POLITECNICO DI TORINO
Repository ISTITUZIONALE

Exact solutions for free vibration analysis of laminated beams by refined layer-wise theory

(Article begins on next page)

17 December 2025

EXACT SOLUTIONS FOR FREE VIBRATION ANALYSIS OF LAMINATED BEAMS BY REFINED LAYER-WISE THEORY

Y. Yan[1], A. Pagani[2], and E. Carrera[2]

[1]College of Mechanics and Materials, Hohai University, 210098, Nanjing, China
e-mail: yanyanghhu@hhu.edu.cn

[2] Mul[2], Department of Mechanical and Aerospace Engineering, Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy.
e-mail: {alfonso.pagani, erasmo.carrera@polito.it}

Keywords: Carrera unified formulation, Close-form solution, Layer-wise approach, Free vibration analysis.

Abstract. *The present work addresses a closed-form solution for the free vibration analysis of simply supported composite laminated beams via a refined one-dimensional (1D) model, which employs the Carrera Unified Formulation (CUF). In the framework of CUF, the 3D displacement field can be expanded as any order of generic unknown variables over the cross section, in the case of beam theories. Particularly, Lagrange expansions of cross-sectional displacement variables in conjunction with Layer-wise (LW) theory are adopted in this analysis. As a consequence, the governing equations can be derived using the principle of virtual work in a unified form and can be solved by a Navier-type, closed-form solution. Numerical investigation is carried out to test the performance of this novel method. The results are compared with those available in the literature as well as the 3D finite element method (FEM) solutions computed by commercial codes. The present CUF model is proved to be able of achieving high accurate results with less computational costs. Besides, they may serve as benchmarks for future assessments in this field.*

1 INTRODUCTION

Determination of vibration characteristics is of crucial importance in the safe design of composite beams subjected to dynamic loads. The classical beam theory under the assumption outlined by Euler-Bernoulli [1] is inadequate to analyze composite structures because it can not capture non- classical vibration modes with couplings between torsion, shear and bending.

Many refined 1D beam models have received wide attention over the last few decades and they can be divided into two categories: Equivalent Single Layer (ESL) and Layer-wise theories. ESL hypothesizes a continuous and differentiable displacement function through the thickness direction. Some of the relevant theories are developed within this framework, e.g., first-order shear deformation theory (FSDT) [2], high-order shear deformation theories (HSDT) [3, 4]. Unfortunately, this approach cannot account for the continuity of the transverse stresses and the zig-zag behavior of the displacements along the thickness. In the domain of LW, a continuous displacement function is adopted for each layer, and a discontinuous derivative of displacement function is imposed at the intra-layer interfaces, see [5, 6].

LW theory can increase the accuracy of results significantly but also computation costs. A unified beam formulation is proposed in this work within the framework of the Carrera Unified Formulation [7], which makes use of Lagrange polynomials expansion (LE) to express the three-dimensional (3D) displacement field via arbitrary order approximation of pure displacement variables at each layer over the cross section, in a LW sense. According to Carrera et al. [8] and Dan et al. [9], the 1D classical FEM shape functions and Navier-type closed-form solution were adopted for the free vibration analysis of laminated beams and isotropic beams, respectively.

In the present paper, the same analytical solution is utilized for the free vibrations of cross-ply composite beam with compact and thin-walled cross sections subjected to the simply supported boundary conditions based on 1D CUF LE model and LW theory. The rest of this paper is structured as follows: (i) a brief introduction of 1D CUF LE theory are given in Section 2; (ii) The equation of motion is obtained using the Navier-type closed-form solution in Section 3; (iii) The numerical results of different assessments considered are presented in Section 4; (iv) some conclusions and remarks of this work are outlined in the last section.

2 1D CUF beam theory

Consider a multi-layer laminated beam in physical coordinate system, as shown in Fig. 1. The generic displacement field, within the framework of CUF theory, can be expanded as arbitrary functions F_τ:

$$\mathbf{u}(x, y, z; t) = F_\tau(x, z)\mathbf{u}_\tau(y; t) \qquad \tau = 1, 2, ..., M \tag{1}$$

where F_τ is a function depending on the x and z coordinates. $\mathbf{u}_\tau$ is the generic displacements vector of axial coordinates y. M is the number of expanded terms, and the repeated subscript, τ, stands for summation.

In this study, Lagrange expansion polynomials are employed as the function F_τ to approximate the displacement field above arbitrarily complex cross sections. The kinematics of a refined beam model approximated with one single nine-node quadratic Lagrange polynomial (L9) are presented here as an illustrative example:

$$u_x = F_1 u_{x_1} + F_2 u_{x_2} + F_3 u_{x_3} + F_4 u_{x_4} + F_5 u_{x_5} + F_6 u_{x_6} + F_7 u_{x_7} + F_8 u_{x_8} + F_9 u_{x_9}$$

$$u_y = F_1 u_{y_1} + F_2 u_{y_2} + F_3 u_{y_3} + F_4 u_{y_4} + F_5 u_{y_5} + F_6 u_{y_6} + F_7 u_{y_7} + F_8 u_{y_8} + F_9 u_{y_9} \tag{2}$$

$$u_z = F_1 u_{z_1} + F_2 u_{z_2} + F_3 u_{z_3} + F_4 u_{z_4} + F_5 u_{z_5} + F_6 u_{z_6} + F_7 u_{z_7} + F_8 u_{z_8} + F_9 u_{z_9}$$
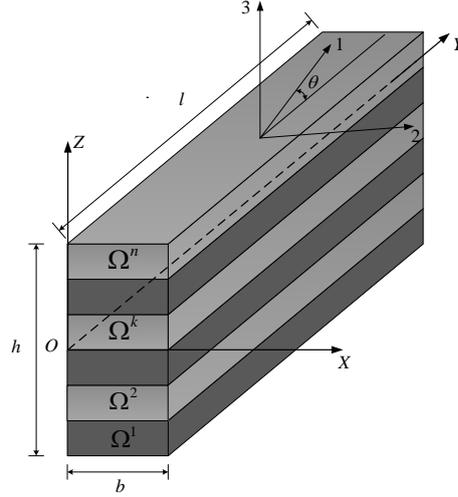
Figure 1: Physical coordinate system for a laminated composite beam.

For the sake of brevity, the explicit expressions of the function F_τ are not reported here, but they can be found in [7].

3 Analytical solution

In the case of simply supported composite beam, the displacement fields are assumed as a sum of harmonic functions:

$$\begin{aligned}
u_{xs}(y;t) &= U_{xs}\sin(\alpha y)e^{i\omega t} \\
u_{ys}(y;t) &= U_{ys}\cos(\alpha y)e^{i\omega t} \\
u_{zs}(y;t) &= U_{zs}\sin(\alpha y)e^{i\omega t}
\end{aligned} \tag{3}$$

where α is:

$$\alpha = \frac{m\pi}{l} \tag{4}$$

U_{xs}, U_{ys} and U_{zs} are the components of the generalized displacements vector. m is the half wave number along the beam axis, ω is the vibrational natural frequency and i is the imaginary unit. A compact form of the equations of motion can be obtained in a matrix form through the variational principle of virtual work:

$$(\mathbf{K}^{\tau s} - \omega^2 \mathbf{M}^{\tau s})\boldsymbol{U} = 0 \tag{5}$$

Eq. (5) is assessed for kth layer and can be assembled into a global algebraic eigensystem in the light of contribution of each layer. LW theory is used to fulfill this procedure, which can be referred to Pagani et al. [10] for the sake of simplicity. In this paper, LW models are implemented by utilizing one or more LE expansions on the cross-sectional domain of each layer, as discussed in the following sections. As a consequence, the theory kinematics can be opportunely varied at layer level by setting the order of LE expansions. This characteristic of LE CUF beam models allows the implementation of higher-order LW models in an easy and straightforward manner.

4 Numerical results

A square cross-section beam, consisting of two-layer [0°/90°] laminates of the same thickness, is considered, see Fig. 2. The dimensions of the beam are: $l/b = 5$m, $b = 0.2$m,
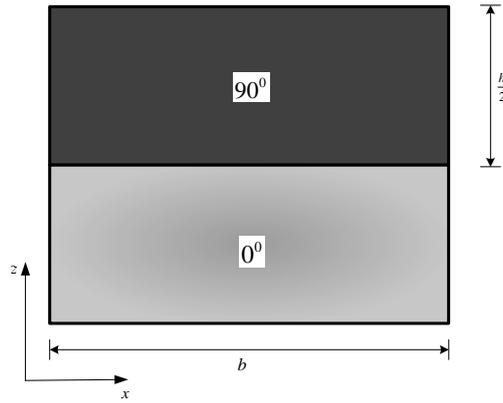
Figure 2: Cross section of a two-layer composite beam.

$h = 0.2$m. The material is assumed to be orthotropic with the following properties: Young modulus: $E_L = 250$ GPa, $E_T = 10$ GPa; Poisson ratio: $\nu_{LT} = \nu_{TT} = 0.33$; material density: $\rho = 2700$ kg/m^3; shear modulus: $G_{LT} = 5$ GPa, $G_{TT} = 2$ GPa, where the subscripts L and T represent the direction parallel and perpendicular to the fibres, respectively.

Unless differently specified, we use the notation $\zeta \times \eta \mathrm{L}\beta$ to denote beams of square cross sections, where ζ and η stand for the number of Lβ elements in the x direction and z direction, and β stands for bilinear (4), second-order (9) and cubic (16) Lagrange polynomials, respectively.

Table 1 shows a list of the first five non-dimensional natural frequencies with one half wave number ($m = 1$). Degrees of freedoms (DOFs) for different models are also reported in the second column. The results obtained by various LE models are compared with the classical beam models, including Euler-Bernoulli beam model (EBBM) and Timoshenko beam model (TBM), and refined closed-form CUF-TE solutions provided by Giunta et al. [11]. Three-dimensional finite element model created by Ansys software also serves as a benchmark for the same assessment, where the quadratic solid element SOLID 186 is used. Two different mesh schemes (coarse mesh and refined mesh) are adopted to ensure the convergency, where FEM 3D$_n$ stands for the n elements, $n \times 10$ elements and n elements along the x-axis, y-axis and z-axis, respectively. The non-dimensional natural frequency ω^* can be computed by the following equation:

$$\omega^* = (\omega l^2/b)\sqrt{\rho/E_T} \tag{6}$$

From Table 1, we can see that lower-order CUF-LE models (1×2L4 and 2×2L4) and refined lower-order CUF-TE model ($N = 2$) yield poor results in mode 4 and mode 5; namely, axial, shear modes and their couplings. Conversely, higher-order models making use of L9 and L16 LW approximation can produce the same results as 3D FEM solutions with less computational costs.

Mode 1 and mode 2 concerning two-layer composite beams obtained by 2×2L16 model are shown in Fig. 3. Out of two graphs, it should be underlined that coupled flexural/torsion appears due to the unsymmetric lamination in consideration.

5   Conclusions

In this paper, a unified closed-form formulation of refined beam models has been extended to the free vibration of simply supported cross-ply composite beams. The analysis has been

Table 1: First five non-dimensional natural frequencies ω^* for a two-layer composite beam $[0°/90°]$ with $m=1$, $L/b = 5$

Model	DOFs	mode 1[a]	mode 2[b]	mode 3[c]	mode 4[d]	mode 5[e]
FEM 3D$_{20}$[g][11]	1037043	4.9357	6.4491	9.0672	33.566	50.448
FEM 3D$_6$[h][11]	33159	4.9387	6.4520	9.0698	33.564	50.441
TBM [11]	10	5.0748	7.5056	-[f]	40.959	-
EBBM [11]	6	6.0098	10.104	-	57.194	-
Refined CUF-TE Theory [11]						
$N = 2$	18	5.0561	6.9642	10.134	37.566	63.563
$N = 10$	198	4.9413	6.4779	9.1134	33.910	50.923
$N = 15$	408	4.9388	6.4663	9.0958	33.803	50.749
$N = 23$	900	4.9375	6.4603	9.0852	33.718	50.640
Present CUF-LE theory						
$1 \times 2L4$	18	5.0529	6.8718	9.7712	36.406	60.331
$2 \times 2L4$	27	5.0528	6.8698	9.7710	36.406	60.305
$1 \times 2L9$	45	5.0186	6.6664	9.4863	33.624	55.646
$2 \times 2L9$	75	5.0185	6.4716	9.1681	33.623	51.560
$1 \times 2L16$	84	4.9359	6.4518	9.0753	33.568	50.736
$2 \times 2L16$	147	4.9358	6.4504	9.0708	33.568	50.564

[a]: Flexural mode on plane yz

[b]: Flexural (plane xy)/torsional mode
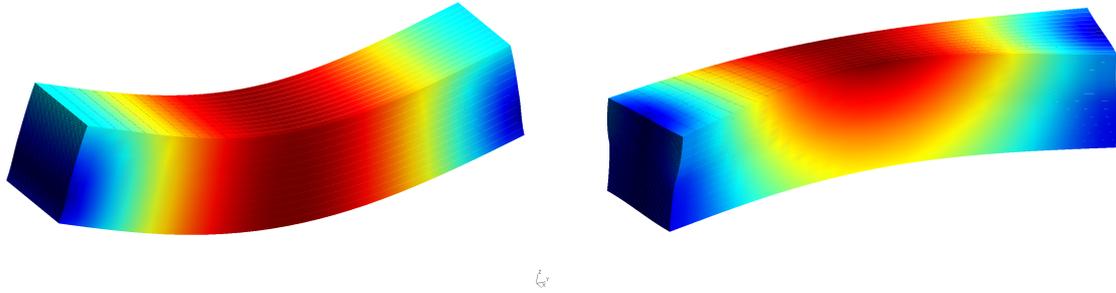
[c]: Torsional mode

[d]: Axial/shear (plane yz) mode

[e]: Shear mode on plane xz

[f]: Mode not provided by the theory

[g]: The number of elements is $20 \times 200 \times 20$

[h]: The number of elements is $6 \times 60 \times 6$

(a) Mode 1, Flexural mode on plane yz of a two-layer laminated beam ($L/b = 5$).

(b) Mode 2, Flexural (plane xy)/torsional mode of a two-layer laminated beam ($L/b = 5$)

Figure 3: Selected mode shapes of two-layer laminated beams of Table 1 via $2 \times 2L16$ model, $m=1$.

performed in the domain of Carrera Unified Formulation, where 3D kinematic fields can be discretized as the expansion of any order of the cross-sectional node displacement unknowns via Lagrange Expansion (LE), being the ability of Layer-wise naturally satisfied. A numerical case has been carried out to demonstrate the accuracy and effectiveness of the proposed methodology in comparison with those from the literature. From these results, the following conclusions can be drawn:

- LE CUF model are considered to yield similar results as 3D FEM results, and more accurately than TE CUF model.

- Non-classical modes such as torsion, shear and axial/shear coupling modes can be detected with higher-order CUF LE model.

Acknowledgments

REFERENCES

[1] W. A. Oldfather, C. A. Ellis, and D. M. Brown, Leonhard euler's elastic curves. *Isis*, **20**, 72–160, 1933.

[2] K. Chandrashekhara, K. Krishnamurthy, and S. Roy, Free vibration of composite beams including rotary inertia and shear deformation. *Composite Structures*, **14**, 269–279, 1990.

[3] A. A. Khdeir and J. N. Reddy, Free vibration of cross-ply laminated beams with arbitrary boundary conditions. *International Journal of Engineering Science*, **32**, 1971–1980, 1994.

[4] J. Li and H. X. Hua, Dynamic stiffness analysis of laminated composite beams using trigonometric shear deformation theory. *Composite Structures*, **89**, 433–442, 2009.

[5] R. P. Shimpi and A. V. Ainapure, Free vibration analysis of two layered cross-ply laminated beams using layer-wise trigonometric shear deformation theory. *Journal of Reinforced Plastics and Composites*, **21**, 1477–1492, 2002.

[6] M. Tahani, Analysis of laminated composite beams using layerwise displacement theories. *Composite Structures*, **79**, 535–547, 2007.

[7] E. Carrera and M. Petrolo, Refined beam elements with only displacement variables and plate/shell capabilities. *Meccanica*, **47**, 537–556, 2012.

[8] E. Carrera, M. Filippi, P. K. R. Mahato, and A. Pagani, Advanced models for free vibration analysis of laminated beams with compact and thin-walled open/closed sections. *Journal of Composite Materials*, **49**, 2085–2101, 2015.

[9] M. Dan, A. Pagani, and E. Carrera, Free vibration analysis of simply supported beams with solid and thin-walled cross-sections using higher-order theories based on displacement variables. *Thin-Walled Structures*, **98**, 478–495, 2016.

[10] A. Pagani, A. G. De Miguel, M. Petrolo, and E. Carrera, Analysis of laminated beams via unified formulation and legendre polynomial expansions. *Composite Structures*, **156**, 78–92, 2016.

[11] G. Giunta, F. Biscani, S. Belouettar, A. J. M. Ferreira, and E. Carrera, Free vibration analysis of composite beams via refined theories. *Composites Part B: Engineering*, **44**, 540–552, 2013.