International Symposium on Recent Advances in
Mechanics, Dynamical Systems and Probability Theory
MDP – 2007
Palermo, June 3-6, 2007

A GENERALIZED FORMULATION FOR CONTACT BETWEEN BEAMS - GEOMETRICAL PRELIMINARIES

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Key words: Contact Mechanics, Beams, Finite Element Method, Consistent Linearization

Abstract. *The currently available formulations for contact between beams are based on the identification of the minimal distance points along the beam axes, followed by some tuning in case of non-circular beam cross-sections. Up to now a suitable implementation within the framework of the Finite Element Method is available both for the frictionless and for the frictional case. The procedure requires the explicit computation of the virtual work contribution due to the contacts. In such a context for solving the problem with implicit schemes, the formulation has also to be consistently linearized. With this respect both the frictionless and the frictional formulation present severe problems. To overcome all the cited problems a generalized formulation is proposed, which deals with contact between circular beams. It has to be remarked that the contact problem is treated first in a completely generic framework, and only in a second step the results are particularized to the FE formulation. For such purpose the centroids of the beams in the 3-D space are considered as parametric functions. The framework for the consistent linearization is developed in a very rigorous and systematic way, providing evidence of the symmetry of the operators. The procedure is quite cumbersome, hence here only the most heavy part, related to the computation of all the fundamental geometrical terms involved, is presented.*

1 INTRODUCTION

The basic problem of contact between three-dimensional beams undergoing large displacements has been properly considered within the framework of the Finite Element method only in the recent years. The interesting contribution for a rod/continuum interaction, proposed by Maker and Laursen [1] in 1994 can be considered as a preliminar starting point.

The formulation of a suitable finite element to deal with contact between frictionless beams has been presented by Wriggers and Zavarise [2] in 1997. The frictional extension

[3] is dated 2000. The formulation is based on the identification of the minimal distance points along the beam axes. It has to be remarked that this formulation can deal with any available beam model. Also, it presents a perfectly symmetric treatment of the two contacting beams, which is for sure an interesting characteristics that permits to avoid several unconsistencies related to the classical "master-slave" approach, commonly used for contact between solids. The resulting equation set has been consistently linearized, with a cumbersome procedure which does not permit to evidence some basic properties, like e.g. the symmetry of the operators. Moreover the frictional formulation is currently restricted to straight beams. The most recent literature presents interesting applications, see e.g. Durville [4], but the problem of the consistent linearization is not addressed anymore.

When a contact relationship is consistently linearized, a dependence both on the geometry and on the contact constitutive law is involved. The biggest effort for the linearization is usually related to the geometrical terms. For this purpose in this paper the contact problem is treated first in a completely generic framework, and only in a second step the results are particularized to the FE discretization. In this framework the centroids of the beams in the 3-D space are considered as parametric functions.

2 CONTACT ELEMENT SETUP

For the self-consistency of the paper, the notation defined first in [2] is here briefly resumed, with suitable modifications. In the following we distinguish between the two beams by using the bar symbol, $\bar{\bullet}$, for one of the two. In any case it has to be remarked that the formulation is completely symmetric, i.e. no distinction between master and slave surfaces, like in the well-known 2D node-to-segment formulation, is required.

We start considering the current configuration, $\mathbf{x}$ and $\bar{\mathbf{x}}$, of the centroids of two beams in a 3D space. Such a configuration is obtained as usual, from the original one, $\mathbf{X}$, $\bar{\mathbf{X}}$, adding the displacement field, $\mathbf{u}$, $\bar{\mathbf{u}}$

$$\mathbf{x}(\xi) = \mathbf{X}(\xi) + \mathbf{u}(\xi) \qquad \overline{\mathbf{x}}\left(\bar{\xi}\right) = \overline{\mathbf{X}}\left(\bar{\xi}\right) + \overline{\mathbf{u}}\left(\bar{\xi}\right) \tag{1}$$

where the classical parameter range $-1 \leq \xi \leq 1$, $-1 \leq \bar{\xi} \leq 1$ has been considered.

2.1 Geometry definition

Contact will eventually take place at the minimal distance location, depicted in Figure (1) . The candidate points along the centroids can be identified solving for the minimal distance

$$d = \min \left\| \mathbf{x}(\xi) - \overline{\mathbf{x}}\left(\bar{\xi}\right) \right\| \tag{2}$$

The solution takes place for suitable values, $\xi = \xi_C$, $\xi = \bar{\xi}_{\bar{C}}$ of the parameters, which identify the candidate contact points, C and $\bar{C}$, along the centroids. To simplify the notation the following equivalence is then used

$$\mathbf{x}(\xi_C) = \mathbf{x}_C \qquad \overline{\mathbf{x}}\left(\bar{\xi}_{\bar{C}}\right) = \overline{\mathbf{x}}_{\bar{C}} \tag{3}$$

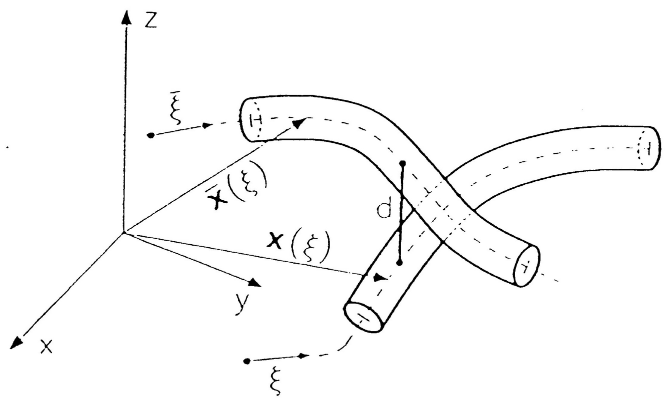

Figure 1: A generic scheme of two contacting beams.

which permits also to write

$$d = \|\mathbf{d}\| = \left\|\mathbf{x}(\xi_C) - \bar{\mathbf{x}}\left(\bar{\xi}_{\bar{C}}\right)\right\| = \|\mathbf{x}_C - \bar{\mathbf{x}}_{\bar{C}}\| \tag{4}$$

where the minimal distance vector, $\mathbf{d}$, i.e. the vector joining the minimal distance points, has been introduced

$$\mathbf{d} = \mathbf{x}_C - \bar{\mathbf{x}}_{\bar{C}} \tag{5}$$

To complete the geometrical setup it is also convenient to introduce the tangent vectors of the centroids at the minimal distance points, which are easily computed as

$$\mathbf{t} = \mathbf{x}_{C,\xi} \qquad \bar{\mathbf{t}} = \bar{\mathbf{x}}_{\bar{C},\bar{\xi}} \tag{6}$$

where the classical notation $\mathbf{x}_{,\xi} = \partial\mathbf{x}/\partial\xi$, and also $\mathbf{x}_{C,\xi} = (\partial\mathbf{x}/\partial\xi)_{\xi=\xi_C}$, has been introduced. This notation will be widely used in the following. The lengths of such tangent vectors are easily defined as

$$t = \|\mathbf{x}_{C,\xi}\| \qquad \bar{t} = \left\|\bar{\mathbf{x}}_{\bar{C},\bar{\xi}}\right\| \tag{7}$$

Equations resume. The vector variables that have been identified and will play a crucial role in the following are the distance vector and the tangent ones

$$\boxed{\mathbf{d} = \mathbf{x}_C - \bar{\mathbf{x}}_{\bar{C}} \qquad \mathbf{t} = \mathbf{x}_{C,\xi} \qquad \bar{\mathbf{t}} = \bar{\mathbf{x}}_{\bar{C},\bar{\xi}}} \tag{8}$$

Moreover the norms of the above vectors are also requested

$$\boxed{d = \|\mathbf{d}\| \qquad t = \|\mathbf{t}\| \qquad \bar{t} = \|\bar{\mathbf{t}}\|} \tag{9}$$

Finally, from the definition (5) and (6) the following equivalences are easily obtained

$$\boxed{\begin{array}{lll} \mathbf{d}_{,\xi} = \mathbf{x}_{C,\xi} = \mathbf{t} & \mathbf{d}_{,\bar{\xi}} = -\mathbf{x}_{\bar{C},\bar{\xi}} = -\bar{\mathbf{t}} & \mathbf{d}_{,\xi\bar{\xi}} = \mathbf{d}_{,\bar{\xi}\xi} = \mathbf{0} \\ \mathbf{d}_{,\xi\xi} = \mathbf{x}_{C,\xi\xi} = \mathbf{t}_{,\xi} & \mathbf{d}_{,\bar{\xi}\bar{\xi}} = -\mathbf{x}_{\bar{C},\bar{\xi}\bar{\xi}} = -\bar{\mathbf{t}}_{,\bar{\xi}} & \mathbf{t}_{,\xi\bar{\xi}} = \bar{\mathbf{t}}_{,\bar{\xi}\xi} = \mathbf{0} \end{array}} \tag{10}$$

2.2 Contact variables

The contact element formulation requires the definition of a set of geometrical variables, identified as g_N, g_T and $\bar{g}_T$. The first one provides the distance between the external surfaces of the beams, hence it is the crucial one to detect if contact takes place or not. The others permit to compute the amount of tangential displacement/sliding along each beam, and are involved only when friction between the beams is considered.

2.2.1 Definition

When the minimal distance points have been identified, the measure of the distance between the external surfaces of the beams is then obtained simply starting from the minimal distance, d, by taking into account the radii of the two circular beams, r and $\bar{r}$, which gives

$$g_N = d - (r + \bar{r}) \tag{11}$$

In this case a restriction to beams with circular cross-section is introduced. Moreover in-plane deformations of the beam section are disregarded, hence both r and $\bar{r}$ are treated as constants. However this limitation does not affect the mainstream of the proposed formulation. Non-circular and deformable sections will result simply into additional contributions.

A completely general framework for the tangential displacement/sliding along the beams is considered for the frictional case. In such formulation the geometry deals with a regularized model, where small tangential displacements are considered also during the stick phase. The classical stick-slip Coulomb model can be recovered as limit case. With reference to Figure (1), we identify with ξ_{C_0} the initial contact position. If a tangential force is then applied, regardless from the stick or the slip phase, the centroid of the beam will assume a new position, identified with ξ_C. Hence the tangential displacement along a centroid can be computed as an integral of the first derivative of the centroid equation

$$g_T = \int_{\xi_{C_0}}^{\xi_C} \|\mathbf{x}_{,\xi}\| \, \mathrm{d}\xi \tag{12}$$

The problem can be greatly simplified if we consider the fact that ξ_C is very close to ξ_{C_0}, i.e.

$$\xi_C - \xi_{C_0} \approx \mathrm{d}\xi \tag{13}$$

Within this hypothesis, for $\xi_{C_0} \leq \xi \leq \xi_C$ the norm of the tangent can be considered as a constant. Also, from (6) we see that $t = \|\mathbf{x}_{C,\xi}\| = \|\mathbf{x}_{,\xi}\|_C$ is the norm of the local tangent vector, which gives

$$g_T = \|\mathbf{t}\| \int_{\xi_{C_0}}^{\xi_{C_0}+\mathrm{d}\xi} \mathrm{d}\xi = t \, \mathrm{d}\xi = t \, (\xi_C - \xi_{C_0}) \tag{14}$$

Hence in such a context the tangential displacement takes place along the local tangent, and it results into a linear equation.

Equations resume. The following geometrical variables constitute hence the whole set needed for the contact element formulation

$$\boxed{g_N = d - (r + \bar{r}) \qquad g_T = t\,(\xi_C - \xi_{C_0}) \qquad \bar{g}_T = \bar{t}\left(\bar{\xi}_{\bar{C}} - \bar{\xi}_{\bar{C}_0}\right)} \tag{15}$$

2.2.2 Variations

The element geometry and the solution method determine the characteristics of the contribution that has to be added to the global equation system in case of active contact. More in detail, the requested terms can be easily determined by writing the virtual work contribution of the contact forces [2, 3] and computing its linearization. For this purpose both virtual variations and linearizations of the geometrical terms have to be computed. It can be easily seen that the terms required are the following

$$\delta g_N,\ \Delta g_N,\ \Delta\delta g_N, \quad \delta g_T,\ \Delta g_T,\ \Delta\delta g_T, \quad \delta\bar{g}_T,\ \Delta\bar{g}_T,\ \Delta\delta\bar{g}_T \tag{16}$$

where the symbols δ and Δ have been introduced, respectively, to distinguish among virtual variation and linearization. Due to the equivalent way to compute virtual variations and linearizations, and due to the similarities for the tangential displacements along the beams, only the effort to compute the following subset of the above terms is really required

$$\delta g_N,\ \Delta\delta g_N, \quad \delta g_T,\ \Delta\delta g_T \tag{17}$$

Considering the first one, starting from the definition (11), where the radii are considered constant, we have

$$\delta g_N = \delta d \tag{18}$$

Considering the second order term we get simply

$$\Delta\delta g_N = \Delta\delta d \tag{19}$$

Also, considering the frictional effects, regardless from the stick or the slip state, as shown in [3, 5], the term ξ_{C_0} has to be treated as a constant, hence from (15) it gives

$$\delta g_T = \delta t\,(\xi_C - \xi_{C_0}) + t\,\delta\xi_C \tag{20}$$

$$\Delta\delta g_T = \Delta\delta t\,(\xi_C - \xi_{C_0}) + \delta t\,\Delta\xi_C + \Delta t\,\delta\xi_C + t\,\Delta\delta\xi_C \tag{21}$$

Equations resume. From the practical point of view the following geometrical variables constitute the whole set that has to be determined for the contact element formulation

$$\boxed{\delta g_N \Rightarrow \delta d \quad \Delta\delta g_N \Rightarrow \Delta\delta d \qquad \delta g_T \Rightarrow \delta t,\ \delta\xi_C \qquad \Delta\delta g_T \Rightarrow \Delta\delta t,\ \Delta\delta\xi_C} \tag{22}$$

The key issue is hence the consistent linearization of the geometry vector norms, d and t, and of the projection parameter, ξ.

3 PRELIMINARIES ON VIRTUAL QUANTITIES

The consistent linearization of the required terms constitutes a hard task. Hence to perform it in a clear and efficient way some preliminary results have to be achieved.

3.1 Contact point coordinates

It has been shown in the previous paragraphs that the solution of the contact problem requires both the virtual variations and the linearizations of the geometrical parameters.

Due to the definition of the current coordinates (1), considering a minimal distance point we have

$$\delta \mathbf{x}_C = \delta \mathbf{X}_C(\xi) + \delta \mathbf{u}_C(\xi, \mathbf{u}) \tag{23}$$

i.e. the current coordinates of a centroid depends both on the parameter and on the displacement field. Hence, in the general case for a node on the centroid we have the following important result

$$\delta \mathbf{x}_C = \delta \mathbf{X}_C + \delta \mathbf{u}_C = \mathbf{X}_{C,\xi}\delta\xi + \mathbf{u}_{C,\xi}\delta\xi + \delta_u \mathbf{u}_C = \mathbf{x}_{C,\xi}\delta\xi + \delta_u \mathbf{u}_C \tag{24}$$

where the important notation δ_u has been introduced to represent the variation only with respect to the displacement field, $\mathbf{u}$. In the following a careful distinction has to be carried out between terms like $\delta\mathbf{u}_C = \mathbf{u}_{C,\xi}\delta\xi + \delta_u\mathbf{u}_C$ and $\delta_u\mathbf{u}_C$.

From the above equation we can easily compute its linearization, which gives

$$\Delta\delta \mathbf{x}_C = \Delta\mathbf{X}_{C,\xi}\delta\xi + \mathbf{X}_{C,\xi}\Delta\delta\xi + \Delta\mathbf{u}_{C,\xi}\delta\xi + \mathbf{u}_{C,\xi}\Delta\delta\xi + \Delta\delta_u\mathbf{u}_C \tag{25}$$

Applying once more the derivation rule outlined in (24), the above equation results in

$$\begin{aligned}\Delta\delta \mathbf{x}_C = {} & \mathbf{X}_{C,\xi\xi}\delta\xi\Delta\xi + \mathbf{X}_{C,\xi}\Delta\delta\xi + \mathbf{u}_{C,\xi\xi}\delta\xi\Delta\xi + \Delta_u\mathbf{u}_{C,\xi}\delta\xi \\ & + \mathbf{u}_{C,\xi}\Delta\delta\xi + \delta_u\mathbf{u}_{C,\xi}\Delta\xi + \Delta_u\delta_u\mathbf{u}_C\end{aligned} \tag{26}$$

and disregarding the higher order terms $\Delta_u\delta_u\mathbf{u}_C$ we have

$$\Delta\delta \mathbf{x}_C = \mathbf{x}_{C,\xi\xi}\delta\xi\Delta\xi + \mathbf{x}_{C,\xi}\Delta\delta\xi + \delta\xi\Delta_u\mathbf{u}_{C,\xi} + \delta_u\mathbf{u}_{C,\xi}\Delta\xi \tag{27}$$

In the same way it is possible to compute also the term $\delta\mathbf{x}_{C,\xi}$, which gives

$$\delta \mathbf{x}_{C,\xi} = \delta \mathbf{X}_{C,\xi} + \delta \mathbf{u}_{C,\xi} = \mathbf{X}_{C,\xi\xi}\delta\xi + \mathbf{u}_{C,\xi\xi}\delta\xi + \delta_u \mathbf{u}_{C,\xi} = \mathbf{x}_{C,\xi\xi}\delta\xi + \delta_u \mathbf{u}_{C,\xi} \tag{28}$$

Concerning the linearization, following the same procedure above employed we have

$$\Delta\delta \mathbf{x}_{C,\xi} = \Delta\mathbf{X}_{C,\xi\xi}\delta\xi + \mathbf{X}_{C,\xi\xi}\Delta\delta\xi + \Delta\mathbf{u}_{C,\xi\xi}\delta\xi + \mathbf{u}_{C,\xi\xi}\Delta\delta\xi + \Delta\delta_u\mathbf{u}_{C,\xi} \tag{29}$$

proceeding as usual and disregarding once more higher order terms, from the above equation we get

$$\begin{aligned}\Delta\delta \mathbf{x}_{C,\xi} = {} & \mathbf{X}_{C,\xi\xi\xi}\delta\xi\Delta\xi + \mathbf{X}_{C,\xi\xi}\Delta\delta\xi + \mathbf{u}_{C,\xi\xi\xi}\delta\xi\Delta\xi + \Delta_u\mathbf{u}_{C,\xi\xi}\delta\xi \\ & + \mathbf{u}_{C,\xi\xi}\Delta\delta\xi + \delta_u\mathbf{u}_{C,\xi\xi}\Delta\xi + \Delta_u\delta_u\mathbf{u}_{C,\xi} \\ = {} & \mathbf{x}_{C,\xi\xi\xi}\delta\xi\Delta\xi + \mathbf{x}_{C,\xi\xi}\Delta\delta\xi + \delta\xi\Delta_u\mathbf{u}_{C,\xi\xi} + \delta_u\mathbf{u}_{C,\xi\xi}\Delta\xi\end{aligned} \tag{30}$$

Equations resume. Results (24, 27, 28, 30) are then summarized using (10) as

$$\boxed{\begin{aligned} \delta \mathbf{x}_C =& \mathbf{t}\delta\xi + \delta_u \mathbf{u}_C \\ \Delta\delta \mathbf{x}_C =& \mathbf{t}_{,\xi}\delta\xi\Delta\xi + \mathbf{t}\Delta\delta\xi + \delta\xi\Delta_u \mathbf{u}_{C,\xi} + \delta_u \mathbf{u}_{C,\xi}\Delta\xi \\ \delta \mathbf{x}_{C,\xi} =& \mathbf{t}_{,\xi}\delta\xi + \delta_u \mathbf{u}_{C,\xi} \\ \Delta\delta \mathbf{x}_{C,\xi} =& \mathbf{t}_{,\xi\xi}\delta\xi\Delta\xi + \mathbf{t}_{,\xi}\Delta\delta\xi + \delta\xi\Delta_u \mathbf{u}_{C,\xi\xi} + \delta_u \mathbf{u}_{C,\xi\xi}\Delta\xi \end{aligned}} \tag{31}$$

It can be easily proved that, due to their structure both $\Delta\delta\mathbf{x}_C$ and $\Delta\delta\mathbf{x}_{C,\xi}$ result into symmetric contributions.

3.2 Geometry vectors

Considering the dependencies for the distance vector and the tangent vectors, from the definition (8) of $\mathbf{d}$, $\mathbf{t}$, $\bar{\mathbf{t}}$, and from a simple application of the results (31), the following equivalences hold for virtual variations and their linearization

$$\boxed{\begin{aligned} \delta\mathbf{d} =& \mathbf{t}\delta\xi - \bar{\mathbf{t}}\delta\bar{\xi} + \delta_u\mathbf{u}_C - \delta_u\bar{\mathbf{u}}_{\bar{C}} \\ \Delta\delta\mathbf{d} =& \mathbf{t}\Delta\delta\xi - \bar{\mathbf{t}}\Delta\delta\bar{\xi} + \mathbf{t}_{,\xi}\delta\xi\Delta\xi - \bar{\mathbf{t}}_{,\bar{\xi}}\delta\bar{\xi}\Delta\bar{\xi} \\ & + (\delta\xi\Delta_u\mathbf{u}_{C,\xi} + \delta_u\mathbf{u}_{C,\xi}\Delta\xi) - \left(\delta\bar{\xi}\Delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}} + \delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\Delta\bar{\xi}\right) \\ \delta\mathbf{t} =& \mathbf{t}_{,\xi}\delta\xi + \delta_u\mathbf{u}_{C,\xi} \\ \Delta\delta\mathbf{t} =& \mathbf{t}_{,\xi\xi}\delta\xi\Delta\xi + \mathbf{t}_{,\xi}\Delta\delta\xi + (\delta\xi\Delta_u\mathbf{u}_{C,\xi\xi} + \delta_u\mathbf{u}_{C,\xi\xi}\Delta\xi) \\ \delta\bar{\mathbf{t}} =& \bar{\mathbf{t}}_{,\bar{\xi}}\delta\bar{\xi} + \delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}} \\ \Delta\delta\bar{\mathbf{t}} =& \bar{\mathbf{t}}_{,\bar{\xi}\bar{\xi}}\delta\bar{\xi}\Delta\bar{\xi} + \bar{\mathbf{t}}_{,\bar{\xi}}\Delta\delta\bar{\xi} + \left(\delta\bar{\xi}\Delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}\bar{\xi}} + \delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}\bar{\xi}}\Delta\bar{\xi}\right) \end{aligned}} \tag{32}$$

4 CONSISTENT LINEARIZATION OF THE GEOMETRY VECTOR NORMS

4.1 Preliminaries: generic vector norm

Considering the norm, d, t and $\bar{t}$ of the above vectors, we outline first the general rule for the virtual variation of a vector norm, $\|\mathbf{v}\|$, which gives

$$\delta v = \delta \|\mathbf{v}\| = \delta\sqrt{\mathbf{v}\cdot\mathbf{v}} = \frac{1}{v}\mathbf{v}\cdot\delta\mathbf{v} \tag{33}$$

The linearization of the above equation results into

$$\Delta\delta\|\mathbf{v}\| = \Delta\left(\frac{1}{v}\mathbf{v}\cdot\delta\mathbf{v}\right) = \frac{1}{v^2}\left[v\left(\Delta\mathbf{v}\cdot\delta\mathbf{v} + \mathbf{v}\cdot\Delta\delta\mathbf{v}\right) - \Delta v\mathbf{v}\cdot\delta\mathbf{v}\right] \tag{34}$$

which can be more suitably expressed using (33) for Δv.

Equation resume. As a summary hence we have

$$\boxed{\delta v = \frac{1}{v}\mathbf{v}\cdot\delta\mathbf{v} \qquad \Delta\delta v = \frac{1}{v}\left(\mathbf{v}\cdot\Delta\delta\mathbf{v} + \Delta\mathbf{v}\cdot\delta\mathbf{v}\right) - \frac{1}{v^3}\delta\mathbf{v}\cdot\mathbf{v}\mathbf{v}\cdot\Delta\mathbf{v}} \tag{35}$$

4.2 Vector norm of the distance vector

With the above results and (32) we can easily compute δd as

$$\delta d = \frac{1}{d}\mathbf{d}\cdot\delta\mathbf{d} = \frac{1}{d}\mathbf{d}\cdot\left(\mathbf{t}\delta\xi - \bar{\mathbf{t}}\delta\bar{\xi} + \delta_u\mathbf{u}_C - \delta_u\bar{\mathbf{u}}_{\bar{C}}\right) \tag{36}$$

Due to the orthogonality conditions $\mathbf{d}\cdot\mathbf{t} = 0$ and $\mathbf{d}\cdot\bar{\mathbf{t}} = 0$, we have

$$\delta d = \frac{1}{d}\mathbf{d}\cdot(\delta_u\mathbf{u}_C - \delta_u\bar{\mathbf{u}}_{\bar{C}}) = \frac{1}{d}\mathbf{d}\cdot\delta_u\hat{\mathbf{u}}_C \tag{37}$$

where, to compact notation the symbol $\delta_u\hat{\mathbf{u}}_C = (\delta_u\mathbf{u}_C - \delta_u\bar{\mathbf{u}}_{\bar{C}})$ has been introduced.

For the second order term, $\Delta\delta d$, due to the simplifications related to the orthogonality conditions, instead of using (35) it is more convenient to directly linearize the above equation, which gives

$$\begin{aligned}\Delta\delta d = &\frac{1}{d}\left[\Delta\mathbf{d}\cdot(\delta_u\mathbf{u}_C - \delta_u\bar{\mathbf{u}}_{\bar{C}}) + \mathbf{d}\cdot\left(\delta_u\mathbf{u}_{C,\xi}\Delta\xi + \Delta_u\delta_u\mathbf{u}_C - \delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\Delta\bar{\xi} - \Delta_u\delta_u\bar{\mathbf{u}}_{\bar{C}}\right)\right]\\ &-\frac{1}{d^2}\left[\Delta d\mathbf{d}\cdot(\delta_u\mathbf{u}_C - \delta_u\bar{\mathbf{u}}_{\bar{C}})\right]\end{aligned} \tag{38}$$

Disregarding higher order terms and using (32) for the expanded version of $\Delta\mathbf{d}$, and (37) for the expanded expression of Δd we get

$$\begin{aligned}\Delta\delta d = &\frac{1}{d}\left[\delta_u\hat{\mathbf{u}}_C\cdot\left(\mathbf{t}\Delta\xi - \bar{\mathbf{t}}\Delta\bar{\xi} + \Delta_u\mathbf{u}_C - \Delta_u\bar{\mathbf{u}}_{\bar{C}}\right) + \mathbf{d}\cdot\left(\delta_u\mathbf{u}_{C,\xi}\Delta\xi - \delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\Delta\bar{\xi}\right)\right]\\ &-\frac{1}{d^3}\left[\delta_u\hat{\mathbf{u}}_C\cdot\mathbf{d}\mathbf{d}\cdot\Delta\hat{\mathbf{u}}_C\right]\end{aligned} \tag{39}$$

or, more suitably

$$\begin{aligned}\Delta\delta d = &\frac{1}{d}\left[\delta_u\hat{\mathbf{u}}_C\cdot\left(\mathbf{t}\Delta\xi - \bar{\mathbf{t}}\Delta\bar{\xi}\right) + \mathbf{d}\cdot\left(\delta_u\mathbf{u}_{C,\xi}\Delta\xi - \delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\Delta\bar{\xi}\right)\right]\\ &+\frac{1}{d}\left[\delta_u\hat{\mathbf{u}}_C\cdot\Delta\hat{\mathbf{u}}_C\right] - \frac{1}{d^3}\left[\delta_u\hat{\mathbf{u}}_C\cdot\mathbf{d}\mathbf{d}\cdot\Delta\hat{\mathbf{u}}_C\right]\end{aligned} \tag{40}$$

4.3 Vector norm of the tangent vector

Concerning the tangent vector, the computation in this case is extremely easy. From (35) and (32) we have immediately

$$\delta t = \frac{1}{t}\mathbf{t}\cdot\delta\mathbf{t} = \frac{1}{t}\mathbf{t}\cdot(\mathbf{t}_{,\xi}\delta\xi + \delta_u\mathbf{u}_{C,\xi}) \tag{41}$$

$$\begin{aligned}\Delta\delta t = &\frac{1}{t}\left[\mathbf{t}\cdot(\mathbf{t}_{,\xi\xi}\delta\xi\Delta\xi + \mathbf{t}_{,\xi}\Delta\delta\xi + \Delta_u\mathbf{u}_{C,\xi\xi}\delta\xi + \delta_u\mathbf{u}_{C,\xi\xi}\Delta\xi)\right.\\ &\left.(\mathbf{t}_{,\xi}\Delta\xi + \Delta_u\mathbf{u}_{C,\xi})\cdot(\mathbf{t}_{,\xi}\delta\xi + \delta_u\mathbf{u}_{C,\xi})\right]\\ &-\frac{1}{t^3}(\mathbf{t}_{,\xi}\Delta\xi + \Delta_u\mathbf{u}_{C,\xi})\cdot\mathbf{t}\mathbf{t}\cdot(\mathbf{t}_{,\xi}\delta\xi + \delta_u\mathbf{u}_{C,\xi})\end{aligned} \tag{42}$$

5 CONSISTENT LINEARIZATION OF THE PROJECTION PARAMETERS

5.1 Preliminaries: variational forms of the orthogonality conditions

We start from the consideration that the minimal distance vector, $\mathbf{d}$, is orthogonal to both the tangent vectors, $\mathbf{t}$ and $\bar{\mathbf{t}}$. Let us assume that a virtual displacement takes place for the centroids. Due to the displacements, the minimal distance points C, and $\bar{C}$ will move along the centroid. The amounts of sliding along the centroids are due the increments $\delta\xi$, $\delta\bar{\xi}$, and such values should permit to satisfy the orthogonality conditions once more in the new position, hence we can write

$$\mathbf{d}\cdot\mathbf{t}=0 \quad \Rightarrow \quad (\mathbf{d}+\delta\mathbf{d})\cdot(\mathbf{t}+\delta\mathbf{t})=0 \quad \Rightarrow \quad \mathbf{d}\cdot\mathbf{t}+\delta\mathbf{d}\cdot\mathbf{t}+\mathbf{d}\cdot\delta\mathbf{t}+\delta\mathbf{d}\cdot\delta\mathbf{t}=0 \quad (43)$$

Applying the original orthogonality conditions to the above result, and disregarding the higher order term we have

$$\delta\mathbf{d}\cdot\mathbf{t}+\mathbf{d}\cdot\delta\mathbf{t}=0 \quad \Rightarrow \quad \delta\left(\mathbf{d}\cdot\mathbf{t}\right)=0 \quad (44)$$

The procedure can be repeated starting from the above result and applying an increment, Δ, which gives

$$\delta\left(\mathbf{d}+\Delta\mathbf{d}\right)\cdot\left(\mathbf{t}+\Delta\mathbf{t}\right)+\left(\mathbf{d}+\Delta\mathbf{d}\right)\cdot\delta\left(\mathbf{t}+\Delta\mathbf{t}\right)=0 \quad (45)$$

$$\delta\mathbf{d}\cdot\mathbf{t}+\delta\mathbf{d}\cdot\Delta\mathbf{t}+\delta\Delta\mathbf{d}\cdot\mathbf{t}+\delta\Delta\mathbf{d}\cdot\Delta\mathbf{t}+\mathbf{d}\cdot\delta\mathbf{t}+\mathbf{d}\cdot\delta\Delta\mathbf{t}+\Delta\mathbf{d}\cdot\delta\mathbf{t}+\Delta\mathbf{d}\cdot\delta\Delta\mathbf{t}=0 \quad (46)$$

Also in this case, using orthogonality conditions in variational form (44) and disregarding higher order terms we have

$$\delta\Delta\mathbf{d}\cdot\mathbf{t}+\delta\mathbf{d}\cdot\Delta\mathbf{t}+\Delta\mathbf{d}\cdot\delta\mathbf{t}+\mathbf{d}\cdot\delta\Delta\mathbf{t}=0 \quad \Rightarrow \quad \Delta\delta\left(\mathbf{d}\cdot\mathbf{t}\right)=0 \quad (47)$$

Equations resume. The results (44) and (47) can be applied to both the centroids. As a summary we get the important results

$$\boxed{\begin{cases}\mathbf{d}\cdot\mathbf{t} &=0\\ \mathbf{d}\cdot\bar{\mathbf{t}} &=0\end{cases} \qquad \begin{cases}\delta\left(\mathbf{d}\cdot\mathbf{t}\right) &=0\\ \delta\left(\mathbf{d}\cdot\bar{\mathbf{t}}\right) &=0\end{cases} \qquad \begin{cases}\Delta\delta\left(\mathbf{d}\cdot\mathbf{t}\right) &=0\\ \Delta\delta\left(\mathbf{d}\cdot\bar{\mathbf{t}}\right) &=0\end{cases}} \quad (48)$$

The above equation set constitute a key tool to compute the contact parameters, as shown in the following.

5.2 Projection parameters

For the computation of the variations of the contact parameters, $\delta\xi$, $\delta\bar{\xi}$ and their linearization, $\Delta\delta\xi$, $\Delta\delta\bar{\xi}$, we use the virtual variation the orthogonality conditions (44) and (47). A straightforward application of the first one provides an equation set which gives us the variation of the parameters as functions of the displacement field. Using (32) for $\delta\mathbf{d}$ and $\delta\mathbf{t}$ we have

$$\begin{cases} \delta\mathbf{d}\cdot\mathbf{t}+\mathbf{d}\cdot\delta\mathbf{t}=\left(\mathbf{t}\delta\xi-\bar{\mathbf{t}}\delta\bar{\xi}+\delta_u\mathbf{u}_C-\delta_u\bar{\mathbf{u}}_{\bar{C}}\right)\cdot\mathbf{t}+\mathbf{d}\cdot\left(\mathbf{t}_{,\xi}\delta\xi+\delta_u\mathbf{u}_{C,\xi}\right) & =0 \\ \delta\mathbf{d}\cdot\bar{\mathbf{t}}+\mathbf{d}\cdot\delta\bar{\mathbf{t}}=\left(\mathbf{t}\delta\xi-\bar{\mathbf{t}}\delta\bar{\xi}+\delta_u\mathbf{u}_C-\delta_u\bar{\mathbf{u}}_{\bar{C}}\right)\cdot\bar{\mathbf{t}}+\mathbf{d}\cdot\left(\bar{\mathbf{t}}_{,\bar{\xi}}\delta\bar{\xi}+\delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\right) & =0 \end{cases} \tag{49}$$

The above problem can be simplified and arranged in matrix form as

$$\begin{bmatrix} \mathbf{t}\cdot\mathbf{t}+\mathbf{d}\cdot\mathbf{t}_{,\xi} & -\mathbf{t}\cdot\bar{\mathbf{t}} \\ \mathbf{t}\cdot\bar{\mathbf{t}} & -\bar{\mathbf{t}}\cdot\bar{\mathbf{t}}+\mathbf{d}\cdot\bar{\mathbf{t}}_{,\bar{\xi}} \end{bmatrix}\begin{bmatrix}\delta\xi\\ \delta\bar{\xi}\end{bmatrix}=\begin{bmatrix} -\left(\delta_u\mathbf{u}_C-\delta_u\bar{\mathbf{u}}_{\bar{C}}\right)\cdot\mathbf{t}-\delta_u\mathbf{u}_{C,\xi}\cdot\mathbf{d} \\ -\left(\delta_u\mathbf{u}_C-\delta_u\bar{\mathbf{u}}_{\bar{C}}\right)\cdot\bar{\mathbf{t}}-\delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\cdot\mathbf{d} \end{bmatrix} \tag{50}$$

If the problem is rewritten in compact form as

$$\begin{bmatrix} a & -b \\ b & \bar{a} \end{bmatrix}\begin{bmatrix}\delta\xi\\ \delta\bar{\xi}\end{bmatrix}=\begin{bmatrix} R_\delta \\ \bar{R}_\delta \end{bmatrix} \tag{51}$$

The solution can be rapidly achieved as

$$\delta\xi=\frac{\bar{a}R_\delta+b\bar{R}_\delta}{D}\qquad \delta\bar{\xi}=\frac{-bR_\delta+a\bar{R}_\delta}{D} \tag{52}$$

where $D=a\bar{a}+b^2$ is the determinant of the matrix of the coefficients.

For the computation of the higher order terms $\Delta\delta\xi$, $\Delta\delta\bar{\xi}$ we use the same approach using the third relation of (48), that can be suitably written also as

$$\begin{cases} \Delta\delta\left(\mathbf{d}\cdot\mathbf{t}\right) & =0 \\ \Delta\delta\left(\mathbf{d}\cdot\bar{\mathbf{t}}\right) & =0 \end{cases} \quad\Rightarrow\quad \begin{cases} \Delta\delta\mathbf{d}\cdot\mathbf{t}+\mathbf{d}\cdot\Delta\delta\mathbf{t} & =-\left(\delta\mathbf{d}\cdot\Delta\mathbf{t}+\Delta\mathbf{d}\cdot\delta\mathbf{t}\right) \\ \Delta\delta\mathbf{d}\cdot\bar{\mathbf{t}}+\mathbf{d}\cdot\Delta\delta\bar{\mathbf{t}} & =-\left(\delta\mathbf{d}\cdot\Delta\bar{\mathbf{t}}+\Delta\mathbf{d}\cdot\delta\bar{\mathbf{t}}\right) \end{cases} \tag{53}$$

The terms can now be expanded using (32), which gives a quite complex equation set

$$\begin{cases} \left(\mathbf{t}\Delta\delta\xi-\bar{\mathbf{t}}\Delta\delta\bar{\xi}+\mathbf{t}_{,\xi}\delta\xi\Delta\xi-\bar{\mathbf{t}}_{,\bar{\xi}}\delta\bar{\xi}\Delta\bar{\xi}\right. \\ \left.+\Delta_u\mathbf{u}_{C,\xi}\delta\xi-\Delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\delta\bar{\xi}+\delta_u\mathbf{u}_{C,\xi}\Delta\xi-\delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\Delta\bar{\xi}\right)\cdot\mathbf{t} \\ +\mathbf{d}\cdot\left(\mathbf{t}_{,\xi\xi}\delta\xi\Delta\xi+\mathbf{t}_{,\xi}\Delta\delta\xi+\Delta_u\mathbf{u}_{C,\xi\xi}\delta\xi+\delta_u\mathbf{u}_{C,\xi\xi}\Delta\xi\right) \quad = \\ -\left(\delta\mathbf{d}\cdot\Delta\mathbf{t}+\Delta\mathbf{d}\cdot\delta\mathbf{t}\right) \\ \\ \left(\mathbf{t}\Delta\delta\xi-\bar{\mathbf{t}}\Delta\delta\bar{\xi}+\mathbf{t}_{,\xi}\delta\xi\Delta\xi-\bar{\mathbf{t}}_{,\bar{\xi}}\delta\bar{\xi}\Delta\bar{\xi}\right. \\ \left.+\Delta_u\mathbf{u}_{C,\xi}\delta\xi-\Delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\delta\bar{\xi}+\delta_u\mathbf{u}_{C,\xi}\Delta\xi-\delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\Delta\bar{\xi}\right)\cdot\bar{\mathbf{t}} \\ +\mathbf{d}\cdot\left(\bar{\mathbf{t}}_{,\bar{\xi}\bar{\xi}}\delta\bar{\xi}\Delta\bar{\xi}+\bar{\mathbf{t}}_{,\bar{\xi}}\Delta\delta\bar{\xi}+\Delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}\bar{\xi}}\delta\bar{\xi}+\delta_u\bar{\mathbf{u}}_{\bar{C},\bar{\xi}\bar{\xi}}\Delta\bar{\xi}\right) \quad = \\ -\left(\delta\mathbf{d}\cdot\Delta\bar{\mathbf{t}}+\Delta\mathbf{d}\cdot\delta\bar{\mathbf{t}}\right) \end{cases} \tag{54}$$

Also in this case the structure of the equation can be reorganized in a suitable matrix form as

$$\begin{bmatrix} \mathbf{t}\cdot\mathbf{t}+\mathbf{d}\cdot\mathbf{t}_{,\xi} & -\mathbf{t}\cdot\bar{\mathbf{t}} \\ \mathbf{t}\cdot\bar{\mathbf{t}} & -\bar{\mathbf{t}}\cdot\bar{\mathbf{t}}+\mathbf{d}\cdot\bar{\mathbf{t}}_{,\bar{\xi}} \end{bmatrix}\begin{bmatrix}\Delta\delta\xi\\ \Delta\delta\bar{\xi}\end{bmatrix}=\begin{bmatrix} R_{\Delta\delta} \\ \bar{R}_{\Delta\delta} \end{bmatrix} \tag{55}$$

Quite interestingly, like in the previous case, the matrix of the coefficients is still the same, hence from (52) the solution is simply obtained as

$$\Delta\delta\xi = \frac{\bar{a}R_{\Delta\delta} + b\bar{R}_{\Delta\delta}}{D} \qquad \Delta\delta\bar{\xi} = \frac{-bR_{\Delta\delta} + a\bar{R}_{\Delta\delta}}{D} \tag{56}$$

6 CONCLUSIONS

The basic set of geometrical parameters that are needed for the Beam-to-Beam contact element formulation has been identified. Starting from their definition, both the virtual variations and the linearizations of the virtual variations have been computed. These results constitutes the crucial data set to build the stiffness matrix and the residual vector of the contact element. For this purpose these vector forms have to be rewritten in matrix form, with a suitable collection of the variational terms and of the constants. Concerning the distance vector, from (37) and (40) we get

$$\boxed{\begin{aligned} \delta d =& \frac{1}{d}\left[\delta_u \hat{\mathbf{u}}_C \cdot \mathbf{d}\right] \\ \Delta\delta d =& \frac{1}{d}\left[\delta_u \hat{\mathbf{u}}_C \cdot \left(\mathbf{t}\Delta\xi - \bar{\mathbf{t}}\Delta\bar{\xi}\right) + \mathbf{d} \cdot \left(\delta_u \mathbf{u}_{C,\xi}\Delta\xi - \delta_u \bar{\mathbf{u}}_{\bar{C},\bar{\xi}}\Delta\bar{\xi}\right)\right] \\ &+ \frac{1}{d}\left[\delta_u \hat{\mathbf{u}}_C \cdot \Delta\hat{\mathbf{u}}_C\right] - \frac{1}{d^3}\left[\delta_u \hat{\mathbf{u}}_C \cdot \mathbf{dd} \cdot \Delta\hat{\mathbf{u}}_C\right] \end{aligned}} \tag{57}$$

Concerning the tangent vector, from (41) and (42) we get

$$\boxed{\begin{aligned} \delta t =& \frac{1}{t}\left[(\delta\xi \mathbf{t}_{,\xi} + \delta_u \mathbf{u}_{C,\xi}) \cdot \mathbf{t}\right] \\ \Delta\delta t =& \frac{1}{t}\left[(\mathbf{t}_{,\xi\xi}\delta\xi\Delta\xi + \mathbf{t}_{,\xi}\Delta\delta\xi + \delta\xi\Delta_u \mathbf{u}_{C,\xi\xi} + \delta_u \mathbf{u}_{C,\xi\xi}\Delta\xi) \cdot \mathbf{t}\right] \\ &+ \frac{1}{t^3}\left[(\delta\xi \mathbf{t}_{,\xi} + \delta_u \mathbf{u}_{C,\xi}) \cdot (\mathbf{t}_{,\xi}\Delta\xi + \Delta_u \mathbf{u}_{C,\xi})\right] \\ &- \frac{1}{t^3}\left[(\delta\xi \mathbf{t}_{,\xi} + \delta_u \mathbf{u}_{C,\xi}) \cdot \mathbf{tt} \cdot (\mathbf{t}_{,\xi}\Delta\xi + \Delta_u \mathbf{u}_{C,\xi})\right] \end{aligned}} \tag{58}$$

Finally, considering the projection parameters, from (52) and (56) we get

$$\boxed{\begin{aligned} \delta\xi &= \frac{\bar{a}R_\delta + b\bar{R}_\delta}{D} & \delta\bar{\xi} &= \frac{-bR_\delta + a\bar{R}_\delta}{D} \\ \Delta\delta\xi &= \frac{\bar{a}R_{\Delta\delta} + b\bar{R}_{\Delta\delta}}{D} & \Delta\delta\bar{\xi} &= \frac{-bR_{\Delta\delta} + a\bar{R}_{\Delta\delta}}{D} \end{aligned}} \tag{59}$$

For a complete solution of the problem some more transformations are needed to get the explicit form of $\delta\xi$ and $\Delta\delta\xi$. The procedure is quite cumbersome, hence it is not detailed here, but it is evident that all the data for such transformation are at this stage of the game available.

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