Stochastic transmission line analysis via polynomial chaos methods: an overview

Stochastic Transmission Line Analysis Via Polynomial Chaos Methods: an Overview

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***Abstract*—The aim of this article is to provide an overview of polynomial chaos (PC) based methods for the statistical analysis of transmission lines. The underlying idea of PC is to represent stochastic line voltages and currents as expansions of predefined orthogonal polynomials. The determination of the expansion coefficients allows obtaining pertinent statistical information and is generally much faster than running, e.g., a Monte Carlo (MC) analysis. There exist several approaches to calculate the PC expansion coefficients. The article briefly reviews virtually all existing methods, whilst focusing on the popular and accurate stochastic Galerkin (SG) method as well as on the recent, more efficient and non-intrusive formulation of the so-called stochastic testing (ST) method. These two techniques are introduced by way of a simple illustrative example, i.e., a single-wire line running above a ground plane. Numerical comparisons in terms of accuracy and efficiency are also provided for a four-wire line.**

***Index Terms*—Multiconductor transmission lines, polynomial chaos, statistical analysis, stochastic Galerkin method, stochastic testing, uncertainty, variability analysis.**

I. Introduction

Signal integrity and EMC assessments often require statistical analyses due to the inherently variable nature of the problem under investigation [1]–[6]. Indeed, design specifications like wire position and routing, or fabrication tolerances, affect the electrical performance of interconnects in a way that is difficult or even impossible to predict deterministically. As a result, even with the most accurate simulation models, the actual performance can largely differ from early-stage predictions due to an unaccounted variability in the design parameters. Therefore, right-the-first-time computer-aided designs require variation-aware simulation models.

The computational inefficiency of traditional tools implemented in many circuit simulators, like the Monte Carlo (MC) method [7], has prompted a wide interest in effective stochastic techniques for circuit modeling and, in particular, for transmission-line design. This article provides an overview of a class of recent methods, based on polynomial chaos (PC) [8], [9] that were proposed for the efficient statistical analysis of transmission line circuits. PC seeks for an approximation of stochastic transmission line responses in terms of an expansion in orthogonal polynomials. The expansion coefficients can be computed using different approaches and directly provide relevant statistical information such as expected value and variance. Moreover, the PC expansion (PCE) can be more generally used as a computationally cheap, yet accurate macromodel to extract other statistical properties like higher-order statistical moments (e.g., skewness and kurtosis) or distribution functions.

The article is organized as follows. Section II states the problem. Section III introduces the general basics of the PC approach. Sections IV and V review the existing methods for the calculation of the PCE coefficients, with specific emphasis on the so-called stochastic Galerkin (SG) and stochastic testing (ST) methods. These are outlined based on an illustrative example. A further application example is provided in Section VI to compare the aforementioned two techniques. Finally, conclusions are drawn in Section VII.

II. Problem Statement

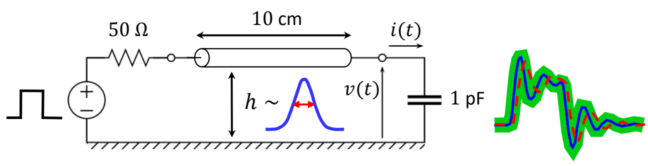

Fig. 1. Illustration of a stochastic transmission line problem: the deterministic excitation of a wire at random height produces a random response.

Consider the situation illustrated in Fig. 1, showing a lossless, dispersion-free single-wire transmission line affected by one random parameter, namely the height h above a perfect conducting ground plane. Owing to this variability, a deterministic excitation produces a stochastic response. Indeed, a different random sample of h yields a different realization of the voltage and current propagating along the line, and a description from a statistical standpoint is therefore necessary.

For convenience, the random parameter is first expressed as a function of a *standardized* random variable (RV) by suitable translation and rescaling. For example, for a Gaussian model, the height is expressed as

$$h = \mu_h + \sigma_h \xi \tag{1}$$

where μ_h and σ_h are the average and standard deviation of h, respectively, whereas ξ is a standard normal RV with zero mean and unit variance.

In the time domain, the stochastic version of the well-known Telegrapher's equations [10], relating the voltage and current propagating along the lossless, dispersion-free wire of Fig. 1, read

$$\frac{\partial}{\partial z}v(z,t,\xi) = -l(\xi)\frac{\partial}{\partial t}i(z,t,\xi) \tag{2a}$$

$$\frac{\partial}{\partial z}i(z,t,\xi) = -c(\xi)\frac{\partial}{\partial t}v(z,t,\xi) \tag{2b}$$

where the pertinent per-unit-length (p.u.l.) inductance l and capacitance c depend on the random parameter ξ as follows:

$$l(\xi) = \frac{\mu}{2\pi}\cosh^{-1}\left(\frac{h}{r_w}\right) = \frac{\mu}{2\pi}\cosh^{-1}\left(\frac{\mu_h + \sigma_h\xi}{r_w}\right) \tag{3a}$$

$$c(\xi) = \frac{2\pi\epsilon}{\cosh^{-1}\left(\frac{h}{r_w}\right)} = \frac{2\pi\epsilon}{\cosh^{-1}\left(\frac{\mu_h + \sigma_h\xi}{r_w}\right)}. \tag{3b}$$

In turn, the voltage and current are also ξ-dependent, and hence, inherently stochastic. For a multiconductor transmission line (MTL), (2) becomes a matrix equation and the discussion that follows is readily generalized. Moreover, the frequency-domain counterpart of (2), at a given angular frequency ω, is readily obtained by considering the phasor representation of the voltage and the current, and by replacing the time-derivatives with $j\omega$. Frequency-dependent losses are included in the frequency domain by considering a complex and frequency-dependent p.u.l. inductance and capacitance [11].

III. The Polynomial Chaos Approach

The traditional approach to cope with a stochastic problem like (2) was to perform deterministic simulations for a large number of realizations of the random parameters, in order to collect samples of the random response [3]. This approach, known as MC, converges very slowly, requiring a number of samples typically on the order of 10^4 and thus making it very inefficient or even prohibitive when a single numerical solution of (2) requires a non-negligible time.

The underlying idea of PC is instead to represent stochastic unknowns (in this case, voltages and currents) as PCEs, e.g.,

$$v(t,\xi) \approx v_0(t)\phi_0(\xi) + v_1(t)\phi_1(\xi) + v_2(t)\phi_2(\xi) + \dots \tag{4}$$

where $v_0(t), v_1(t), v_2(t), \dots$ are the yet-to-be-determined PCE coefficients and $\phi_0, \phi_1, \phi_2, \dots$ are polynomials of increasing degree in the standardized variable ξ. They are orthonormal according to the inner product defined as

$$\langle \phi_m, \phi_n \rangle = \int_{-\infty}^{+\infty} \phi_m(\xi)\phi_n(\xi)w(\xi)d\xi = \delta_{mn} \tag{5}$$

meaning that the above integral is non-zero only when $m = n$. The function $w(\xi)$ appearing in the integrand is the probability density function (PDF) of ξ, which is defined by the statistical model ascribed to the random parameters in the problem. For the sake of illustration, this article focusses on the common case of Gaussian variability, for which the above PDF is $w(\xi) = e^{-\xi^2/2}/\sqrt{2\pi}$. The corresponding orthonormal polynomials that satisfy (5) are the normalized Hermite polynomials, the first three being

$$\phi_0 = 1 \tag{6a}$$

$$\phi_1 = \xi \tag{6b}$$

$$\phi_2 = (\xi^2 - 1)/\sqrt{2}. \tag{6c}$$

The advantage of using a PCE is that its coefficients (e.g., $v_0(t), v_1(t), v_2(t), \dots$ in (4)) are usually computed much faster than running a MC simulation. The representation (4) is then used as an analytical and computationally-cheap macromodel to rapidly obtain a large number of samples of the actual response, from which statistical information such as stochastic moments or distribution functions can be computed. Moreover, the first two statistical moments (e.g., the expected value, or mean, and the variance) are readily given by

$$\mathrm{E}\{v(t)\} = v_0(t) \tag{7}$$

i.e., the zero-order coefficient, and

$$\mathrm{Var}\{v(t)\} = v_1^2(t) + v_2^2(t) + \dots \tag{8}$$

i.e., the sum of the squares of all remaining higher-order coefficients, respectively. A second-order expansion turns out to be sufficiently accurate in most cases. Hence, in the remainder of this article, polynomials of higher degree will be neglected in the PCE (4) if not otherwise specified. It is also worth noting that, for a given stochastic problem, the PCE coefficients are *deterministic* quantities (i.e., they *do not* depend on ξ).

The generalization to multiple *independent* RVs is relatively straightforward and it amounts to considering multivariate basis functions obtained as product combinations of the univariate polynomials, along with the corresponding multidimensional extension of the integral (5). On the contrary, for dependent RVs the framework becomes much more involved [9]. In the following sections, the two main state-of-the-art techniques for the determination of the PCE coefficients in (4), namely the SG method and the class of stochastic collocation (SC) methods, are outlined. For the sake of brevity, only the developments for the first equation (2a) are explicitly discussed. The dual equation (2b) is treated in an analogous fashion.

IV. Stochastic Galerkin Method

The first application of the SG method to transmission line analysis was proposed in [12]. The method starts by representing the p.u.l. parameters as PCEs like (4), i.e.,

$$\begin{aligned} l(\xi) &\approx l_0\phi_0(\xi) + l_1\phi_1(\xi) + l_2\phi_2(\xi) \\ &= l_0 + l_1 \cdot \xi + l_2 \cdot (\xi^2 - 1)/\sqrt{2}. \end{aligned} \tag{9}$$

For the simple case of Fig. 1, the coefficients of the above expansion are readily computed by projection of the analytical expression of the p.u.l. parameters (3) onto the basis functions:

$$l_k = \langle l, \phi_k \rangle = \int_{-\infty}^{+\infty} l(\xi)\phi_k(\xi)e^{-\xi^2/2}/\sqrt{2\pi}d\xi \tag{10}$$

for $k = 0, 1, 2$. The above integral can be evaluated using any numerical integration technique.

For general cylindrical structures, possibly including dielectric coating and/or a shield, an efficient numerical scheme that also accounts for proximity effects was proposed in [13]. For all other cases (e.g., layered structures), a numerical projection must be implemented. This is typically done by means of Gauss quadratures [14], [15].

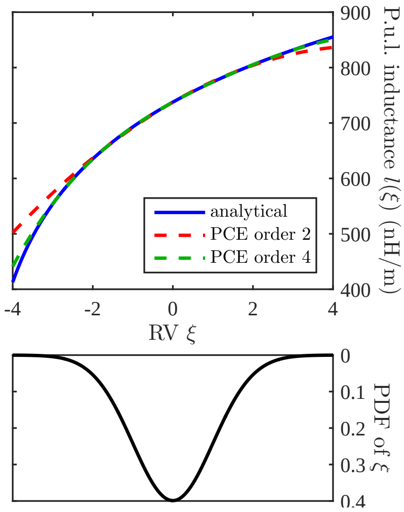

Fig. 2. Approximation of the wire p.u.l. inductance by means of a PCE. Bottom panel: PDF of the random parameter ξ; top left panel: actual value as a function of ξ (solid blue line) and approximations obtained with a second-order (dashed red line) and fourth-order (dashed green line) PCE; right panel: actual PDF of the inductance (solid blue line) and second- and fourth-order PCE approximations (dashed red and green lines, respectively).

The second-order PCE coefficients of the p.u.l. inductance and capacitance (3) computed for $r_w = 0.5$ mm, $\mu_h = 1$ cm and $\sigma_h = 0.2$ cm by means of the method in [13] are

$$l_0 = 733.37,\ l_1 = 41.79,\ l_2 = -6.05 \quad [\text{nH/m}] \tag{11a}$$
$$c_0 = 15.224,\ c_1 = -0.894,\ c_2 = 0.199 \quad [\text{pF/m}]. \tag{11b}$$

Fig. 2 shows in the top left panel the corresponding parametric approximation provided by the PCE (9) (dashed red line) of the actual value of the p.u.l. inductance (3a) (solid blue line) as a function of ξ. The bottom panel shows the PDF of ξ. It is interesting to note that the PCE becomes progressively inaccurate for larger values of $|\xi|$, which are however unlikely to occur. Hence, the overall accuracy in reproducing the PDF of the inductance (top right panel) is satisfactory. As expected, the approximation obtained with a higher-order expansion (dashed green line) exhibits even better accuracy owing to the convergence properties of the PCE. This simple example illustrates the key idea underlying the use of PC.

Substituting the second-order PCEs of the voltage, current, and p.u.l. parameters into (2), leads to

$$\begin{aligned}&\tfrac{\partial}{\partial z}v_0\phi_0 + \tfrac{\partial}{\partial z}v_1\phi_1 + \tfrac{\partial}{\partial z}v_2\phi_2 \\ &\quad = -(l_0\phi_0 + l_1\phi_1 + l_2\phi_2)(\tfrac{\partial}{\partial t}i_0\phi_0 + \tfrac{\partial}{\partial t}i_1\phi_1 + \tfrac{\partial}{\partial t}i_2\phi_2)\end{aligned} \tag{12}$$

where the arguments have been omitted for brevity of notation.

Galerkin testing, i.e., the multiplication of both the left- and right-hand side of (12) by ϕ_m and projection using (5), allows to eliminate the ξ-dependency, thus obtaining the deterministic equation

$$\begin{aligned}\tfrac{\partial}{\partial z}v_m = \ & -(l_0\alpha_{00m} + l_1\alpha_{10m} + l_2\alpha_{20m})\tfrac{\partial}{\partial t}i_0 \\ & -(l_0\alpha_{01m} + l_1\alpha_{11m} + l_2\alpha_{21m})\tfrac{\partial}{\partial t}i_1 \\ & -(l_0\alpha_{02m} + l_1\alpha_{12m} + l_2\alpha_{22m})\tfrac{\partial}{\partial t}i_2.\end{aligned} \tag{13}$$

The orthogonality of the polynomials has been utilized to simplify the l.h.s., and the scalar coefficients

$$\alpha_{knm} = \langle \phi_k\phi_n, \phi_m\rangle = \int_{-\infty}^{+\infty} \phi_k(\xi)\phi_n(\xi)\phi_m(\xi)w(\xi)d\xi \tag{14}$$

have been introduced in the r.h.s.. It should be noted that for standard polynomials the above integral is often known in closed form as a function of the triplet of indices k, n, m.

Calculating and collecting all three equations for $m = 0, 1, 2$ yields the following deterministic MTL-like equation in the PCE coefficients

$$\frac{\partial}{\partial z}\begin{bmatrix} v_0(z,t) \\ v_1(z,t) \\ v_2(z,t)\end{bmatrix} = -\boldsymbol{L}\frac{\partial}{\partial t}\begin{bmatrix} i_0(z,t) \\ i_1(z,t) \\ i_2(z,t)\end{bmatrix} \tag{15a}$$
$$\frac{\partial}{\partial z}\begin{bmatrix} i_0(z,t) \\ i_1(z,t) \\ i_2(z,t)\end{bmatrix} = -\boldsymbol{C}\frac{\partial}{\partial t}\begin{bmatrix} v_0(z,t) \\ v_1(z,t) \\ v_2(z,t)\end{bmatrix}. \tag{15b}$$

The new p.u.l. *matrices* are

$$\boldsymbol{L} = l_0\boldsymbol{A}_0 + l_1\boldsymbol{A}_1 + l_2\boldsymbol{A}_2 \tag{16a}$$
$$\boldsymbol{C} = c_0\boldsymbol{A}_0 + c_1\boldsymbol{A}_1 + c_2\boldsymbol{A}_2 \tag{16b}$$

with “auxiliary” matrices $[\boldsymbol{A}_k]_{mn} = \alpha_{knm}$. For the normalized Hermite polynomials (6), these auxiliary matrices are

$$\boldsymbol{A}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix} \boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0\end{bmatrix} \boldsymbol{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 2\sqrt{2}\end{bmatrix}. \tag{17}$$

It should be noted that, with respect to (2), the SG equation (15) is deterministic but larger, with an augmentation factor equaling the number of PCE coefficients. It describes an equivalent deterministic MTL whose voltages and currents correspond to the PCE coefficients of the original, stochastic line voltages and currents.

Combining the PCE coefficients in (11) with (17), the following augmented matrices are obtained for the example of Fig. 1

$$\boldsymbol{L} = \begin{bmatrix} 733.37 & 41.79 & -6.05 \\ 41.79 & 724.81 & 59.10 \\ -6.05 & 59.10 & 716.25\end{bmatrix} \text{nH/m} \tag{18a}$$
$$\boldsymbol{C} = \begin{bmatrix} 15.224 & -0.894 & 0.199 \\ -0.894 & 15.506 & -1.264 \\ 0.199 & -1.264 & 15.787\end{bmatrix} \text{pF/m}. \tag{18b}$$

A. Terminations and Solution of the SG Problem

The solution of a MTL equation like (15) requires proper boundary conditions. These are given by the current-voltage relations of the components attached to the terminations. The determination of the equivalent terminations for (15) from the terminations of the original line (2) deserves a special discussion. For linear and deterministic loads, independent sources are preserved on the terminals of the line associated to the zero-order PCE coefficients only, whereas the very same load is replicated on all terminals [15]. Hence, in this case, the

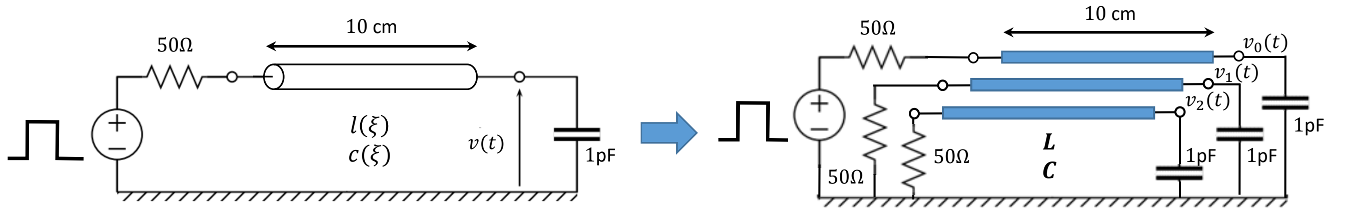

Fig. 3. Illustration of the SG method: from a stochastic line a deterministic and augmented MTL is constructed, whose voltages and currents correspond to the PCE of the original quantities. Linear and deterministic loads are replicated on all conductors, whereas independent sources appear only on the "zero-order" conductor.

overall equivalent circuit for the line of Fig. 1 is as illustrated in Fig. 3.

The inclusion of stochastic terminations requires a Galerkin projection of the terminal equations, leading to a more complex equivalent circuit that couples the various PCE coefficients [16]. Finally, for the case of nonlinear loads, the new terminal equations are always coupled, even for the deterministic case [16]–[19].

Once the SG problem is fully determined with the pertinent terminal conditions, the MTL of Fig. 3 is readily analyzed using any conventional tool, as for example the standard frequency-domain solution [11], [12], FDTD schemes [17], or implementing the circuit in SPICE-type simulators [15]. It should be noted that the occurrence of possible non-physical entries in the equivalent p.u.l. matrices (e.g., negative inductance values or positive off-diagonal capacitance entries, cfr. (18)) does not pose any issue on passivity. It was proven that the augmented p.u.l. matrices still describe passive transmission lines as they remain positive definite [20].

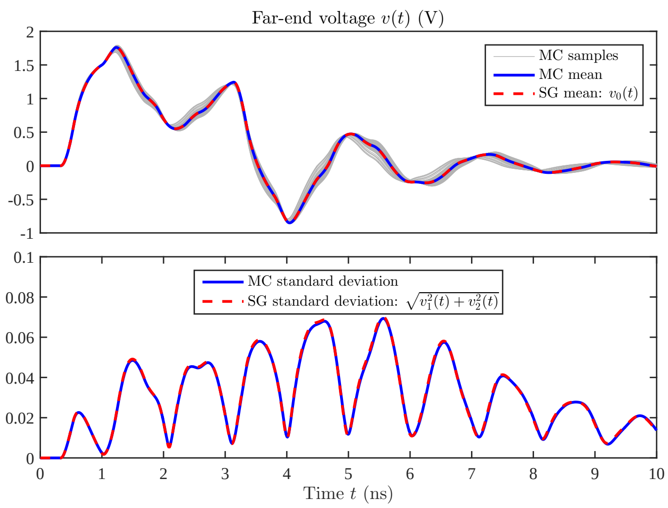

Fig. 4. Stochastic analysis of the voltage transmitted to the far-end termination of the wire of Fig. 1. Top panel: samples from a MC simulation (gray lines), mean computed over these samples (solid blue line) and zero-order PCE coefficient (dashed red line). Bottom panel: standard deviation computed from both the MC samples and the PCE coefficients.

In the top panel, Fig. 4 shows the stochastic variation (gray area) of the voltage at the far-end termination of the line of Fig. 1 obtained from a MC simulation by considering an input voltage source that produces a pulse of amplitude 1 V, a duration of 3 ns, and with rise/fall times of 200 ps. The corresponding mean computed over 1000 samples (solid blue line) is compared against the estimation provided by the zero-order PCE coefficient v_0 (dashed red line), which is obtained by simulating the equivalent SG line in HSPICE. The bottom panel provides a comparison on the standard deviation obtained with the two approaches, showing excellent agreement. The simulation times are 162.4 s and 0.29 s for MC and the SG approach, respectively. Hence, for this example, the latter provides a speed-up factor of $560\times$.

V. Stochastic Collocation Methods

The SG method is elegant and usually very accurate. Nonetheless, it has an important limitation in that it requires a modification of the original problem. It is therefore referred to as being an *intrusive* technique, although in most cases the same deterministic solver (e.g., SPICE), as used to solve the original problem, can still be used for the SG problem as well.

Stochastic collocation (SC) methods are alternative non-intrusive *sampling-based* approaches that only require the computation of some samples of the stochastic response, from which the PCE coefficients are reconstructed with a post-processing manipulation. Hence, SC methods are often much easier to implement. Nevertheless, earlier implementations required a number of samples that was growing faster than the augmentation factor of the SG method as the number of random parameters increased, thus suggesting that a trade-off in computational efficiency existed between the two. The SC methods include the pseudo-spectral SC method, which calculates the PCE coefficients by projection as in (10), with the integral computed numerically, typically by means of a Gauss quadrature [9]. A rigorous attempt to compare the pseudo-spectral SC and the SG method shows that the former is an approximation of the latter [21]. Specifically, they coincide when the p.u.l. parameters are represented exactly by PCEs of order one, as is the case when the relation with the stochastic parameters is linear. More advanced implementations utilize sparse quadrature grids [9], thereby (partially) alleviating the demand in number of samples [22].

Alternative approaches that (loosely) fall within the category of SC methods are those based on linear regression. They retrieve the PCE coefficients by solving (4) in a least-square sense based on a given set of samples of ξ and corresponding realizations of the stochastic response. Earlier implementations used an overdetermined set of samples [23], whereas a recent implementation uses an optimal subset of Gauss quadrature nodes whose number equals the number of unknowns [24].

In the following section, another approach that also requires as few samples as the number of sought-for PCE coefficients and can be in fact included in the class of SC methods is discussed.

A. *Non-intrusive Stochastic Testing*

This recent approach was proposed in [25], [26], based on a non-intrusive reformulation of the ST method originally presented in [27]. The ST enforces the PCE (4) to be exact at a fixed set of points $\xi = \{\xi_0, \xi_1, \xi_2, \ldots\}$ equaling the number of PCE coefficients, thus yielding the following square system of equations:

$$\begin{aligned} v(t,\xi_0) &= a_{00}v_0(t) + a_{01}v_1(t) + a_{02}v_2(t) \\ v(t,\xi_1) &= a_{10}v_0(t) + a_{11}v_1(t) + a_{12}v_2(t) \\ v(t,\xi_2) &= a_{20}v_0(t) + a_{21}v_1(t) + a_{22}v_2(t) \end{aligned} \tag{19}$$

where the notation $a_{mk} = \phi_k(\xi_m)$ is used. The above system of equations is then inverted to retrieve the PCE coefficients:

$$\begin{bmatrix} v_0(t) \\ v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} v(t,\xi_0) \\ v(t,\xi_1) \\ v(t,\xi_2) \end{bmatrix}. \tag{20}$$

Hence, this approach merely amounts to calculating the stochastic response for some predefined samples of the random parameters, and then *transforming* these responses into the unknown PCE coefficients by pre-multiplication with a matrix containing the polynomials evaluated at these samples. It allows to considerably increase the number of random parameters that can be effectively accounted for, up to a few tens [26].

In the univariate case, the points $\{\xi_0, \xi_1, \xi_2, \ldots\}$ coincide with the nodes of a Gauss quadrature rule [14] of the same order as the PCE. These are in turn the zeros of the first higher-order polynomial excluded in the PCE. For the example considered, the ST points are the zeros of $\phi_3 = \xi^3 - 3\xi$, i.e., $\xi_0 = 0$, $\xi_1 = -\sqrt{3}$, and $\xi_2 = +\sqrt{3}$. The corresponding matrix to be inverted in (20) is

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1/\sqrt{2} \\ 1 & -\sqrt{3} & 2/\sqrt{2} \\ 1 & +\sqrt{3} & 2/\sqrt{2} \end{bmatrix}. \tag{21}$$

For the multivariate case, the ST algorithm suitably selects the samples among the full tensor-product quadrature grid [27].

Fig. 5 collects the results for the ST-based simulation of the line of Fig. 1. First, the stochastic problem is simulated in HSPICE for the aforementioned three values of the RV ξ, corresponding to $h = 1$ cm, $h = 0.654$ cm, and $h = 1.346$ cm. The resulting responses are shown in the top panel of Fig. 5. These curves are then converted into the sought-for PCE coefficients via the transformation (20), leading to the blue curves in the central panel (v_0, yielding the mean) and in the bottom panel (v_1 and v_2, yielding the standard deviation). The same quantities previously computed with the SG method are shown by the dashed red lines for comparison, without any appreciable difference.

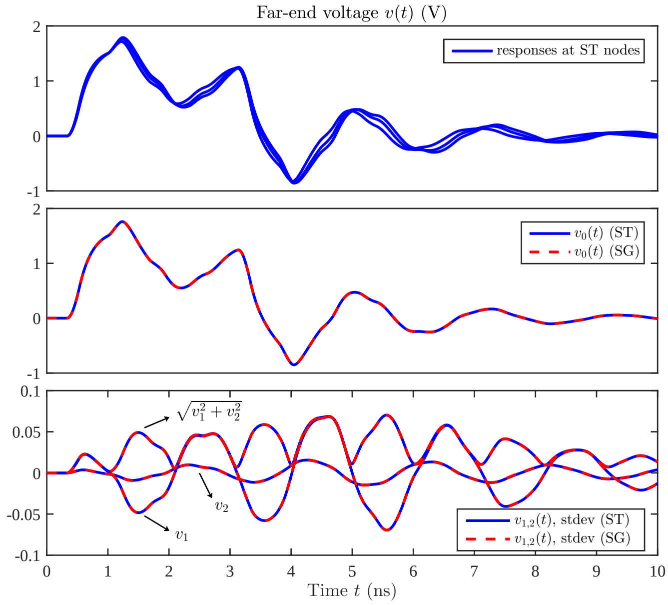

Fig. 5. ST-based simulation of the wire of Fig. 1. The three responses at the ST points (top panel) are converted into the zero-order (central panel), first- and second-order (bottom panel) PCE coefficients (blue lines). For comparison, the results previously computed with the SG-based simulation are also shown (dashed red lines).

The simulation time for the ST is 0.65 s, therefore slightly slower than the SG simulation of the equivalent augmented line. However, as shown in the next section, the ST largely outperforms the SG method when the number of random parameters increases.

VI. Application Example

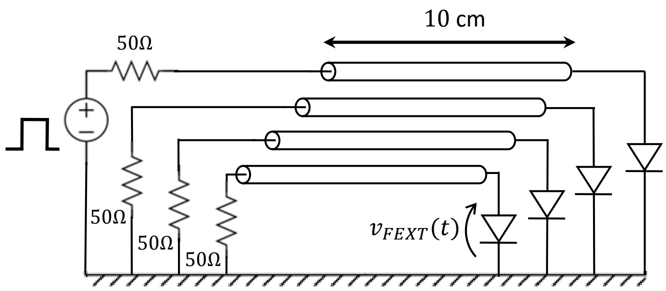

Fig. 6. Application example consisting of a four-wire MTL running above a perfect ground plane and terminated with diodes.

As a conclusive example, the multiconductor line of Fig. 6 is considered, which consists of four replicas of the wire line of Fig. 1 at a distance of 1 cm each and terminated with diodes. Seven independent random parameters, namely the three wire-to-wire separations and the four wire heights, are assumed to be Gaussian distributed with a standard deviation of 10% of the mean.

In the top panel, Fig. 7 shows the variation of the far-end crosstalk voltage v_{FEXT} on the furthest (outermost) conductor (gray lines) as well as the corresponding mean computed from

10000 MC samples (solid blue line), by means of the SG-augmented equivalent line and the model in [19] for the non-linear terminations (dashed red line), and with the ST method (dotted green line). The three means are indistinguishable from each other.

The bottom panel shows the standard deviation. A close-up around 10 ns (top-right corner) allows to better appreciate the accuracy of the three methods. The convergence of MC for 100, 1000 and 10000 samples is shown by the dotted, dashed and solid blue lines, respectively. For this second-order moment, there is also some difference in accuracy between the ST and the SG method, the latter being in very good agreement with the MC result based on the largest sample size.

The above comparison confirms both the slow convergence of MC and the superior accuracy of the SG method over the ST one. On the other hand, the SG method takes 502 s as opposed to the mere 10.4 s required by the ST, which therefore achieves a speed-up of 330× against MC (3432 s) and a further, impressive speed-up of $48.3\times$ against the SG method. The efficiency of the SG is here reduced by the larger number of RVs and by the presence of nonlinear terminations, which require a model that couples the voltage and current PCE coefficients, as discussed in Section IV-A. The SG simulation of a similar structure terminated by linear loads, e.g., capacitors instead of diodes (results not shown here), takes 205 s only.

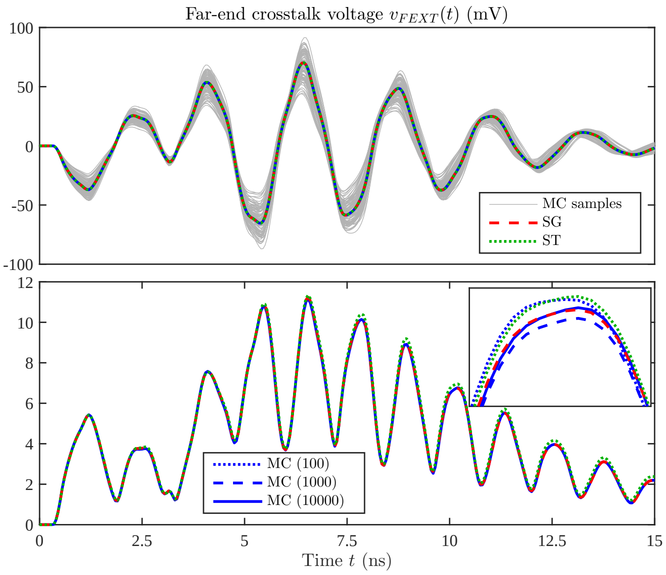

Fig. 7. Stochastic analysis of the far-end crosstalk voltage in the line of Fig. 6. Top panel: samples from a MC simulation (gray lines), mean computed over these samples (solid blue line) and zero-order PCE coefficient from the SG (dashed red line) and ST (dotted green line) simulations. Bottom panel: standard deviation computed based on 100 (dotted blue), 1000 (dashed blue) and 10000 (solid blue line) MC samples, as well as from the SG (dashed red line) and ST (dotted green line) simulations.

Finally, Fig. 8 shows the PDFs of the crosstalk voltage computed at 5.4 ns and 6.5 ns. Owing to the limited number of samples, the information from the MC analysis (gray bars) can be computed only with a limited resolution. In stark contrast to this, the PCE of the crosstalk voltage can be evaluated at an arbitrarily high number of samples in negligible time, thus allowing to obtain accurate and smooth PDFs. The results for the SG and ST simulations in Fig. 8 (solid red and dashed green lines, respectively), in very good agreement, are computed based on 10^7 samples. This would be definitely infeasible with MC!

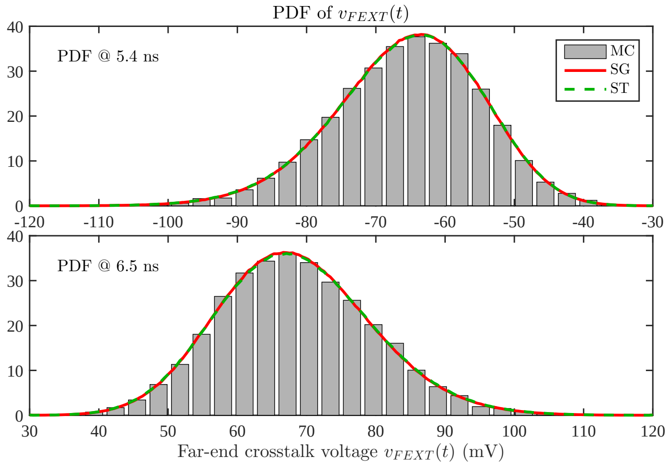

Fig. 8. PDF of the far-end crosstalk voltage $v_{FEXT}(t)$ at time $t = 5.4$ ns (top panel) and $t = 6.5$ ns (bottom panel). The result from MC analysis (gray bars) is compared against the SG (solid red line) and ST (dashed green line) results.

VII. Conclusions

This article provides an overview of the state-of-the-art methods for the statistical transmission line analysis based on the PC framework. The discussion focuses in particular on the intrusive SG method and on the non-intrusive ST method, the latter being a recent and very effective technique belonging to the class of SC methods. These two approaches are introduced based on a simple illustrative example, for which step-by-step calculations are provided. A more complex example with nonlinear terminations allows comparing the performance in terms of accuracy and efficiency for a larger number of random parameters, showing that the ST method is slightly less accurate but much faster than the SG method. In general, PC methods are much faster than MC. However, the ST method scales much better with the random space dimensionality than the SG method, especially when nonlinear components are included. Whilst this article concentrates on signal integrity and crosstalk investigations in simple source-line-load configurations, it is worth mentioning that PC methods are readily applied to study more complex transmission-line network topologies [15] as well as the stochastic illumination of MTLs by external fields [28].

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