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This paper focuses on the problem of combining multi-agent preference orderings of different alternatives into a single fused ordering, when the agents' importance is expressed through a rank-ordering and not a set of weights. An enhanced version of the algorithm proposed by Yager in (Fuzzy Sets and Systems, 117(1): 1-12, 2001) is presented. The main advantages of the new algorithm are that: (i) it better reflects the multi-agent preference orderings and (ii) it is more versatile, since it admits preference orderings with omitted or incomparable alternatives. The description of the new algorithm is supported by a realistic example.

Keywords: Decision making; Multi-agent; Preference ordering; Fusion; Ordinal semi-democratic; Partial ordering.

1. Introduction

A general problem, which may concern practical contexts of different nature, is to aggregate multi-agent orderings of different alternatives into a single fused ordering. Let us assume that there are M decision-making agents D_1, D_2, …, D_M, each of which defines an ordering of n alternatives a, b, c, etc.. This decision-making problem is fairly general [1, 2, 3] and can be applied to a variety of real-life contexts, ranging from *multi-criteria decision aiding* [4] to *social choice* [5, 6] and *voting theory* [7, 8].

The problem becomes more specific if the importance hierarchy of agents is expressed through a rank-ordering and not a set of weights defined on a *ratio* scale. This decision-making framework can be denominated as "ordinal semi-democratic"; the adjective "semi-democratic" indicates that agents do not necessarily have the same importance, while "ordinal" indicates that their hierarchy is defined by a crude ordering. The set of the possible solutions to the problem may range between the two extremes of (i) *full dictatorship*—in which the fused ordering coincides with the preference ordering by the most important agent (dictator)—and (ii) *full democracy*—where all agents' orderings are considered as equi-important.

Some years ago, Yager [9] proposed an algorithm to address the problem of interest in a relatively simple, fast and automatable way. Unfortunately, this algorithm (hereafter abbreviated as YA, which stands for "Yager's Algorithm") has two major limitations: (i) the resulting fused ordering may sometimes not reflect the preference ordering for the majority of agents and (ii) it is applicable to *linear* orderings only, without incomparabilities and omissions of the alternatives of interest. For details, we refer the reader to [9, 10].

The objective of this paper is to enhance the YA so as to overcome its limitations and adapt to less stringent preference orderings. A new algorithm, denominated as "Enhanced (Yager's) Algorithm" (hereafter abbreviated as EYA), will be proposed.

The remainder of the paper is organized into two sections. Sec. 2 illustrates the EYA by presenting a realistic example. Sec. 3 summarizes the original contributions of the paper and its practical implications, limitations and suggestions for future research.

2. Enhanced Yager's Algorithm (EYA)

The EYA can be decomposed in three phases, which are individually described in the following sub-sections:

- construction, normalization and reorganization of preference vectors;
- definition of the reading sequence;
- construction of the fused ordering.

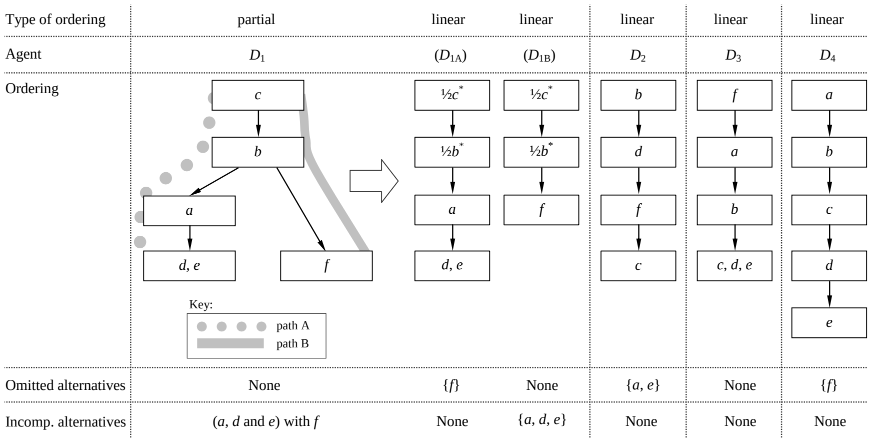
2.1. *Construction, normalization and reorganization of preference vectors*

The YA is applicable to *linear* orderings only, where no alternatives are omitted and any two alternatives are comparable [9]. A generic linear ordering can be diagrammed as an acyclic line or chain of elements containing the alternatives of interest, linked by arrows depicting the strict preference relationship. In this conventional representation, the most preferred alternatives are positioned at the top. Two generic alternatives are always comparable, since there exist a path from the first to the second one (or vice versa) that is directed downwards.
The EYA is more versatile since admits orderings with omitted and/or incomparable alternatives, i.e., orderings that, according to the Mathematics' Order theory, are classified as *partial* [11]. This type of ordering can be diagrammed as a graph with branches, which determine different possible paths

from the element(s) at the top to that one(s) at the bottom. If two alternatives are not comparable, there exists no direct path from the first to the second one (or *viceversa*).

The first step of this phase is to transform each (partial) ordering with incomparabilities into a set of linear sub-orderings. Precisely, a partial ordering can be artificially split into p linear sub-orderings, corresponding to the possible paths from the top to the bottom element(s). Obviously, the number of paths depends on the configuration of the relevant graph (e.g., amount and position of the branches). For the purpose of example, let us consider the preference orderings illustrated in Fig. 1, in which the agents' importance ordering is assumed to be $D_4 > (D_2 \sim D_3) > D_1$. It can be noticed that the (partial) ordering by agent D_1 includes $p = 2$ possible paths (A and B); therefore, this ordering is turned into two linear sub-orderings, D_{1A} and D_{1B}.

Each alternative in the sub-orderings is associated with a conventional number of occurrences, fractionalized with respect to the number of sub-orderings where the alternative is present. E.g., for c and b, the fractional number of occurrences is 1/2 as these alternatives are contained in both the sub-orderings D_{1A} and D_{1B}. The relative importance associated with each linear sub-ordering is that of the relevant source (partial) ordering.



(*) Coefficient "½" means that the alternative of interest has a (fractional) number of occurrences in that vector element equals 1/2.

Fig. 1. Graphical representation of the preference orderings by four fictitious agents (D_1 to D_4). The alternatives of interest are a, b, c, d, e and f. The ordering by D_1 has two paths, therefore it is turned into two linear sub-orderings (D_{1A} and D_{1B}). The agents' importance ordering is assumed to be $D_4 > (D_2 \sim D_3) > D_1$.

Next, linear (sub-)orderings are turned into preference vectors, according to the following convention. We place the alternatives as they appear in the ordering, with the most preferred one(s) in the top positions. If at any point $t > 1$ alternatives are tied (i.e., indifferent), we place them in the same element and then place the null set ("Null") in the next $t – 1$ lower positions. Although there are six total alternatives (a, b, c, d, e and f), some of them may be omitted in a certain vector; therefore the number of elements (n_i) can change from a vector to one other. Table 1 exemplifies the construction of the preference vectors from the orderings in Fig. 1. Each vector element is associated with a relative-position indicator, which represents the cumulative relative frequency $f_{i,j}$—i.e., the ratio between the position (j) of an element, starting from the bottom, and n_i.

Table 1. Construction of preference vectors for the linear (sub-)orderings in Fig. 1.

Agent	D_{1A}		D_{1B}		D_2		D_3		D_4	
Orderings	$c>b>a>(d\sim e)$		$c>b>f$		$b>d>f>c$		$f>a>b>(c\sim d\sim e)$		$a>b>c>d>e$	
No. of alternatives (n_i)	5		3		4		6		5	
Omitted alternative(s)	$\{f\}$		$\{a, d, e\}$		$\{a, e\}$		Null		$\{f\}$	
Preference vectors	$f_{1A,j}$	Elem.	$f_{1B,j}$	Elem.	$f_{2,j}$	Elem.	$f_{3,j}$	Elem.	$f_{4,j}$	Elem.
	1.00	$\{½c\}$	1.00	$\{½c\}$	1.00	$\{b\}$	1.00	$\{f\}$	1.00	$\{a\}$
	0.80	$\{½b\}$	0.67	$\{½b\}$	0.75	$\{d\}$	0.83	$\{a\}$	0.80	$\{b\}$
	0.60	$\{a\}$	0.33	$\{f\}$	0.50	$\{f\}$	0.67	$\{b\}$	0.60	$\{c\}$
	0.40	$\{d, e\}$			0.25	$\{c\}$	0.50	$\{c, d, e\}$	0.40	$\{d\}$
	0.20	Null					0.33	Null	0.20	$\{e\}$
							0.17	Null		

$f_{i,j} = j/n_i$ is the cumulative relative frequency referring to the j-th element of an i-th vector.

Before being reorganized, vectors should be normalized in terms of length, i.e., turned into new vectors with the same number of elements. We define the set F^*, given by the *union* of the $f_{i,j}$ values relating to the vectors of interest, sorted in ascending order:

$$F^* = sort\left(\bigcup_{\forall i} F_i\right) \tag{1}$$

in which $F_i = \{f_{i,1}, f_{i,2}, \ldots, f_{i,n_i}\}$ is the set of $f_{i,j}$ indicators relating to a certain i-th preference vector and the "*sort*" operator represents the ascending order permutation. For example, considering the five preference vectors in Table 1, it is obtained:

$$F^* = \{0.17, 0.20, 0.25, 0.33, 0.40, 0.50, 0.60, 0.67, 0.75, 0.80, 0.83, 1.00\}. \tag{2}$$

Being independent on a particular i-th vector, the elements in F^* can be conventionally renamed as f_j^* (without subscript "i"):

$$F^* = \{f_1^*, f_2^*, ...\}. \quad (3)$$

Next, we define:

$$n_T = card(F^*), \quad (4)$$

i.e., the total number of elements of F^*; e.g., considering the ordered set in Eq. 2, $n_T = 12$. Obviously, $n_i \leq n_T \ \forall i$.

Each *i*-th preference vector can be now normalized by adding a "Null" element for each f_j^* value that is included in the set F^* but not included in F_i, following a decreasing sequence. Fig. 2(a) exemplifies this mechanism for the preference vectors in Table 1.

(a) Normalized vectors

	D_{1A}	D_{1B}	D_2	D_3	D_4
f_j^*	Elem.	Elem.	Elem.	Elem.	Elem.
1.00	{½c}	{½c}	{b}	{f}	{a}
0.83	Null	Null	Null	{a}	Null
0.80	{½b}	Null	Null	Null	{b}
0.75	Null	Null	{d}	Null	Null
0.67	Null	{½b}	Null	{b}	Null
0.60	{a}	Null	Null	Null	{c}
0.50	Null	Null	{f}	{c, d, e}	Null
0.40	{d, e}	Null	Null	Null	{d}
0.33	Null	{f}	Null	Null	Null
0.25	Null	Null	{c}	Null	Null
0.20	Null	Null	Null	Null	{e}
0.17	Null	Null	Null	Null	Null

(b) Reorganized (aggregated) vectors

D_4		(D_2 ~ D_3)		(D_{1A} ~ D_{1B})	
S	Elem.	S	Elem.	S	Elem.
34	{a}	35	{b, f}	36	{c}
31	Null	32	{a}	33	Null
28	{b}	29	Null	30	{½b}
25	Null	26	{d}	27	Null
22	Null	23	{b}	24	{½b}
19	{c}	20	Null	21	{a}
16	Null	17	{c, d, e, f}	18	Null
13	{d}	14	Null	15	{d, e}
10	Null	11	Null	12	{f}
7	Null	8	{c}	9	Null
4	{e}	5	Null	6	Null
1	Null	2	Null	3	Null

Fig. 2. (a) Normalization of the preference vectors in Table 1 (the elements highlighted in grey have been added to normalize the number of elements of each vector to $n_T = 12$). (b) Construction of reorganized vectors; Vectors are sorted in decreasing order with respect to the agents' importance and S are the relevant sequence numbers.

Next, the normalized vectors are sorted in decreasing order with respect to the agents' importance and (ii) the equi-important vectors (e.g., D_{1A}, D_{1B} and D_2, D_3 in the example) are aggregated into a single one, through a level-by-level union of their elements. Going back to the example in Fig. 1, the resulting reorganized vectors are three, for simplicity denominated as D_4, (D_2 ~ D_3) and (D_{1A} ~ D_{1B}) (see Fig. 2(b)).

2.2. *Definition of the reading sequence*

The object of this phase is determining a sequence for the element-by-element reading of the reorganized vectors. Precisely, the sequence is based on a lexicographical order based on two dimensions: (i) f_j^* values (in increasing order) and (ii) relative importance of the agent (in decreasing order). Fig. 2(b) reports the full sequence numbers (S) associated with each element of the reorganized vectors.

2.3. *Construction of the fused ordering*

The construction of the fused ordering is gradual: alternatives are progressively included into a gradual ordering, which is initially Null. A k-th alternative is included at the top of the gradual ordering when—during the element-by-element reading sequence—its number of occurrences (O_k) reaches a certain threshold, i.e.:

$$T_{k,x} = x \cdot O_k^T, \tag{5}$$

being x a conventional percentage of the total number of occurrences (O_k^T) in the reorganized vectors' elements. Table 2 shows the $T_{k,x}$ values related to the alternatives; x was conventionally set to 50%.

Table 2. Thresholds for the selection of the alternatives; x was conventionally set to 50%.

Alternatives	a	b	c	d	e	f
O_k^T	3	4	4	4	3	3
$T_{k,50\%}$	1.5	2	2	2	1.5	1.5

If the $T_{k,x}$ thresholds related to multiple alternatives are reached at the same moment, they will be considered as indifferent. Considering the reorganized vectors in Fig. 2(b) and the thresholds in Table 2, the fused preference ordering is $a > b > c > (d \sim e) > f$. Table 3 shows the step-by-step results; the last columns contains the gradual construction of the fused ordering.

It is worth remarking that a k-th alternative is included in the lower positions of the fused ordering when a predetermined portion (x) of its occurrences (not just a single one, as suggested by Yager [9]) are in a lower position of the individual preference orderings.

3. Concluding remarks

This paper proposed an enhanced version of the YA, which has two main advantages: (i) it is more versatile, since it admits preference orderings with omitted or incomparable alternatives, and (ii) it better reflects the multi-agent preference orderings, since it is based on a gradual construction of the fused ordering. Also, it is automatable and can be applied to a variety of practical contexts.

Table 3. Step-by-step construction of the fused ordering for the reorganized orderings in Fig. 2(b).

Step (S)	Element	Occurrences (O_k) *a*	*b*	*c*	*d*	*e*	*f*	Residual alternatives	Gradual ordering
0	-	-	-	-	-	-	-	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
1	Null	0	0	0	0	0	0	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
2	Null	0	0	0	0	0	0	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
3	Null	0	0	0	0	0	0	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
4	{*e*}	0	0	0	0	1	0	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
5	Null	0	0	0	0	1	0	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
6	Null	0	0	0	0	1	0	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
7	Null	0	0	0	0	1	0	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
8	{*f*}	0	0	0	0	1	1	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
9	Null	0	0	0	0	1	1	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
10	Null	0	0	0	0	1	1	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
11	Null	0	0	0	0	1	1	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
12	{*f*}	0	0	0	0	1	2	{*a*, *b*, *c*, *d*, *e*}	*f*
13	{*d*}	0	0	0	1	1	2	{*a*, *b*, *c*, *d*, *e*}	*f*
14	Null	0	0	0	1	1	2	{*a*, *b*, *c*, *d*, *e*}	*f*
15	{*d*, *e*}	0	0	0	2	2	2	{*a*, *b*, *c*}	(*d*~*e*)>*f*
16	Null	0	0	0	2	2	2	{*a*, *b*, *c*}	(*d*~*e*)>*f*
17	{*c*, *d*, *e*, *f*}	0	0	1	3	3	3	{*a*, *b*, *c*}	(*d*~*e*)>*f*
18	Null	0	0	1	3	3	3	{*a*, *b*, *c*}	(*d*~*e*)>*f*
19	{*c*}	0	0	2	3	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
20	Null	0	0	2	3	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
21	{*a*}	1	0	2	3	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
22	Null	1	0	2	3	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
23	{*b*}	1	1	2	3	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
24	{½*b*}	1	1.5	2	3	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
25	Null	1	1.5	2	3	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
26	{*d*}	1	1.5	2	4	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
27	Null	1	1.5	2	4	3	3	{*a*, *b*}	*c*>(*d*~*e*)>*f*
28	{*b*}	1	2.5	2	4	3	3	{*a*}	*b*>*c*>(*d*~*e*)>*f*
29	Null	1	2.5	2	4	3	3	{*a*}	*b*>*c*>(*d*~*e*)>*f*
30	{½*b*}	1	3	2	4	3	3	{*a*}	*b*>*c*>(*d*~*e*)>*f*
31	Null	1	3	2	4	3	3	{*a*}	*b*>*c*>(*d*~*e*)>*f*
32	{*a*}	2	3	2	4	3	3	Null	*a*>*b*>*c*>(*d*~*e*)>*f*
End	-	-	-	-	-	-	-	-	-

The fused ordering is constructed without overlooking the higher positions of the agents' preference orderings; e.g., in the example illustrated in Table 3, the fused ordering is determined after having read more than the 80% of the non-null vector elements (i.e., fourteen out of seventeen).

It can be shown that the EYA provides a fused ordering which is reasonably consistent with the agents' preference orderings, even in situations in which some alternatives are characterized by relatively large fluctuations (e.g., consider the alternatives *c* and *f* in the example in Fig. 1).

A potentially controversial aspect of the new algorithm is the mechanism for aggregating and/or comparing elements from different preference vectors.

The underlying assumption is that the degree of preference of the alternatives in different preference vectors mainly depends on their relative position, depicted by $f_{i,j}$ indicators.

Future research go in several directions: (i) quantitative analysis of the robustness of the algorithm with respect to small variations in the preference orderings and/or in the $T_{k,x}$ thresholds, (ii) application of the algorithm to various decision-making frameworks [12], and (iii) revision of the logic for aggregating the preference vectors, introducing preference/indifference thresholds related to the $f_{i,j}$ indicators.

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