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# Rayleigh Quotient Based Analysis of MIMO Linear Receivers

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*Abstract*—**The statistics of indefinite quadratic forms in Gaussian vectors are of particular relevance as they often occur in signal processing, wireless communications, information theory and adaptive filter theory. Very recently their distribution has been characterized in closed form in [1]. In this work, we extend such results to the case of indefinite quadratic form with random kernel matrix. We focus on Rayleigh quotients in random matrices commonly met in MIMO communications. As an instance of practical application of our findings, the gap between the multiuser efficiency of a MIMO linear Minimum Mean-Squared Error (MMSE) receiver and the corresponding efficiency in the Zero Forcing (ZF) case, is statistically characterized in the finite-size setting for the first time.**

## I. Introduction

Since the pioneering investigation in [2], Minimum Mean-Squared Error (MMSE) and Zero Forcing (ZF) receivers have been proven to give different performance in terms of outage and error probability even at asymptotically high Signal to Noise Ratio (SNR). In [2], assuming independent stream decoding in a multi-antenna link affected by Rayleigh fading, the authors have focused on the high-SNR statistics of the per-stream SNR gap between the output of a MIMO MMSE filter and a ZF filter. A non-asymptotic analysis of such a gap has been developed in [3], for different interfering stream fading models and for arbitrary SNR values.

Specifically, the analysis in [3] relies upon early results in the characterization of (in general matrix-valued) quadratic forms in normal vectors [4], and it is extended to the case of randomly distributed kernel matrix. In particular, kernel matrices considered in [3] represent the impact of interfering streams random covariance, within a single-stream decoding setting. According to the findings in [2], error and outage probability offset due to the mentioned SNR gap, have been mainly investigated in [3]. Furthermore, by assuming the desired and the interfering streams to follow far-field heterogeneous fading laws, the authors have paved the way to the analysis of a multiuser setting. In this framework, a key parameter is the multiuser efficiency, which, for linear receivers, takes the form of, e.g., [5, Eq. (18)]. While error and outage probability can be evaluated as functions of the above mentioned SNR gap (see e.g., [2, Eqq.(39),(52)]), multiuser efficiency characterization boils down to the study of a ratio of quadratic forms, otherwise known as Rayleigh quotient in linear algebra [6].

A closed-form expression for the cumulative density function (cdf) and the corresponding probability density function (pdf) of a ratio of quadratic forms have been provided only very recently, as a by-product of a deep investigation on quadratic forms[1] in indefinite matrices [1]. We herein further extend the results in [1], by assuming the kernel matrix of the quadratic form at the numerator of the ratio to be random. By doing so, we achieve our starting goal of providing an expression for the pdf of the multiuser efficiency gap between MMSE and ZF receivers. Numerical results, referring to a spatially correlated Rayleigh fading scenario, complement and validate our analysis. We also remark that the proof technique is of independent interest for further applications in both the signal processing and the multivariate statistics domain, and provide a further instance of application of multiple integral simplification lemmas appearing in the milestone paper [7].

The paper is organized as follows. The next section introduces the system model and the definition of the multiuser efficiency gap. Section III illustrates the derivation of the cdf expression for the standard Rayleigh quotient of a random matrix, and includes remarks on the distribution of a generalized Rayleigh quotient. Numerical results are presented in Section IV, while conclusions are provided in Section VI.

*Notation*: Boldface uppercase and lowercase letters denote matrices and vectors, respectively. $\mathbf{I}$ is the identity matrix. The determinant and the conjugate transpose of the generic matrix $\mathbf{A}$ are denoted by $|\mathbf{A}|$ and $\mathbf{A}^{\mathsf{H}}$, respectively, while $\mathbf{A}_{i,j}$ is the $(i,j)$-th element of $\mathbf{A}$. Moreover, $\mathbb{E}_a[\cdot]$ represents the average operator with respect to the random variable $a$. For any $m \times m$ Hermitian matrix $\mathbf{A}$ with eigenvalues $\mathbf{a} = [a_1, \dots, a_m]$, the Vandermonde determinant is defined as: $\mathcal{V}(\mathbf{A}) = \prod_{1 \le k < \ell \le m}(a_k - a_\ell)$. For simplicity, the Vandermonde determinant $\mathcal{V}(\mathbf{A})$ is also denoted as $\mathcal{V}(\mathbf{a})$.

The complex multivariate Gamma function is defined as [9]: $\Gamma_p(q) = \pi_p \prod_{\ell=1}^{p}(q-\ell)!$, with $p$ and $q$ non-negative integers such that $p \le q$, and $\pi_p = \pi^{p(p-1)/2}$. $f_a(a)$ denotes the pdf of the scalar random variable $a$ (for the pdf of random matrices, we drop the subscript). Finally, $a \sim b$ indicates that two variables $a$ and $b$ share the same distribution.

[1]For sake of compactness, herein we only discussed the (very limited) number of references on indefinite quadratic forms. For a detailed literature analysis, we refer the interested reader to [1, Sec.I.A,B], which surveys the state-of-art and discusses the different approaches to quadratic forms characterization.

## II. Multiuser Efficiency Gap

Multiuser efficiency is a merit figure widely investigated in early analysis of wireless multiuser systems, since its very first definition in [10] for the optimal receiver. Recently, it has been gaining new attention, due to its interesting possible connection with compressed sensing problems [11]. In the context of linear receivers, multiuser efficiency is defined as the ratio between the achieved Signal to Interference and Noise Ratio (SINR) and the corresponding SNR in absence of other users interference, for each independently decoded user. We focus on a MIMO system with $n_t$ transmitters and $n_r \geq n_t$ receivers, represented by the input-output relationship:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}\,, \tag{1}$$

where the received signal vector $\mathbf{y}$ is of length $n_r$, $\mathbf{H}$ is the $n_r \times n_t$ random channel matrix, $\mathbf{x}$ is a random input vector of size $n_t$ with covariance $\mathbb{E}[\mathbf{x}\mathbf{x}^{\mathsf{H}}] = \mathcal{E}_s/n_t\mathbf{I}$, and $\mathbf{n}$ represents Gaussian noise with covariance $\mathbb{E}[\mathbf{n}\mathbf{n}^{\mathsf{H}}] = \mathcal{N}_0\mathbf{I}$.

In case of independent stream decoding, the output SINR corresponding to the $k$-th transmitted signal stream can be expressed for the MMSE and, respectively, for the ZF receiver as [12, Ch. 6]:

$$\gamma_k^{\text{mmse}} = \frac{1}{\left[\left(\mathbf{I}+\delta\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}\right]_{k,k}} - 1, \quad \gamma_k^{\text{zf}} = \frac{\delta}{\left[\left(\mathbf{H}^{\mathsf{H}}\mathbf{H}\right)^{-1}\right]_{k,k}}\,, \tag{2}$$

where $\delta = \frac{\mathcal{E}_s}{n_t\mathcal{N}_0}$. The difference of the above quantities is referred to as *SINR gap* and is defined as [2], [13]

$$\nu_k = \gamma_k^{\text{mmse}} - \gamma_k^{\text{zf}}\,. \tag{3}$$

The SINR gap is a non-decreasing function of the SNR, accounting for the energy nulled out by the ZF but not by the MMSE receiver.

The multiuser efficiency achieved by the $k$-th stream is given by the ratio between the corresponding SINR and the SNR in absence of other-stream interference, i.e.,[5, Eq. (18)]

$$\eta_k^{\text{mmse}} = \frac{\gamma_k^{\text{mmse}}}{\delta\|\mathbf{h}_k\|^2}\,, \quad \eta_k^{\text{zf}} = \frac{\gamma_k^{\text{zf}}}{\delta\|\mathbf{h}_k\|^2} \tag{4}$$

where $\mathbf{h}_k$ is the $k$-th column of $\mathbf{H}$. In analogy with the SINR gap, one can naturally define the *multiuser efficiency gap* from (3) and (4) as follows

$$\mu_k = \eta_k^{\text{mmse}} - \eta_k^{\text{zf}} = \frac{\nu_k}{\delta\|\mathbf{h}_k\|^2}\,. \tag{5}$$

Let $\mathbf{H}_k$ be the $n_r \times (n_t - 1)$ matrix obtained by removing the column $\mathbf{h}_k$ from $\mathbf{H}$. Let $\mathbf{H}_k = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k{}^{\mathsf{H}}$ be the singular value decomposition of $\mathbf{H}_k$, where $\mathbf{U}_k$ is an $n_r \times n_r$ unitary matrix. Moreover, let $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_{n_t-1}]$ be the $n_t - 1$ non-zero eigenvalues of $\mathbf{H}_k\mathbf{H}_k{}^{\mathsf{H}}$.

Then the multiuser efficiency gap can be rewritten as the ratio of two quadratic forms as proposed in [2, Eq.(26)], i.e.,

$$\mu_k = \frac{\mathbf{h}_k{}^{\mathsf{H}}\mathbf{A}\mathbf{h}_k}{\|\mathbf{h}_k\|^2}\,, \tag{6}$$

where $\mathbf{A} = \mathbf{U}_k\mathbf{L}\mathbf{U}_k{}^{\mathsf{H}}$, and the $n_r \times n_r$ diagonal matrix $\mathbf{L}$ is given by

$$\mathbf{L} = \operatorname{diag}\Bigg(\underbrace{\alpha_1, \ldots, \alpha_{n_t-1}}_{\boldsymbol{\alpha}}, \underbrace{0, \ldots, 0}_{n_r-n_t+1}\Bigg)$$

and

$$\alpha_i = (1 + \delta\lambda_i)^{-1} \qquad i = 1, \ldots, n_t - 1\,. \tag{7}$$

For the sake of simplicity, in the following we assume a square channel matrix, i.e. $n_t = n_r = n$. In such a case the matrix $\mathbf{A}$ has eigenvalues $[\boldsymbol{\alpha}, 0]$, with $\alpha_i > 0$ for $i = 1, \ldots, n_t - 1$.

Being both the numerator and the denominator in (6) quadratic forms in $\mathbf{h}_k$, $\mu_k$ can be expressed as a Rayleigh quotient, whose main properties are summarized in the next subsection, as a preliminary stage toward its characterization.

## III. Rayleigh Quotient Characterization

Given a square Hermitian matrix $\mathbf{A}$ of size $n$, with ordered eigenvalues $a_1 \geq \ldots \geq a_n$, and a complex vector $\mathbf{v}$ of length-$n$, the ratio

$$r(\mathbf{A}, \mathbf{v}) = \frac{\mathbf{v}^{\mathsf{H}}\mathbf{A}\mathbf{v}}{\|\mathbf{v}\|^2} \tag{8}$$

is defined as Rayleigh quotient of $\mathbf{A}$ with respect to $\mathbf{v}$. A relevant relationship between $r$ and the spectrum of $\mathbf{A}$ holds [6], namely

$$a_n \leq r(\mathbf{A}, \mathbf{v}) \leq a_1\,.$$

Rayleigh quotients with deterministic $\mathbf{A}$ and $\mathbf{v}$ have been largely studied [6]. A first compact characterization of the pdf and of the cdf of $r(\mathbf{A}, \mathbf{v})$, when $\mathbf{v}$ is random, has been recently derived in [1, Sec.VII.A], under various assumptions on the pdf. The whole analysis in [1] assumes a deterministic matrix $\mathbf{A}$. Some of the possible applications in wireless communications, among which multiuser efficiency characterization, call instead for the study of $r(\mathbf{A}, \mathbf{v})$ when both $\mathbf{A}$ and $\mathbf{v}$ are randomly distributed. We hereinafter recall the result from [1] and, later, we detail the procedure to get a statistical characterization of (8). We do so under Gaussian assumption for $\mathbf{v}$ as well as some assumptions on the statistics of $\mathbf{A}$, tailored to the multiuser efficiency problem at hand.

Let us consider the cdf of the Rayleigh quotient, i.e., $F_r(z) = \mathbb{P}(r \leq z)$, with $z \geq 0$, and $\mathbf{A}$ a random square Hermitian matrix with eigenvalues $\mathbf{a} = [a_1, \ldots, a_n]$. Following [1], one easily gets

$$\begin{aligned} F_{r|\mathbf{A}}(z) &= \mathbb{P}\left(r \leq z|\mathbf{A}\right) \\ &= \mathbb{P}\left(\frac{\mathbf{v}^{\mathsf{H}}\mathbf{A}\mathbf{v}}{\|\mathbf{v}\|^2} \leq z|\mathbf{A}\right) \\ &= \mathbb{P}\left(\mathbf{v}^{\mathsf{H}}\left(\mathbf{A} - z\mathbf{I}\right)\mathbf{v} \leq 0|\mathbf{A}\right)\,. \end{aligned} \tag{9}$$

It is evident that Rayleigh quotient characterization boils down to the study of the sign probability of the quadratic form $\mathbf{v}^{\mathsf{H}}\left(\mathbf{A} - z\mathbf{I}\right)\mathbf{v}$. Under the assumption of $\mathbf{v}$ following a standard complex Gaussian multivariate distribution, with uncorrelated elements, the cdf of $r(\mathbf{A},\mathbf{v})$ coincides with the cdf of $\mathbf{v}^{\mathsf{H}}(\mathbf{A}-z\mathbf{I})\,\mathbf{v}$ evaluated at 0. Using [1, Eq.(48)], we therefore write:

$$F_{r|\mathbf{A}}(z)=1-\sum_{\ell=1}^{n}\frac{(a_\ell-z)^n\,u\,(a_\ell-z)}{\prod_{j=1,j\neq\ell}^{n}(a_\ell-a_j)\,|a_\ell-z|} \tag{10}$$

where $u(\cdot)$ is the Heaviside step function. Eq. (10) holds only when the eigenvalues of $\mathbf{A}$ are all distinct[2], which is our case since we assume $\mathbf{a}=[\boldsymbol{\alpha},0]$ and the eigenvalues $\boldsymbol{\alpha}$ to be positive and distinct with probability 1. For $\mathbf{a}=[\boldsymbol{\alpha},0]$ and $z>0$, we can then rewrite (10) as

$$F_{r|\mathbf{A}}(z)=F_{r|\boldsymbol{\alpha}}(z)=1-\sum_{\ell=1}^{n-1}\frac{(\alpha_\ell-z)^{n-1}\,u\,(\alpha_\ell-z)}{\alpha_\ell\prod_{j=1,j\neq\ell}^{n-1}(\alpha_\ell-\alpha_j)} \tag{11}$$

Observe that in our case (11) only depends on $\boldsymbol{\alpha}$ therefore we can write $F_{r|\mathbf{A}}(z)=F_{r|\boldsymbol{\alpha}}(z)$.

Taking into account the randomness of the matrix $\mathbf{A}$, the cdf of $r(\mathbf{A},\mathbf{v})$ can be obtained by averaging $F_{r|\boldsymbol{\alpha}}(z)$ over the distribution of $\boldsymbol{\alpha}$. In practice

$$\begin{aligned}F_r(z)&=\mathbb{E}_{\boldsymbol{\alpha}}\left[\mathbb{P}\left(r<z|\boldsymbol{\alpha}\right)\right]\\&=\int_{\mathcal{A}^{n-1}}F_{r|\boldsymbol{\alpha}}(z)f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})\,\mathrm{d}\boldsymbol{\alpha}\,.\end{aligned} \tag{12}$$

where $f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$ is the distribution of the unordered eigenvalues $\boldsymbol{\alpha}$ and $\mathcal{A}\subset\mathbb{R}^+$ is the support of the generic eigenvalue $\alpha$. Of course, the expression of the integral in (12) will depend on the postulated $f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$, which, in turn, is tailored to the specific application one has in mind. By using the definition of the Vandermonde determinant, it is easy to show that

$$\mathcal{V}(\boldsymbol{\alpha})=(-1)^{n-1-\ell}\mathcal{V}(\boldsymbol{\alpha}_\ell)\prod_{j=1,j\neq\ell}^{n-1}(\alpha_\ell-\alpha_j)$$

where $\boldsymbol{\alpha}_\ell=[\alpha_1,\ldots,\alpha_{\ell-1},\alpha_{\ell+1},\ldots,\alpha_n]$ for $\ell=1,\ldots,n-1$. Then we can rewrite the expression of $F_{r|\boldsymbol{\alpha}}(z)$ as

$$F_{r|\boldsymbol{\alpha}}(z)=1-\sum_{\ell=1}^{n-1}\frac{(-1)^{n-1-\ell}\mathcal{V}(\boldsymbol{\alpha}_\ell)}{\alpha_\ell\mathcal{V}(\boldsymbol{\alpha})}\,(\alpha_\ell-z)^{n-1}\,u(\alpha_\ell-z)\,. \tag{13}$$

The expectation with respect to $\boldsymbol{\alpha}$, is accomplished in the following proposition, assuming a quite general[3] model for the distribution of $\boldsymbol{\alpha}$.

*Proposition 3.1:* Let us consider the Rayleigh quotient $r(\mathbf{A},\mathbf{v})$ in (8) where $\mathbf{A}$ is an Hermitian random matrix with eigenvalues $\mathbf{a}=[\boldsymbol{\alpha},0]$ and $\mathbf{v}$ is a standard complex Gaussian vector of size $n$ with uncorrelated entries. Assume that the joint law of the $n-1$ (distinct) non-zero, unordered, positive eigenvalues $\boldsymbol{\alpha}=[\alpha_1,\ldots,\alpha_{n-1}]$ of $\mathbf{A}$, can be written as

$$f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})=K|\boldsymbol{\Phi}(\boldsymbol{\alpha})|\mathcal{V}(\boldsymbol{\alpha})\prod_{i=1}^{n-1}\psi(\alpha_i)\,, \tag{14}$$

where $\boldsymbol{\Phi}(\boldsymbol{\alpha})$ is an $(n-1)\times(n-1)$ matrix whose generic $i,j$-th entry can be expressed as $\phi_i(\alpha_j)$, and $K$ is a normalization constant. Then the cdf and the pdf of $r(\mathbf{A},\mathbf{v})$ admit the following closed-form expressions

$$F_r(z)\;=\;1-K(n-1)!\sum_{h=1}^{n-1}(-1)^{n-1+h}|\mathbf{F}_h|\mathcal{I}_{h,n}(z)\,, \tag{15}$$

$$f_r(z)\;=\;K(n-1)(n-1)!\sum_{h=1}^{n-1}(-1)^{n-1+h}|\mathbf{F}_h|\mathcal{I}_{h,n-1}(z)\,. \tag{16}$$

In (15) and (16), $\mathbf{F}_h$ is a square matrix of size $(n-2)\times(n-2)$ and generic entry

$$(\mathbf{F}_h)_{i,j}=\begin{cases}\int_{\mathcal{A}}\alpha^{j-1}\phi_i(\alpha)\psi(\alpha)\,\mathrm{d}\alpha & 1\le i<h\\ \int_{\mathcal{A}}\alpha^{j-1}\phi_{i+1}(\alpha)\psi(\alpha)\,\mathrm{d}\alpha & h\le i\le n-2\end{cases} \tag{17}$$

while

$$\mathcal{I}_{h,q}(z)=\int_{[z,+\infty)\cap\mathcal{A}}\phi_h(\alpha)\psi(\alpha)\frac{(\alpha-z)^{q-1}}{\alpha}\,\mathrm{d}\alpha\,. \tag{18}$$

*Proof:* We first observe that the term $|\boldsymbol{\Phi}(\boldsymbol{\alpha})|$ can be rewritten by using the Laplace expansion over the $\ell$-th row of $\boldsymbol{\Phi}(\boldsymbol{\alpha})$, i.e.,

$$|\boldsymbol{\Phi}(\boldsymbol{\alpha})|=\sum_{h=1}^{n-1}(-1)^{h+\ell}\phi_h(\alpha_\ell)D_{h,\ell}(\boldsymbol{\alpha}_\ell)$$

where $D_{h,\ell}$ denotes the $h,\ell$-th co-factor (see e.g. [6, (0.3.1)]) in the determinant of $\boldsymbol{\Phi}(\boldsymbol{\alpha})$. We evaluate $F_r(z)$ by plugging (14) and (13) into (12). By doing so, we get

$$\begin{aligned}F_r(z)\;=\;&1-K\sum_{\ell=1}^{n-1}\sum_{h=1}^{n-1}(-1)^{n-1+h}\cdot\\&\int_{\mathcal{A}^{n-2}}\mathcal{V}(\boldsymbol{\alpha}_\ell)D_{h,\ell}(\boldsymbol{\alpha}_\ell)\prod_{i=1,i\neq\ell}^{n-1}\psi(\alpha_i)\,\mathrm{d}\boldsymbol{\alpha}_\ell\\&\int_{\mathcal{A}}\phi_h(\alpha_\ell)\psi(\alpha_\ell)\frac{(\alpha_\ell-z)^{n-1}}{\alpha_\ell}u(\alpha_\ell-z)\,\mathrm{d}\alpha_\ell\end{aligned} \tag{19}$$

where all terms depending on $\alpha_\ell$ are collected in the second integral. The integral over $\boldsymbol{\alpha}_\ell$ can be performed by using the result in [7, Corollary I], thus obtaining

$$\int\mathcal{V}(\boldsymbol{\alpha}_\ell)D_{h,\ell}(\boldsymbol{\alpha}_\ell)\prod_{i=1,i\neq\ell}^{n-1}\psi(\alpha_i)\,\mathrm{d}\boldsymbol{\alpha}_\ell=(n-2)!|\mathbf{F}_h|$$

where the elements of $(\mathbf{F}_h)_{i,j}$ are given in (17)[4].

The integral over $\alpha_\ell$ can be immediately rewritten as in (18). Since both $|\mathbf{F}_h|$ and $\mathcal{I}_{h,n}(z)$ are independent of $\ell$, the sum over

[2]For the case of multiple eigenvalues, the reader is referred to the more general derivation provided in [1, Eq.(30)].

[3]We indeed consider matrix $\mathbf{A}$ to belong to a so-called *polynomial ensemble* of random matrices. This ensemble contains Wishart matrices and, in general, matrices largely adopted in wireless communications since early MIMO analysis [14, and references therein].

[4]Notice that, by definition of co-factor, the matrix whose determinant we denote by $D_{h,\ell}(\boldsymbol{\alpha}_\ell)$ is obtained from $\boldsymbol{\Phi}(\boldsymbol{\alpha})$, by deleting its $h$-th row and its $\ell$-th column.

$\ell$ reverts to a factor $n-1$. Then (15) follows. The density $f_r(z)$ is readily obtained by taking the derivative of $F_r(z)$ in (19).

$$\begin{aligned} f_r(z) &= \frac{\mathrm{d}}{\mathrm{d}z}F_r(z) \\ &= -K(n-1)!\sum_{h=1}^{n-1}(-1)^{n-1+h}|\mathbf{F}_h|\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{I}_{h,n}(z) \\ &= K(n-1)(n-1)!\sum_{h=1}^{n-1}(-1)^{n-1+h}|\mathbf{F}_h|\mathcal{I}_{h,n-1}(z) \end{aligned}$$

The last equality follows from the definition of $\mathcal{I}_{h,n}(z)$ and from the rules of differentiation under integral sign. ■

## IV. Exploitation of the analytical result

Our previous proposition has an immediate application in the multiuser efficiency characterization. For this to be effective, it is sufficient to pick a fading law to particularize (14) and henceforth evaluate the integrals in (18). Note that, as largely observed in the literature (see e.g., [14], [15]), (14) embodies most of the micro-wave fading models in the MIMO case. Mirroring the assumptions of Proposition 3.1, we assume the intended stream to be affected by uncorrelated Rayleigh fading, and investigate different fading laws affecting the interfering streams.

### A. Rayleigh-faded interferers

For sake of simplicity, we start the analysis from the case of uncorrelated Rayleigh-faded interference. This translates in $\mathbf{h}_k$ in (6) following a standard multivariate complex Gaussian distribution, with uncorrelated elements. The joint distribution of the $n-1$ non-zero random eigenvalues of $\mathbf{H}_k\mathbf{H}_k{}^{\mathsf{H}}$ can be instead written as [19]

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \kappa\mathcal{V}^2(\boldsymbol{\lambda})\prod_{i=1}^{n-1}\mathrm{e}^{-\lambda_i}\lambda_i\,, \tag{20}$$

where

$$\kappa = \frac{\pi_{n-1}^2}{(n-1)!\Gamma_{n-1}(n-1)\Gamma_{n-1}(n)}\,.$$

We now need to map the distribution of $\boldsymbol{\lambda}$ into that of $\boldsymbol{\alpha}$ by using (7) which, solved for $\lambda_i$, provides

$$\lambda_i = \frac{1}{\delta\alpha_i} - \frac{1}{\delta} \tag{21}$$

for $i = 1, \dots, n-1$. We then obtain [16, Chap. 7]

$$f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = |\mathbf{J}|f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) \tag{22}$$

where $|\mathbf{J}|$ is the absolute value of the determinant of the Jacobian matrix characterizing the change of variable from $\boldsymbol{\lambda}$ to $\boldsymbol{\alpha}$. In practice $\mathbf{J}_{ij} = \frac{\mathrm{d}\lambda_i}{\mathrm{d}\alpha_j}$. By differentiating (21), we obtain $\frac{\mathrm{d}\lambda_i}{\mathrm{d}\alpha_i} = -\frac{1}{\delta\alpha_i^2}$. Since $\frac{\mathrm{d}\lambda_i}{\mathrm{d}\alpha_j} = 0$ for $i \neq j$, we have

$$|\mathbf{J}| = \frac{1}{\delta^{n-1}\prod_{i=1}^{n-1}\alpha_i^2}\,.$$

Moreover, we observe that

$$\begin{aligned} \mathcal{V}(\boldsymbol{\lambda}) &= \prod_{1\le i<j<n-1}(\lambda_j - \lambda_i) \\ &= \prod_{1\le i<j<n-1}\left(\frac{1}{\delta\alpha_j} - \frac{1}{\delta} - \frac{1}{\delta\alpha_i} + \frac{1}{\delta}\right) \\ &= \prod_{1\le i<j<n-1}\frac{\alpha_i - \alpha_j}{\delta\alpha_i\alpha_j} \\ &= \frac{(-1)^{(n-1)(n-2)/2}\mathcal{V}(\boldsymbol{\alpha})}{\delta^{(n-1)(n-2)/2}\prod_{i=1}^{n-1}\alpha_i^{n-2}}\,. \end{aligned} \tag{23}$$

By substituting these results in (22), we get

$$f_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = \frac{\kappa\mathrm{e}^{(n-1)/\delta}}{\delta^{n(n-1)}}\mathcal{V}^2(\boldsymbol{\alpha})\prod_{i=1}^{n-1}\frac{\mathrm{e}^{-1/\alpha_i\delta}}{\alpha_i^{2n-1}}(1-\alpha_i)\,. \tag{24}$$

At last, we observe that $\lambda_i \in \mathbb{R}^+$ for $i = 1, \dots, n-1$ and, by the transformation law given in (7), we have $\alpha_i \in \mathcal{A} = [0,1]$. By comparing (24) to (14), we identify the following terms

$$|\boldsymbol{\Phi}(\boldsymbol{\alpha})| = \mathcal{V}(\boldsymbol{\alpha}),\ \phi_i(\alpha_j) = \alpha_j^{i-1},\ \psi(\alpha_i) = \frac{\mathrm{e}^{-1/\alpha_i\delta}}{\alpha_i^{2n-1}}(1-\alpha_i).$$

Moreover,

$$K = \frac{\kappa\mathrm{e}^{(n-1)/\delta}}{\delta^{n(n-1)}}\,.$$

Upon replacement, where needed, of the above expressions into (17) and (18), one obtains

$$(\mathbf{F}_h)_{i,j} = \begin{cases} \displaystyle\int_0^1(1-\alpha)\frac{\mathrm{e}^{-\frac{1}{\alpha\delta}}}{\alpha^{2n-i-j+1}}\,\mathrm{d}\alpha & 1 \le i < h \\ \displaystyle\int_0^1(1-\alpha)\frac{\mathrm{e}^{-\frac{1}{\alpha\delta}}}{\alpha^{2n-i-j}}\,\mathrm{d}\alpha & h \le i \le n-2 \end{cases} \tag{25}$$

while

$$\mathcal{I}_{h,q}(z) = \int_z^1(1-\alpha)(\alpha-z)^{q-1}\frac{\mathrm{e}^{-\frac{1}{\alpha\delta}}}{\alpha^{2n-h+1}}\,\mathrm{d}\alpha\,. \tag{26}$$

### B. Spatially correlated interferers

Moving to a slightly less homogeneous scenario, we postulate that, while the useful signal stream undergoes uncorrelated Rayleigh fading, interference is coming from a bunch of spatially correlated users. This occurs, e.g., when interfering transmitters are located in close proximity to each other, but well spatially separated from the useful signal source. In particular, for $n_t = n_r = n$, the distribution of the unordered eigenvalues $\boldsymbol{\lambda}$ is given by

$$f_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}) = \kappa\mathcal{V}(\boldsymbol{\lambda})|\mathbf{E}|\prod_{i=1}^{n-1}\lambda_i\,, \tag{27}$$

with [9]

$$\kappa = \frac{\pi_{n-1}}{(n-1)!\Gamma_{n-1}(n)\mathcal{V}(\boldsymbol{\Theta})|\boldsymbol{\Theta}|^2}\,.$$

In the above expressions, $\boldsymbol{\Theta}$ is the common covariance of each column of $\mathbf{H}_k$, whose eigenvalues are denoted by $\{\theta_1, \dots, \theta_{n-1}\}$, and the generic entry of the matrix $\mathbf{E}$ is given by $\mathbf{E}_{i,j} = \mathrm{e}^{-\lambda_i/\theta_j}$. Under these assumptions on the fading model, the density of $\boldsymbol{\alpha}$ can be written as

$$f(\boldsymbol{\alpha}) = \frac{(-1)^{(n-2)(n-1)/2}\kappa}{\delta^{(n-1)(n+2)/2}}\mathcal{V}(\boldsymbol{\alpha})|\widetilde{\mathbf{E}}|\prod_{i=1}^{n-1}\frac{1-\alpha_i}{\alpha_i^{n+1}}\,, \tag{28}$$

with

$$\widetilde{\mathbf{E}}_{i,j} = \exp\left(\frac{\alpha_i - 1}{\delta\alpha_i\theta_j}\right).$$

By comparing this expression to (14), we identify the following terms:

$$\boldsymbol{\Phi}(\boldsymbol{\alpha}) = \widetilde{\mathbf{E}},\ \phi_i(\alpha_j) = \exp\left(\frac{\alpha_i - 1}{\delta\alpha_i\theta_j}\right),\ \psi(\alpha_i) = \frac{1-\alpha_i}{\alpha_i^{n+1}}.$$

Moreover,

$$K = \kappa\frac{(-1)^{(n-1)(n-2)/2}}{\delta^{(n-1)(n+2)/2}}\,.$$

As a consequence, $F_r(z)$ can be determined by evaluating

$$(\mathbf{F}_h)_{i,j} = \begin{cases} \mathrm{e}^{\frac{1}{\delta\theta_i}}\displaystyle\int_0^1 \frac{1-\alpha}{\alpha^{n-j+2}}\mathrm{e}^{-\frac{1}{\theta_i\alpha\delta}}\,\mathrm{d}\alpha & 1 \le i < h \\ \mathrm{e}^{\frac{1}{\delta\theta_{i+1}}}\displaystyle\int_0^1 \frac{1-\alpha}{\alpha^{n-j+2}}\mathrm{e}^{-\frac{1}{\theta_{i+1}\alpha\delta}}\,\mathrm{d}\alpha & h \le i \le n-2 \end{cases} \tag{29}$$

while

$$\mathcal{I}_{h,q}(z) = \mathrm{e}^{\frac{1}{\delta\theta_h}}\int_z^1 \frac{\mathrm{e}^{-\frac{1}{\theta_h\alpha\delta}}(\alpha - z)^{q-1}}{\alpha^{n+2}}(1-\alpha)\,\mathrm{d}\alpha\,. \tag{30}$$

## V. Results

We now validate our analysis through numerical simulation. Figure 1 shows the cdf of the Rayleigh quotient in the presence of uncorrelated Rayleigh-faded interferers. Analytical results are denoted by lines (either solid or dashed), while numerical results are represented by circle-shaped markers. The agreement between analytical and numerical results is excellent, for any value of the number of antennas $n$ and of the SNR $E_s/N_0$. Furthermore, as expected, low values of the Rayleigh quotient become more likely as $E_s/N_0$ increases, as well as, given $E_s/N_0$, for a smaller number of antennas. Figures 2 and 3 depict the cdf and pdf, respectively, of the Rayleigh quotient for the case where interferers are correlated and Rayleigh faded. Here we set $\theta_i = \frac{2i}{n(n-1)}$, i.e., such that $\sum_{i=1}^{n-1}\theta_i = 1$. We observe that when the SNR increases the difference between the performance of ZF and MMSE receivers tend to vanish, i.e., smaller values of $r$ are more likely. A similar effect occurs as the number of antenna decreases.

## VI. Conclusion

We provided closed-form statistics for the Rayleigh quotient of a random matrix with respect to a Gaussian uncorrelated vector, in the complex case. As an immediate application of our result, the gap in terms of multiuser efficiency achievable by MMSE and ZF receivers is statistically characterized for the first time in closed form, and for a large class of fading distributions. Assuming the intended stream to be Rayleigh-faded, we first analyze a totally homogeneous case, where all interfering streams undergo uncorrelated Rayleigh fading; then we move to a correlated Rayleigh fading for the interference. Numerical results validate our analysis in both cases. Exploitation of our results on Rayleigh quotient in both cognitive radio strategy analysis, as well as in stochastic networks performance evaluation, are subject of our ongoing work.

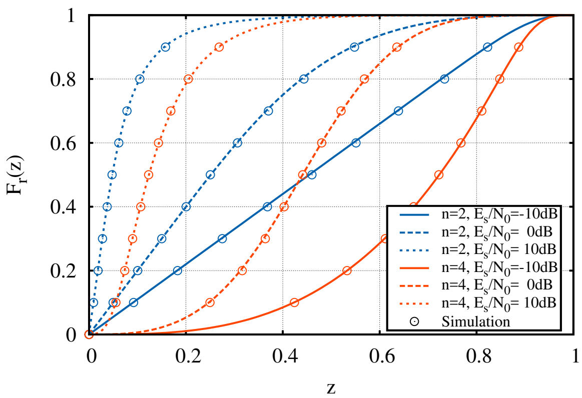


Fig. 1. Cumulative distribution function of the Rayleigh quotient under Rayleigh-faded uncorrelated interferers.

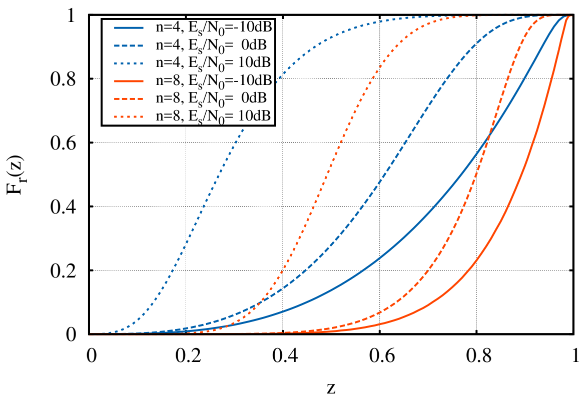


Fig. 2. Cumulative distribution function of the Rayleigh quotient under Rayleigh-faded correlated interferers.

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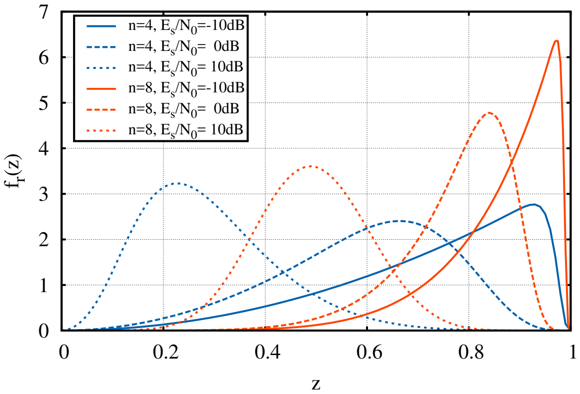


Fig. 3. Probability density function of the Rayleigh quotient under Rayleigh-faded correlated interferers.


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