into a (squared) matrix $\boldsymbol{W}$, whose construction is illustrated in the “Appendix” section, and $\boldsymbol{X}$ can be estimated as:

$$\hat{\boldsymbol{X}} = \left(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{B}. \tag{21}$$

Combining Eqs. 21 and 14, the final (ratio) scaling $\hat{\boldsymbol{Y}}$ can be obtained as:

$$\hat{\boldsymbol{Y}} = \hat{\boldsymbol{Y}}[\hat{\boldsymbol{X}}] = \hat{\boldsymbol{Y}}[(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A})^{-1} \cdot \boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{B}]. \tag{22}$$

Next, the uncertainty related to the elements in $\hat{\boldsymbol{Y}} = [\hat{y}_1, \hat{y}_2, \cdots]^T \in R^{n \times 1}$ can be determined by applying the relationship:

$$\Sigma_{\boldsymbol{Y}} = \boldsymbol{J}_{\hat{\boldsymbol{X}}} \cdot [(\boldsymbol{A}^T \cdot \boldsymbol{W} \cdot \boldsymbol{A})^{-1}] \cdot \boldsymbol{J}^T_{\hat{\boldsymbol{X}}}. \tag{23}$$

where $\boldsymbol{J}_{\hat{\boldsymbol{X}}} \in R^{(q-1)\times(q-1)}$ is a Jacobian matrix containing the partial derivatives related to the equations of the system in Eq. 14, with respect to the elements of $\hat{\boldsymbol{X}}$. Assuming that the $p_{i,j}$ and $\hat{y}_i$ values are approximately normally distributed, a 95% confidence interval related to each $\hat{y}_i$ value can be computed as:

$$\hat{y}_i \pm U_{\hat{y}_i} = \hat{y}_i \pm 2 \cdot \sigma_{\hat{y}_i} \quad \forall i, \tag{24}$$

$U_{\hat{y}_i}$ being the so-called *expanded uncertainty* of $\hat{y}_i$ with a coverage factor k = 2 and $\sigma_{\hat{y}_i} = \sqrt{\Sigma_{\boldsymbol{Y},(i,i)}}$ (JCGM 100:2008, 2008).

APPLICATION EXAMPLE

The proposed technique is applied to the design of a civilian aircraft seat. The goal is to prioritize the customer requirements (CRs) in Table 1 (i.e., objects), according to their importance (i.e., attribute) for m = 20 regular air passengers (i.e., judges).

Table 2 contains m “complete” linear orderings by judges, assuming that they have no difficulty in managing both the regular and dummy objects. These orderings are then translated into a number of paired-comparison relationships (i.e., $C_2^{12} = 66$ for each preference ordering, resulting in total 66·20 =1320 paired-comparison relationships) and the LCJ is applied, producing the scaling in Table 5(a) (see also the graphical representation in Figure 5). These results are already referred to the conventional scale (y), which is included in the range [0, 100].

Table 1 – List of the major CRs related to an aircraft seat, from the perspective of passengers.

Abbr.	**Description**
o_1	Comfortable (does not give you back ache)
o_2	Enough leg room
o_3	Comfortable when you recline
o_4	Does not hit person behind when you recline
o_5	Comfortable seat belt
o_6	Seat belt feels safe
o_7	Arm rests not too narrow
o_8	Arm rest folds right away
o_9	Does not make you sweat
o_{10}	Does not soak up a spilt drink
o_{11}	Hole in tray for coffee cup
o_{12}	Magazines can be easily removed from rack

Table 2 – Linear (or "complete") preference orderings used in the application example.

Judges	Linear preference orderings
j_1	$o_M > (o_1 \sim o_2) > (o_5 \sim o_6 \sim o_7) > o_3 > o_4 > (o_{10} \sim o_8) > (o_{11} \sim o_9) > (o_Z \sim o_{12})$
j_2	$(o_1 \sim o_5 \sim o_7 \sim o_M) > (o_9 \sim o_6) > (o_2 \sim o_{11} \sim o_8) > (o_4 \sim o_{12} \sim o_{10}) > o_3 > o_Z$
j_3	$(o_1 \sim o_M) > (o_3 \sim o_2) > (o_6 \sim o_5) > o_7 > (o_8 \sim o_9 \sim o_4) > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$
j_4	$(o_1 \sim o_2 \sim o_7 \sim o_M) > (o_3 \sim o_5) > o_6 > (o_8 \sim o_9 \sim o_{10}) > (o_{11} \sim o_{12}) > o_4 > o_Z$
j_5	$(o_1 \sim o_2 \sim o_5 \sim o_M) > (o_3 \sim o_6 \sim o_9 \sim o_4) > (o_7 \sim o_8) > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$
j_6	$o_M > (o_1 \sim o_5 \sim o_6) > o_7 > (o_2 \sim o_9 \sim o_3) > (o_{11} \sim o_8) > (o_4 \sim o_{10}) > (o_Z \sim o_{12})$
j_7	$(o_2 \sim o_7 \sim o_M) > o_1 > (o_5 \sim o_8 \sim o_6) > (o_9 \sim o_{10} \sim o_3) > o_4 > (o_{12} \sim o_Z \sim o_{11})$
j_8	$(o_1 \sim o_5 \sim o_M) > (o_6 \sim o_7 \sim o_9 \sim o_{12} \sim o_2) > (o_8 \sim o_{11}) > (o_3 \sim o_4) > (o_Z \sim o_{10})$
j_9	$(o_1 \sim o_4 \sim o_M) > (o_5 \sim o_7 \sim o_9) > o_2 > o_6 > (o_{10} \sim o_8) > (o_{11} \sim o_{12} \sim o_3) > o_Z$
j_{10}	$(o_2 \sim o_5 \sim o_6 \sim o_M) > o_7 > o_1 > o_3 > o_{11} > o_9 > (o_8 \sim o_{10}) > (o_{12} \sim o_Z \sim o_4)$
j_{11}	$(o_1 \sim o_M) > (o_6 \sim o_2) > (o_7 \sim o_5) > o_3 > o_4 > o_9 > o_{11} > o_{12} > (o_{10} \sim o_Z \sim o_8)$
j_{12}	$(o_2 \sim o_7 \sim o_M) > o_1 > (o_9 \sim o_3) > (o_8 \sim o_6 \sim o_4) > o_5 > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$
j_{13}	$(o_1 \sim o_2 \sim o_5 \sim o_8 \sim o_M) > (o_7 \sim o_3 \sim o_6) > (o_9 \sim o_{11} \sim o_4) > (o_{12} \sim o_Z \sim o_{10})$
j_{14}	$(o_2 \sim o_5 \sim o_6 \sim o_M) > o_1 > o_3 > o_9 > o_7 > o_4 > (o_{10} \sim o_{11}) > (o_{12} \sim o_Z \sim o_8)$
j_{15}	$(o_2 \sim o_6 \sim o_7 \sim o_M) > o_1 > o_9 > o_8 > (o_{11} \sim o_{12} \sim o_5) > o_3 > o_4 > (o_Z \sim o_{10})$
j_{16}	$(o_1 \sim o_M) > (o_2 \sim o_6) > o_7 > o_9 > o_5 > (o_4 \sim o_8) > o_{11} > o_3 > (o_{12} \sim o_Z \sim o_{10})$
j_{17}	$(o_1 \sim o_2 \sim o_M) > (o_7 \sim o_5) > (o_8 \sim o_9 \sim o_{10} \sim o_4) > (o_6 \sim o_{11} \sim o_{12}) > (o_Z \sim o_3)$
j_{18}	$(o_1 \sim o_2 \sim o_3 \sim o_M) > o_9 > (o_7 \sim o_8 \sim o_4 \sim o_5) > o_{10} > (o_{11} \sim o_6) > (o_Z \sim o_{12})$
j_{19}	$(o_1 \sim o_2 \sim o_6 \sim o_9 \sim o_M) > (o_3 \sim o_7 \sim o_4 \sim o_5) > o_8 > o_{11} > (o_{12} \sim o_{10}) > o_Z$
j_{20}	$o_M > (o_1 \sim o_6) > o_7 > (o_5 \sim o_9 \sim o_2) > (o_3 \sim o_{11} \sim o_8) > (o_4 \sim o_{12}) > o_{10} > o_Z$

Total no. of usable paired-comparison relationships: 1320 of 1320;
o_1 to o_{12} are the regular objects, while o_Z to o_M are the dummy objects;
">" and "~" respectively depict the *strict preference* and *indifference* relationships;
$(o_i \sim o_j \sim \ldots)$ is a generic block containing indifferent objects.

Table 5(a) also shows that, consistently with the considerations in section "Methodology", the results of the LCJ are identical to those obtained by applying the OLS method to the same orderings. On the other hand, the application of the GLS method produces a very close – although non-identical – result. The difference stems from the fact that – unlike LCJ and OLS – the GLS takes into account the uncertainties related to $\hat{p}_{ij}$ values. The GLS solution is therefore superior from both a conceptual and practical point of view.

To study the effectiveness of the GLS in the presence of partial orderings, we have intentionally "degraded" the linear orderings in Table 2, replacing some of the relationships of strict preference (">" and "<") and indifference ("~"), with incomparability relationships ("||"). Precisely, two types of partial orderings have been generated according to the following "degradation criteria" (see also the example in Figure 4):

Type-t&b preference orderings, in which the relationships between the *t* more preferred (top) objects – and any other tied object – and those between the *b* less preferred (bottom) objects – and any other tied object – have been preserved ("*t&b*" stands for "top and bottom"). The remaining objects are allocated at an intermediate hierarchical level, which is certainly lower than the top block and higher than the bottom block.

Type-t preference orderings, in which the relationships between the *t* more preferred (top) objects – and any other tied object – have been preserved. The remaining objects are allocated at a lower hierarchical level, with mutual relationships of incomparability. In this case, the bottom block is empty.

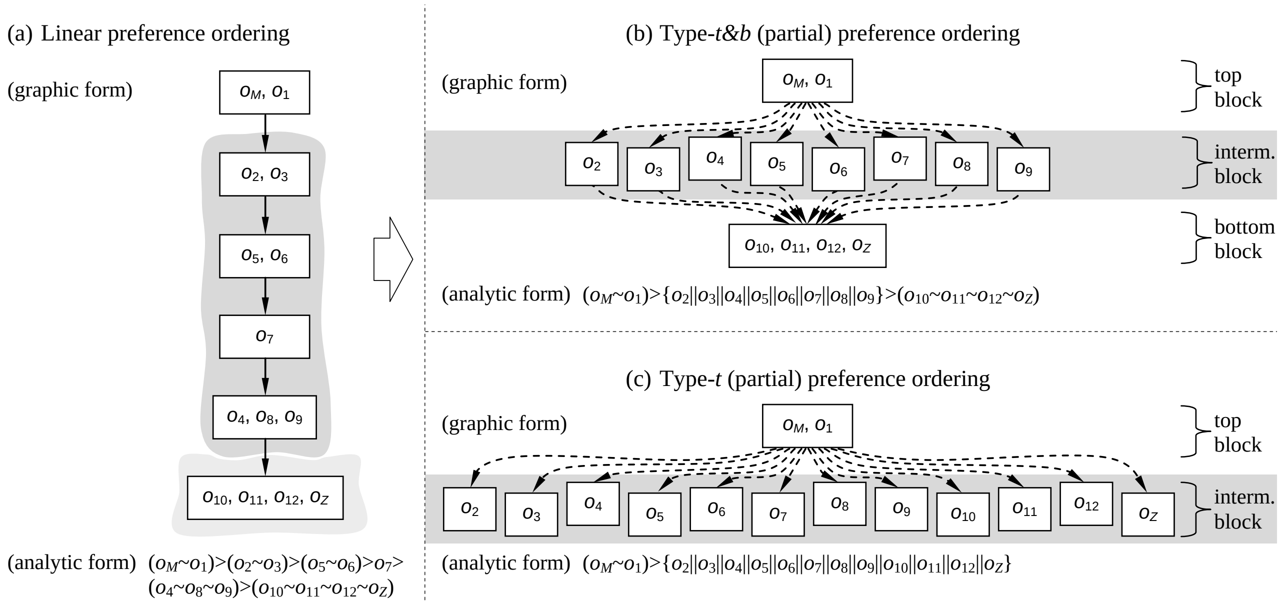


Figure 4 – Example of degradation of the linear preference ordering by j_5 (see Table 5) into a type-*t&b* and type-*t* partial preference ordering. In this case, $t = b = 1$.

Table 3 and Table 4 report the resulting type-*t&b* and type-*t* orderings, which are obtained by degrading the linear orderings in Table 2. In this case, the "level of degradation" is significant since $t = b = 1$; as a rough indicator of this level, we can consider the portion of *usable* paired-comparison relationships that can be obtained from the resulting partial preference orderings: i.e., 1150 out of 1320 for type-*t&b* orderings and 726 out of 1320 for type-*t* orderings.

Table 5 and Figure 5 contain the results of the application of the LCJ, OLS and GLS (where applicable) to the paired-comparison relationships that result from the preference orderings in Table 2, Table 3 and Table 4. In all cases, the resulting (x) scaling has been turned into a (y) scaling, through the transformation in Eq. 13.

The GLS results that are obtained for linear preference orderings (in Table 5(a)) can be used as "gold standard" to evaluate the goodness of the GLS results for degraded (partial) orderings. As for type-*t&b* (partial) orderings, results (in Table 5(b) and Figure 5) are – quite surprisingly – very close to those related to linear orderings, both in terms of accuracy and dispersion. As for type-*t* (partial) orderings, results (in Table 5(c) and Figure 5) worsen considerably, especially for the less preferred objects (see the very wide uncertainty bands). This is probably due by the relatively small amount of usable paired-comparison relationships that concern the less preferred objects.

Table 3 – Type-*t&b* (partial) preference orderings that are obtained by "degrading" the linear orderings in Table 2; both *t* and *b* values have been set to 1.

Judges	Type-*t&b* (partial) preference orderings
j_1	$o_M > \{o_1 \parallel o_2 \parallel o_5 \parallel o_6 \parallel o_7 \parallel o_3 \parallel o_4 \parallel o_{10} \parallel o_8 \parallel o_{11} \parallel o_9\} > (o_Z \sim o_{12})$
j_2	$(o_1 \sim o_5 \sim o_7 \sim o_M) > \{o_9 \parallel o_6 \parallel o_2 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{12} \parallel o_{10} \parallel o_3\} > o_Z$
j_3	$(o_1 \sim o_M) > \{o_3 \parallel o_2 \parallel o_6 \parallel o_5 \parallel o_7 \parallel o_8 \parallel o_9 \parallel o_4\} > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$
j_4	$(o_1 \sim o_2 \sim o_7 \sim o_M) > \{o_3 \parallel o_5 \parallel o_6 \parallel o_8 \parallel o_9 \parallel o_{10} \parallel o_{11} \parallel o_{12} \parallel o_4\} > o_Z$
j_5	$(o_1 \sim o_2 \sim o_5 \sim o_M) > \{o_3 \parallel o_6 \parallel o_9 \parallel o_4 \parallel o_7 \parallel o_8\} > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$
j_6	$o_M > \{o_1 \parallel o_5 \parallel o_6 \parallel o_7 \parallel o_2 \parallel o_9 \parallel o_3 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{10}\} > (o_Z \sim o_{12})$
j_7	$(o_2 \sim o_7 \sim o_M) > \{o_1 \parallel o_5 \parallel o_8 \parallel o_6 \parallel o_9 \parallel o_{10} \parallel o_3 \parallel o_4\} > (o_{12} \sim o_Z \sim o_{11})$
j_8	$(o_1 \sim o_5 \sim o_M) > \{o_6 \parallel o_7 \parallel o_9 \parallel o_{12} \parallel o_2 \parallel o_8 \parallel o_{11} \parallel o_3 \parallel o_4\} > (o_Z \sim o_{10})$
j_9	$(o_1 \sim o_4 \sim o_M) > \{o_5 \parallel o_7 \parallel o_9 \parallel o_2 \parallel o_6 \parallel o_{10} \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_3\} > o_Z$
j_{10}	$(o_2 \sim o_5 \sim o_6 \sim o_M) > \{o_7 \parallel o_1 \parallel o_3 \parallel o_{11} \parallel o_9 \parallel o_8 \parallel o_{10}\} > (o_{12} \sim o_Z \sim o_4)$
j_{11}	$(o_1 \sim o_M) > \{o_6 \parallel o_2 \parallel o_7 \parallel o_5 \parallel o_3 \parallel o_4 \parallel o_9 \parallel o_{11} \parallel o_{12}\} > (o_{10} \sim o_Z \sim o_8)$
j_{12}	$(o_2 \sim o_7 \sim o_M) > \{o_1 \parallel o_9 \parallel o_3 \parallel o_8 \parallel o_6 \parallel o_4 \parallel o_5\} > (o_{11} \sim o_{12} \sim o_Z \sim o_{10})$
j_{13}	$(o_1 \sim o_2 \sim o_5 \sim o_8 \sim o_M) > \{o_7 \parallel o_3 \parallel o_6 \parallel o_9 \parallel o_{11} \parallel o_4\} > (o_{12} \sim o_Z \sim o_{10})$
j_{14}	$(o_2 \sim o_5 \sim o_6 \sim o_M) > \{o_1 \parallel o_3 \parallel o_9 \parallel o_7 \parallel o_4 \parallel o_{10} \parallel o_{11}\} > (o_{12} \sim o_Z \sim o_8)$
j_{15}	$(o_2 \sim o_6 \sim o_7 \sim o_M) > \{o_1 \parallel o_9 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_5 \parallel o_3 \parallel o_4\} > (o_Z \sim o_{10})$
j_{16}	$(o_1 \sim o_M) > \{o_2 \parallel o_6 \parallel o_7 \parallel o_9 \parallel o_5 \parallel o_4 \parallel o_8 \parallel o_{11} \parallel o_3\} > (o_{12} \sim o_Z \sim o_{10})$
j_{17}	$(o_1 \sim o_2 \sim o_M) > \{o_7 \parallel o_5 \parallel o_8 \parallel o_9 \parallel o_{10} \parallel o_4 \parallel o_6 \parallel o_{11} \parallel o_{12}\} > (o_Z \sim o_3)$
j_{18}	$(o_1 \sim o_2 \sim o_3 \sim o_M) > \{o_9 \parallel o_7 \parallel o_8 \parallel o_4 \parallel o_5 \parallel o_{10} \parallel o_{11} \parallel o_6\} > (o_Z \sim o_{12})$
j_{19}	$(o_1 \sim o_2 \sim o_6 \sim o_9 \sim o_M) > \{o_3 \parallel o_7 \parallel o_4 \parallel o_5 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_{10}\} > o_Z$
j_{20}	$o_M > \{o_1 \parallel o_6 \parallel o_7 \parallel o_5 \parallel o_9 \parallel o_2 \parallel o_3 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{12} \parallel o_{10}\} > o_Z$

Total no. of usable paired-comparison relationships: 1150 of 1320;
o_1 to o_{12} are the regular objects, while o_Z to o_M are the dummy objects;
">", "~" and "$\parallel$" respectively depict the *strict preference*, *indifference* and *incomparability* relationships;
$\{o_i \parallel o_j \parallel \ldots\}$ is a generic block containing incomparable objects;
$(o_i \sim o_j \sim \ldots)$ is a generic block containing indifferent objects.

Table 4 – Type-*t* (partial) preference orderings that are obtained by "degrading" the linear orderings in Table 2; *t* values have been set to 1.

Judges	Type-*t* (partial) preference orderings
j_1	$o_M > \{o_1 \parallel o_2 \parallel o_5 \parallel o_6 \parallel o_7 \parallel o_3 \parallel o_4 \parallel o_{10} \parallel o_8 \parallel o_{11} \parallel o_9 \parallel o_Z \parallel o_{12}\}$
j_2	$(o_1 \sim o_5 \sim o_7 \sim o_M) > \{o_9 \parallel o_6 \parallel o_2 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{12} \parallel o_{10} \parallel o_3 \parallel o_Z\}$
j_3	$(o_1 \sim o_M) > \{o_3 \parallel o_2 \parallel o_6 \parallel o_5 \parallel o_7 \parallel o_8 \parallel o_9 \parallel o_4 \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_{10}\}$
j_4	$(o_1 \sim o_2 \sim o_7 \sim o_M) > \{o_3 \parallel o_5 \parallel o_6 \parallel o_8 \parallel o_9 \parallel o_{10} \parallel o_{11} \parallel o_{12} \parallel o_4 \parallel o_Z\}$
j_5	$(o_1 \sim o_2 \sim o_5 \sim o_M) > \{o_3 \parallel o_6 \parallel o_9 \parallel o_4 \parallel o_7 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_{10}\}$
j_6	$o_M > \{o_1 \parallel o_5 \parallel o_6 \parallel o_7 \parallel o_2 \parallel o_9 \parallel o_3 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{10} \parallel o_Z \parallel o_{12}\}$
j_7	$(o_2 \sim o_7 \sim o_M) > \{o_1 \parallel o_5 \parallel o_8 \parallel o_6 \parallel o_9 \parallel o_{10} \parallel o_3 \parallel o_4 \parallel o_{12} \parallel o_Z \parallel o_{11}\}$
j_8	$(o_1 \sim o_5 \sim o_M) > \{o_6 \parallel o_7 \parallel o_9 \parallel o_{12} \parallel o_2 \parallel o_8 \parallel o_{11} \parallel o_3 \parallel o_4 \parallel o_Z \parallel o_{10}\}$
j_9	$(o_1 \sim o_4 \sim o_M) > \{o_5 \parallel o_7 \parallel o_9 \parallel o_2 \parallel o_6 \parallel o_{10} \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_3 \parallel o_Z\}$
j_{10}	$(o_2 \sim o_5 \sim o_6 \sim o_M) > \{o_7 \parallel o_1 \parallel o_3 \parallel o_{11} \parallel o_9 \parallel o_8 \parallel o_{10} \parallel o_{12} \parallel o_Z \parallel o_4\}$
j_{11}	$(o_1 \sim o_M) > \{o_6 \parallel o_2 \parallel o_7 \parallel o_5 \parallel o_3 \parallel o_4 \parallel o_9 \parallel o_{11} \parallel o_{12} \parallel o_{10} \parallel o_Z \parallel o_8\}$
j_{12}	$(o_2 \sim o_7 \sim o_M) > \{o_1 \parallel o_9 \parallel o_3 \parallel o_8 \parallel o_6 \parallel o_4 \parallel o_5 \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_{10}\}$
j_{13}	$(o_1 \sim o_2 \sim o_5 \sim o_8 \sim o_M) > \{o_7 \parallel o_3 \parallel o_6 \parallel o_9 \parallel o_{11} \parallel o_4 \parallel o_{12} \parallel o_Z \parallel o_{10}\}$
j_{14}	$(o_2 \sim o_5 \sim o_6 \sim o_M) > \{o_1 \parallel o_3 \parallel o_9 \parallel o_7 \parallel o_4 \parallel o_{10} \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_8\}$
j_{15}	$(o_2 \sim o_6 \sim o_7 \sim o_M) > \{o_1 \parallel o_9 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_5 \parallel o_3 \parallel o_4 \parallel o_Z \parallel o_{10}\}$
j_{16}	$(o_1 \sim o_M) > \{o_2 \parallel o_6 \parallel o_7 \parallel o_9 \parallel o_5 \parallel o_4 \parallel o_8 \parallel o_{11} \parallel o_3 \parallel o_{12} \parallel o_Z \parallel o_{10}\}$
j_{17}	$(o_1 \sim o_2 \sim o_M) > \{o_7 \parallel o_5 \parallel o_8 \parallel o_9 \parallel o_{10} \parallel o_4 \parallel o_6 \parallel o_{11} \parallel o_{12} \parallel o_Z \parallel o_3\}$
j_{18}	$(o_1 \sim o_2 \sim o_3 \sim o_M) > \{o_9 \parallel o_7 \parallel o_8 \parallel o_4 \parallel o_5 \parallel o_{10} \parallel o_{11} \parallel o_6 \parallel o_Z \parallel o_{12}\}$
j_{19}	$(o_1 \sim o_2 \sim o_6 \sim o_9 \sim o_M) > \{o_3 \parallel o_7 \parallel o_4 \parallel o_5 \parallel o_8 \parallel o_{11} \parallel o_{12} \parallel o_{10} \parallel o_Z\}$
j_{20}	$o_M > \{o_1 \parallel o_6 \parallel o_7 \parallel o_5 \parallel o_9 \parallel o_2 \parallel o_3 \parallel o_{11} \parallel o_8 \parallel o_4 \parallel o_{12} \parallel o_{10} \parallel o_Z\}$

Total no. of usable paired-comparison relationships: 726 of 1320;
o_1 to o_{12} are the regular objects, while o_Z to o_M are the dummy objects;
">", "~" and "$\parallel$" respectively depict the *strict preference*, *indifference* and *incomparability* relationships;
$\{o_i \parallel o_j \parallel \ldots\}$ is a generic block containing incomparable objects;
$(o_i \sim o_j \sim \ldots)$ is a generic block containing indifferent objects.

Table 5 – Results of the application of several techniques (i.e., LCJ, OLS and GLS) to the orderings in Table 2; GLS solutions are associated with relevant extended uncertainties (according to Eq. 24).

Objects	**(a) Linear/complete orderings**		**(b) type-*t&b* (partial)**	**(b) type-*t* (partial)**
	LCJ=OLS	GLS	GLS	GLS
o_Z	0.0	0.0 ±7.2	0.0 ±7.2	0.0 ±24.1
o_M	100.0	100.0 ±6.7	100.0 ±6.7	100.0 ±8.8
o_1	93.6	90.2 ±8.7	87.8 ±8.9	84.2 ±12.3
o_2	85.1	83.4 ±8.1	86.5 ±9.0	82.6 ±12.6
o_3	46.1	47.2 ±7.6	46.0 ±10.6	21.8 ±26.3
o_4	37.7	37.7 ±7.9	47.4 ±10.6	23.4 ±25.7
o_5	73.7	71.1 ±7.7	73.9 ±9.2	64.5 ±15.0
o_6	68.5	67.7 ±7.5	67.0 ±9.7	54.3 ±17.3
o_7	73.6	70.5 ±7.7	70.9 ±9.3	59.6 ±15.9
o_8	40.3	41.6 ±7.8	40.9 ±10.1	20.6 ±26.8
o_9	55.3	55.1 ±7.8	50.5 ±11.9	20.9 ±26.7
o_{10}	19.3	20.9 ±8.3	17.7 ±9.9	0.0 ±36.2
o_{11}	23.7	26.9 ±8.3	29.0 ±10.6	0.0 ±36.2
o_{12}	13.4	14.8 ±8.7	11.9 ±9.8	0.0 ±36.2

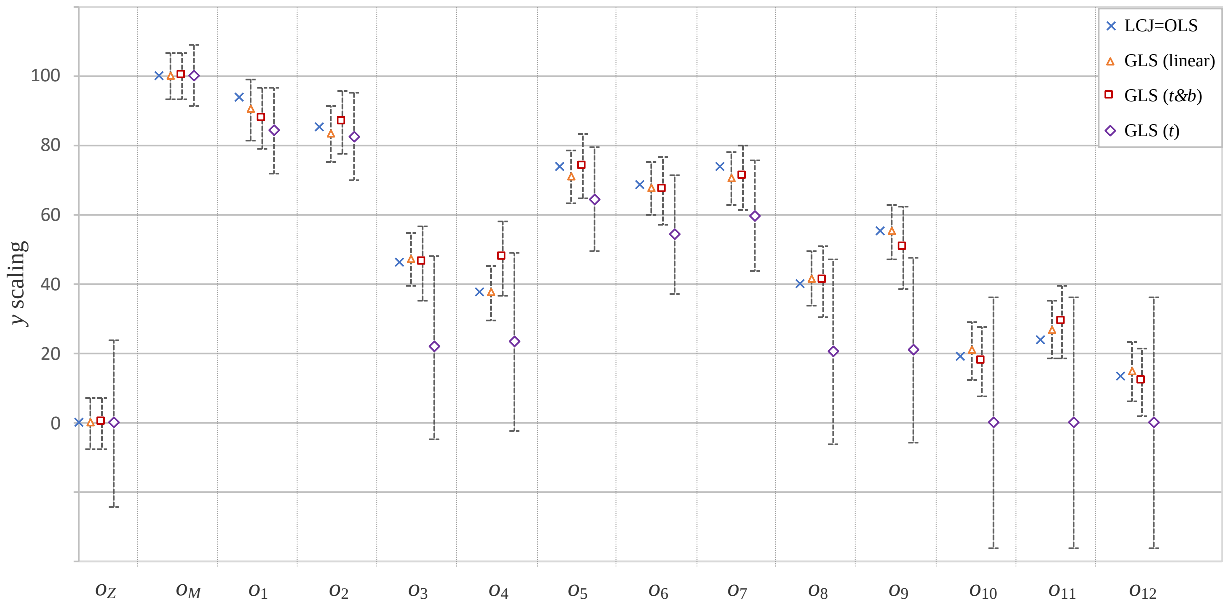


Figure 5 – Graphical representation of the results in Table 5.

These preliminary results have a significant practical implication: even when adopting a simplified response mode like type-*t&b* partial orderings, relatively accurate results can be obtained. On the contrary, the adoption of type-*t* partial orderings produces acceptable results only for the more preferred objects.

CONCLUSIONS

The proposed technique allows to fuse multiple *partial* preference orderings into a ratio scaling with a relevant uncertainty estimation. Apart from regular objects, these orderings will also include two dummy objects, to univocally represent the zero and the maximum-possible degree of the attribute on a conventional ratio scale. This technique represents an important improvement over the technique proposed in (Franceschini and Maisano, 2018), whose application is limited to *linear* orderings exclusively.

From a technical point of view, the proposed technique is based on the formulation of a system of equations – borrowing the underlying postulates/assumptions of the LCJ – and its solution through the GLS method. From a practical point of view, the new response mode makes the technique more versatile and adaptable to a variety of contexts in which the concentration effort of judges cannot realistically be too high (e.g., telephone or street interviews).

Based on the above considerations, the proposed technique reasonably represents an appropriate response to the previously formulated research question: "How could the *ZM*-technique be modified, so as to (1) make the response more user-friendly and reliable and (2) determine a (statistically sound) estimate of the uncertainty related to the solution?".

Preliminary results show that the technique is largely automatable, computationally efficient and provides relatively accurate results, even when preference orderings are significantly "incomplete". Additionally, it seems that much better results can be obtained when partial orderings contain both the more and the less preferred elements (i.e., type-*t&b* orderings).

Regarding the future, we will test the new technique in a more organic way. Precisely, we plan to investigate the accuracy of the solution depending on various factors, such as (i) "level of degradation" of the (partial) preference orderings, (ii) number of judges, (iii) number of objects, (iv) degree of agreement between judges, etc.

REFERENCES

Aludaat, K.M., Alodat, M.T. (2008), "A note on approximating the normal distribution function", Applied Mathematical Sciences, Vol. 2, No. 9, pp. 425-429.

Arbuckle, J., Nugent, J.H. (1973), "A general procedure for parameter estimation for the law of comparative judgement", British Journal of Mathematical and Statistical Psychology, Vol. 26, No. 2, pp. 240-260.

Coaley, K. (2014), An introduction to psychological assessment and psychometrics (2nd edn.), Sage, London.

DeVellis, R.F. (2016), Scale development: Theory and applications (4th edn.), Sage, London.

Edwards, A.L. (1957), Techniques of Attitude Scale Construction, Irvington Publishers, New York.

Franceschini, F., Galetto, M., Maisano, D. (2007), Management by Measurement: Designing Key Indicators and Performance Measurement Systems, Springer, Berlin.

Franceschini, F., Maisano, D. (2018), "A new proposal to improve the customer competitive benchmarking in QFD", to appear in Quality Engineering, doi: 10.1080/08982112.2018.1437178.

Gulliksen, H. (1956), "A least squares solution for paired comparisons with incomplete data", Psychometrika, Vol. 21, pp. 125-134.

Harzing, A.W., Baldueza, J., ... and Zander, L. (2009), "Rating versus ranking: What is the best way to reduce response and language bias in cross-national research?", International Business Review, Vol. 18, No. (4), pp. 417-432.

JCGM 100:2008 (2008), Evaluation of measurement data - Guide to the expression of uncertainty in measurement, BIPM, Paris.

Kariya T., Kurata H. (2004), Generalized least squares, John Wiley & Sons, New York.

Keeney, R.L., Raiffa, H. (1993), Decisions with multiple objectives: preferences and value trade-offs, Cambridge university press, Cambridge.

Lagerspetz, E. (2016), Social choice and democratic values, Springer, Heidelberg.

Lenartowicz, T., Roth, K. (2001), "Does subculture within a country matter? A cross-cultural study of motivational domains and business performance in Brazil", Journal of International Business Studies, Vol. 32, No. 2, pp. 305-325.

Montag, E.D. (2006), "Empirical formula for creating error bars for the method of paired comparison", Journal of Electronic Imaging, Vol. 15, No. 1, 010502.

Nederpelt, R., Kamareddine, F. (2004), Logical Reasoning: A First Course, King's College Publications, London.

Paruolo, P., Saisana, M., Saltelli, A. (2013). "Ratings and rankings: voodoo or science?", Journal of the Royal Statistical Society: Series A (Statistics in Society), Vol. 176, No. 3, pp. 609-634.

Roberts, F.S. (1979), Measurement theory: with applications to decisionmaking, utility, and the social sciences, Encyclopedia of Mathematics and its Applications, vol. 7, Addison-Wesley, Reading, MA.

Ross, S.M. (2014), Introduction to probability and statistics for engineers and scientists, Academic Press, Amsterdam.

Thurstone, LL. (1927), "A law of comparative judgments", Psychological Review, Vol. 34, No. 4, p. 273.

APPENDIX

This section illustrates in more detail the application of the GLS method to the problem of interest. From an operational point of view, the GLS requires the definition of a (squared) weight matrix (***W***), which encapsulates the uncertainty related to the equations of the system. A practical way to define ***W*** is to apply the *Multivariate Law of Propagation of Uncertainty* (MLPU) to the system in Eq. 10, referring to the input variables affected by uncertainty (Kariya and Kurata, 2004); these variables can be collected in the column vector $\boldsymbol{\xi}$. Precisely, ***W*** can be determined propagating the uncertainty of the elements in $\boldsymbol{\xi}$ to the equations of the system:

$$\boldsymbol{W} = \left[\boldsymbol{J}_{\xi} \cdot \Sigma_{\xi} \cdot \boldsymbol{J}_{\xi}^{T}\right]^{-1}, \tag{A1}$$

where $\boldsymbol{J}_{\xi}$ is the Jacobian matrix containing the partial derivatives of the first members of Eq. 10, with respect to the elements in $\boldsymbol{\xi}$, and $\boldsymbol{\Sigma}_{\xi}$ is the covariance matrix of $\boldsymbol{\xi}$.

By applying the GLS method to the system in Eq. 11, a final estimate of ***X*** can be obtained as (Kariya and Kurata, 2004):

$$\hat{\boldsymbol{X}} = \left(\boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1} \cdot \boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{B}. \tag{A2}$$

The uncertainty of the solution can be estimated through a covariance matrix $\Sigma_{\boldsymbol{X}}$, which can be obtained by applying the following relationship:

$$\Sigma_{\boldsymbol{X}} = \left(\boldsymbol{A}^{T} \cdot \boldsymbol{W} \cdot \boldsymbol{A}\right)^{-1}. \tag{A3}$$

$\Sigma_{\boldsymbol{X}}$ – unlike the homologous matrix resulting from the OLS method (in Eq. 19) – is of considerable practical use, since it is obtained by propagating the real uncertainty of input data.

Focussing on the problem of interest, the vector containing the input variables affected by uncertainty is $\boldsymbol{\xi} = [\ldots, p_{ij}, \ldots]^{T} \in R^{(q-1)\times 1}$. On the other hand, the partial derivatives in the

Jacobian matrix $\boldsymbol{J}_{\xi} \in R^{(q-1)\times(q-1)}$ can be determined in a closed form, by approximating terms $\Phi^{-1}(1-\hat{p}_{ij})$ (see Eq. 10) through the following formula (Aludaat and Alodat, 2008):

$$\Phi^{-1}(1-\hat{p}_{i,j}) \approx k\sqrt{\frac{-\ln[1-(1-2\cdot\hat{p}_{i,j})^2]}{\sqrt{\pi/8}}} \qquad \begin{cases} 0 \le \hat{p}_{i,j} \le 0.5 \rightarrow k = 1 \\ 0.5 < \hat{p}_{i,j} \le 1 \rightarrow k = -1 \end{cases}, \tag{A4}$$

from which:

$$\begin{aligned} &\frac{\partial\left[\Phi^{-1}(1-\hat{p}_{i,j})\right]}{\partial\hat{p}_{i,j}} \approx \left|\frac{\sqrt{2}\cdot(2\cdot\hat{p}_{i,j}-1)}{\sqrt{-2\cdot\sqrt{2\cdot\pi}\cdot\ln(-4\cdot\hat{p}_{i,j}^2+4\cdot\hat{p}_{i,j})}\cdot\hat{p}_{i,j}\cdot\left(1-\hat{p}_{i,j}\right)}\right| && \text{for } \hat{p}_{i,j} \ne 0.5 \\ &\frac{\partial\left[\Phi^{-1}(1-\hat{p}_{i,j})\right]}{\partial\hat{p}_{i,j}} \approx 2.506628 && \text{for } \hat{p}_{i,j} = 0.5 \end{aligned}. \tag{A5}$$

The matrix $\boldsymbol{\Sigma}_{\xi} \in R^{(q-1)\times(q-1)}$ diagonally contains the variances related to the input variables, i.e., $\hat{p}_{ij}$ terms:

$$\sigma_{p_{ij}}^2 = \frac{\hat{p}_{ij}\cdot(1-\hat{p}_{ij})}{m_{ij}}. \tag{A6}$$

The relevant covariances can be neglected, upon the reasonable assumption that the estimates of different p_{ij} values are (statistically) independent from each other.

Next, it is possible to determine the matrix $\boldsymbol{W}$ (Eq. A1) and, subsequently, $\hat{\boldsymbol{X}}$ (Eq. A2) with the relevant uncertainty (Eq. A3); this solution is defined on an interval scale (x), as illustrated in the "Background information" section.

Through the transformation in Eq. 14, the x scaling can be transformed into a new one (y), which is included in the conventional range [0, 100]. The uncertainty related to the elements in $\hat{\boldsymbol{Y}} = \left[\hat{y}_1, \hat{y}_2, \cdots\right]^T \in R^{n\times 1}$ can be determined by applying the relationship:

$$\Sigma_{\boldsymbol{Y}} = \boldsymbol{J}_{\hat{\boldsymbol{X}}} \cdot \Sigma_{\boldsymbol{X}} \cdot \boldsymbol{J}_{\hat{\boldsymbol{X}}}^T, \tag{A7}$$

where $\boldsymbol{J}_{\hat{\boldsymbol{X}}} \in R^{(q-1)\times(q-1)}$ is the Jacobian matrix containing the partial derivatives related to the equations of the system in Eq. 14, with respect to the elements of $\hat{\boldsymbol{X}}$ (demonstration omitted).