Conformal geometry of timelike curves in the (1+2)-Einstein universe

(Article begins on next page)

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CONFORMAL GEOMETRY OF TIMELIKE CURVES
IN THE $(1+2)$-EINSTEIN UNIVERSE

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ABSTRACT. We study the conformal geometry of timelike curves in the $(1+2)$-Einstein universe, the conformal compactification of Minkowski 3-space defined as the quotient of the null cone of $\mathbb{R}^{2,3}$ by the action by positive scalar multiplications. The purpose is to describe local and global conformal invariants of timelike curves and to address the question of existence and properties of closed trajectories for the conformal strain functional. Some relations between the conformal geometry of timelike curves and the geometry of knots and links in the 3-sphere are discussed.

1. INTRODUCTION

Conformal Lorentzian geometry has played an important role in general relativity since the work of H. Weyl [52]. In the 1980s, it has been at the basis of the development of twistor approach to gravity by Penrose and Rindler [42] and it is one of the main ingredients in the recently proposed cyclic cosmological models in general relativity [40, 41, 50]. It also plays a role in the regularization of the Kepler problem [21, 26, 27, 29], in conformal field theory [48], and in Lie sphere geometry [5, 7]. For what concerns in particular the geometry of curves, while the subject of conformal Lorentzian invariants of null curves has received some attention [5, 3, 51], that of timelike curves seems to have been little studied before.

In this paper we investigate the geometry of timelike curves in $\mathcal{E}^{1,2}$, the conformal compactification of Minkowski 3-space defined as the space of oriented null lines of $\mathbb{R}^{2,3}$ through the origin. This study is intended as a preliminary step to understand the 4-dimensional case, which is that of physical interest. Despite some formal similarities, there are substantial differences between the conformal Riemannian and Lorentzian case: the Lorentzian space $\mathcal{E}^{1,2}$ has the topology of $S^1 \times S^2$, which is not simply connected; the global Lorentzian metrics on $\mathcal{E}^{1,2}$ are never maximally symmetric; the universal covering of the conformal group of $\mathcal{E}^{1,2}$ does not admit finite dimensional representations and has a center which is discrete, but not finite [2, 45]. Following [3, 16], we call $\mathcal{E}^{1,2}$ the $(1+2)$-Einstein universe.[1] A motivation for this terminology is that the universal covering of $\mathcal{E}^{1,2}$, $\mathbb{R} \times S^2$, endowed with the product metric $-dt^2 + g_{S^2}$, provides a static solution of Einstein's equation with a positive cosmological constant. This solution was proposed by Einstein himself as a model of a closed static universe in [11]. In addition, the pseudo-Riemannian geometries of the standard Friedmann–Lemaitre–Robertson–Walker cosmological models can be realized as subgeometries of $\mathcal{E}^{1,2}$ [22].

The purposes of this paper are threefold. The first is to describe local and global conformal differential invariants of a timelike curve. The second purpose is to address the question of existence and properties of closed trajectories for the variational problem

[1]Actually, $\mathcal{E}^{1,2}$ is the double covering of the space that in [3, 16] is called Einstein universe.

defined by the conformal strain functional, the Lorentzian analogue of the conformal arclength functional in Möbius geometry [25, 30, 31, 34]. The Lagrangian of the strain functional depends on third-order jets and shares many similarities with the relativistic models for massless or massive particles based on higher-order action functionals, a topic which has been much studied over the past twenty years [15, 19, 23, 32, 33, 35, 36, 43]. The last purpose is to establish a connection between the conformal geometry of timelike curves in the Einstein universe and the geometry of transversal knots in the unit 3-sphere.

From a physical point of view, the relevant objects are the lifts of timelike curves to the universal covering of $\mathcal{E}^{1,2}$ and their global conformal invariants. The compactified model has the advantage of having a matrix group as its restricted conformal group, which simplifies the use of the geometric methods based on the transformation group and eases the computational aspects.

The paper is organized as follows. In Section 2, we collect some background material about conformal Lorentzian geometry. For the geometry of the Einstein universe, we mainly follow [3].

In Section 3, we study the conformal geometry of timelike curves in the Einstein universe. We define the infinitesimal conformal strain, which is the Lorentzian analogue of the conformal arc element of a curve in S^3 [24, 28, 30, 31, 34], and the notion of a conformal vertex. An explicit description of curves all of whose points are vertices is given in Proposition 1. Next, we define the concept of osculating conformal cycle and give a geometric characterization of conformal vertices in terms of the analytic contact between the curve and its osculating cycle (Proposition 2). We then prove the existence of a canonical conformal frame field along a generic timelike curve (i.e., a timelike curve without vertices) and define the two conformal curvatures, which are the main local conformal invariants of a generic curve (Theorem 3). As a byproduct, some elementary consequences are derived (Propositions 4, 5 and 6 and Corollary 7).

In Section 4, the canonical conformal frame is used to investigate generic timelike curves with constant conformal curvatures. We exhibit explicit parameterizations of such curves in terms of elementary functions (Theorems 8 and 9) and discuss their main geometric properties. These are the Lorentzian counterparts of analogous results for curves with constant conformal curvatures in S^3 [47].

In Section 5, we use the canonical frame to compute the Euler-Lagrange equations of the conformal strain functional (Theorem 10). Consequently we show that the conformal curvatures of the extrema can be expressed in terms of Jacobi's elliptic functions. As a byproduct, we show that the conformal equivalence classes of critical curves depend on two real constants and prove that there exist countably many distinct conformal equivalence classes of closed trajectories for the strain functional (Theorem 11).

In Section 6, we establish a connection between the conformal geometry of timelike curves and the geometry of transversal knots in the S^3 via the directrices of a generic timelike curve. These are immersed curves in S^3, everywhere transverse to the canonical contact distribution, which are built using the symplectic lift of the canonical conformal frame. If the directrices of a generic closed timelike curve are simple, then their linking and Bennequin numbers [17] provide three global conformal invariants, different in general from the Maslov index of the curve. These invariants are computed for the directrices of a special class of closed timelike curves with constant conformal curvatures (Proposition 12). It is still an open question how the local symplectic invariants [1, 8] of the directrices can be related to the strain and the conformal curvatures. A more difficult problem is to understand how the classical and non-classical invariants of transversal knots of S^3 [12, 13, 14, 17] are related to the conformal geometry of closed timelike curves.

2. Conformal Lorentzian geometry

2.1. **The automorphism group** $A^\uparrow_+(2,3)$. Consider $\mathbb{R}^5$ with a nondegenerate scalar product $\langle\cdot,\cdot\rangle$ of signature $(2,3)$ and a volume form $dV \in \Lambda^5(\mathbb{R}^5)$. Let $\mathcal{N}^2$ denote the cone of all isotropic bivectors, i.e., the non-zero decomposable bivectors $X \wedge Y \in \Lambda_2(\mathbb{R}^5)$, such that $\langle X, X\rangle = \langle Y, Y\rangle = \langle X, Y\rangle = 0$. Choose an oriented spacelike 3-dimensional vector subspace $\mathbb{V}^3_+ \subset \mathbb{R}^5$ and a positive-oriented orthogonal basis V_1, V_2, V_3 of $\mathbb{V}^3_+$. We call an isotropic bivector $X \wedge Y$ *future directed* if $dV(X, Y, V_1, V_2, V_3) > 0$ and denote by $\mathcal{N}^2_+$ the half-cone of all future-directed isotropic bivectors. Let $\mathbb{R}^{2,3}$ denote $\mathbb{R}^5$ with the scalar product $\langle\cdot,\cdot\rangle$, the orientation induced by dV, and the time-orientation determined by the half-cone $\mathcal{N}^2_+$. The 10-dimensional Lie group $A^\uparrow_+(2,3)$ of linear isometries of $\mathbb{R}^{2,3}$ preserving the given orientation and time-orientation is referred to as the *automorphism group* of $\mathbb{R}^{2,3}$. Its Lie algebra is the vector space

$$\mathfrak{a}(2,3) = \{f \in \mathrm{End}(\mathbb{R}^{2,3}) \,:\, \langle f(X), Y\rangle + \langle X, f(Y)\rangle = 0, \,\forall\, X, Y \in \mathbb{R}^{2,3}\},$$

equipped with the commutator as a Lie bracket. Given a basis $\mathcal{B} = (B_0, B_1, B_2, B_3, B_4)$ of $\mathbb{R}^{2,3}$ and an endomorphism $f \in \mathrm{End}(\mathbb{R}^{2,3})$, let $M_\mathcal{B}(f)$ be the 5×5 matrix representing f with respect to $\mathcal{B}$. Similarly, let $G_\mathcal{B}$ be the symmetric matrix representing the scalar product $\langle\cdot,\cdot\rangle$ with respect to $\mathcal{B}$. For every choice of $\mathcal{B}$, the map $\chi_\mathcal{B} : F \in A^\uparrow_+(2,3) \mapsto M_\mathcal{B}(F) \in \mathrm{GL}(5, \mathbb{R})$ is a faithful matrix representation of $A^\uparrow_+(2,3)$. We say that:

- $\mathcal{B}$ is a *Möbius basis* if $\mathcal{B}$ is positive-oriented, $G_\mathcal{B} = -(E^0_4 + E^4_0) - E^1_1 + E^2_2 + E^3_3 =: \mathtt{m}$, $\mathtt{m} = (\mathtt{m}_{ij})$, and the isotropic bivector $B_0 \wedge (B_1 + B_2)$ is future-oriented;[2]
- $\mathcal{B}$ is a *Poincaré basis* if $\mathcal{B}$ is positive-oriented, $G_\mathcal{B} = -E^0_0 - E^1_1 + E^2_2 + E^3_3 + E^4_4$, and the isotropic bivector $(B_0 - B_4) \wedge (B_1 + B_2)$ is future-oriented;
- $\mathcal{B}$ is a *Lie basis* if $\mathcal{B}$ is positive-oriented, $G_\mathcal{B} = -(E^0_4 + E^4_0) - (E^3_1 + E^1_3) + E^2_2$, and the isotropic bivector $B_0 \wedge B_1$ is future-oriented.

We choose and fix a reference Möbius basis $\mathbf{M}^o = (M^o_0, \ldots, M^o_4)$ of $\mathbb{R}^{2,3}$, referred to as the *standard Möbius basis*. The image of $A^\uparrow_+(2,3)$ under the faithful representation determined by $\mathbf{M}^o$ is a connected closed subgroup of $\mathrm{SL}(5, \mathbb{R})$, denoted by $M^\uparrow_+(2,3)$. Let

$$(2.1) \qquad \begin{cases} T_{\mathrm{mp}} = \frac{1}{\sqrt{2}}(E^0_0 + E^4_0) + E^1_1 + E^2_2 + E^3_3 + \frac{1}{\sqrt{2}}(E^4_4 - E^0_4), \\ T_{\mathrm{m}\ell} = E^0_0 - \frac{1}{\sqrt{2}}(E^1_1 + E^3_1) + E^2_2 + \frac{1}{\sqrt{2}}(E^3_1 - E^3_3) + E^4_4. \end{cases}$$

Then, $\mathcal{B}$ is a Möbius basis if and only if $\mathcal{B}\cdot T_{\mathrm{mp}}$ is a Poincaré basis if and only if $\mathcal{B}\cdot T_{\mathrm{m}\ell}$ is a Lie basis; in particular, $\mathbf{P}^o = (P^o_0, \ldots, P^o_4) = \mathbf{M}^o \cdot T_{\mathrm{mp}}$ is referred to as the *standard Poincaré basis* and $\mathbf{L}^o = (L^o_0, \ldots, L^o_4) = \mathbf{M}^o \cdot T_{\mathrm{m}\ell}$ the *standard Lie basis* of $\mathbb{R}^{2,3}$. Differentiating the $\mathbb{R}^{2,3}$-valued maps $\mathcal{M}_j : F \in A^\uparrow_+ \mapsto F(M^o_j) \in \mathbb{R}^{2,3}$, $j = 0, \ldots, 4$, yields $d\mathcal{M}_j = \sum_i \mu^i_j \mathcal{M}_i$, where μ^i_j are left-invariant 1-forms. The conditions $\langle \mathcal{M}_j, \mathcal{M}_i\rangle = \mathtt{m}_{ji}$ imply that $\mu = (\mu^i_j)$ takes values in the Lie algebra $\mathfrak{m}(2,3)$ of $M^\uparrow_+(2,3)$. Choosing

$$M^0_0 = E^0_0 - E^4_4, \quad M^0_1 = E^0_1 - E^4_1, \quad M^0_2 = E^0_2 + E^4_2, \quad M^0_3 = E^0_3 + E^4_3, \quad M^1_2 = E^1_2 + E^2_1,$$
$$M^1_3 = E^1_3 + E^3_1, \quad M^2_3 = E^2_3 - E^3_2, \quad M^4_1 = E^4_1 - E^1_0, \quad M^4_2 = E^4_2 + E^2_0, \quad M^4_3 = E^4_3 + E^3_0$$

as a basis of $\mathfrak{m}(2,3)$, we can write

$$\mu = \mu^0_0 M^0_0 + \mu^1_0 M^0_1 + \mu^2_0 M^0_2 + \mu^3_0 M^0_3 + \mu^2_1 M^1_2$$
$$+ \mu^3_1 M^1_3 + \mu^3_2 M^2_3 + \mu^1_4 M^4_1 + \mu^2_4 M^4_2 + \mu^3_4 M^4_3,$$

where the left-invariant 1-forms μ^0_0, μ^1_0, μ^2_0, μ^3_0, μ^2_1, μ^3_1, μ^3_2, μ^1_4, μ^2_4, μ^3_4 are linearly independent and span $\mathfrak{a}(2,3)^*$, the dual of the Lie algebra $\mathfrak{a}(2,3)$. If $F \in A^\uparrow_+(2,3)$, the vectors $\mathcal{M}_0(F), \ldots, \mathcal{M}_4(F)$ constitute a Möbius basis of $\mathbb{R}^{2,3}$, whose dual basis is denoted by $\mathcal{M}^0(F), \ldots, \mathcal{M}^4(F)$. The Maurer-Cartan form of $A^\uparrow_+(2,3)$ can be written as $\mu = \sum_{i,j} \mu^i_j \mathcal{M}^j \otimes \mathcal{M}_i$. It satisfies the Maurer-Cartan equations $d\mu = -\frac{1}{2}[\mu, \mu]$.

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[2]E^a_b, $0 \le a, b \le 4$, denotes the elementary matrix with 1 in the (a, b) place and 0 elsewhere.
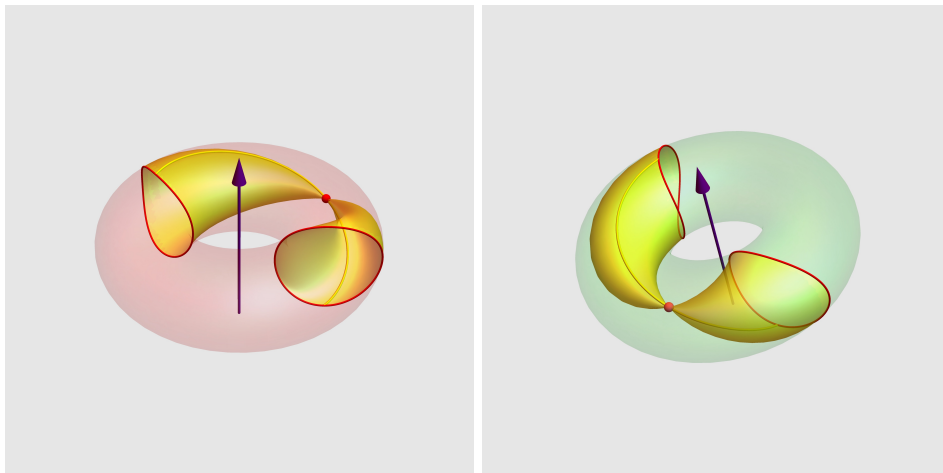
FIGURE 1. The two adS chambers and a null light-cone. The Einstein universe is obtained by identifying the two boundaries. The wall, i.e., the surface given by the identification of the two boundaries, is a totally umbilical torus of signature $(1,1)$.

2.2. **The $(1+2)$-Einstein universe and its conformal group.** Consider the orthogonal direct sum decomposition $\mathbb{R}^{2,3} = \mathbb{V}^2_- \oplus \mathbb{V}^3_+$ into the oriented, negative definite 2-dimensional subspace $\mathbb{V}^2_- = [P^o_0 \wedge P^o_1]$ and the oriented 3-dimensional spacelike subspace $\mathbb{V}^3_+ = [P^o_2 \wedge P^o_3 \wedge P^o_4]$. Let $(x_0, \ldots, x_4)$ be the Cartesian coordinates of the standard Poincaré basis and denote by $\mathcal{E}^{1,2}$ the 3-dimensional submanifold of $\mathbb{R}^{2,3}$ defined by the equations $x_0^2 + x_1^2 = 1$ and $x_2^2 + x_3^2 + x_4^2 = 1$. As a manifold, $\mathcal{E}^{1,2}$ is the Cartesian product $S^1_- \times S^2_+$ of the unit circle of $\mathbb{V}^2_-$ with the 2-dimensional unit sphere of $\mathbb{V}^3_+$. The scalar product on $\mathbb{R}^{2,3}$ induces a Lorentzian pseudo-metric g_e on $\mathcal{E}^{1,2}$. The normal bundle of $\mathcal{E}^{1,2}$ is spanned by the restrictions of the vector fields $\mathbf{n}_1 = x_0 \partial_{x_0} + x_1 \partial_{x_1}$ and $\mathbf{n}_2 = x_2 \partial_{x_2} + x_3 \partial_{x_3} + x_4 \partial_{x_4}$. Thus, contracting dV with $\mathbf{n}_1$ and $\mathbf{n}_2$, we get a volume form on $\mathcal{E}^{1,2}$ which in turn defines an orientation on $\mathcal{E}^{1,2}$. The vector field $-x_1 \partial_{x_0} + x_0 \partial_{x_1}$ is tangent to $\mathcal{E}^{1,2}$ and induces a nowhere vanishing timelike vector field $\mathbf{t}$ on $\mathcal{E}^{1,2}$. We time-orient $\mathcal{E}^{1,2}$ by requiring that $\mathbf{t}$ is future-oriented.

Definition 1. The Lorentzian manifold $(\mathcal{E}^{1,2}, g_e)$, with the above specified orientation and time-orientation, is called the $(1+2)$-*Einstein universe*. The Einstein universe is a homogeneous Lorentzian manifold and its restricted isometry group is a 4-dimensional maximal compact subgroup of $\mathrm{A}^\uparrow_+(2,3)$, isomorphic to $SO(2) \times SO(3)$.

For each non-zero vector $X \in \mathbb{R}^{2,3}$, we denote by $[X]$ the oriented line spanned by X (i.e., the ray of X). The set of all null rays, denoted by $\mathcal{M}^{1,2}$ is a manifold and the map

$$\Phi : \mathcal{E}^{1,2} \ni X \mapsto [X] \in \mathcal{M}^{1,2} \tag{2.2}$$

is a diffeomorphism. This allows us to identify $\mathcal{E}^{1,2}$ with $\mathcal{M}^{1,2}$ and to transfer to $\mathcal{M}^{1,2}$ the oriented, time-oriented conformal Lorentzian structure of $\mathcal{E}^{1,2}$. We will make no distinction between the two models and the context will make clear which of them is being used. Using the above identification, the automorphism group $\mathrm{A}^\uparrow_+(2,3)$ acts effectively and transitively on the left of $\mathcal{E}^{1,2} \cong \mathcal{M}^{1,2}$ by $F \cdot X = [F(X)]$, for each $F \in \mathrm{A}^\uparrow_+(2,3)$ and $[X] \in \mathcal{M}^{1,2}$. The action preserves the oriented, time-oriented conformal Lorentzian structure of $\mathcal{E}^{1,2}$. It is a classical result that every restricted conformal transformation of the Einstein universe is induced by a unique element of $\mathrm{A}^\uparrow_+(2,3)$ [10, 16]. For this reason, we call $\mathrm{A}^\uparrow_+(2,3)$ the

(restricted) *conformal group of the Einstein universe.* The map

$$\tau : \mathcal{E}^{1,2} \ni X \mapsto (x_0 x_2 - x_1(x_3 + 2), x_1 x_2 + x_0(x_3 + 2), x_4) \in \mathbb{R}^3$$

is a 2:1 branched covering onto the toroid $\mathrm{T} \subset \mathbb{R}^3$ swept out by the rotation around the z-axis of the unit disk in the xz-plane centered at $(2, 0, 0)$. Thus, the Einstein universe can be identified with the quotient space $\mathrm{T} \sqcup \mathrm{T}/ \sim$ of the disjoint union $\mathrm{T} \sqcup \mathrm{T} = \mathrm{T} \times \{-1, 1\}$ of two copies of the toroid T modulo the equivalence relation defined by $[(P, \epsilon)]_\sim = \{(P, \epsilon)\}$, if $P \in \mathrm{Int}(\mathrm{T})$, and $[(P, \epsilon)]_\sim = \{(P, \pm\epsilon)\}$, if $P \in \partial\mathrm{T}$. In what follows we will use the "toroidal" projection τ to visualize and clarify the geometrical content of the results.

2.3. Conformal embeddings of Lorentzian space forms.

As a model for *anti-de Sitter 3-space*, we consider the hyperquadric of $\mathbb{R}^4$

$$\mathbb{M}^{(1,2)}_{-1} = \{(x, y) \in \mathbb{R}^4 : -(x^1)^2 - (x^2)^2 + (y^1)^2 + (y^2)^2 = -1\}$$

equipped with the Lorentzian structure induced by the neutral scalar product

$$(\mathbf{x}, \widetilde{\mathbf{x}})_{(2,2)} = -x^1 \widetilde{x}^1 - x^1 \widetilde{x}^1 + y^1 \widetilde{y}^1 + y^2 \widetilde{y}^2.$$

Then, $\mathbb{M}^{(1,2)}_{-1}$ can be embedded in the Einstein universe by the conformal map

$$\mathbf{j}_{adS} : (x, y) \mapsto \frac{1}{\sqrt{(x^1)^2 + (x^2)^2}} \left(x^1 P_0^o + x^2 P_1^o + P_2^o + y^1 P_3^o + y^2 P_4^o\right).$$

The image is the open subset $\mathcal{T}_+ = \{X \in \mathcal{E}^{1,2} : \langle X, P_2^o \rangle > 0\}$, the *positive adS-chamber*. The boundary of $\mathcal{T}_+$ is the *adS-wall*, i.e., the timelike embedded torus

$$\mathcal{T}_\infty = \{X \in \mathcal{E}^{1,2} : \langle X, P_2^o \rangle = 0\} \subset \mathcal{E}^{1,2}.$$

The complement of $\mathcal{T}_+ \cup \mathcal{T}_\infty$ is the *negative adS-chamber* $\mathcal{T}_-$, another copy of the anti-de Sitter space inside $\mathcal{E}^{1,2}$. The restriction of the branched covering τ to each of the two adS-chambers is a smooth diffeomorhism onto the interior of T, while the restriction of τ to the adS-wall is a diffeomorphism onto $\partial\mathrm{T}$ (see Figure 1).

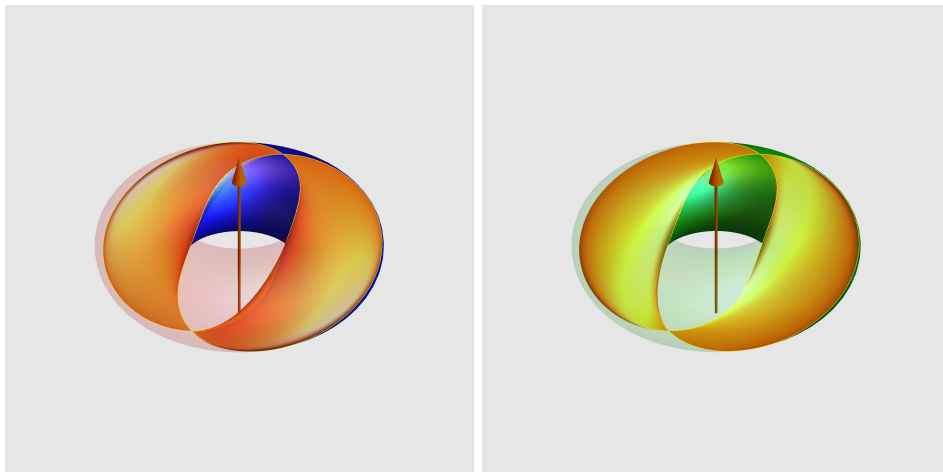FIGURE 2. The Minkowski chamber. Its boundary is a totally umbilical null cone of the Einstein universe. The two surfaces colored in blue and green are identified.

Let $\mathbb{M}^{(1,2)}_0$ be *Minkowski 3-space*, i.e., the affine space $\mathbb{R}^3$ with the Lorentzian scalar product

$$(\mathbf{x}, \widetilde{\mathbf{x}})_{(1,2)} = -x^1 \widetilde{x}^1 + x^2 \widetilde{x}^2 + x^3 \widetilde{x}^3.$$

For each point $\mathbf{x} = {}^t(x_1, x_2, x_3) \in \mathbb{M}_0^{(1,2)}$, let $^*\mathbf{x} = (-x_1, x_2, x_3)$. The map

$$\mathbf{j}_m(\mathbf{x}) = \Phi^{-1}\left(\left[M_0^o + x_1 M_1^o + x_2 M_2^o + x_3 M_3^o + \frac{{}^*\mathbf{x}\mathbf{x}}{2} M_4^o\right]\right)$$

is a conformal embedding whose image, the *positive Minkowski-chamber*, is the open subset $\mathcal{M}_+ = \{X \in \mathcal{E}^{1,2} : -\langle X, P_0^o \rangle > \langle X, P_4^o \rangle\}$. Its boundary, the *Minkowski-wall*, is the light cone $\mathcal{M}_\infty = \{X \in \mathcal{E}^{1,2} : -\langle X, P_0^o \rangle = \langle X, P_4^o \rangle\}$. The complement of $\mathcal{M}_+ \cup \mathcal{M}_\infty$ is the *negative Minkowski-chamber*, another copy of Minkowski space inside $\mathcal{E}^{1,2}$ (see Figure 2). Next, consider *de Sitter 3-space*, that is, the quadric

$$\mathbb{M}_1^{(1,2)} = \left\{{}^t(w^1, w^2, w^3, w^4) \in \mathbb{R}^{1,3} : -(w^1)^2 + (w^2)^2 + (w^3)^2 + (w^4)^2 = 1\right\} \subset \mathbb{R}^4$$

equipped with the Lorentzian structure induced by the scalar product

$$(\mathbf{w}, \widetilde{\mathbf{w}})_{(1,3)} = -w^1 \widetilde{w}^1 + w^2 \widetilde{w}^2 + w^3 \widetilde{w}^3 + w^4 \widetilde{w}^4.$$

Then, $\mathbb{M}_1^{(1,2)}$ is mapped into $\mathcal{M}^{1,2}$ by the conformal embedding

$$\mathbf{j}_{dS} : {}^t(w^1, w^2, w^3, w^4) \mapsto \frac{1}{\sqrt{1 + (w^1)^2}}(P_0^o + w^1 P_1^o + w^2 P_2^o + w^3 P_3^o + w^4 P_4^o).$$

The image of $\mathbf{j}_{dS}$ is the *positive dS-chamber* $\mathcal{S}_+ = \{X = \sum_{j=0}^4 x^j P_j^o \in \mathcal{E}^{1,2} : x^0 > 0\}$. Its boundary is the disjoint union of two spacelike surfaces $\mathcal{S}_\infty^\pm = \{X = \sum_{j=0}^4 x^j P_j^o \in \mathcal{E}^{1,2} : x^0 = 0, \ x^1 = \pm 1\}$, the *dS-walls* of $\mathcal{E}^{1,2}$. Each wall is a totally umbilical 2-dimensional spacelike sphere embedded in $\mathcal{E}^{1,2}$. The complement of $\mathcal{S}_+ \cup \mathcal{S}_\infty^+ \cup \mathcal{S}_\infty^-$ is the *negative dS-chamber* $\mathcal{S}_- = \{X = \sum_{j=0}^4 x^j P_j^o \in \mathcal{E}^{1,2} : x^0 < 0\}$, which is another copy of de Sitter space inside the Einstein universe (see Figure 3).
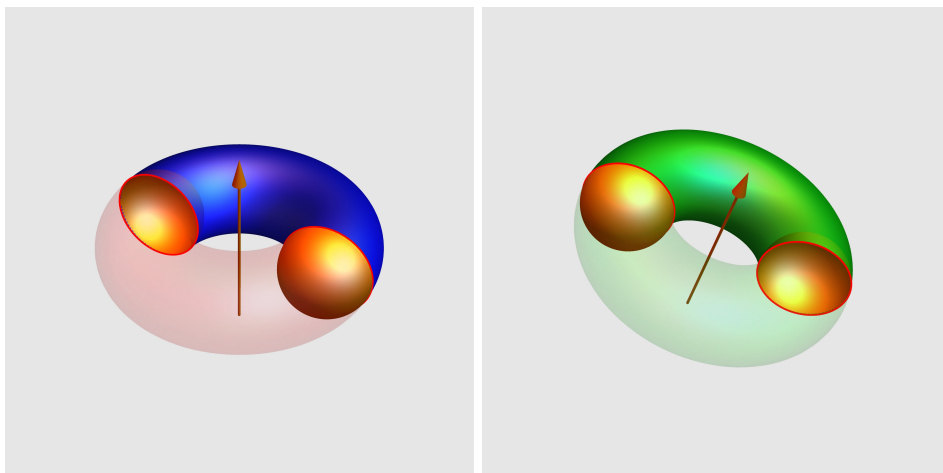
FIGURE 3. The de Sitter chamber. The boundary consists of two disjoint spacelike totally umbilical spheres of the Einstein universe. The surfaces colored in blue and green are identified.

3. CONFORMAL GEOMETRY OF TIMELIKE CURVES

Let $\gamma : I \subset \mathbb{R} \to \mathcal{E}^{1,2}$ be a parametrization of a smooth curve of the Einstein universe. We write $\gamma = \eta + \beta$, where $\eta : I \to S_-^1 \subset \mathbb{V}_-^2 = [P_0^o \wedge P_1^o]$ and $\beta : I \to S_+^2 \subset \mathbb{V}_+^3 = [P_2^o \wedge P_3^o \wedge P_4^o]$ are smooth curves, referred to as the *time-* and *space-component* of γ, respectively. In particular, γ is timelike if and only if $-\langle \eta', \eta' \rangle > \langle \beta', \beta' \rangle \geq 0$ and is future-directed if and only if $\eta' = \varrho_\eta j \eta$, where ϱ_η is a strictly positive function and

$j : \mathbb{V}_- \to \mathbb{V}_-$ is the counterclockwise rotation of an angle $\pi/2$ in the oriented, negative definite Euclidean plane $\mathbb{V}_-^2$. Henceforth, we only consider future-directed timelike curves.

Definition 2. The *Maslov index* of a closed timelike curve is the degree is its time-component.

Definition 3. If $\gamma : I \to \mathcal{E}^{1,2}$ is a timelike curve, then $\gamma|_t, \gamma'|_t, \gamma''|_t$ span an oriented 3-dimensional vector subspace of $\mathbb{R}^{2,3}$ of signature $(-, -, +)$. Such a subspace, denoted by $\mathcal{T}|_t$, is called the *conformal osculating space* of γ at $\gamma(t)$. Its orthogonal complement $\mathcal{N}|_t$ is the *normal conformal space* of γ at $\gamma(t)$. The orthogonal projection onto the normal spaces is denoted by $\pi_\mathcal{N}|_t$. The continuous function

$$(3.1) \qquad \upsilon_\gamma = \sqrt[4]{\frac{\langle \pi_\mathcal{N}(\gamma'''), \pi_\mathcal{N}(\gamma''') \rangle}{|\langle \gamma', \gamma' \rangle|}}$$

is called the *strain density* of γ. The exterior differential form $\sigma_\gamma := \upsilon_\gamma dt$ is the *infinitesimal strain*. In the proof of Theorem 3 below, we will show that σ_γ is invariant under the action of the conformal group and changes of parameter. The *strain* is a function $u_\gamma : I \to \mathbb{R}$, such that $du_\gamma = \sigma_\gamma$. By construction, the strain is a non-decreasing function of class C^1, uniquely defined up to an additive constant. A point $\gamma(t_0)$ is called a *conformal vertex* if the infinitesimal strain vanishes at t_0. A timelike curve without conformal vertices is said *generic*. If $\upsilon_\gamma = 0$, γ is said a *conformal cycle*.

Remark 1. The strain and the infinitesimal strain are dimensionless quantities. The infinitesimal strain is the Lorentzian analogue of the conformal arc element of a curve in the 3-dimensional round sphere [24, 30, 34].

Definition 4. Two timelike curves $\gamma : I \to \mathcal{E}^{1,2}$ and $\widetilde{\gamma} : \widetilde{I} \to \mathcal{E}^{1,2}$ are said to be *conformally equivalent* to each other if there exist a change of parameters $f : I \to \widetilde{I}$ and a restricted conformal transformation $F \in \mathrm{A}_+^\uparrow(2,3)$ such that $\gamma = F \cdot \widetilde{\gamma} \circ f$.

Proposition 1. *A conformal cycle is equivalent to the centerline of the positive adS-chamber.*

Proof. It is quite easy to see that the normal and osculating spaces of a conformal cycle are constants. By possibly applying a restricted conformal transformation, we may assume that the osculating space coincides with $[M_0^o, M_1^o, M_4^o]$. Then, using the identification of $\mathcal{E}^{1,2}$ with $\mathcal{M}^{1,2}$, we can write $\gamma = [x_0 M_0^o + x_1 M_1^o + x_4 M_4^o]$, where x_0, x_1 and x_4 are smooth functions, such that $2x_0 x_4 - x_1^2 = 0$. Hence, there exists a smooth function $\phi : I \to \mathbb{R}$, such that $x_0(t) = (\cos\phi(t) + 1)/\sqrt{2}$, $x_1(t) = \sin\phi(t)$ and $x_4(t) = (\cos\phi(t) - 1)/\sqrt{2}$. Since γ is a rank-one map, the derivative of ϕ is nowhere vanishing. Taking ϕ as a new parameter, we obtain $\gamma(\phi) = \cos(\phi)P_1^o + \sin(\phi)P_1^0$. This proves the result. $\square$

Remark 2. A conformal cycle consists of the null rays belonging to the intersection of the null cone of $\mathbb{R}^{2,3}$ with a 3-dimensional linear subspace of type $(2,1)$. Therefore, the totality of conformal cycles can be identified with $G_{2,1}(\mathbb{R}^{2,3})$, the 6-dimensional Grassmannian of 3-dimensional timelike subspaces of type $(2,1)$. This also shows that any conformal cycle is invariant under a 3-dimensional group of restricted conformal transformations isomorphic to $\mathrm{O}_+^\uparrow(2,1)$.

Given a timelike curve $\gamma : I \to \mathcal{E}^{1,2}$, the totality of null rays belonging to $\mathcal{T}|_{t_0}$ is the *osculating cycle* of γ at $\gamma(t_0)$. We give two characterizations of conformal vertices using osculating cycles. The proof of the results relies on simple computations and is omitted.

Proposition 2. *Let γ be a timelike curve. Then:*
- *The osculating cycle of γ at $\gamma(t_0)$ has second order analytic contact with γ at $\gamma(t_0)$. In addition, $\gamma(t_0)$ is a conformal vertex if and only if the order of analytic contact is strictly bigger than 2.*

- $\gamma(t_0)$ *is a conformal vertex if and only if t_0 is a stationary point of the curve*

$$I \ni t \mapsto [M_0|_{t_0}, M_1|_{t_0}, M_4|_{t_0}] \in G_{2,1}(\mathbb{R}^{2,3}).$$

Remark 3. The conformal strain density of a curve at a point $\gamma(t)$ measures the infinitesimal distorsion of the curve from its osculating cycle.
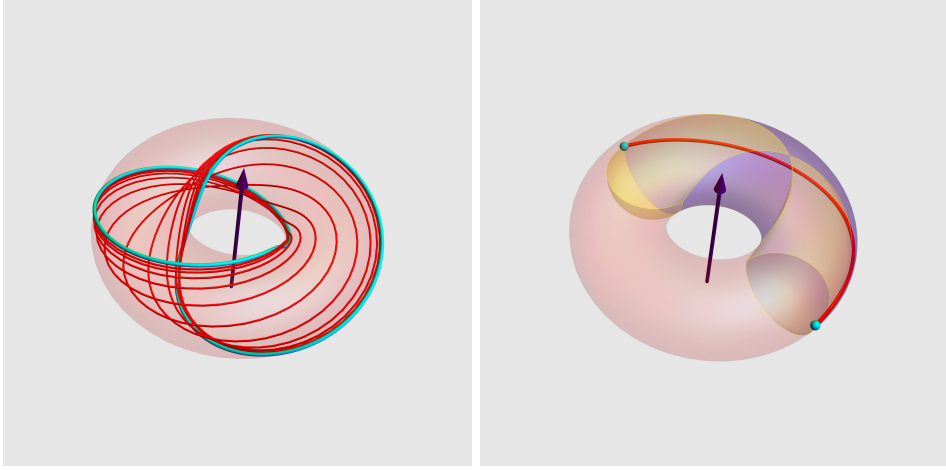

FIGURE 4. Curves with constant curvatures of the Classes C_1 (on the left) and C_3 (on the right), with parameters $a = 0.5$, $b = 0.2$ and $a = 1$, $b = 1.42009$, respectively.

3.1. **The canonical conformal frame of a generic timelike curve.** Let $\mathcal{M}$ be the manifold of Möbius frames of $\mathbb{R}^{2,3}$. The map $A_+^\uparrow(2,3) \to \mathcal{M}$, $F \mapsto (F(M_0^o), \dots, F(M_4^o))$, gives an explicit identification of the conformal group with the frame manifold $\mathcal{M}$. Consider a timelike curve $\gamma : I \to \mathcal{M}^{1,2}$. A *conformal frame* along γ is a lift $\mathbf{M} : I \to \mathcal{M}$ of γ to $\mathcal{M}$, i.e., a smooth map from the open interval I to $\mathcal{M}$, such that $M_0|_t = r(t) \cdot \gamma(t)$, where $r(t) > 0$, for each $t \in I$. Given a frame along γ, we put $\mathbf{M}^*(\mu) = \mathfrak{m}dt$, where $\mathfrak{m} = (m_j^i)$ is a smooth map with values in the Lie algebra $\mathfrak{m}(2,3)$. We can write

$$\mathfrak{m} = \begin{pmatrix} m_0^0 & m_1^0 & m_2^0 & m_3^0 & 0 \\ m_0^1 & 0 & m_2^1 & m_3^1 & -m_1^0 \\ m_0^2 & m_2^1 & 0 & -m_2^3 & m_2^0 \\ m_0^3 & m_3^1 & m_2^3 & 0 & m_3^0 \\ 0 & -m_0^1 & m_0^2 & m_0^3 & -m_0^0 \end{pmatrix}.$$

Slightly abusing notation, we omit $\mathbf{M}^*$ and write $\mu_j^i = m_j^i dt$ for the pull-back of forms.

Theorem 3. *Let $\gamma : I \to \mathcal{E}^{1,2}$ be a generic, future-oriented, timelike curve. Then there exists a unique conformal frame $\mathbf{M} : I \to \mathcal{M}$ along γ, the* canonical conformal frame, *such that*

$$(\mu_j^i) = \begin{pmatrix} 0 & -h & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & h \\ 0 & 0 & 0 & -k & 1 \\ 0 & 0 & k & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \sigma_\gamma,$$

where σ_γ is the infinitesimal strain of γ and $h, k : I \to \mathbb{R}$ are smooth functions, called the conformal curvatures of γ.

Proof. If $\mathbf{M}$ and $\widehat{\mathbf{M}}$ are two conformal frames along γ, then $\widehat{\mathbf{M}} = \mathbf{M}\mathbf{X}$, where $\mathbf{X}$ is a smooth map taking values in the subgroup $M_+^\uparrow(2,3)_0 = \{\mathbf{X} \in M_+^\uparrow(2,3) : X_0^1 = X_0^2 = X_0^3 = X_0^4 = 0\}$ of $M_+^\uparrow(2,3)$. The elements of $M_+^\uparrow(2,3)_0$ can be written as

$$\mathbf{X}(r, A, u) = \begin{pmatrix} r & {}^*uA & \frac{{}^*uu}{2r} \\ 0 & A & u/r \\ 0 & 0 & 1/r \end{pmatrix},$$

where $A \in O_+^\uparrow(1,2)$, $r > 0$, $u = {}^t(u^1, u^2, u^3)$ and ${}^*u = (-u^1, u^2, u^3)$. The two maps $\mathfrak{m}$ and $\widehat{\mathfrak{m}}$ are related by

$$(3.2) \qquad \widehat{\mathfrak{m}} = \mathbf{X}^{-1}\mathbf{X}' + \mathbf{X}^{-1}\mathfrak{m}\mathbf{X}.$$

A conformal frame along γ is of *first order* if $M_0 \wedge M_0'$ is a positive multiple of $M_0 \wedge M_1$. It is easily seen that first order frames do exist along any timelike curve. If $\mathbf{M} : I \to \mathcal{M}$ is a first order conformal frame along a future-directed timelike curve, then $m_0^1 > 0$ and $\mu_0^2 = \mu_0^3 = 0$. If $\mathbf{M}$ is a first order conformal frame, then any other is given by $\widetilde{\mathbf{M}} = \mathbf{M}\mathbf{X}$, where $\mathbf{X}$ is a smooth map with values in the subgroup

$$(3.3) \qquad M_+^\uparrow(2,3)_1 = \{X \in M_+^\uparrow(2,3)_0 : X_1^1 = 1, X_1^2 = X_1^3 = X_1^4 = 0\}$$

of $M_+^\uparrow(2,3)_0$. Thus, we may write

$$\mathbf{X} = \mathbf{X}(r, \theta, u, \hat{u}) = \begin{pmatrix} r & -u & {}^t\hat{u}R(\theta) & \frac{-u^2 + {}^t\hat{u}\hat{u}}{2r} \\ 0 & 1 & 0 & u/r \\ 0 & 0 & R(\theta) & \hat{u}/r \\ 0 & 0 & 0 & 1/r \end{pmatrix},$$

where $r : I \to \mathbb{R}^+$, θ, $u : I \to \mathbb{R}$, and $\hat{u} = {}^t(u^2, u^3) : I \to \mathbb{R}^2$ are smooth maps and

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Using (3.2), we find $\tilde{\mu}_0^1 = r\mu_0^1$ and $(\tilde{\mu}_1^2, \tilde{\mu}_1^3) = (\mu_1^2 - u^2\mu_0^1, \mu_1^3 - u^3\mu_0^1)R(\theta)$. From this we see that any timelike curve admits a *second order* conformal frame, that is, a first order conformal frame such that $\mu_1^2 = \mu_1^3 = 0$. In addition, if $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ are second order conformal frames along γ, then $\widetilde{\mathbf{M}} = \mathbf{M}\mathbf{Y}$, where $\mathbf{Y}$ is a smooth map with values in the subgroup $M_+^\uparrow(2,3)_2 = \{X \in M_+^\uparrow(2,3)_1 : X_4^2 = 1, X_4^3 = 0\}$ of $M_+^\uparrow(2,3)_1$. Then, $\mathbf{Y}$ can be written as $\mathbf{Y}(r, \theta, u) = \mathbf{X}(r, \theta, u, 0)$, where $r > 0$, and θ, u are smooth real-valued functions. From this we infer that

$$(3.4) \qquad (\tilde{\mu}_4^2, \tilde{\mu}_4^3) = r^{-1}(\mu_4^2, \mu_4^3)R(\theta).$$

Therefore, the differential form

$$(3.5) \qquad \mathcal{Q}_\gamma = \left((\mu_4^2)^2 + (\mu_4^3)^2\right)(\mu_0^1)^2 = \left((m_4^2)^2 + (m_4^3)^2\right)(m_0^1)^2 dt^4$$

is smooth and well defined, i.e., it does not depend on the choice of the second order frame. By construction, $\mathcal{Q}_\gamma$ is positive semidefinite, in the sense that $\mathcal{Q}_\gamma = Q_\gamma dt^4$, where Q_γ is a smooth non-negative function. We now prove that $\sqrt[4]{Q_\gamma}$ coincides with the strain density (3.1) of the curve. To this purpose, we choose a second order frame field along γ such that $M_0 = \gamma$ and $\gamma' = \mathrm{v}M_1$, where $\mathrm{v} = \sqrt{|\langle\gamma', \gamma'\rangle|} = m_0^1$ is the speed of γ. From the Frenet equations $\mathbf{M}' = \mathbf{M}\mathfrak{m}$ for $\mathbf{M}$, we obtain $\gamma'' = \mathrm{v}'M_1 + \mathrm{v}M_1' = \mathrm{v}'M_1 + \mathrm{v}(m_1^0 M_0 - \mathrm{v}M_4)$. Differentiating this equation and using again the Frenet equations yield $\gamma''' \equiv -\mathrm{v}^2(m_4^2 M_2 + m_4^3 M_3)$, modulo $\{M_0, M_1, M_4\}$. This implies

$$(3.6) \qquad \pi_\mathcal{N}(\gamma''') = -\mathrm{v}^2(m_4^2 M_2 + m_4^3 M_3), \quad \langle\pi_\mathcal{N}(\gamma'''), \pi_\mathcal{N}(\gamma''')\rangle = \mathrm{v}^4((m_4^2)^2 + (m_4^3)^2),$$

which combined with (3.5) proves the claim. Now, using (3.4) and the fact that $(m_4^2)^2 + (m_4^3)^2 > 0$, it follows that there exist second order frames along γ, such that $\mu_4^2 = \mu_0^1$, $\mu_4^3 = 0$. These frames are said of *third order*. If $\mathbf{M}$ and $\widetilde{\mathbf{M}}$ are third order frames along γ, then $\widetilde{\mathbf{M}} = \mathbf{M}\mathbf{Y}(u)$, where $\mathbf{Y}(u) = \mathbf{X}(1, 0, u, 0)$ and $u : I \to \mathbb{R}$ is a smooth function.

Then, from (3.2) is follows that $\tilde{\mu}_0^0 = \mu_0^0 + u\mu_0^1$. Therefore, we may single out a unique third order frame such that $\mu_0^0 = 0$, which satisfies the required properties. $\square$
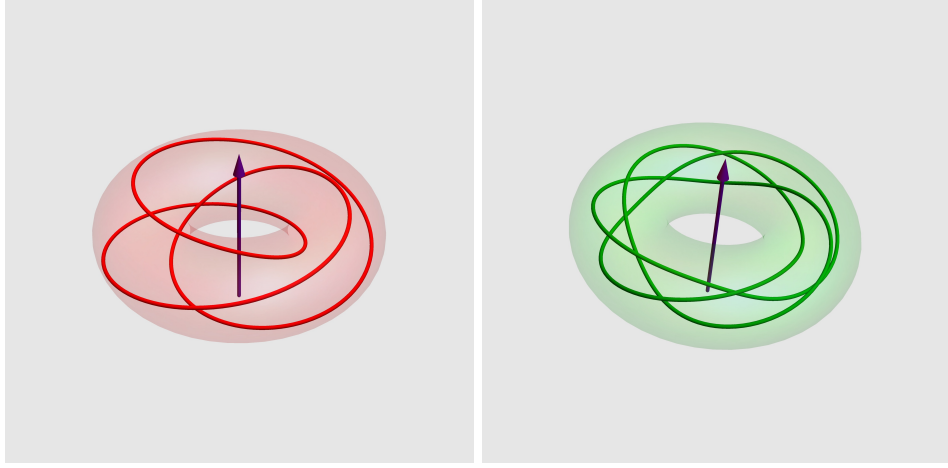

FIGURE 5. Curves with constant curvatures of the Classes C_{2i} (on the left) and C_{2ii} (on the right), with parameters $a = 0.5$, $b = 2/3$ and $a = 0.9$, $b = 5/3$, respectively.

The following are standard but relevant consequences of the existence of a canonical frame.

Proposition 4. *A generic timelike curve can be parametrized is such a way that σ_γ coincides with the differential of the independent variable. In this case, we say that the curve is* parametrized by conformal parameter*, which is usually denoted by u. The conformal parameter is uniquely defined up to a constant, $u \to u + c$.*

Remark 4. Let γ and $\widetilde{\gamma}$ be two generic timelike and future-directed curves, parametrized by conformal parameter. If they are equivalent to each other, the change of parameter is a shift of the independent variable, so that $\gamma(u) = F \cdot \widetilde{\gamma}(u + c)$, where c is a real constant and $F \in \mathrm{A}_+^\uparrow(2,3)$ is a conformal transformation. The conformal curvatures are related by $k(u) = \widetilde{k}(u + c)$ and $h(u) = \widetilde{h}(u + c)$, for each $u \in I$.

Remark 5. From now on, when considering a generic timelike curve parametrized by conformal parameter, we suppose that its interval of definition I be maximal. In this case, I is called the *proper interval* of γ.

Proposition 5. *Two timelike curves $\gamma, \widetilde{\gamma}$ parametrized by conformal parameter are equivalent to each other if and only if $I = \widetilde{I} + c$, $k(u) = \widetilde{k}(u + c)$, and $h(u) = \widetilde{h}(u + c)$, for some constant c.*

Definition 5. Given two smooth functions $k, h : I \to \mathbb{R}$, let $\mathcal{K}(h, k)$ denote the $\mathfrak{m}(2,3)$-valued map defined by $\mathcal{K}(h, k) := M_2^0 - M_1^4 - kM_3^2 - hM_1^0$. The curvatures and the canonical frame can be used to build the *curvature operator* of γ, i.e., the map $\mathfrak{K}_\gamma : I \to \mathfrak{a}(2,3)$ defined by $\mathbf{M}\,\mathcal{K}(h, k)\,\mathbf{M}^*$, where k, h are the conformal curvatures, $\mathbf{M}$ is the canonical conformal frame and $\mathbf{M}^*|_t = {}^t(M^0|_t, \ldots, M^4|_t)$ is the dual basis of $\mathbf{M}|_t$, for each $t \in I$.

Proposition 6. *If $k, h : I \to \mathbb{R}$ are two smooth functions, then there is a generic timelike curve γ, parametrized by conformal parameter, whose canonical conformal frame $\mathbf{M}$ satisfies the* conformal Frenet equations *$\mathbf{M}' = \mathbf{M}\,\mathcal{K}(k, h)$. By construction, k and h are the conformal curvatures of γ, which is unique, up to a restricted conformal transformation.*

As a consequence of the previous results, we have the following.

Corollary 7. *The first conformal curvature k vanishes identically if and only if there exists a restricted conformal transformation $F \in A_+^\uparrow(2,3)$, such that the trajectory of $F \circ \gamma$ belongs to the adS-wall of the Einstein universe.*

Proof. The conformal curvature k is identically zero if and only if M_3, the third vector of the canonical moving frame, is constant. This is equivalent to the existence of a restricted conformal transformation $F \in A_+^\uparrow(2,2)$, such that the null ray $F \circ \gamma(u)$ is orthogonal to the third vector of the standard Poincaré basis of $\mathbb{R}^{2,3}$, i.e., if and only if $F \circ \gamma(u) \in \mathcal{T}_\infty$. $\square$

FIGURE 6. Curves with constant curvatures of the Classes C_4 (on the left) and C_5 (on the right), with parameters $a = 0.99$, $b = 7.01722$ and $b = 1.001$, respectively.

4. TIMELIKE CURVES WITH CONSTANT CONFORMAL CURVATURES

If we act on a generic timelike curve with a time-preserving and orientation-reversing conformal transformation, the first curvature changes sign. Therefore, without loss of generality, we may assume $k \geq 0$. Consequently, the conformal equivalence classes of generic timelike curves with constant conformal curvatures can be parametrized in terms of two real parameters (k,h) belonging to the half plane $\mathbb{R}_+^2 = \{(k,h) \in \mathbb{R}^2 : k \geq 0\}$. In principle, since any timelike curve with constant curvatures is an orbit of a 1-parameter subgroup of $A_+^\uparrow(2,3)$ generated by the curvature operator $\mathfrak{K}_\gamma$, the explicit determination of timelike curves with constant curvatures can be reduced to compute $\mathrm{Exp}(t\mathfrak{K}_\gamma)$. However, the parametrizations obtained in this way lack in general a direct geometric interpretation suitable for the description of curve trajectories. Some additional work is required. We classify the generic homogeneous curves with $k > 0$ in terms of the stratification of $\mathbb{R}_+^2$ determined by the orbit-type of $\mathcal{K}(k,h)$. There are nine classes, namely:

$$(4.1) \quad \begin{cases} C_1 = \{(k,h) \in \mathbb{R}_+^2 \ : \ -2 < k^2 - 2h < 2, k > 0\}, \\ C_{2.i} = \{(k,h) \in \mathbb{R}_+^2 \ : \ k^2 - 2h < -2, k > 0\}, \\ C_{2.ii} = \{(k,h) \in \mathbb{R}_+^2 \ : \ 2 < k^2 - 2h, h > -\frac{1}{2k^2}, k > 1\}, \\ C_3 = \{(k,h) \in \mathbb{R}_+^2 \ : \ 0 < k < 1, -\frac{1}{2k^2} < h < \frac{k^2-2}{2}\}, \\ C_4 = \{(k,h) \in \mathbb{R}_+^2 \ : \ h < -\frac{1}{2k^2}, k > 0\} \end{cases}$$

and

$$(4.2) \quad \begin{cases} C_5 = \{(k,h) \in \mathbb{R}_+^2 \ : \ k > 1, h = -\frac{1}{2k^2}\}, \\ C_6 = \{(k,h) \in \mathbb{R}_+^2 \ : \ 0 < k < 1, h = -\frac{1}{2k^2}\}, \\ C_{7.i} = \{(k,h) \in \mathbb{R}_+^2 \ : \ k^2 - 2h = -2, k > 0\}, \\ C_{7.ii} = \{(k,h) \in \mathbb{R}_+^2 \ : \ k^2 - 2h = 2, k > 0\}, \\ C_8 = \{(k,h) \in \mathbb{R}_+^2 \ : \ k^2 - 2h = 2, 0 < k < 1\}, \\ C_9 = \{(1, -1/2)\}. \end{cases}$$

The curvature operators of homogeneous curves of the first four classes are regular elements of the Lie algebra $\mathfrak{a}(2,3)$. In the other cases, the curvature operators are exceptional elements of $\mathfrak{a}(2,3)$. For this reason, the homogeneous curves of the first four classes are said *regular*, while those belonging to the last five classes are said *exceptional*. We now describe explicit parametrizations of homogeneous timelike curves in terms of elementary functions. For the classes of regular curves, we have the following.
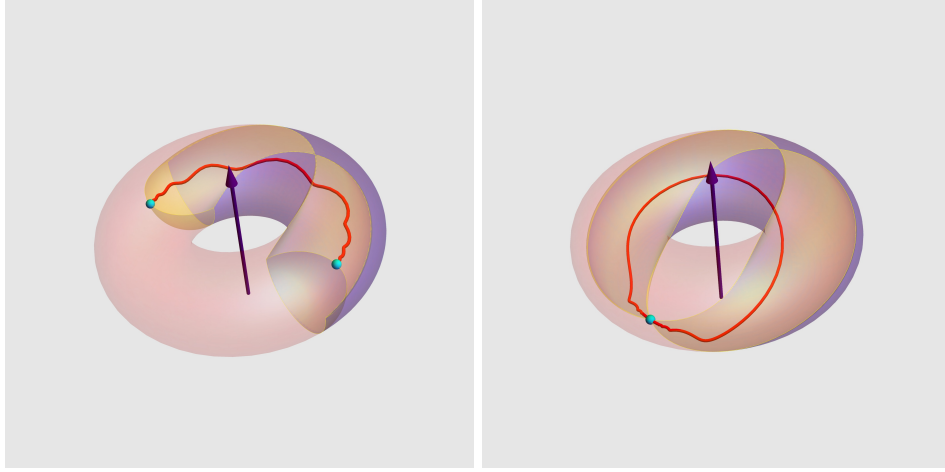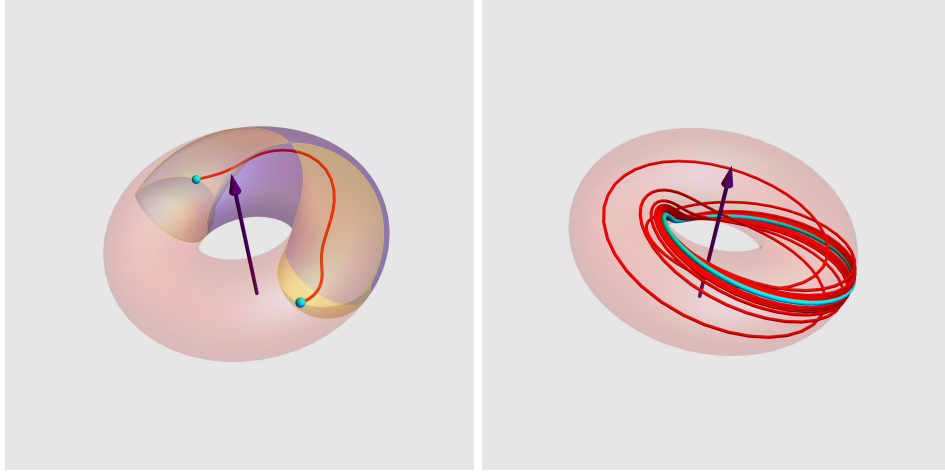

FIGURE 7. Curves with constant curvatures of the Classes C_6 (on the left) and C_{7i} (on the right), with parameters $b = 0.999$ and $b = 1.5$, respectively.

Theorem 8. *Let γ be a regular homogeneous curve. Then the following hold true:*

- *If $\gamma \in C_1$, there exists a unique Lie basis $(L_0, \dots, L_4)$ and a unique (a, b) belonging to the open domain $\Omega_1 = \{(a, b) \in (-1, 1) \times \mathbb{R} : b > 0, (1 + b + a(b - 1))(b - 1 + a(1 + b)) < 0\}$ such that, after a change of the independent variable, the curve is parametrized by $\mathbb{R} \ni t \mapsto [\gamma_{(a,b)}(t)]$, where*

$$(4.3) \quad \gamma_{(a,b)}(t) = x_0(t)L_0 + x_1(t)L_1 - \sqrt{2(1 - a^2)}L_2 + x_3(t)L_3 + x_4(t)L_4$$

  and

$$x_0(t) = e^{b(t - \frac{\pi}{4})}(\cos(t) + a\sin(t)), \quad x_1(t) = -e^{-b(t - \frac{\pi}{4})}(\sin(t) + a\cos(t)),$$
$$x_3(t) = e^{b(t - \frac{\pi}{4})}(a\cos(t) - \sin(t)), \quad x_4(t) = e^{-b(t - \frac{\pi}{4})}(\cos(t) - a\sin(t)).$$

- *If $\gamma \in C_{2.i}$, there exists a unique Poincaré basis $(P_0, \dots, P_4)$ and a unique element (a, b) of $\Omega_{2.i} = (0, 1) \times (0, 1)$ such that, after a change of variable, the curve is parametrized by $\mathbb{R} \ni t \mapsto [\gamma_{(a,b)}(t)]$, where*

$$(4.4) \quad \gamma_{(a,b)}(t) = \cos(t)P_0 + \sin(t)P_1 + aP_2 + \sqrt{1 - a^2}\cos(bt)P_3 - \sqrt{1 - a^2}\sin(bt)P_4.$$

- If $\gamma \in \mathrm{C}_{2.ii}$, there exists a unique Poincaré basis $(P_0, \dots, P_4)$ and a unique (a, b) belonging to open domain $\Omega_{2.ii} = \{(a, b) \in \mathbb{R}^2 : a \in (0, 1), b > 1, (1 - a^2)b^2 < 1\}$ such that, after a change of variable, the curve is parametrized by $\mathbb{R} \ni t \mapsto [\gamma_{(a,b)}(t)]$, where

$$(4.5) \qquad \gamma_{(a,b)}(t) = \cos(t)P_0 + \sin(t)P_1 - aP_2 + \sqrt{1 - a^2}\cos(bt)P_3 + \sqrt{1 - a^2}\sin(bt)P_4.$$

- If γ belongs to C_3, there exists a unique Lie basis $(L_0, \dots, L_4)$ and a unique element (a, b) of the domain $\Omega_3 = \{(a, b) \in \mathbb{R}^2 : a > 1/4, b > 1, 4a(b^2 - 1) - (b^2 + 1) > 0\}$ such that, after a change of variable, the curve is parametrized by $\mathbb{R} \ni t \mapsto [\gamma_{(a,b)}(t)]$, where

$$(4.6) \qquad \gamma_{(a,b)}(t) = \frac{1 + 4a}{4}e^t L_0 + e^{-bt}L_1 + L_2 + \frac{1 - 4a}{4}e^{bt}L_3 + e^{-t}L_4.$$

- If $\gamma \in \mathrm{C}_4$, there exists a unique Poincaré basis $(P_0, \dots, P_4)$ and a unique (a, b) in the open domain $\Omega_4 = \{(a, b) \in \mathbb{R}^2 : a \in (0, 1), b > 0, a^2 + (a^2 - 1)b^2 > 0\}$ such that, after a change of variable, the curve is parametrized by $\mathbb{R} \ni t \mapsto [\gamma_{(a,b)}(t)]$, where

$$(4.7) \quad \gamma_{(a,b)}(t) = P_0 + a\sinh(t)P_1 - a\cosh(t)P_2 + \sqrt{1 - a^2}\cos(bt)P_3 + \sqrt{1 - a^2}\sin(bt)P_4.$$

Proof. All the curves in the statement are orbits of a 1-parameter group of conformal transformations, so that the scalar products $\langle \gamma^{(n)}, \gamma^{(m)} \rangle$ are constants. Let $c = \sqrt{-\langle \gamma', \gamma' \rangle}$ and consider the nowhere vanishing vector field along γ defined by

$$\Gamma_4 = -\frac{1}{c^2}\gamma'' + \frac{\langle \gamma'', \gamma'' \rangle}{2c^4}\gamma.$$

Next, consider the constant

$$R = \sqrt[4]{\frac{\langle \Gamma_4', \Gamma_4' \rangle}{c^2} + \frac{\langle \Gamma_4', \gamma' \rangle \langle \gamma'', \gamma'' \rangle}{c^6} + \frac{\langle \gamma'', \gamma'' \rangle^2 \langle \gamma', \gamma' \rangle}{4c^{10}}}$$

and put

$$M_0 = R\gamma, \quad M_1 = \frac{1}{c}\gamma', \quad M_2 = \frac{1}{R^2 c}\Gamma_4' + \frac{1}{2c^4 R^2}\gamma', \quad M_4 = \frac{1}{R}\Gamma_4.$$

It is now a computational matter to check that $\langle M_i, M_j \rangle = \mathtt{m}_{ij}$, $i, j = 0, 1, 2, 4$. Then, if we let M_3 be the unique spacelike vector field along γ, such that $\mathbf{M}|_t = (M_0, M_1, M_2, M_3, M_4)|_t$ is a Möbius basis, for each $t \in \mathbb{R}$, we obtain a conformal moving frame satisfying $\mathbf{M}' = \upsilon \cdot \mathbf{M} \mathcal{K}(k, h)$, where

$$(4.8) \qquad \upsilon = Rc, \quad k = \frac{1}{Rc}\sqrt{\langle M_2' - R^2 c\gamma, M_2' - R^2 c\gamma \rangle}, \quad h = -\frac{\langle \gamma'', \gamma'' \rangle}{2c^4 R^2}.$$

This shows that $\mathbf{M}$ is the canonical conformal frame along γ and, in addition, that the conformal curvatures of γ are the constants k and h in (4.8).

Let (a, b) be an element of Ω_1 and let $\gamma_{(a,b)}$ be given as in (4.3). From (4.8), we compute the following expressions for the conformal curvatures of $\gamma_{(a,b)}$,

$$k_{(a,b)} = \sqrt{\frac{(1 + b^2)(1 - a^2)}{2b(1 + a^2)}}, \quad h_{(a,b)} = \frac{1 - 6b^2 + b^4 + 8ab(b^2 - 1) - a^2(b^4 - 6b^2 + 1)}{4b(1 + a^2)(1 + b^2)}.$$

The map $(a, b) \mapsto (k_{(a,b)}, h_{(a,b)})$ is a diffeomorphism of Ω_1 onto C_1, and hence each homogeneous curve of the first class is parametrized by $\gamma_{(a,b)}$, for a unique $(a, b) \in \Omega_1$.

Let $(a, b) \in \Omega_{2.i}$ and $\gamma_{(a,b)}$ be given as in (4.4). Then, using (4.8), we obtain

$$k_{(a,b)} = \frac{a\sqrt{b}}{\sqrt[4]{(1 - b^2)^2(1 - a^2)}}, \quad h_{(a,b)} = \frac{1 - b^4(1 - a^2)}{2b(1 - b^2)\sqrt{1 - a^2}}.$$

Since the map $(a, b) \mapsto (k_{(a,b)}, h_{(a,b)})$ is a diffeomorphism of $\Omega_{2.i}$ onto $\mathrm{C}_{2.i}$, we deduce that each homogeneous curve of the first type of the second class is parametrized by $\gamma_{(a,b)}$, for a unique $(a, b) \in \Omega_{2.i}$.

Let (a, b) be an element of $\Omega_{2.ii}$ and $\gamma_{(a,b)}$ be given as in (4.5). Then, using (4.8), we obtain

$$(4.9) \qquad k_{(a,b)} = \frac{a\sqrt{b}}{\sqrt[4]{(b^2 - 1)^2(1 - a^2)}}, \quad h_{(a,b)} = \frac{1 - b^4(1 - a^2)}{2b(b^2 - 1)\sqrt{1 - a^2}}.$$

The map $(a, b) \mapsto (k_{(a,b)}, h_{(a,b)})$ is a diffeomorphism of $\Omega_{2.ii}$ onto $\mathrm{C}_{2.ii}$, so that every homogeneous curve of the second type of the second class is parametrized by $\gamma_{(a,b)}$, for a unique $(a, b) \in \Omega_{2.ii}$.

If $(a, b) \in \Omega_3$ the conformal curvatures of $\gamma_{(a,b)}$ are given by

$$k_{(a,b)} = \frac{\sqrt{2b}}{\sqrt[4]{(b^2 - 1)^2(16a^2 - 1)}}, \quad h_{(a,b)} = \frac{(4a - 1)b^4 - 4a - 1}{2b(b^2 - 1)\sqrt{16a^2 - 1}}.$$

The map $(a, b) \mapsto (k_{(a,b)}, h_{(a,b)})$ is a diffeomorphism of Ω_3 onto C_3 and hence each homogeneous curve of the third class is parametrized by $\gamma_{(a,b)}$, for a unique $(a, b) \in \Omega_3$.

If $(a, b) \in \Omega_4$, the conformal curvatures of $\gamma_{(a,b)}$ are given by

$$k_{(a,b)} = \frac{\sqrt{b}}{\sqrt[4]{a^2(1 - a^2)(1 + b^2)^2}}, \quad h_{(a,b)} = \frac{a^2(b^4 - 1) - b^4}{2ab\sqrt{1 - a^2}(1 + b^2)}.$$

The map $(a, b) \mapsto (k_{(a,b)}, h_{(a,b)})$ is a diffeomorphism of Ω_4 onto C_4, so that each homogeneous curve of the third class is parametrized by $\gamma_{(a,b)}$, for a unique $(a, b) \in \Omega_4$. $\qquad \square$

Remark 6. The curves of the first class can be trapped in an adS-chamber, but not in a Minkowski or a dS-chamber. They have two distinct asymptotic closed null curves lying in an adS-wall (see Figure 4). The curves of the second class may be trapped in an adS-chamber. If b is a rational number m/n, the curves are closed torus knots of type (n, m). Otherwise they are "irrational" lines of a homogeneous torus. They cannot be trapped in a Minkowski or a dS-chamber. The integer m is the liking number of the toroidal projection with the z-axis, while n is the linking number with the centerline of the toroid (see Figure 5). The homogeneous curves of the third class can be trapped in the intersection of three chambers of different types, anti-de Sitter, de Sitter, or Minkowski. They have two limit points lying on the intersection of the walls of the three chambers trapping the curve (see Figure 4). Also the homogeneous curves of the fourth class are trapped in the intersection of three chambers of different types. They have two limit points belonging to the adS-chamber. One of them lies in the intersection of the Minkowski-wall with one of the dS-walls, while the other lies in the intersection of the Minkowski-wall with the other dS-wall (see Figure 6).

Arguing as in the proof of Theorem 8, we can prove the following result for the classes of exceptional curves.

Theorem 9. *Let γ be an exceptional homogeneous curve. Then the following hold true:*

- *If $\gamma \in \mathrm{C}_5$, there is a unique Möbius basis $(M_0, \dots, M_4)$ and a unique $b > 1$ such that, after a change of variable, the curve is parameterized by $\mathbb{R} \ni t \mapsto [\gamma_{(b)}(t)]$, where*

$$(4.10) \qquad \gamma_{(b)}(t) = \frac{1}{2}(1 - b^2 t^2)M_0 + bt M_1 + \cos(t)M_2 - \sin(t)M_3 + M_4.$$

- *If $\gamma \in \mathrm{C}_6$, there exists a unique Möbius basis $(M_0, \dots, M_4)$ and a unique positive $b < 1$ such that, after a change of variable, the curve is parameterized by $\mathbb{R} \ni t \mapsto [\gamma_{(b)}(t)]$, where*

$$(4.11) \qquad \gamma_{(b)}(t) = \frac{1}{2}(1 + b^2 t^2)M_0 + \sinh(t)M_1 + \cosh(t)M_2 + +bt M_3 + +M.$$

- If $\gamma \in C_{7.i}$, there exists a unique Lie basis $(L_0, \ldots, L_4)$ and a unique $b > 1$ such that, after a change of variable, the curve is parameterized by $\mathbb{R} \ni t \mapsto [\gamma_{(b)}(t)]$, where

$$(4.12) \qquad \gamma_{(b)}(t) = x_0(t)L_0 - x_1(t)L_1 + \sqrt{\frac{2(b^2-1)}{b}}L_2 - \sin(t)L_3 + \cos(t)L_4$$

and $x_0(t) = \frac{(b^2-1)\cos(t) - t\sin(t)}{b}$, $x_1(t) = \frac{t\cos(t) + (b^2-1)\sin(t)}{b}$.

- If $\gamma \in C_{7.ii}$, there exists a unique Lie basis $(L_0, \ldots, L_4)$ and a unique $b > 1$ such that, after a change of variable, the curve is parameterized by $\mathbb{R} \ni t \mapsto [\gamma_{(b)}(t)]$, where

$$(4.13) \qquad \gamma_{(b)}(t) = x_0(t)L_0 - b\sin(t)L_1 - \sqrt{2b(1+b^2)}L_2 + x_3(t)L_3 + b\cos(t)L_4$$

and $x_0(t) = (1+b^2)\cos(t) + t\sin(t)$, $x_3(t) = t\cos(t) - (1+b^2)\sin(t)$.

- If $\gamma \in C_8$, there exists a unique Lie basis $(L_0, \ldots, L_4)$ and a unique $b \in (0, 1)$ such that, after a change of variable, the curve is parameterized by $\mathbb{R} \ni t \mapsto [\gamma_{(b)}(t)]$, where

$$(4.14) \quad \gamma_{(b)}(t) = (b-t)e^t L_0 - \sqrt{1-b}\,e^t L_1 + 2\sqrt[4]{b^2(1-b)}L_2 - (b+t)e^{-t}L_3 + \sqrt{1-b}\,e^{-t}L_4.$$

- If $\gamma \in C_9$, there exists a unique Lie basis $(L_0, \ldots, L_4)$ such that, after a change of variable, the curve is parameterized by $\mathbb{R} \ni t \mapsto [\widetilde{\gamma}(t)]$, where

$$(4.15) \qquad \widetilde{\gamma}(t) = \frac{t^4 + 6t^2 - 3}{24}L_0 + \frac{t(t^2+3)}{6}L_1 + \frac{1+t^2}{2}L_2 + tL_3 - L_4.$$
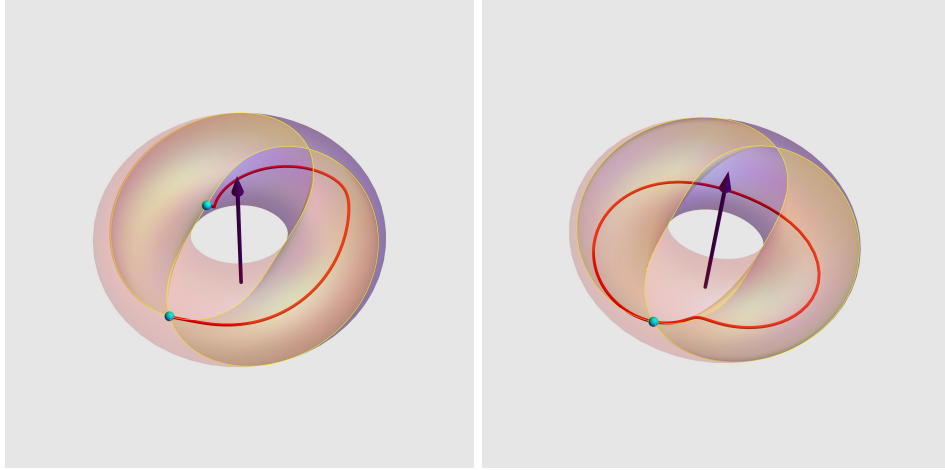
FIGURE 8. Curves with constant curvatures of the Classes C_8 (on the left) and C_9 (on the right).

Remark 7. The curves of the fifth class are equivalent to timelike round helices of Minkowski space (with a timelike axis). They are trapped in the intersection of a Minkowski-chamber with an adS-chamber. They spiral toward one of the two vertices of the Minkowski-wall and in the interior of the adS-chamber. Such curves cannot be trapped in any dS-chamber (see Figure 6). The curves of the sixth class are equivalent to timelike "hyperbolic" helices of Minkowski space (with a spacelike axis). They are trapped in the intersection of three chamber of different types and have two distinct limit points lying in the interior of the adS-chamber and on the intersections of the Minokowski-wall with one of the dS-walls (see Figure 7). The curves of the seventh class can be trapped in an adS-chamber but not

in a Minkowski or dS-chamber. They have one asymptotic closed null curve lying in the adS-wall (see Figure 7). The curves of the eighth class can be trapped in the intersections of an adS-chamber with a Minkowski-chamber but cannot be trapped in a dS-chamber. They have two distinct limit points lying in the adS-wall (see Figure 8). One of the limit points is the vertex of the Minkowski wall. The curves of the ninth class are all equivalent to each other; they can be trapped in the intersections of an adS-chamber with a Minkowski-chamber but cannot be trapped in a dS-chamber. They have a unique limit point, one of the two vertices of the Minkowski wall. Such curves close up smoothly at infinity (see Figure 8).

5. The conformal strain functional

5.1. **The Euler-Lagrange equations.** A variation of a curve $\gamma : \mathrm{I} \subset \mathbb{R} \to \mathcal{E}^{1,2}$ is a map $\Gamma : \mathrm{I} \times (-\epsilon, \epsilon) \to \mathcal{E}^{1,2}$ such that $\Gamma(u, 0) = \gamma(u)$, for every $u \in \mathrm{I}$. Given $t \in (-\epsilon, \epsilon)$, we write $\gamma_{[t]}$ to denote the curve $\mathrm{I} \ni u \mapsto \Gamma(u, t)$. If $\Gamma(u, t) = \gamma(u)$, for every u outside a closed subinterval $\mathrm{K} \subset \mathrm{I}$ and for every $t \in (-\epsilon, \epsilon)$, we say that the variation is compactly supported. The smallest of all such closed subintervals is the support of the variation. If γ is generic and timelike and Γ is compactly supported then, up to choosing ϵ sufficiently small, the curves $\gamma_{[t]}$ are generic and timelike, for every t. In the latter case we say that the variation is admissible. Given a closed interval $\mathrm{K} \subset \mathrm{I}$, the *total strain* of the timelike arc $\gamma(\mathrm{K})$ is the integral

$$\mathcal{S}_{\mathrm{K}}(\gamma) = \int_{\mathrm{K}} \upsilon_{\gamma}(u) du.$$

We say that γ is a critical curve of the conformal strain functional if

$$\frac{d}{dt} \left(\mathcal{S}_{\mathrm{K}}(\gamma_{[t]}) \right)|_{t=0} = 0,$$

for every closed sub-interval $\mathrm{K} \subset \mathrm{I}$ and for every compactly supported variation γ with support contained in K.

Theorem 10. *A generic timelike curve parametrized by conformal parameter is a critical curve of the conformal strain functional if and only if*

$$(5.1) \qquad\qquad \ddot{k} - k^3 + 2kh = 0, \quad \dot{h} - 3k\dot{k} = 0.$$

Proof. First, we prove that any timelike generic curve satisfying (5.1) is a critical point of the conformal strain functional. So, assume that (5.1) is satisfied and let Γ be an admissible variation. Let $\upsilon_{[t]}$, $k_{[t]}$ and $h_{[t]}$ denote the strain density and the conformal curvatures of $\gamma_{[t]}$ and set $\mathfrak{v}(u, t) = \upsilon_{[t]}(u)$, $\mathfrak{k}(u, t) = k_{[t]}(u)$, $\mathfrak{h}(u, t) = h_{[t]}(u)$. These functions satisfy $\mathfrak{v}(u, t) = 1$, $\mathfrak{k}(u, t) = k(u)$ and $\mathfrak{h}(u, t) = h(u)$, for every $(u, t) \notin \mathrm{K} \times (-\epsilon, \epsilon)$, where K denotes the support of the variation. Note that since γ is parametrized by conformal parameter, then $\mathfrak{v}(u, 0) = 1$, for every $u \in I$. Consider the map $\tilde{\mathbf{M}} : I \times (-\epsilon, \epsilon) \to \mathcal{M}$, such that $\tilde{\mathbf{M}}(u, t) = \mathbf{M}_{[t]}(u)$, where $\mathbf{M}_{[t]}$ is the canonical frame along $\gamma_{[t]}$, and let $\mathcal{U}$, $\mathcal{V} : I \times (-\epsilon, \epsilon) \to \mathfrak{m}(2, 3)$ denote the maps such that $\partial_u \tilde{\mathbf{M}} = \tilde{\mathbf{M}} \mathcal{U}$ and $\partial_t \tilde{\mathbf{M}} = \tilde{\mathbf{M}} \mathcal{V}$. From the Maurer–Cartan equations, it follows that

$$(5.2) \qquad\qquad \partial_u \mathcal{V} - \partial_t \mathcal{U} + [\mathcal{U}, \mathcal{V}] = 0.$$

By construction, we have $\mathcal{U} = \mathfrak{v}(M_2^0 - M_1^4 - \mathfrak{k}M_3^2 - \mathfrak{h}M_1^0)$. Let

$$\mathcal{V} = V_0^0 M_0^0 + V_0^1 M_1^0 + V_0^2 M_2^0 + V_0^3 M_3^0 + V_1^2 M_2^1$$
$$+ V_1^3 M_3^1 + V_2^3 M_3^2 + V_4^1 M_1^4 + V_4^2 M_2^4 + V_4^3 M_3^4.$$

Using (5.2), we compute

$$(5.3)\quad\begin{cases} V_0^0 = \frac{1}{\mathfrak{v}}(\partial_t\mathfrak{v} + \partial_u V_4^1), \\ V_0^0 = \frac{1}{\mathfrak{v}}(\partial_u V_0^2 - \partial_t\mathfrak{v}) - (\mathfrak{h}V_1^2 + \mathfrak{k}V_0^3), \\ V_1^2 = \frac{1}{\mathfrak{v}}\partial_u V_4^2 - \mathfrak{k}V_4^3, \\ V_1^3 = \frac{1}{\mathfrak{v}}\partial_u V_4^3 + \mathfrak{k}V_4^2, \\ V_0^3 = -\frac{1}{\mathfrak{v}}\partial_u V_1^3 - \mathfrak{k}V_1^2 - \mathfrak{h}V_4^3. \end{cases}$$

From the third, fourth and fifth equations of (5.3), it follows that

$$(5.4)\qquad V_0^3 = -\left(\frac{1}{\mathfrak{v}^2}\partial_{uu}^2 V_4^3 - \frac{1}{\mathfrak{v}^3}\partial_u\mathfrak{v}\partial_u V_4^3 + \frac{2}{\mathfrak{v}}\mathfrak{k}\partial_u V_4^2 + \frac{1}{\mathfrak{v}}\partial_u\mathfrak{k}V_4^2 + (\mathfrak{h} - \mathfrak{k}^2)V_4^3\right).$$

Let $v_j^i : I \to \mathbb{R}$ be the functions defined by $v_j^i(u) = V_j^i(u, 0)$. Evaluating (5.3) and (5.4) at $(u, 0)$ yields

$$(5.5)\quad\begin{cases} \partial_t\mathfrak{v}|_{(u,0)} = v_0^0 - \dot{v}_4^1, \\ v_0^0 = \frac{1}{2}\left(\dot{v}_0^2 + \dot{v}_4^1 - (hv_1^2 + kv_0^3)\right), \\ v_1^2 = \dot{v}_4^2 - kv_4^3, \\ v_0^3 = -(\ddot{v}_4^3 + 2k\dot{v}_4^2 + \dot{k}v_4^2 + (h - k^2)v_4^3). \end{cases}$$

Note that the supports of the functions v_j^i are contained in K. Using (5.5) and integrating by parts yields

$$(5.6)\qquad \frac{d}{dt}\left(\int_K \mathfrak{v}(u, t)du\right)\bigg|_{t=0} = \int_K v_0^0(u)du = -\frac{1}{2}\int_K(hv_1^2 + kv_0^3)du.$$

Using again (5.5), we have

$$(5.7)\quad\begin{aligned} \int_K(hv_1^2 + kv_0^3)du &= \int_K\left((k^3 - 2kh)v_4^3 + (h - 2k^2)\dot{v}_4^2 - k\dot{k}v_4^2 - k\ddot{v}_4^3\right)du \\ &= -\int_K(\ddot{k} - k^3 + 2kh)v_4^3 du - \int_K(\dot{h} - 3k\dot{k})v_4^2 du. \end{aligned}$$

From (5.6) and (5.7), it follows that γ is a critical curve if (5.1) is satisfied.

Conversely, it suffices to prove that for each $u_0 \in I$ there exists a closed interval $K \subset I$, containing u_0, such that, for every pair of smooth functions $v, w : I \to \mathbb{R}$ whose supports are contained in K, there exists a compactly supported variation satisfying $v_4^1 = 0$, $v_4^2 = v$ and $v_4^3 = w$. Since this property is local and invariant under the action of the conformal group, we may assume that the trajectory of the curve is contained in $\mathbf{j}_m(\mathbb{M}_0^{(1,2)})$. We can then write

$$\gamma = [M_0^o + \sum_{j=1}^3 \alpha^j M_j^o + \frac{{}^*\alpha\alpha}{2}M_4^o],$$

where $\alpha = {}^t(\alpha^1, \alpha^2, \alpha^3)$ is a future-directed timelike curve of Minkowski 3-space. Next, let

$$(5.8)\quad\begin{cases} T = \frac{1}{\sqrt{(\dot{\alpha}^1)^2 - (\dot{\alpha}^2)^2 - (\dot{\alpha}^3)^2}}\,{}^t(\dot{\alpha}^1, \dot{\alpha}^2, \dot{\alpha}^3), \\ N = \frac{1}{\sqrt{(\dot{\alpha}^1)^2 - (\dot{\alpha}^2)^2}}\,{}^t(\dot{\alpha}^2, \dot{\alpha}^1, 0), \\ B = T_\alpha \times N_\alpha, \end{cases}$$

where "$\times$" denotes the vector cross product in $\mathbb{M}_0^{(1,2)}$. By posing

$$(5.9)\quad\begin{cases} F_0 = M_0^o + \sum_{j=1}^3 \alpha^j M_j^o + \frac{{}^*\alpha\alpha}{2}M_4^o,, \\ F_1 = \sum_{j=1}^3 T^j M_j^o + {}^*\alpha T M_4^o, \\ F_2 = \sum_{j=1}^3 N^j M_j^o + {}^*\alpha N M_4^o, \\ F_3 = \sum_{j=1}^3 B^j M_j^o + {}^*\alpha B M_4^o, \\ F_4 = M_4^o, \end{cases}$$

the map $\mathbf{F} = (\mathrm{F}_0, \ldots, \mathrm{F}_4) : I \to \mathcal{M}$ is a first order conformal frame field, the Minkowski frame along γ. Thus, there exists a unique smooth map $X : I \to M_+^{\uparrow}(2,3)_1$, such that $\mathbf{M} = \mathbf{F} \cdot X$ is the canonical frame.[3] Let

$$(5.10) \qquad X = \begin{pmatrix} r & * & * & * & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & \cos\vartheta & -\sin\vartheta & * \\ 0 & 0 & \sin\vartheta & \cos\vartheta & * \\ 0 & 0 & 0 & 0 & r^{-1} \end{pmatrix},$$

where $r, \vartheta : I \to \mathbb{R}$ are smooth functions. Next, let

$$(5.11) \qquad \phi_2 = \frac{1}{r}(v\cos\vartheta - w\sin\vartheta), \quad \phi_3 = \frac{1}{r}(v\sin\vartheta + w\cos\vartheta)$$

and define $\upsilon : \mathrm{I} \to \mathbb{M}_0^{(1,2)}$ by $\upsilon = \phi_2 N_\alpha + \phi_3 B_\alpha$. Since $\mathrm{Supp}(\upsilon) \subset \mathrm{K}$, then

$$(5.12) \qquad \tilde{\alpha} : \mathrm{I} \times (-\epsilon, \epsilon) \to \mathbb{M}_0^{(1,2)}, (u,t) \mapsto \alpha(u) + t\upsilon(u),$$

is a compactly supported variation of α, and hence

$$\Gamma : (u,t) \mapsto [M_0^o + \sum_{j=1}^{3} \widetilde{\alpha}^j M_j^o + \frac{{}^*\widetilde{\alpha}\widetilde{\alpha}}{2} M_4^o]$$

is a compactly supported variation of γ. Without loss of generality, we may assume that all the curves $\gamma_{[t]}$ are generic and timelike. For each $t \in (-\epsilon, \epsilon)$, let $\mathbf{F}_{[t]} : I \to \mathcal{M}$ be the Minkowski first order frame along $\gamma_{[t]}$ and define $\widetilde{\mathbf{F}} : I \times (-\epsilon, \epsilon) \to \mathcal{M}$ by $\widetilde{\mathbf{F}}(u,t) = \mathbf{F}_{[t]}(u)$. Then, for every $t \in (-\epsilon, \epsilon)$, there exists a unique map $\tilde{X}_{[t]} : I \to M_+^{\uparrow}(2,3)_1$, such that $\mathbf{M}_{[t]} = \mathbf{F}_{[t]} \cdot \tilde{X}_{[t]}$ is the canonical frame along $\gamma_{[t]}$ and that $\tilde{X}_{[0]} = X$. If we put $\tilde{\mathbf{M}}(u,t) = \mathbf{M}_{[t]}(u)$, then $V(u) = \mathcal{V}(u,0) = (\widetilde{\mathbf{M}}^{-1}\partial_t\widetilde{\mathbf{M}})|_{(u,0)} = U(u) + W(u)$, where $U(u) = X(u)^{-1} \cdot (\widetilde{\mathbf{F}}^{-1}\partial_t\widetilde{\mathbf{F}})|_{(u,0)} \cdot X(u)$ and $W(u) = (\tilde{X}^{-1}\partial_t\tilde{X})|_{(u,0)}$. Let $V_0 = {}^t(v^0, v^1, v^2, v^3, 0)$, $U_0 = {}^t(u^0, u^1, u^2, u^3, 0)$, and $W_0 = {}^t(w^0, w^1, w^2, w^3, 0)$ be the first column vectors of V, U and W, respectively. Since $\tilde{X}$ take values in the subgroup $M_+^{\uparrow}(2,1)_1$, then $u^1 = u^2 = u^3 = 0$ and $v^1 = w^1, v^2 = w^2, v^3 = w^3$. Taking into account (5.10), we have

$$(5.13) \qquad \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} = r \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \cdot \begin{pmatrix} m^1 \\ m^2 \\ m^3 \end{pmatrix},$$

where m^1, m^2 and m^3 are the second, third and fourth components of the first column vector of $(\widetilde{\mathbf{F}}^{-1}\partial_t\widetilde{\mathbf{F}})|_{(u,0)}$. According to the definition of $\tilde{\mathbf{F}}$, it follows that m^1, m^2 and m^3 are the components of υ with respect to the frame (T, N, B). Then, in view of (5.12), we get

$$(5.14) \qquad \begin{pmatrix} m^1 \\ m^2 \\ m^3 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi^2 \\ \phi^3 \end{pmatrix} = \frac{1}{r} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v \\ w \end{pmatrix}.$$

From (5.13) and (5.14), it follows that $v^1 = m^1 = 0$, $v^2 = v$ and $v^3 = w$, which proves the required result. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 8. A generic timelike curve with constant curvatures can be a critical point of the strain functional if and only if $k^3 = 2kh$. Thus, assuming $k > 0$, these curves belong to the classes C_1, C_4 or C_9.

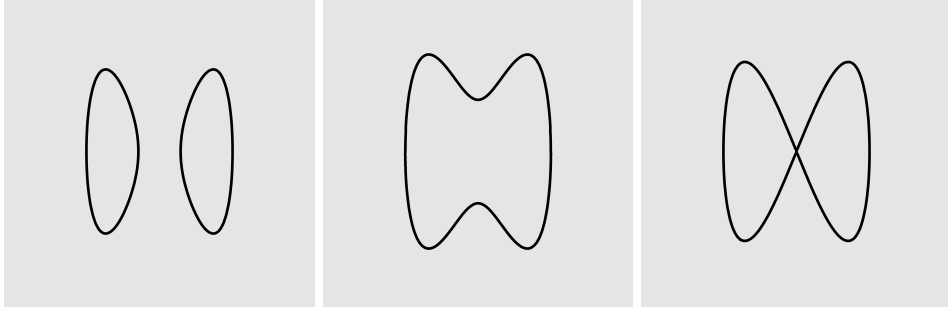[3]Here, $M_+^{\uparrow}(2,3)_1$ is the closed subgroup defined in (3.3).

FIGURE 9. Phase portraits of the first, second and third type.

Remark 9. Within the general scheme of Griffiths' formalism of the calculus of variations [20], using the Euler–Lagrange equations and the existence of the canonical frame, one can build a dynamical system defined on an appropriate momentum space in such a way that the projections of its trajectories on the Einstein universe are the critical curves of the variational problem. More specifically, the momentum space Y is the Cartesian product $\mathcal{M} \times \mathbb{R}^3$ with fiber coordinates $(k, \dot{k}, h)$ equipped with the projection $\pi_Y : (\mathbf{M}, k, \dot{k}, h) \in Y \to [M_0] \in \mathcal{E}^{1,2}$. The exterior differential 1-forms μ_0^0, μ_0^1, μ_0^2, μ_0^3, μ_1^2, μ_2^3, μ_2^3, μ_4^1, μ_4^2, μ_4^3 and dk, $d\dot{k}$, dh define an absolute parallelism on Y. Let $\partial_{\mu_0^0}$, $\partial_{\mu_0^1}$, $\partial_{\mu_0^2}$, $\partial_{\mu_0^3}$, $\partial_{\mu_1^2}$, $\partial_{\mu_1^3}$, $\partial_{\mu_2^3}$, $\partial_{\mu_4^1}$, $\partial_{\mu_4^2}$, $\partial_{\mu_4^3}$ and ∂_k, $\partial_{\dot{k}}$, ∂_h be the vector fields of the parallelism and let ξ be the vector field

$$\xi = \partial_{\mu_0^1} + \partial_{\mu_4^2} + h\partial_{\mu_4^1} + k\partial_{\mu_2^3} + \dot{k}\partial_k + (k^3 - 2kh)\partial_{\dot{k}} + 3k\dot{k}\partial_h.$$

The integral curves of ξ are maps $\mathbb{R} \to Y$, $u \mapsto (\mathbf{M}(u), k(u), \dot{k}(u), h(u))$, where

- $\gamma : u \in \mathbb{R} \to [M_0(u)] \in \mathcal{E}^{1,2}$ is a critical curve of the conformal strain functional, parameterized by the natural parameter;
- k, h are its conformal curvatures and $\dot{k}$ is the derivative of k;
- $\mathbf{M} : \mathbb{R} \to \mathcal{M}$ is the canonical conformal frame field along γ.

In principle then, the problem is reduced to the integration of the vector field ξ. It is important to note that ξ is the characteristic vector field of the $A_+^\uparrow(2,3)$-invariant contact form

$$\frac{1}{2}(\mu_0^1 + \mu_4^2 + (k^2 - h)\mu_0^2 - \dot{k}\mu_0^3 + k\mu_1^3).$$

5.2. **The conformal curvatures of the extrema.** From now on we consider critical curves with non-constant curvatures. Then (5.1) implies that γ is a critical curve if and only if there exists $(e_1, e_2) \in \mathcal{D} = \{(a, b) : a < b, \ b > 0\}$ such that

(5.15) $$\dot{k}^2 = -(k^2 - e_1)(k^2 - e_2), \quad h = \frac{3}{2}k^2 - \frac{1}{2}(e_1 + e_2).$$

We say that e_1, e_2 are the *parameters* of the critical curve.

Definition 6. The *phase portrait* of a critical curve with parameters e_1 and e_2 is the real algebraic curve $\mathcal{F}_{e_1,e_2} \subset \mathbb{R}^2$ defined by the equation $y^2 + (x^2 - e_1)(x^2 - e_2) = 0$.

Three possibilities may occur depending on whether $0 < e_1 < e_2$, $e_1 < 0 < e_2$, or $e_1 = 0 < e_2$. Correspondingly, we have a partition of $\mathcal{D}$ into three regions: $\mathcal{D}_1 = \{(e_1, e_2) \in \mathcal{D} : 0 < e_1 < e_2\}$, $\mathcal{D}_2 = \{(e_1, e_2) \in \mathcal{D} : e_1 < 0 < e_2\}$ and $\mathcal{D}_3 = \{(e_1, e_2) \in \mathcal{D} : e_1 = 0 < e_2\}$.

Phase-Type 1. If $(e_1, e_2) \in \mathcal{D}_1$, the phase portrait $\mathcal{F}_{e_1,e_2}$ is the real part of a smooth elliptic curve. It consists of two ovaloids, one of them belongs to the half plane $x > 0$ while the other is the mirror image of the first by the reflection about the y-axis (see Figure 9). Thus, the first conformal curvature is either strictly positive or else strictly

negative. By possibly acting with a time-preserving and orientation-reversing conformal transformation, we may assume $k > 0$. We put

$$(5.16) \qquad m_1 = \frac{e_2 - e_1}{e_2}, \quad p_1 = e_2.$$

From (5.15) it follows that the conformal curvatures of a critical curve with parameters $(e_1, e_2) \in \mathcal{D}_1$ are

$$(5.17) \qquad \begin{cases} k(u) = \sqrt{p_1(1 - m_1)}\,\mathrm{nd}(\sqrt{p_1}u + c, m_1), \\ h(u) = \frac{1}{2}(3(1 - m_1)\mathrm{nd}(\sqrt{p}u + c, m_1)^2 + (m_1 - 2), \end{cases}$$

where $\mathrm{nd}(\cdot, m)$ is the Jacobi nd-function with parameter m. By possibly shifting the independent variable, we can assume $c = 0$.

Phase-Type 2. If $(e_1, e_2) \in \mathcal{D}_2$, then $\mathcal{F}_{e_1, e_2}$ is the real part of a smooth elliptic curve. Unlike the previous case, $\mathcal{F}_{e_1, e_2}$ is connected and symmetric with respect to the reflections about the coordinate axes (see Figure 9). Given $(e_1, e_2) \in \mathcal{D}_2$, let

$$(5.18) \qquad m_2 = \frac{e_2}{e_2 - e_1} \in (0, 1), \quad p_2 = e_2 - e_1 > 0.$$

Consequently, the conformal curvatures can be written as

$$(5.19) \qquad \begin{cases} k(u) = \sqrt{p_2(1 - m_2)m_2}\,\mathrm{sd}(\sqrt{p_2}u + c, m_2), \\ h(u) = \frac{1}{2}p_2(3(1 - m_2)\,\mathrm{sd}(\sqrt{p_2}u + c, m_2)^2 + 1 - 2m_2), \end{cases}$$

where $\mathrm{sd}(\cdot, m)$ is the Jacobi sd-function with parameter m and c is a suitable constant. Again, by possibly shifting the independent variable, we take $c = 0$.

Phase-Type 3. If $e_1 = 0$ and $e_2 > 0$, the phase portrait is a rational Lemniscate, symmetric with respect to the reflections about the coordinate axes, with the double point located at the origin (see Figure 9). By possibly acting by a time-preserving and orientation-reversing conformal transformation, we may assume that $k > 0$. We then have

$$(5.20) \qquad \begin{cases} k(u) = \frac{2 + e_2 \cdot \exp(\sqrt{e_2}(u+c))}{e_2 + \exp(2\sqrt{e_2}(u+c))}, \\ h(u) = \frac{6e_2^2 \cdot \exp(2\sqrt{e_2}(u+c))}{(\exp(2\sqrt{e_2}(u+c)) + e_2)^2} - \frac{1}{2}e_2, \end{cases}$$

where c is a constant that can be put equal to zero.

Remark 10. Let γ be a critical curve with conformal curvatures k, h as in (5.17), (5.19), or (5.20). Let $\mathcal{K}(h, k) = M_2^0 - M_1^4 - kM_3^2 - hM_1^0$, as in Section 3.1, and define $\mathcal{H} := -M_1^0 - M_2^4 - kM_3^1 + k'M_3^0 + (h - k^2)M_2^0$. A direct computation shows that $\mathcal{H}$ satisfies the Lax equation $\mathcal{H}' = [\mathcal{H}, \mathcal{K}]$. Let $\mathbf{M}$ be the canonical frame field along γ. Since $\mathbf{M}' = \mathbf{M}\mathcal{K}$, it follows from the Lax equation that $(\mathbf{M}\mathcal{H}\mathbf{M}^{-1})' = 0$, which implies the existence of a constant element $\mathfrak{m}_\gamma \in \mathfrak{m}(2, 3)$, the *momentum* of γ, such that $\mathfrak{m}_\gamma = \mathbf{M}\mathcal{H}\mathbf{M}^{-1}$, along γ. This is a consequence of the Noether conservation theorem and of the invariance of the strain functional under the action of the conformal group.

Once the curvatures of the extrema are known, the determination of their trajectories amounts to the integration of the 5×5 linear system $X' = \mathcal{K}(h, k)X$, where h, k are as in (5.17), (5.19), or (5.20). Such systems can be integrated in terms of elliptic functions and elliptic integrals. The integration of timelike trajectories is based on the study of their phase types and of the orbit types of their momenta. The detailed analysis produces fifteen cases, to be treated one by one.

Theorem 11. *There exist countably many distinct conformal equivalence classes of closed critical curve of the the conformal strain functional.*

Proof. Suppose (e_1, e_2) belongs to $\mathcal{D}^* = \{(a, b) \in \mathcal{D} : b - a > 2, a + b < -\sqrt{(a - b)^2 - 4}\}$. In this case, the phase portrait is of the second type, so the curvatures are as in (5.19)
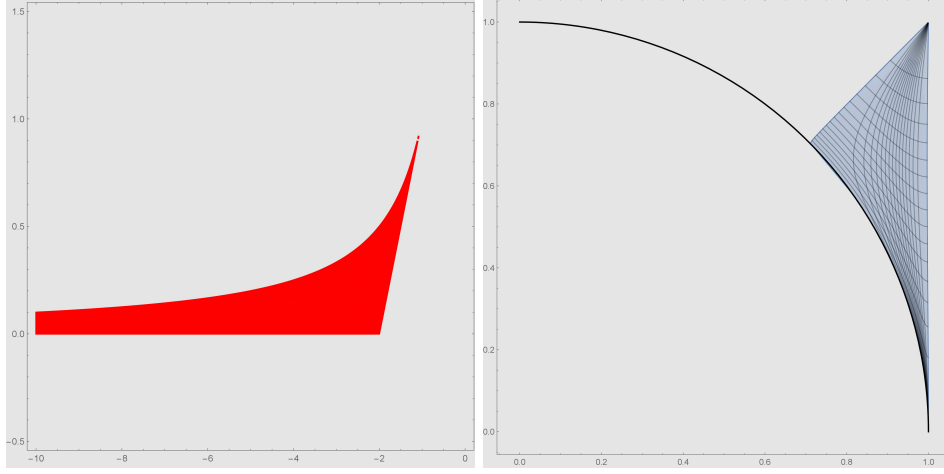
FIGURE 10. The domain $\mathcal{D}^*$ (on the left) and the plot of the map Ψ.

and the momentum belongs to the Lie algebra of a maximal compact Abelian subgroup. Let

$$\xi_1 = \sqrt{\frac{1}{2} \mid (e_1 + e_2) + \sqrt{(e_2 - e_1)^2 - 4} \mid}, \quad \xi_2 = \sqrt{\frac{1}{2} \mid (e_1 + e_2) - \sqrt{(e_2 - e_1)^2 - 4} \mid}$$

and, for $j \in \{1, 2\}$, consider the function

$$\Phi_j(u) = \frac{\xi_j}{p(m-1) + \xi_j} \left(u + \frac{\sqrt{p}(m-1)}{\xi_j^2} \Pi\left(\frac{m\xi_j^2 + mp(m-1)}{\xi_j^2}, \text{am}(\sqrt{p}u, m), m \right) \right),$$

where $\text{am}(\cdot, m)$ is the Jacobi amplitude and

$$\Pi(x, \phi, m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m\sin^2(\theta)}(1 - x\sin^2(\theta))}$$

is the complete elliptic integral of the third kind. We now consider the mappings

$$R = \cos(\Phi_1)(E_0^0 + E_1^1) - \sin(\Phi_1)(E_2^1 - E_2^1) + E_2^2$$
$$+ \cos(\Phi_2)(E_3^3 + E_4^4) - \sin(\Phi_2)(E_4^3 - E_3^4)$$

and

$$^tW = \left(\frac{\sqrt{k^2 + \xi_1^2}}{\xi_1 \sqrt[4]{(e_2 - e_1)^2 - 4}}, 0, -\frac{k}{\sqrt{1 + e_1 e_2}}, \frac{-\sqrt{k^2 + \xi_2^2}}{\xi_2 \sqrt[4]{(e_2 - e_1)^2 - 4}}, 0 \right).$$

Then, a timelike critical curve with parameters $(e_1, e_2) \in \mathcal{D}^*$ is equivalent to

$$\gamma : \mathbb{R} \ni u \mapsto [T_{mp} \cdot R(u) \cdot W(u)] \in \mathcal{M}^{1,2}.$$

Let

$$\Psi_j = \frac{1}{2\pi} \left(\Phi_j(\frac{4K(m)}{\sqrt{p}}) - \Phi_j(0) \right), \quad (j = 1, 2),$$

where $K(m)$ is the complete elliptic integral of the first kind with parameter m, and we consider the map $\Psi : \mathcal{D}^* \to \mathbb{R}^2$, $(e_1, e_2) \mapsto (\Psi_1(e_1, e_2), \Psi_2(e_1, e_2))$. Since W is periodic with minimal period $\omega = 4K(m)/\sqrt{p}$, it follows that a critical curve with parameters $(e_1, e_2) \in \mathcal{D}^*$ is closed if and only if $\Psi(e_1, e_2) \in \mathbb{Q}^2$. In fact, Ψ is a bijection onto the open domain $\widetilde{\mathcal{D}} = \{(x, y) \in \mathbb{R}^2 : 0 < x < y, 1 - y^2 < x < 1\}$, so that the conformal equivalence classes of closed trajectories of this type are in one-to-one correspondence with the countable set $\widetilde{\mathcal{D}} \cap \mathbb{Q}^2$ (see Figure 10). $\qquad\square$

Remark 11. The inverse of the period map Ψ can be computed by numerical methods. In addition, also the closed trajectories with parameters belonging to $\mathcal{D}^*$ can effectively be computed (see Figure 11). The above theorem is the Lorentzian counterpart of an analogous result for the conformal arclength functional for curves in S^3 [31, 34].
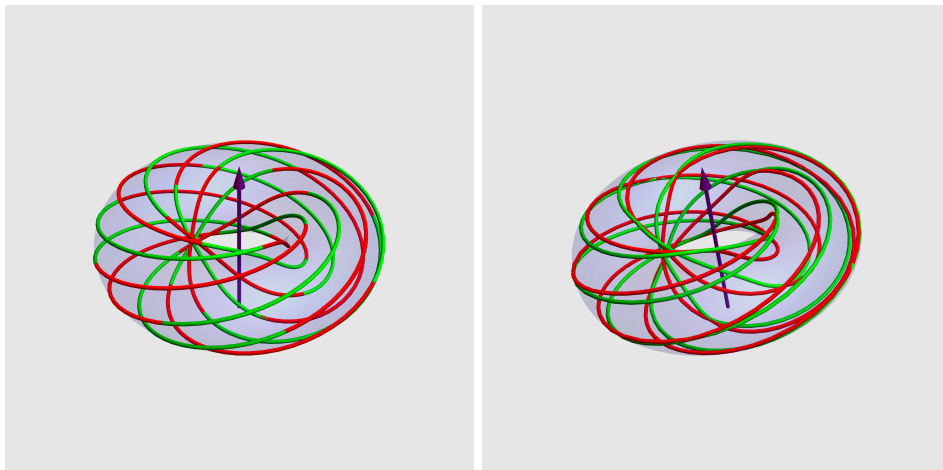

FIGURE 11. The "toroidal projections" of closed critical curves with $e_1 \approx -1.98638$, $e_2 \approx 0.0275109$, $\Psi(e_1, e_2) = (3/4, 2/3)$ (on the left) and $e_1 \approx -1.74929$, $e_2 \approx 0.283545$, $\Psi(e_1, e_2) = (4/5, 3/4)$ (on the right). The arcs pictured in red are trapped in the positive adS chamber and the arcs pictured in green are trapped in the negative adS chamber.

6. TIMELIKE CURVES AND TRANSVERSAL KNOTS IN S^3

In this section we establish and discuss a connection between the conformal Lorentzian geometry of timelike curves and the geometry of transversal knots in the unit 3-sphere. We begin by recalling some background material about knots and links, the symplectic group, and the standard contact structure of the 3-sphere.

6.1. **Preliminaries and notation.** Let M be the unit sphere S^3 or the Euclidean space $\mathbb{R}^3$. On $S^3 \subset \mathbb{R}^4$, we fix the orientation defined by the volume form $\mathrm{Vol}|_p = -i_p^*(dy^1 \wedge dy^2 \wedge dy^3 \wedge dy^4)$, where $(y_1, \ldots, y_4)$ are the standard coordinates of $\mathbb{R}^4$ and $i : S^3 \hookrightarrow \mathbb{R}^4$ is the standard inclusion. In $\mathbb{R}^3$ we choose the natural orientation defined by the volume form $\mathrm{Vol} = dx^1 \wedge dx^2 \wedge dx^3$.

A *knot* $\mathrm{K} \subset M$ is a simple, closed oriented curve. Let $\mathrm{N(K)}$ be the normal bundle of the knot. For each $\epsilon > 0$, we put $\mathrm{N}_\epsilon(\mathrm{K}) = \{(p, \overrightarrow{X}) \in \mathrm{N}_p(\mathrm{K}) : \|\overrightarrow{X}\| < \epsilon\}$. If ϵ is sufficiently small, there exists a tubular neighborhood $\mathcal{U} \subset M$ of K and a diffeomorphism Φ of $\mathrm{N}_\epsilon(\mathrm{K})$ onto $\mathcal{U}$. The orientations of M and K determine an orientation on $\mathrm{N}_p(\mathrm{K})$, for every $p \in \mathrm{K}$. Fix $p_0 \in \mathrm{K}$, a positive-oriented orthogonal basis $(\overrightarrow{V}_1, \overrightarrow{V}_2)$ of $\mathrm{N}_{p_0}(\mathrm{K})$ and a positive number $\eta \in (0, \epsilon)$. Let $\Gamma_{p_0}(\eta) \subset \mathrm{N}_\epsilon(\mathrm{K})$ be the oriented circle $\{(p_0, \eta(\cos(\theta)\overrightarrow{V}_1 + \sin(\theta)\overrightarrow{V}_2) : \theta \in \mathbb{R}\}$. Then, the homology class of $\Phi(\Gamma_{p_0}(\eta))$ is a generator of the first homology group $\mathrm{H}_1(M \setminus \mathrm{K}, \mathbb{Z})$ that we use to identify $\mathrm{H}_1(M \setminus \mathrm{K}, \mathbb{Z})$ and $\mathbb{Z}$.

Definition 7. A *link* of M is an ordered pair $(\mathrm{K}, \widehat{\mathrm{K}})$ of disjoint knots. Its *linking number*, denoted by $\mathrm{Lk}(\mathrm{K}, \widehat{\mathrm{K}})$, is the integer that corresponds to the homology class of $\widehat{\mathrm{K}}$ in $\mathrm{H}_1(M \setminus \mathrm{K}) \cong \mathbb{Z}$.

Remark 12. Let $(K, \widehat{K})$ be a link of S^3 and $\mathtt{P} \in S^3$ be a point such that $\mathtt{P} \notin K \cup \widehat{K}$. Denote by st : $S^3 \setminus \{\mathtt{P}\} \to \mathbb{R}^3$ the composition of the stereographic projection from $\mathtt{P}$ with the reflection of $\mathbb{R}^3$ with respect to the coordinate xy-plane. This is an orientation-preserving conformal mapping such that $\mathrm{Lk}(K, \widehat{K}) = \mathrm{Lk}(\mathrm{st}(K), \mathrm{st}(\widehat{K}))$. Thus, in principle, the computation of the linking number of a spherical link can be reduced to that of the linking number of a link in $\mathbb{R}^3$. The latter can be evaluated by means of its link diagram. More precisely, given a link $(K, \widehat{K})$ in $\mathbb{R}^3$, we fix a plane and consider the orthogonal projections K_o and $\widehat{K}_o$ of the knots onto this plane. For a generic choice of the plane, the curves K_o and $\widehat{K}_o$ are immersed and intersect each other transversely in a finite number of crossing points $p_1, \ldots, p_r$. Let $\gamma, \widehat{\gamma} : \mathbb{R} \to \mathbb{R}^3$ be smooth parametrizations of K and $\widehat{K}$, respectively, with the same minimal period $\omega > 0$. Correspondingly, we have smooth parametrizations γ_o and $\widehat{\gamma}_o$ of K_o and $\widehat{K}_o$. If $t_j, \widetilde{t}_j \in [0, \omega)$ are chosen so that $\gamma_o(t_j) = \widehat{\gamma}_o(\widetilde{t}_j) = p_j$, the index of p_j is defined by

$$\epsilon(p_j) = \mathrm{sgn}((\gamma(t_j) - \widehat{\gamma}(\widetilde{t}_j)) \cdot (\gamma_o'|_{t_j} \times \widehat{\gamma}_o'|_{\widetilde{t}_j}))$$

and the linking number is given by

$$\mathrm{Lk}(K, \widehat{K}) = \frac{1}{2} \sum_{j=1}^{r} \epsilon(p_j).$$

Example 1. Let p, q be two positive integers, such that $q > p > 0$ and $\gcd(p, q) = 1$. Denote by $\mathrm{K}_{p,q}, \mathrm{K}_{p,q}^* \subset \mathbb{R}^3$ the torus knots of type (p, q) parametrized by

(6.1)
$$\begin{cases} \Gamma_{p,q} : \mathbb{R} \ni u \mapsto \left((1 + \frac{\cos(pu)}{2}) \cos(qu), (1 + \frac{\cos(pu)}{2}) \sin(qu), -\frac{\sin(pu)}{2}, \right), \\ \Gamma_{p,q}^* : \mathbb{R} \ni u \mapsto \left((1 + \frac{\cos(pu)}{4}) \cos(qu), (1 + \frac{\cos(pu)}{4}) \sin(qu), -\frac{\sin(pu)}{4}, \right). \end{cases}$$

Their projections onto the xy-plane intersect transversely in $2pq$ distinct points, each of which has index 1 (see Figure 12). From this we infer that $\mathrm{Lk}(\mathrm{K}_{p,q}, \mathrm{K}_{p,q}^*) = pq$.
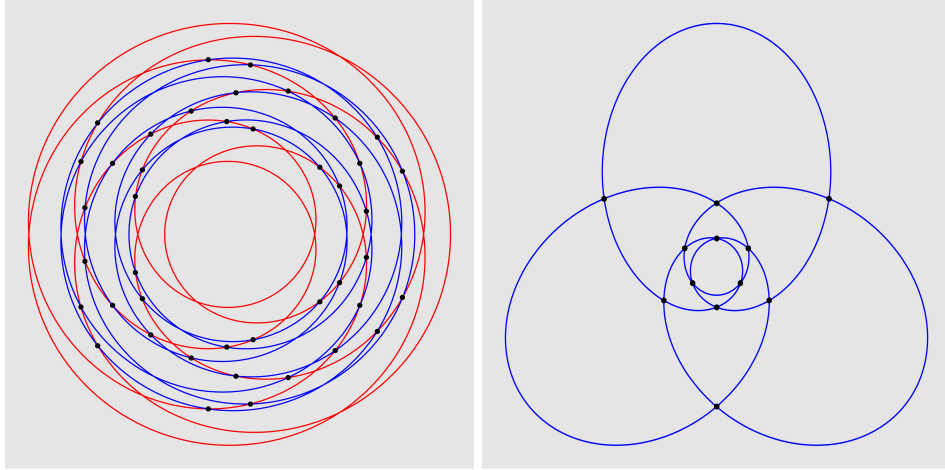
FIGURE 12. The link diagram of $(\mathrm{K}_{3,7}, \mathrm{K}_{3,7}^*)$ (on the left) and the writhe diagram of $\check{\mathrm{K}}_{3,5}$ (on the right).

The notion of linking number can be adapted for defining the linking number of a vector filed along a knot.

Definition 8. Let $K \subset M$ be a knot and $\overrightarrow{\mathbf{X}} : K \to T(M)$ be a vector field along K. Suppose that $\overrightarrow{\mathbf{X}}$ is transverse to K and denote by $\overrightarrow{\mathbf{X}}^v$ the orthogonal projection of $\overrightarrow{\mathbf{X}}$ onto the normal bundle of K. Let $\eta > 0$ be a positive real number such that $\eta \|\overrightarrow{\mathbf{X}}^v|_p\| < \epsilon$, for every $p \in K$, and let $K + \eta\overrightarrow{\mathbf{X}}$ be the knot $K \ni p \mapsto \Phi[(p, \eta\overrightarrow{\mathbf{X}}^v|_p)]$. Then $(K, K + \eta\overrightarrow{\mathbf{X}})$ is a link and its linking number, denoted by $\mathrm{Lk}_K(\overrightarrow{\mathbf{X}})$, is said to be the *linking number of the vector field* $\overrightarrow{\mathbf{X}}$ *along* K. If $M = S^3$ and $\mathtt{P} \in S^3 \setminus K$, then $\mathrm{Lk}_K(\overrightarrow{\mathbf{X}}) = \mathrm{Lk}_{\mathrm{st}(K)}(\mathrm{st}_*(\overrightarrow{\mathbf{X}}))$.

The linking number of a vector field can be used to define the Bennequin number of a transverse knot in S^3 and the self-linking number of a knot in $\mathbb{R}^3$.

On the 3-sphere, consider the contact form $\zeta = x_1 dx_3 + x_2 dx_4 - x_3 dx_1 - x_4 dx_2$ and the corresponding contact distribution $\mathcal{D} = \{(p, \overrightarrow{\mathbf{X}}) \in \mathrm{T}_p(S^3) : \zeta|_p(\overrightarrow{\mathbf{X}}) = 0\}$. Note that $\zeta \wedge d\zeta$ is the volume form used to define the orientation S^3. The contact distribution is parallelized by the unit vector fields

$$(6.2) \qquad \begin{cases} \overrightarrow{\mathbf{E}}_1 : \mathtt{X}(x_1, x_2, x_3, x_4) \in S^3 \mapsto (-x_4, x_3, -x_2, x_1) \in T_{\mathtt{P}}(S^3), \\ \overrightarrow{\mathbf{E}}_2 : \mathtt{X}(x_1, x_2, x_3, x_4) \in S^3 \mapsto (-x_2, x_1, x_4, -x_3) \in T_{\mathtt{P}}(S^3). \end{cases}$$

In particular, it admits nowhere vanishing global sections.

Definition 9. Let $K \subset S^3$ be a knot everywhere transverse to the contact distribution $\mathcal{D}$ and let $\overrightarrow{\mathbf{X}}$ be a nowhere vanishing cross section of $\mathcal{D}$. The *Bennequin number of* K, denoted by $\mathbf{b}(K)$, is the integer $\mathrm{Lk}_K(\overrightarrow{\mathbf{X}})$ [18, 13]. It is easily seen that the definition is independent of the section, so that we can choose indifferently $\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{E}}_1$ or $\overrightarrow{\mathbf{X}} = \overrightarrow{\mathbf{E}}_2$. It is important to note that the Bennequin number is invariant under transversal isotopies.

Let K be a knot without inflection points and let $(\overrightarrow{\mathbf{T}}, \overrightarrow{\mathbf{N}}, \overrightarrow{\mathbf{B}})$ be its Frenet frame field.

Definition 10. The linking number $\mathrm{Lk}_K(\overrightarrow{\mathbf{N}})$ of the unit normal along K is called the *self-linking number* of K and will be denoted by $\mathrm{SL}(K)$.

In the literature, the Bennequin number of a transverse knot of S^3 is sometime referred to as the self-linking number of the knot. To avoid confusion, we will not follow this practice. The self-linking number is invariant only under regular isotopies, i.e., isotopies through simple closed curves without inflection points. In this respect, it is a geometric invariant, but not a topological one. Examples of torus knots in Euclidean space of the same type but with different self-linking numbers are given in [2].

Remark 13. In some cases, the self-linking number can be computed in terms of the *integral writhe* of the knot relative to a fixed nonzero vector $\overrightarrow{v} \in \mathbb{R}^3$ in general position with respect to K (i.e., the projection K_o of K onto a plane orthogonal to $\overrightarrow{v}$ is an immersed curve with ordinary double points). We recall that the integral writhe of K with respect to $\overrightarrow{v}$, denoted by $\mathrm{Wr}_K(\overrightarrow{v})$, is the sum of the indices of the double points of K_o. The index of a double point $p \in K_o$ can be defined as in Remark 12. Namely, we take a parametrization $\gamma_o : \mathbb{R} \to [\overrightarrow{v}]^\perp \subset \mathbb{R}^3$ of K_o, induced by a periodic parametrization $\gamma : \mathbb{R} \to \mathbb{R}^3$ of K. By possibly replacing $\overrightarrow{v}$ with $-\overrightarrow{v}$, we may assume that γ_o is compatible with the counterclockwise orientation of K_o with respect to normal unit vector $\overrightarrow{v}$. If p is a double point of K_o, let $t, t' \in [0, \omega)$ such that $\gamma_o(t) = \gamma_o(t') = p$ and $\gamma_3(t) > \gamma_3(t')$ (i.e., $\gamma(t) - \gamma(t')$ is a positive multiple of the unit vector $\overrightarrow{v}$). Then, the index of p is $\epsilon(p) = \mathrm{sgn}(\overrightarrow{v} \cdot (\gamma'|_t \times \gamma'|_{t'}))$ and the integral writhe of K with respect to $\overrightarrow{v}$ is defined by

$$\mathrm{Wr}_K(\overrightarrow{v}) = \sum_{p \in K_o^*} \epsilon(p),$$

where K_o^* is the set of double points of K_o. Note that the integral writhe coincides with $\mathrm{Lk}_K(\overrightarrow{v})$, the linking number of the constant vector field $\overrightarrow{v}$ along the knot K. If the projection K_o is a locally convex curve, then

$$\mathrm{SL}(K) = \mathrm{Wr}_K(\overrightarrow{v}).$$

The proof of this assertion can be found in [44].

Example 2. Consider the torus knot $\check{K}_{p,q} \subset \mathbb{R}^3$ of type (q,p) parametrized by

$$(6.3) \qquad \check{\Gamma}_{p,q}(t) = \frac{1}{4\cos(pt) - 5}\left(3\sin(qt), 4\sin(pt), -3\cos(qt)\right).$$

It is a computational matter to check that $\check{K}_{p,q}$ has no inflection points and that its projection onto the xz-plane is locally convex and possesses $pq - p$ double points, all of them with index 1 (see Figure 12). From this we can conclude that $\mathrm{SL}(\check{K}_{p,q}) = pq - p$.

The self-linking number can be used for computing the linking number of a nowhere vanishing vector field $\overrightarrow{\mathbf{X}}$ normal to a knot $K \subset \mathbb{R}^3$. In fact, by the Călugăreanu–Fuller–Pohl formula [6, 18, 37, 44, 46], we have

$$(6.4) \qquad \mathrm{Lk}_K(\overrightarrow{\mathbf{X}}) = \mathrm{SL}(K) + \Theta_K(\overrightarrow{\mathbf{X}}),$$

where $\Theta_K(\overrightarrow{\mathbf{X}})$ is the *rotation number* of $\overrightarrow{\mathbf{X}}$ along K, that is, the degree of the map

$$\psi_{K,\overrightarrow{\mathbf{X}}} : K \ni p \mapsto (\overrightarrow{\mathbf{X}}|_p \cdot \overrightarrow{\mathbf{N}}|_p, \overrightarrow{\mathbf{X}}|_p \cdot \overrightarrow{\mathbf{B}}|_p) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

Actually, to compute the rotation number one can replace $\overrightarrow{\mathbf{B}}$ and $\overrightarrow{\mathbf{N}}$ by $\overrightarrow{S}_b = \gamma' \times \gamma''$ and $\overrightarrow{S}_n = -\gamma' \times \overrightarrow{S}_b$, where γ is any parametrization of the knot. Note that the rotation number and the linking number of $\overrightarrow{\mathbf{X}}$ depend only on the direction of the vector field and not on its norm. For this reason, in the literature it is often assumed that $\overrightarrow{\mathbf{X}}$ is a unit normal vector field along the knot.

6.2. **The symplectic group and the standard contact structure of S^3.** Let $\omega(\mathbf{X}, \mathbf{Y}) = \mathbf{X}^1\mathbf{Y}^3 + \mathbf{X}^2\mathbf{Y}^4 - \mathbf{X}^3\mathbf{Y}^1 - \mathbf{X}^4\mathbf{Y}^2$ be the standard symplectic form of $\mathbb{R}^4$, let $\mathrm{Sp}(4,\mathbb{R})$ be the linear symplectic group of ω, and let $\mathfrak{sp}(\mathbb{R}^4, \omega)$ denote its Lie algebra. The elements of $\mathfrak{sp}(\mathbb{R}^4, \omega)$ are the 4×4 matrices of the form $\mathbf{A}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & -{}^t\mathbf{a} \end{pmatrix}$, where $\mathbf{a} = (a_j^i)$, $\mathbf{b} = (b_j^i)$, $\mathbf{c} = (c_j^i) \in \mathbb{R}(2,2)$, and $\mathbf{c} = {}^t\mathbf{c}$, $\mathbf{b} = {}^t\mathbf{b}$. There are two transitive actions of the symplectic group playing an essential role in our discussion. The first is the action on the unit sphere $S^3 \subset \mathbb{R}^4$ defined by

$$\mathrm{Sp}(4,\mathbb{R}) \times S^3 \ni (\mathbf{X}, \mathbf{x}) \mapsto \mathbf{X} \star \mathbf{x} = |\mathbf{X} \cdot \mathbf{x}|^{-1}\mathbf{X} \cdot \mathbf{x} \in S^3.$$

Such an action preserves the standard oriented contact structure on S^3 defined by the 1-form ζ. Next, consider the 6-dimensional vector space $\Lambda^2(\mathbb{R}^4)$ of skew-symmetric 2-forms of $\mathbb{R}^4$. The linear symplectic group acts on the left on $\Lambda^2(\mathbb{R}^4)$ by $\mathbf{X} \star \alpha(\mathbf{x}, \mathbf{y}) = \alpha(\mathbf{X}^{-1} \cdot \mathbf{x}, \mathbf{X}^{-1} \cdot \mathbf{y})$, for every $\mathbf{X} \in \mathrm{Sp}(\mathbb{R}^4, \omega)$, $\alpha \in \Lambda^2(\mathbb{R}^4)$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^4$. The vector space $\Lambda^2(\mathbb{R}^4)$ can be equipped with the $\mathrm{Sp}(\mathbb{R}^4, \omega)$-invariant neutral scalar product (α, β), defined by $(\alpha, \beta)\omega^2 = \alpha \wedge \beta$, $\forall \alpha, \beta \in \Lambda^2(\mathbb{R}^4)$. By construction, ω is a spacelike vector so that its polar space $[\omega]^\perp$ is a 5-dimensional invariant subspace of signature $(2,3)$. Let $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ be the canonical basis of $\mathbb{R}^4$ and $(\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \mathbf{e}^4)$ its dual basis. Then, the elements

$$\begin{aligned} \mathbf{E}_0 &= \mathbf{e}^1 \wedge \mathbf{e}^2, & \mathbf{E}_1 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}^1 \wedge \mathbf{e}^4 - \mathbf{e}^2 \wedge \mathbf{e}^3), \\ \mathbf{E}_2 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}^1 \wedge \mathbf{e}^4 + \mathbf{e}^2 \wedge \mathbf{e}^3), & \mathbf{E}_3 &= \tfrac{1}{\sqrt{2}}(\mathbf{e}^1 \wedge \mathbf{e}^3 - \mathbf{e}^2 \wedge \mathbf{e}^4), \\ \mathbf{E}_4 &= -\mathbf{e}^3 \wedge \mathbf{e}^4 \end{aligned}$$

constitute a basis of $[\omega]^\perp$ such that $(\mathbf{E}_i, \mathbf{E}_j) = \mathfrak{m}_{ij}$. Consequently, $\mathbb{R}^{2,3}$ can be identified with $[\omega]^\perp$ by the linear isometry

$$\widehat{\varrho} : \mathbb{R}^{2,3} \ni \sum_{j=0}^4 x^j M_j^o \mapsto \sum_{j=0}^4 x^j \mathbf{E}_j \in [\omega]^\perp \subset \Lambda^2(\mathbb{R}^4).$$

For $\mathbf{X} \in \mathrm{Sp}(\mathbb{R}^4, \omega)$, let $\varrho(\mathbf{X})$ be the automorphism of $\mathbb{R}^{2,3}$ defined by $V \mapsto \widehat{\varrho}^{-1}(\mathbf{X} \cdot \widehat{\varrho}(V))$. The map $\varrho : \mathrm{Sp}(\mathbb{R}^4, \omega) \to \mathrm{A}_+^\uparrow(2,3)$, $\mathbf{X} \mapsto \varrho(\mathbf{X})$, is a 2:1 group covering homomorphism whose

kernel is the center $\mathcal{Z} = \{\pm\mathrm{Id}_{4\times4}\} \cong \mathbb{Z}_2$ of $\mathrm{Sp}(4,\mathbb{R})$. The corresponding induced Lie algebra isomorphism is given by $\varrho_* : \mathfrak{sp}(\mathbb{R}^4,\omega) \to \mathfrak{a}(2,3)$, $\mathtt{A}(\mathbf{a},\mathbf{b},\mathbf{c}) \mapsto \sum_{i,j} \widehat{\varrho}_*(\mathbf{a},\mathbf{b},\mathbf{c})^i_j M^o_i \otimes M^j_o$, where $\widehat{\varrho}_*(\mathbf{a},\mathbf{b},\mathbf{c}) \in \mathfrak{m}(2,3)$ is defined by

$$\begin{aligned}\widehat{\varrho}_*(\mathbf{a},\mathbf{b},\mathbf{c}) = &- (a_1^1 + a_2^2)M_0^0 + a_2^2 M_2^1 - (a_2^2 + a_1^2)M_3^1 + (a_2^2 - a_1^2)M_3^2 \\ &- \frac{1}{\sqrt{2}}(c_1^1 + c_2^2)M_1^0 + \frac{1}{\sqrt{2}}(c_1^1 - c_2^2)M_2^0 + \sqrt{2}c_2^1 M_3^0 \\ &+ \frac{1}{\sqrt{2}}(b_1^1 + b_2^2)M_1^4 + \frac{1}{\sqrt{2}}(b_1^1 - b_2^2)M_2^4 + \sqrt{2}b_2^1 M_3^4.\end{aligned}$$

Remark 14 (A Lagrangian model for the Einstein universe). The symplectic form ω induces an invariant pairing between $\mathbb{R}_4$, the dual of $\mathbb{R}^4$, and $\mathbb{R}^4$ given by

$$\mathbb{R}_4 \ni \mathtt{x} = x_1\mathbf{e}^1 + x_2\mathbf{e}^2 + x_3\mathbf{e}^3 + x_4\mathbf{e}^4 \mapsto \mathtt{x}^\natural = x_3\mathbf{e}_1 + x_4\mathbf{e}_2 - x_1\mathbf{e}_3 - x_2\mathbf{e}_4 \in \mathbb{R}^4.$$

The linear isometry $\widehat{\varrho} : \mathbb{R}^{2,3} \to [\omega]^\perp \subset \Lambda^2(\mathbb{R}^4)$ takes the null vectors of $\mathbb{R}^{2,3}$ to the decomposable 2-forms orthogonal to ω. Any $[X] \in \mathcal{M}^{1,2}$ determines a unique oriented line $[\widehat{\varrho}(X)] \subset [\omega]^\perp$ spanned by a decomposable 2-form $\widehat{\varrho}(X) = \alpha(X) \wedge \beta(X)$, such that $\widehat{\varrho}(X) \wedge \omega = 0$. The latter identity means that the 2-plane $\widehat{\varrho}(X)^\perp = \{\mathtt{w} \in \mathbb{R}^4 : i_{\mathtt{w}}\widehat{\varrho}(X) = 0\}$ is Lagrangian. Such a plane is oriented by the basis $(\alpha(X)^\natural, \beta(X)^\natural)$, so that $\mathcal{M}^{2,1}$ can be identified with $\mathcal{L}_+^2(\mathbb{R}^4,\omega)$, the manifold of oriented Lagrangian planes of $(\mathbb{R}^4,\omega)$, by the map $\mathcal{M}^{1,2} \ni [X] \mapsto [\alpha(X)^\natural \wedge \beta(X)^\natural] \in \mathcal{L}_+^2(\mathbb{R}^4,\omega)$.
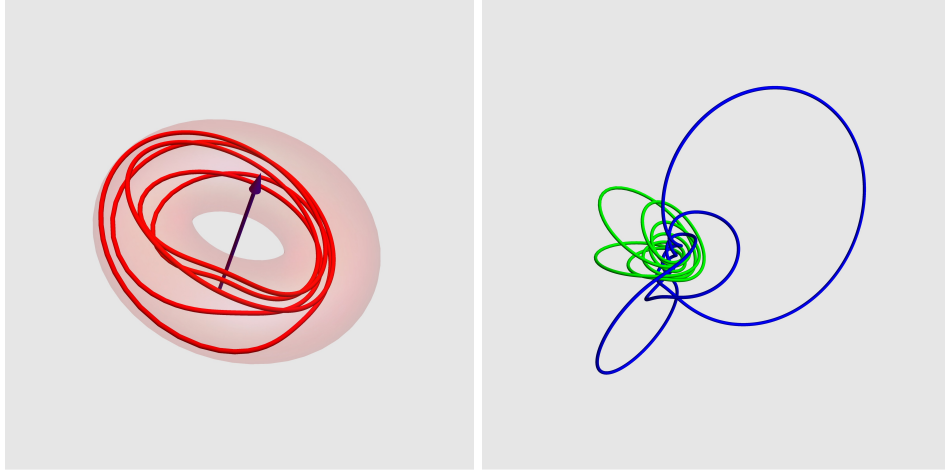
FIGURE 13. A closed homogenoeus curve with $b = 2/5$ trapped in the adS chamber (on the left) and the stereographic projections of its directrices from the point $(1,0,0,0)$.

6.3. **Directrices.** Let $\gamma : I \to \mathcal{E}^{1,2}$ be a generic timelike curve parametrized by the conformal parameter and let $\mathbf{M} : I \to \mathcal{M}$ be its canonical frame. Using the identification of $\mathcal{M}$ with $\mathrm{A}_+^\uparrow(2,3)$, let $\widetilde{\mathbf{M}} : \mathbb{R} \to \mathrm{Sp}(4,\mathbb{R})$ be a lift of $\mathbf{M}$ to $\mathrm{Sp}(4,\mathbb{R})$. Then, it follows that $\Gamma = |\widetilde{\mathbf{M}}_3|^{-1}\widetilde{\mathbf{M}}_3$ and $\Gamma^* = |\widetilde{\mathbf{M}}_4|^{-1}\widetilde{\mathbf{M}}_4$ are smooth maps into the 3-sphere, which are referred to as the *directrices* of γ. Using the isomorphism ϱ_*, it is easily seen that $\widetilde{\mathbf{M}}$ satisfies $\widetilde{\mathbf{M}}' = \widetilde{\mathbf{M}}\widetilde{\mathcal{K}}(h,k)$, where k and h are the conformal curvatures of γ and $\widetilde{\mathcal{K}}(h,k)$ is

the $\mathfrak{sp}(4, \mathbb{R})$-valued map

$$(6.5) \qquad \widetilde{\mathcal{K}}(h, k) = \begin{pmatrix} 0 & -k/2 & -1/\sqrt{2} & 0 \\ k/2 & 0 & 0 & -1/\sqrt{2} \\ (h-1)/\sqrt{2} & 0 & 0 & -k/2 \\ 0 & (h+1)/\sqrt{2} & k/2 & 0 \end{pmatrix}.$$

This implies that the directrices are immersed curves of S^3, everywhere transverse to the contact distribution of S^3. If γ is a simple closed curve, with minimal period ℓ, then Γ and Γ^* are periodic, with minimal period $\widetilde{\ell} = \ell/\sigma^*$, where $\sigma^* \in \{1, 1/2\}$ is the *symplectic spin* of γ. Note that σ^* is invariant by regular timelike homotopies (i.e., through generic timelike curves). Consequently, Γ and Γ^* originate smooth immersions of the circle $\mathbb{R}/\widetilde{\ell}\mathbb{Z}$ into the 3-sphere. If the trajectories of Γ and Γ^* are simple and disjoint, one can compute the linking number $\mathrm{Lk}(\Gamma, \Gamma^*)$ and the Bennequin numbers $\mathbf{b}(\Gamma)$ and $\mathbf{b}(\Gamma^*)$. These three integers provide global conformal invariants of γ, different in general from the Maslov index of γ. The next Proposition explains how to compute these invariants for a closed homogeneous timelike curve of the class $C_{2.i}$ considered in Section 4 (see Figure 13).

Proposition 12. *Let γ be a closed homogeneous curve of the second class and of the first type, with parameters $a \in (0, 1)$ and $0 < b = m/n < 1$. Then, the following hold true:*

(1) *n is its Maslov index;*
(2) *the curve has symplectic spin $1/2$;*
(3) *if p and q are the relatively prime positive integers defined by $p/q = (n-m)/(n+m)$, then its directrices Γ and Γ^* are transversal positive torus knots of type (q, p) such that $\mathrm{Lk}(\Gamma, \Gamma^*) = pq$ and $\mathbf{b}(\Gamma) = \mathbf{b}(\Gamma^*) = pq - (p+q)$.*

Proof. (1) The assertion follows immediately from formula (4.5) of Theorem 8.

(2) Consider the parametrization of γ given by the natural parameter so that, up to the action of the conformal group,

$$\gamma(u) = [\mathbf{M}^o \cdot \mathrm{Exp}(u\mathcal{K}(h, k)) \cdot {}^t(1, 0, 0, 0)],$$

where h and k are the constants defined as in (4.9) and $\mathcal{K}(h, k)$ is the element of $\mathfrak{m}(2, 3)$ defined by $\mathcal{K}(h, k) := M_2^0 - M_1^4 - kM_3^2 - hM_1^0$. This is a simple element of $\mathfrak{m}(2, 3)$ with eigenvalues $0, \pm i\lambda_1, \pm i\lambda_2$, where

$$\lambda_1 = \frac{\sqrt{b(1 + b^2(a^2 - 1))}}{(1 - a^2)^{1/4}\sqrt{1 - b^2}}, \quad \lambda_2 = \frac{\sqrt{1 + b^2(a^2 - 1)}}{(1 - a^2)^{1/4}\sqrt{b(1 - b^2)}}.$$

Since $\lambda_1/\lambda_2 = b = m/n$, the minimal period of γ (i.e., the total strain of the curve) is

$$(6.6) \qquad \omega(a, m/n) = \frac{2\pi n}{\lambda_2} = 2\pi \frac{(1 - a^2)^{1/4}\sqrt{m(n^2 - m^2)}}{\sqrt{n(n^2 - (1 - a^2)m^2)}}.$$

Let $\widetilde{\mathcal{K}}(h, k)) \in \mathfrak{sp}(4, \mathbb{R})$ be given as in (6.5). Then, putting

$$\mathrm{C} = \mathrm{Exp}(u\widetilde{\mathcal{K}}(h, k)) \cdot \mathbf{e}_3, \quad \mathrm{C}^* = \mathrm{Exp}(u\widetilde{\mathcal{K}}(h, k)) \cdot \mathbf{e}_4,$$

the two directrices of γ are parameterized by

$$\Gamma : u \to \|\mathrm{C}(u)\|^{-1}\mathrm{C}(u), \quad \Gamma^* : u \to \|\mathrm{C}^*(u)\|^{-1}\mathrm{C}^*(u),$$

respectively. The matrix $\widetilde{\mathcal{K}}(h, k)$ is a simple element and has four purely imaginary distinct eigenvalues, $\pm i\widetilde{\lambda}_1, \pm i\widetilde{\lambda}_2$, where

$$\widetilde{\lambda}_1 = \frac{\sqrt{(1 + b)(1 + (a^2 - 1)b^2)}}{2(1 - a^2)^{1/4}\sqrt{b(1 - b)}}, \quad \widetilde{\lambda}_2 = \frac{\sqrt{(1 - b)(1 + (a^2 - 1)b^2)}}{2(1 - a^2)^{1/4}\sqrt{b(1 + b)}}.$$

Thus, $\widetilde{\lambda}_1/\widetilde{\lambda}_2 = (n+m)/(n-m)$. This implies that the two directrices have minimal period $\omega_*(a, m/n) = 2\pi(n + m)/\widetilde{\lambda}_2(n - m) = 2\omega(a, m/n)$. This proves that the symplectic spin of the curve is $1/2$.

(3) We first write the directrices of γ in a canonical form. To this end, we consider the eigenvectors $\mathtt{U}_1$ and $\mathtt{U}_2$ of the eigenvalues $-i\widetilde{\lambda}_1$ and $-i\widetilde{\lambda}_2$, respectively, and denote by $\mathtt{V}_j$ and $\mathtt{W}_j$, $j=1,2$, their real and imaginary parts. Then, the matrix $\mathtt{B}$ with column vectors

$$\mathtt{B}_1 = \mathtt{V}_1/\sqrt{-\omega(\mathtt{V}_1,\mathtt{W}_1)}, \quad \mathtt{B}_2 = \mathtt{V}_2/\sqrt{-\omega(\mathtt{V}_2,\mathtt{W}_2)},$$
$$\mathtt{B}_3 = \mathtt{W}_1/\sqrt{-\omega(\mathtt{V}_1,\mathtt{W}_1)}, \quad \mathtt{B}_4 = \mathtt{W}_2/\sqrt{-\omega(\mathtt{V}_2,\mathtt{W}_2)}$$

belongs to $\mathrm{Sp}(4,\mathbb{R})$ and a direct computation shows that

$$(6.7) \qquad \begin{cases} \mathtt{B}\cdot\Gamma(u) = \left(\frac{-2A\sin(qu)}{1+A^2}, \frac{(1-A^2)\sin(pu)}{1+A^2}, \frac{2A\cos(qu)}{1+A^2}, \frac{(A^2-1)\cos(pu)}{1+A^2}\right), \\ \mathtt{B}\cdot\Gamma^*(u) = \left(\frac{(1-A^2)\cos(qu)}{1+A^2}, \frac{2A\cos(pu)}{1+A^2}, \frac{(1-a^2)\sin(qu)}{1+A^2}, \frac{2A\sin(pu)}{1+A^2}\right), \end{cases}$$

where

$$A = \frac{\sqrt{1-\sqrt{1+a^2}}}{\sqrt{2}-\sqrt{1+\sqrt{1+a^2}}} \in (1+\sqrt{2},+\infty).$$

Since all our considerations are invariant under the action of the symplectic group, we may assume that the directrices are the transversal knots $\widehat{\mathrm{K}}_{A,p,q}$ and $\widehat{\mathrm{K}}^*_{A,p,q}$ of S^3 parameterized by the maps $\Gamma_{A,p,q}$ and $\Gamma_{A,p,q}$ in the left hand side of (6.7). Since the "point at infinity" $\mathtt{P} = {}^t(0,0,0,1)$ is not contained in by $\widehat{\mathrm{K}}_{A,p,q}\cup\widehat{\mathrm{K}}^*_{A,p,q}$, we may consider their images $\mathrm{K}_{A,p,q}$ and $\mathrm{K}^*_{A,p,q} \subset \mathbb{R}^3$ under the conformal map $\mathrm{st}: S^3\setminus\{\mathtt{P}\}\to\mathbb{R}^3$.[4] It is now an easy task to check that the link $(\mathrm{K}_{A,p,q}, \mathrm{K}^*_{A,p,q})$ is isotopic to $(\mathrm{K}_{p,q}, \mathrm{K}^*_{p,q})$, where $\mathrm{K}_{p,q}$ and $\mathrm{K}^*_{p,q}$ are the torus knots defined as in (6.1). This implies

$$\mathrm{Lk}(\widehat{\mathrm{K}}_{A,p,q}, \widehat{\mathrm{K}}^*_{A,p,q}) = \mathrm{Lk}(\mathrm{K}_{A,p,q}, \mathrm{K}^*_{A,p,q}) = \mathrm{Lk}(\mathrm{K}_{p,q}, \mathrm{K}^*_{p,q}) = pq.$$

For fixed integers p,q, the knots $\widehat{\mathrm{K}}_{A,p,q}$, $A > 1$, are transversal to the contact distribution and transversally isotopic to each other. Thus, the Bennequin numbers do no depend on the parameter A. Let $(\overrightarrow{\mathbf{E}}_1, \overrightarrow{\mathbf{E}}_2)$ be the parallelization of the contact distribution of S^3 defined as in (6.2) and let $\overrightarrow{\mathbf{Y}}_{p,q}$ be the vector field

$$\overrightarrow{\mathbf{Y}}_{p,q} : \mathtt{P}(x_1,x_2,x_3,x_4) \mapsto (-qx_3, -px_4, qx_1, px_2) \in \mathrm{T}_{\mathtt{P}}(S^3).$$

Then, $(\overrightarrow{\mathbf{Y}}_{p,q}, \overrightarrow{\mathbf{E}}_1, \overrightarrow{\mathbf{E}}_2)$ is a parallelization of the 3-sphere and $\widehat{\mathrm{K}}_{A,p,q}$ is the trajectory of $\overrightarrow{\mathbf{Y}}_{p,q}$ passing through the point ${}^t(0,0,2A/(1+A^2),(A^2-1)/(1+A^2))$. Since $\overrightarrow{\mathbf{Y}}_{p,q}$ is tangent to $\widehat{\mathrm{K}}_{A,p,q}$ and transversal to the contact distribution, the vector field

$$\overrightarrow{\mathbf{Z}}_{p,q} = \overrightarrow{\mathbf{E}}_1 - \frac{\overrightarrow{\mathbf{Y}}_{p,q}\cdot\overrightarrow{\mathbf{E}}_1}{\|\overrightarrow{\mathbf{Y}}_{p,q}\|^2}\overrightarrow{\mathbf{Y}}_{p,q}$$

is normal to $\widehat{\mathrm{K}}_{A,p,q}$ and, in addition, $\overrightarrow{\mathbf{E}}_2, \overrightarrow{\mathbf{Z}}_{p,q}$ span a 2-dimensional distribution everywhere transverse to $\widehat{\mathrm{K}}_{A,p,q}$. Thus, the vector fields $\overrightarrow{\mathbf{X}}_{p,q} = \mathrm{st}_*(\overrightarrow{\mathbf{Z}}_{p,q})$ and $\overrightarrow{\mathbf{V}}_{p,q} = \mathrm{st}_*(\overrightarrow{\mathbf{E}}_2)$ generate a plane field distribution transverse to $\mathrm{K}_{A,p,q}$. Taking into account that st is a conformal map, the vector field $\overrightarrow{\mathbf{X}}_{p,q}$ is normal to $\mathrm{K}_{A,p,q}$. Choosing $\epsilon > 0$ sufficiently small, the image of the map

$$(p,\theta) \in \mathrm{K}_{A,p,q}\times\mathbb{R} \mapsto p + \epsilon(\cos(\theta)\overrightarrow{\mathbf{X}}_{p,q} + \sin(\theta)\overrightarrow{\mathbf{V}}_{p,q}) \in \mathbb{R}^3$$

is an embedded torus whose inner region contains $\mathrm{K}_{A,p,q}$. Since the map

$$(p,r) \in \mathrm{K}_{A,p,q}\times[0,1] \mapsto p + \epsilon(\cos(r\pi/2)\overrightarrow{\mathbf{X}}_{p,q} + \sin(r\pi/2)\overrightarrow{\mathbf{V}}_{p,q})$$

is a smooth homotopy (in $\mathbb{R}^3\setminus\mathrm{K}_{A,p,q}$) between $\mathrm{K}_{A,p,q} + \epsilon\overrightarrow{\mathbf{V}}_{p,q}$ and $\mathrm{K}_{A,p,q} + \epsilon\overrightarrow{\mathbf{X}}_{p,q}$, we obtain

$$\mathbf{b}(\widehat{\mathrm{K}}_{A,p,q}) = \mathrm{Lk}_{\widehat{\mathrm{K}}_{A,p,q}}(\overrightarrow{\mathbf{E}}_2) = \mathrm{Lk}_{\mathrm{st}(\widehat{\mathrm{K}}_{A,p,q})}(\mathrm{st}_*(\overrightarrow{\mathbf{E}}_2)) = \mathrm{Lk}_{\mathrm{K}_{A,p,q}}(\overrightarrow{\mathbf{V}}_{p,q}) = \mathrm{Lk}_{\mathrm{K}_{A,p,q}}(\overrightarrow{\mathbf{X}}_{p,q}).$$

---

[4]The map st is the composition of the stereographic projection form the point at infinity and the reflection about the xy-plane of $\mathbb{R}^3$.

By construction, the vector field $\overrightarrow{\mathbf{X}}_{p,q}$ is a positive multiple of the normal projection $\overrightarrow{\mathbf{W}}_{p,q}$ of $\mathrm{st}_*(\overrightarrow{\mathbf{E}}_1)$ along $\mathrm{K}_{A,p,q}$ so that

$$\mathrm{Lk}_{\mathrm{K}_{A,p,q}}(\overrightarrow{\mathbf{W}}_{p,q}) = \mathrm{Lk}_{\mathrm{K}_{A,p,q}}(\overrightarrow{\mathbf{X}}_{p,q}).$$

This implies that $\mathbf{b}(\widehat{\mathrm{K}}_{A,p,q})$ coincides with $\mathrm{Lk}_{\mathrm{K}_{A,p,q}}(\overrightarrow{\mathbf{W}}_{p,q})$. The latter is equal to

$$\mathrm{SL}(\mathrm{K}_{A,p,q}) + \Theta_{\mathrm{K}_{A,p,q}}(\overrightarrow{\mathbf{W}}_{p,q}).$$

Since everything is independent of the parameter A, we assume $A = 3$. Then, $\widehat{\mathrm{K}}_{A,p,q}$ is the torus knot parameterized by the map $\check{\Gamma}_{p,q}$ defined as in (6.3). From this we get $\mathrm{SL}(\mathrm{K}_{A,p,q}) = pq - p$. On the other hand, $\Theta_{\mathrm{K}_{A,p,q}}(\overrightarrow{\mathbf{W}}_{p,q})$ is the degree of the map

$$\phi : \mathbb{R}/2\pi\mathbb{Z} \ni [u] \mapsto (\overrightarrow{\mathbf{W}}_{p,q}|_u \cdot \overrightarrow{\mathbf{P}}_n|_u, \overrightarrow{\mathbf{W}}_{p,q}|_u \cdot \overrightarrow{\mathbf{P}}_b|_u) \in \mathbb{R}^2 \setminus \{(0,0)\},$$

where $\overrightarrow{\mathbf{P}}_b|_u = \check{\Gamma}'_{p,q} \times \check{\Gamma}''_{p,q}|_u$ and $\overrightarrow{\mathbf{P}}_n|_u = -\check{\Gamma}'_{p,q} \times \overrightarrow{\mathbf{P}}_b|_u$. The components of ϕ can be computed by any software of symbolic computation. As a result ϕ can be written as $\phi([u]) = (\mathrm{S}_1(u) + \mathrm{S}_2(u))\,{}^t(\cos(qu), \sin(qu))$, where S_1 is a off-diagonal 2×2 matrix with positive entries and S_2 is a periodic map into $\mathbb{R}(2,2)$ of period 2π such that

$$\det(\mathrm{S}_1(u) + r\mathrm{S}_2(u)) < 0, \quad \forall u \in \mathbb{R},\ r \in [0,1].$$

Consequently, ϕ is homotopic to $\mathbb{R}/2\pi\mathbb{Z} \ni [u] \mapsto {}^t(\cos(qu), -\sin(qu))$, and hence

$$\Theta_{\mathrm{K}_{A,p,q}}(\overrightarrow{\mathbf{W}}_{p,q}) = -q.$$

This proves that $\mathbf{b}(\widehat{\mathrm{K}}_{A,p,q}) = pq - p - q$, as claimed.

It is now easy to check that $\Gamma_{\sqrt{3},p,q}$ and $\Gamma^*_{\sqrt{3},p,q}$ are congruent to each other with respect to the symplectic group. In particular, $\widehat{\mathrm{K}}_{\sqrt{3},p,q}$ and $\widehat{\mathrm{K}}^*_{\sqrt{3},p,q}$ have the same Bennequin number. Taking into account that the Bennequin number is independent of the parameter A, provided that $A > 1$, we obtain

$$\mathbf{b}(\Gamma^*_{A,p,q}) = \mathbf{b}(\Gamma^*_{\sqrt{3},p,q}) = \mathbf{b}(\Gamma_{\sqrt{3},p,q}) = pq - p - q,$$

which proves the required result. $\qquad\square$

Remark 15. The proposition above implies as a consequence that the conformal equivalence class of a closed homogeneous timelike curve of type C_{2i} is determined by a geometrical invariant, the total strain, and two topological invariants of its directrices, the Bennequin number and the linking number. In addition, the proposition shows that the directrices of the homogeneous knots of the class C_{2i} are representatives of the contact isotopy classes of transversal torus knots of type (p,q) with maximal Bennequin number [12, 14].

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