ON THE GROUP OF THE FIBONACCI NUMBERS

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Abstract Here we will show that the numbers of Fibonacci are forming a group. Each number is represented by a 2x2 symmetric matrix and the operation of the group is the product of matrices. This approach allows to define the negaFibonacci numbers by means of the inverse of the Fibonacci matrices.

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The Fibonacci numbers are a sequence of integers characterized by the fact that every number, after the first two, is the sum of the two preceding ones. Therefore, we have the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …. and so on.

The recurrence relation is given by:

$$F_n = F_{n-1} + F_{n-2}$$

with $F_0 = 0$, $F_1 = 1$. Then $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, etc.

The item of Wikipedia, about the Fibonacci numbers [1], gives them also in the form:

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = M \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

However, we find also in [1] the matrices:

$$(1) \qquad M^n = \underbrace{M \cdots M}_{n} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Therefore, we have: $M^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $M^2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $M^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, etc.

Let us consider the group of these symmetric matrices and discuss it.

Let us remember that a group is a set *A* having an operation • which is combining the elements of *A*. That is, the operation combines any two elements *a,b* to form another element of the group denoted *a*•*b*. To qualify (*A*,•) as a group, the set and operation must satisfy the following requirements. *Closure*: For all *a*,*b* in *A*, the result of the operation *a*•*b* is also in *A*. *Associativity:* For all *a*,*b* and *c* in *A,* it holds (*a*•*b*)•*c* = *a*•(*b*•*c*). *Identity element*: An element *e* exists in *A*, such that for all elements *a* in *A*, it is *e*•*a* = *a*•*e* = *a*. *Inverse element:* For each *a* in *A*, there exists an element *b* in *A* such that *a*•*b* = *b*•*a* = *e*, where *e* is the identity (the notation is inherited from the multiplicative operation). A further requirement is the c*ommutativity:* For all *a*,*b* in *A*, *a*•*b* = *b*•*a*. In this case, the group is known as an Abelian group.
For the set of the matrices (1), the operation is the product of the matrices. Is it *commutative*? The answer is positive.

$$M^n M^m = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} = \begin{pmatrix} (F_{n+1}F_{m+1}+F_nF_m) & (F_{n+1}F_m+F_nF_{m-1}) \\ (F_nF_{m+1}+F_{n-1}F_m) & (F_nF_m+F_{n-1}F_{m-1}) \end{pmatrix}$$

Being $F_{m+1}=F_m+F_{m-1}$ and $F_{n+1}=F_n+F_{n-1}$, we can see that the product gives a symmetric matrix:

$$M^n M^m = \begin{pmatrix} (F_{n+1}F_{m+1}+F_nF_m) & (F_nF_m+F_{n-1}F_m+F_nF_{m-1}) \\ (F_nF_m+F_nF_{m-1}+F_{n-1}F_m) & (F_nF_m+F_{n-1}F_{m-1}) \end{pmatrix}$$

And also:

$$M^n M^m = \begin{pmatrix} (F_{n+1}F_{m+1}+F_nF_m) & (F_{m+1}F_n+F_mF_{n-1}) \\ (F_{n+1}F_m+F_nF_{m-1}) & (F_nF_m+F_{n-1}F_{m-1}) \end{pmatrix} \quad (2)$$

The same for:

$$M^m M^n = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} (F_{m+1}F_{n+1}+F_mF_n) & (F_{m+1}F_n+F_mF_{n-1}) \\ (F_mF_{n+1}+F_{m-1}F_n) & (F_mF_n+F_{m-1}F_{n-1}) \end{pmatrix} \quad (3)$$

From (2) and (3):

$$M^m M^n = M^n M^m$$

We can tell that the product of two Fibonacci symmetric matrices A and B is a symmetric matrix, because A and B commute.
Let us consider the matrices again:

$$M^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

and evaluate the determinant, to obtain the Cassini identity.

Because the determinant of a matrix product of square matrices equals the product of their determinants, we have:

$$(-1)^n = F_{n+1} F_{n-1} - F_n^2 \quad (4)$$

(4) is the Cassini's Identity.
Let us discuss the *closure*. It means that, if we have any product of two Fibonacci matrices, we have another Fibonacci matrix. Actually:

$$M^m M^n = M^{m+n} = \begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix} = \begin{pmatrix} (F_{m+1} F_{n+1} + F_m F_n) & (F_{m+1} F_n + F_m F_{n-1}) \\ (F_m F_{n+1} + F_{m-1} F_n) & (F_m F_n + F_{m-1} F_{n-1}) \end{pmatrix} \quad (5)$$

From (5) we have other relations among Fibonacci numbers.

The *identity* element is: $M^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The *inverse* element is obtained in the following manner:

$$(M^n)^{-1} M^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = M^0$$

Therefore:

$$a = \frac{F_{n-1}}{F_{n+1} F_{n-1} - F_n^2} \qquad b = \frac{F_n}{-F_{n+1} F_{n-1} + F_n^2} \qquad c = \frac{F_n}{-F_{n+1} F_{n-1} + F_n^2} \qquad d = \frac{F_{n+1}}{F_{n+1} F_{n-1} - F_n^2}$$

Let us calculate some inverses:

$$(M^1)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \quad (M^2)^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad (M^3)^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix} \quad (M^4)^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \text{ etc.}$$

So we can easily see that we have here the "negaFibonacci" numbers: 0, 1, -1, 2, -3, 5, -8, 13, -21 , … etc. In [1], these numbers are given as:

$$F_{-n} = (-1)^{n+1} F_n$$

From [1], it seems that these numbers were defined by Ref.2 (in fact, I was not able to find a copy of the article mentioned by Wikipedia).

If we use the matrices, the negaFibonacci are the inverse of them.

Let us conclude considering the *associativity*, that is $(M^m M^n) M^k = M^m (M^n M^k)$

$$(M^m M^n) M^k = M^{m+n} M^k = M^{m+n+k}$$
$$M^m (M^n M^k) = M^m M^{n+k} = M^{m+n+k}$$

Here we have seen that the numbers of Fibonacci, represented by 2x2 symmetric matrices, are forming a group. The operation of the group is the product of matrices. The negaFibonacci numbers are defined by means of the inverse of the Fibonacci matrices.

References

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