A coupled stress and energy criterion within finite fracture mechanics

A COUPLED STRESS AND ENERGY CRITERION WITHIN FINITE FRACTURE MECHANICS

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ABSTRACT

The aim of the present paper is to introduce a new failure criterion within the framework of Finite Fracture Mechanics. Criteria assuming that failure of quasi-brittle materials is affected by the stress or energy flux acting over a finite distance from the crack tip are widely used inside the scientific community. The novelty of the present approach relies on the assumption that this finite distance is not a material constant but a structural parameter. Its value is determined by a condition of consistency of both energy and stress approaches. The model is general. In order to check its validity, two problems are addressed, namely the strength prediction of (i) three point bending tests of specimens with various sizes and crack depths and (ii) V-notched specimens with different notch opening angles. Comparisons with experimental data are then provided. For the former problem we used data from the literature on high strength concrete specimens; for the latter one, *ad hoc* experiments have been carried out on polystyrene specimens. For both the geometries the fit provided is excellent, thus confirming the soundness of the present approach.

Introduction

When dealing with brittle or quasi-brittle materials, two main failure criteria are generally taken into account. The former is a stress criterion:

$$\sigma = \sigma_u \tag{1}$$

i.e. failure takes place if, at least in one point, the stress σ reaches the tensile strength σ_u. The latter is an energy criterion:

$$\mathcal{G} = \mathcal{G}_F \tag{2}$$

It states that failure happens if the crack driving force $\mathcal{G}$ equals the crack resistance $\mathcal{G}_F$. $\mathcal{G}_F$ is the so-called fracture energy, i.e. the energy necessary to create the unit fracture surface. As is well known, according to Irwin's relationship, the energy criterion can be expressed equivalently in terms of stress-intensity factor (SIF) K_I and fracture toughness K_{Ic}:

$$K_I = K_{Ic} \tag{3}$$

For the sake of simplicity, only mode I crack propagation will be dealt with in the following.

The stress criterion provides good results for crack-free bodies, whereas the energy criterion is physically sound for bodies containing a sufficiently large crack. Otherwise both the criteria fail. In fact, the stress criterion provides a null failure load for a body containing a crack, the stress field being singular in front of the crack tip. On the other hand, the energy criterion provides an infinite failure load for a crack-free body, the stress-intensity factor being zero in absence of a crack. Hence the criteria (1-3) work satisfactorily for the extreme cases (i.e. no crack or large crack) but are no longer valid for the intermediate cases, such as, for instance, short cracks or sharp notches.

In order to overcome this drawback, several failure criteria have been proposed in the literature. Concerning numerical applications to quasi-brittle materials, we may cite the fictitious crack model introduced by Hillerborg et al. [1], which takes into account both the tensile strength and the fracture energy of the given material. Of course, in order to describe the structural behavior by the fictitious crack model, a code must be implemented and the results are obtained numerically.

Easier criteria can be put forward by introducing a material length Δ, which allows one to get analytical results, at least with sufficiently simple geometries, or to couple the failure criterion with a linear elastic analysis performed through a computer simulation such as the finite element method. The task of the characteristic length Δ is to take into account the fracture toughness for the stress-based criterion or the tensile strength for the energy-based criterion. More in detail, according to the stress criterion failure is achieved when the average stress ahead of the crack tip over a segment of length Δ_S reaches the critical value σ_u, whereas, according to the energy criterion, failure is achieved when the energy released in a crack extension of length Δ_E reaches the critical value $\mathcal{G}_F\,\Delta_E$.

The physical interpretation of these criteria is that fracture does not propagate continuously but by a finite crack extension, whose length is a material constant. Hence, the framework is a finite fracture mechanics (FFM) approach. From a physical point of view, the finite crack extension can be interpreted as a fracture process zone, in front of the crack tip, where the crack propagates as soon as critical conditions are reached.

The FFM criteria have been widely used inside the Scientific Community. Note that the stress criterion dates back to Neuber [2] and Novozhilov [3]. On the other hand, for details about the energy criterion, we refer to the recent papers by Pugno & Ruoff [4] and by Taylor, Cornetti & Pugno [5], who suggested the name Finite Fracture Mechanics. These two papers have demonstrated that this simplified approach can be used for all classes of materials, at all size scales from point defects in carbon nanotubes to large notches in engineering structures.

The coupled stress and energy FFM criterion

As mentioned above, the classical failure criteria (1-2) can be modified introducing an internal length, Δ, whose value is determined imposing the fulfillment of the limit cases: long crack failure load for the stress criterion and crack-free failure load for the energy criterion. Nevertheless, the two approaches remain distinct and the fulfillment of one of them usually implies the violation of the other one. In other words, when applying the average stress criterion, the energy released in the crack extension is not always $\mathcal{G}_F\,\Delta$. On the other hand, when applying the energy criterion, the resultant of the stresses acting on the crack extension can differ from the product $\sigma_u\,\Delta$. This incongruence may lead to paradoxes (Leguillon [6]).

In order to overcome this inconsistency, we can proceed removing the hypothesis that Δ is a material constant. Note that the crack is still propagating by finite steps, i.e. we remain in the framework of FFM. The value of the finite crack extension is determined by the fulfillment of both the stress and energy criteria, i.e., when both of the following equations are satisfied:

$$\begin{cases} \displaystyle\int_a^{a+\Delta_{SE}} \sigma_y(x)\,\mathrm{d}x = \sigma_u\,\Delta_{SE} \\ \displaystyle\int_a^{a+\Delta_{SE}} K_I^2(a)\,\mathrm{d}a = K_{Ic}^2\,\Delta_{SE} \end{cases} \tag{4}$$

This means that failure takes place whenever there is a segment of length Δ_{SE} over which the stress resultant is equal to $\sigma_u\Delta_{SE}$, and, contemporarily, the energy available for that crack extension is equal to $\mathcal{G}_F\,\Delta_{SE}$. Eqns (4) represent a system of two equations in the two unknowns: σ_f, the failure load, and Δ_{SE}, the crack extension. We propose that, while each single equation represents only a necessary condition for failure, the fulfillment of both of them represents a necessary and sufficient condition for fracture to propagate. From a physical point of view, the criterion expressed in eqn (4) is equivalent to state that fracture is energy-driven but a sufficiently high stress field must act in order to trigger crack propagation.

The present criterion is close to the one proposed in Leguillon [6]. But Leguillon's formulation differs from ours because it is based on the point-wise stress criterion. To our knowledge, equations (4) have not been proposed in the literature yet.

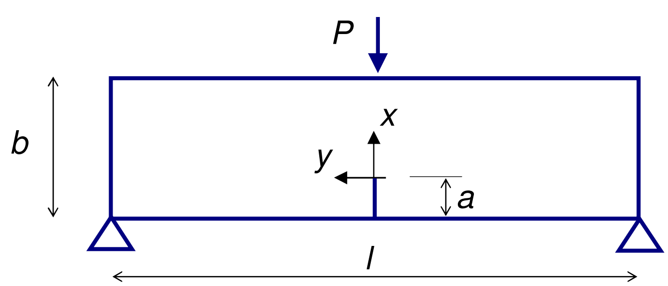

Figure 1. Cracked three point bending specimen.

Size effect for cracked and uncracked TPB specimens

In this section, the coupled criterion (4) will be applied to the strength prediction of cracked and uncracked TPB specimens of slenderness l/b equal to 4 (see fig.1). For this particular geometry, the main difficulty is due to the fact that only the asymptotic stress field in front of the crack tip is known analytically, whereas the whole stress field along the ligament is required in order to detect the size effect. Therefore we performed a finite element analysis for a given set of relative crack depth values and obtained the stress function S_α by interpolation. The subscript α reminds us that the stress function changes with the relative crack depth $\alpha = a/b$:

$$\sigma_y = \sigma \; S_\alpha(x/b) \tag{5}$$

where $\sigma = 6P / bt$ is the nominal maximum stress in the y direction (t being the beam thickness), reached at the bottom of the beam middle section of an uncracked specimen. Note that $x = 0$ corresponds to the crack tip.

The shape function for TPB cracked beams is not known in a closed form, but an analytical expression with error less than 0.5% for any value of a is available in the stress-intensity factor handbooks (Tada et al. [7]):

$$K_\mathrm{I} = \sigma\sqrt{\pi a} \; F(a/b) \tag{6}$$

The coupled criterion can be applied substituting eqns (5) and (6) in the system (4). Then, in order to get an equation in a unique unknown, we can raise the second equation to the square and take the ratio side by side so as to eliminate the nominal failure stress σ_f, giving:

$$\frac{\displaystyle\int_a^{a+\Delta_{SE}} \pi a \, F^2(a/b)\,\mathrm{d}a}{\left[\displaystyle\int_0^{\Delta_{SE}} S_\alpha(x/b)\,\mathrm{d}x\right]^2} = \frac{K_{Ic}^2}{\sigma_u^2 \, \Delta_{SE}} \tag{7}$$

In order to get a dimensionless formulation of the problem, we denote by Δ_S the finite crack advancement according to the FFM stress criterion, which is equal to $\frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma_u}\right)^2$ (see, e.g., Taylor [8]). The dimensionless quantities $\beta = b / \Delta_S$ (i.e. the dimensionless structural size) and $\varepsilon = \Delta_{SE} / b$ are now introduced. Note that, except for a factor, β coincides with $1/s^2$, where s is the brittleness number proposed by Carpinteri [9] since 1981 to describe the brittle to ductile transition in quasi-brittle material failure. Eqn (7) then becomes:

$$2\varepsilon\beta \int_\alpha^{\alpha+\varepsilon} t \, F^2(t)\,\mathrm{d}t = \left[\int_0^\varepsilon S_\alpha(t)\,\mathrm{d}t\right]^2 \tag{8}$$

where the unknown is the variable ε. Eqn (8) can be solved numerically: the solution is a function of α, the relative crack depth, and of β, the (dimensionless) structural size. Hence, the nominal failure stress can be deduced from the first equation of the system (4) as follows:

$$\frac{\sigma_f}{\sigma_u} = \frac{\varepsilon}{\displaystyle\int_0^\varepsilon S_\alpha(t)\,\mathrm{d}t} \tag{9}$$

The results are plotted in the bi-logarithmic graph of fig. 2, where the dimensionless strength vs. the dimensionless size is drawn for different values of the relative crack depth. It is seen that, at small sizes, all the curves show the same slope ¼, valid for cracked as well as for uncracked structures. On the other hand the behavior at large size is completely different: the uncracked structure strength tends to a constant value (the tensile strength σ_u) whereas the cracked structure strength decreases with the ½ LEFM slope.

Observing fig. 2, it is interesting to point out that the coupled criterion (4) is able to catch the concave-convex transition when passing from uncracked to cracked specimens in the log-log plot. While for the cracked structures the slope increases from ¼ to ½ (in absolute value), for the uncracked structures it decreases from ¼ to 0. Furthermore, note that the shape of the curve describing the size effect of uncracked structures strictly resembles that of the Multi-Fractal Scaling Law proposed by Carpinteri [10].

In order to have a feedback for the proposed model, we present a comparison with the data obtained by Karihaloo et al. [11]. He tested TPB high strength concrete specimens of various sizes and with different relative crack depths. His data are plotted in fig. 2 together with the predictions of the coupled FFM model. The values of the tensile strength and of the fracture toughness were determined by a best-fit procedure: we found 8.27 MPa for strength and 1.52 MPa√m for toughness. As expected, these values are a lower bound for the apparent tensile strengths of uncracked specimens (8.28 MPa being the minimum value) and an upper bound for the apparent fracture toughness of cracked specimens (1.48 MPa √m being its maximum value).

It can be seen from fig. 2 that the predictive accuracy of the coupled criterion is very good. A great advantage of the present approach is that only two parameters are needed to fit all the data. This feature is shared by the cohesive crack model (see, for a review, Carpinteri et al. [12]), which, however, requires a specific computer code in order to be implemented. As a drawback, it should be highlighted that the results presented herein are restricted to the TPB geometry. Applications to other shapes would require suitable solutions of the system (4), which can be obtained either analytically or numerically.

Finally, it is interesting to observe that, analyzing only the data relative to the uncracked specimens and using the coupled criterion, the best-fit procedure provides a fracture toughness equal to 1.50 MPa√m. This value is very close to the one obtained applying LEFM to the large cracked specimens (1.48 MPa√m). In other words, the method presented herein is able to provide the fracture toughness using test data from uncracked specimens, as long as the range of specimen sizes is broad enough. This implies that the mechanism of fracture is the same, whether or not a pre-crack is present.

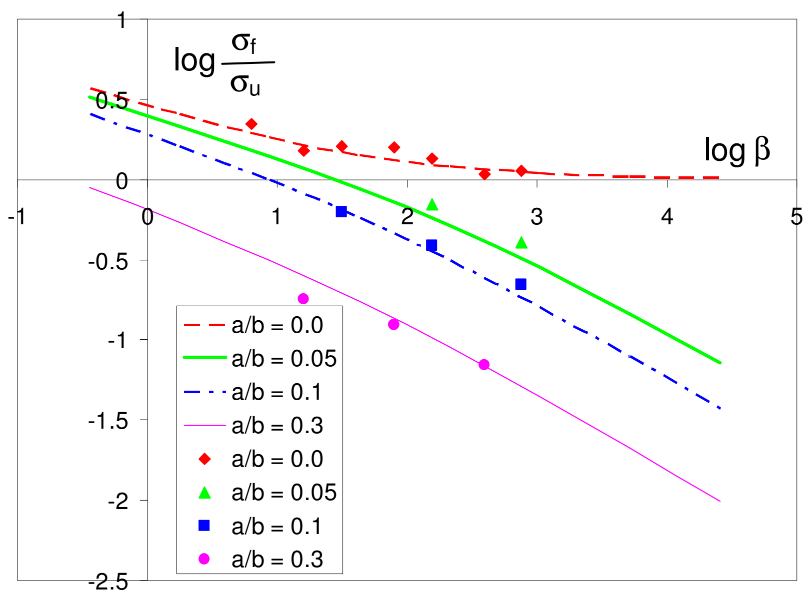

Figure 2. Bi-logarithmic plot of nominal failure stress vs. specimen size for various relative crack depths and comparison with experimental data on high strength concrete specimens by Karihaloo et al. [11].

TPB V- notched specimens

In this section, the coupled criterion (4) is applied to the strength prediction of TPB specimens with a V-notch at mid-span (see fig.3a). In this case we are interested in the variation of the strength along with the opening angle ω. We assume the material to be brittle enough (i.e. the structural size to be large enough) so that brittle collapse proceeds ductile collapse (Carpinteri [13]). In this case (and differently from what shown in the previous section), only the asymptotic stress field is required for FFM to apply:

$$\sigma_y(x) = \frac{K_I^*}{(2\pi x)^{1-\lambda}} \tag{10}$$

where K_I^* is the generalized stress-intensity factor and λ is an exponent which is a function of the angle according to the classical work by Williams [14]. Observe that, since the order of the stress field singularity changes along with ω, also the physical dimensions of K_I^* change, being intermediate between those of a SIF (ω =0, λ = ½) and of a stress ($\omega = \pi$, λ = 1).

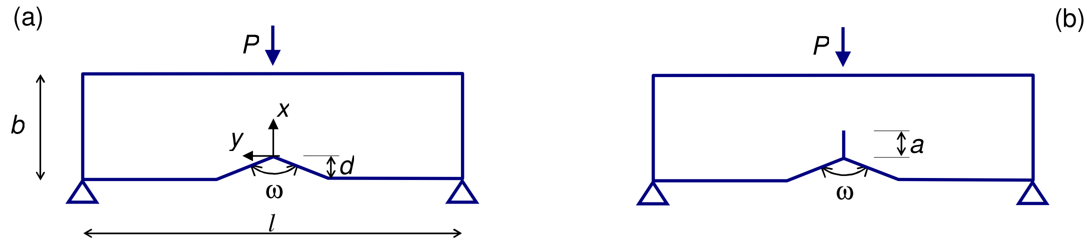

Figure 3. TPB V-notched specimen (a) and crack at the V-notch tip (b).

Eqn (10) is sufficient to apply the FFM stress criterion (Seweryn [15], Carpinteri & Pugno [16]). In order to apply the coupled criterion, also the SIF for a short crack of length a stemming from the notch tip is required (see fig. 3b). This function is not available in the literature. Nevertheless it can be derived using suitable weight functions provided by Tada et al. [7]. In their SIF handbook there is an approximate solution (better than 2%) for a crack at a V-notch tip loaded by a pair of forces acting on the crack lips (see fig. 4). Introducing the variable $m = 1 - \omega / 2\pi$, the SIF is provided by:

$$K_I = P\sqrt{\frac{2}{\pi a}}\, F\left(\frac{x}{a}, m\right) \tag{11}$$

where:

$$F\left(\frac{x}{a}, m\right) = \frac{f(m) + g(m)\dfrac{x}{a} + h(m)\left(\dfrac{x}{a}\right)^2}{\sqrt{1-\dfrac{x}{a}}} \tag{12}$$

For functions $f(m)$, $g(m)$ and $h(m)$, see Tada et al. [7].

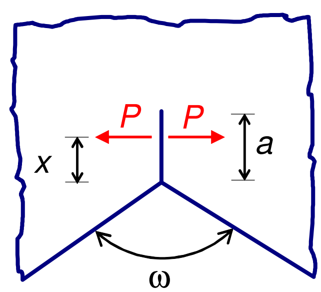

Figure 4. Crack at a V-notch tip loaded by a pair of forces acting on the crack lips

The principle of superposition can now be invoked to determine the SIF of the geometry of fig. 3b. It provides:

$$K_I(a) = \int_0^a \sigma_y(x)\sqrt{\frac{2}{\pi a}}\, F\left(\frac{x}{a}, m\right) \mathrm{d}x \tag{13}$$

Substituting eqn (10) yields:

$$K_I(a) = \psi(\omega) \frac{K_I^*}{(2\pi)^{1-\lambda}} a^{\lambda - 1/2} \tag{14}$$

where the expression of the function ψ is given by solving the integral in eqn (13):

$$\psi(m) = \sqrt{\frac{2}{\pi}} \int_0^1 \frac{F(t,m)}{t^{1-\lambda}} \mathrm{d}t = \sqrt{\frac{2}{\pi}} \left[f(m)\, B(\lambda, \frac{1}{2}) + g(m)\, B(\lambda+1, \frac{1}{2}) + h(m)\, B(\lambda+2, \frac{1}{2}) \right] \tag{15}$$

B is the Beta function. It is worthwhile to observe that eqn (14) encompasses the limit case of an edge crack, $K_I \propto \sqrt{a}$ ($\omega = \pi$, $\lambda = 1$) and a pre-existing crack, $K_I \propto a^0$ = constant ($\omega = 0$, $\lambda = 1/2$). Furthermore note that a formula close to eqn (14) providing the crack driving force for a short crack at a V-notch tip has been recently derived by Leguillon [6]. He exploited a different technique (i.e. the theory of asymptotic matching) without providing an analytic expression as we give here by means of eqn (15).

We are now able to apply the coupled FFM criterion. We need to substitute eqns (10) and (14) into the system (4). Analytical passages provide the values of the finite crack extension Δ_{SE}:

$$\Delta_{SE} = \frac{2}{\lambda\, \psi^2} \left(\frac{K_{Ic}}{\sigma_u} \right)^2 \tag{16}$$

and the critical value of the generalized SIF:

$$K_{Ic}^* = \lambda^{\lambda} \left[\frac{\pi}{\psi^2} \right]^{1-\lambda} \frac{(2K_{Ic})^{2(1-\lambda)}}{(\sigma_u)^{1-2\lambda}} = \xi(\omega) \frac{(2K_{Ic})^{2(1-\lambda)}}{(\sigma_u)^{1-2\lambda}} \tag{17}$$

where function $\xi(\omega)$ is introduced for the sake of simplicity. Observe that both the finite crack extension Δ_{SE} and the generalized SIF K_{Ic}^* are function of the material parameters (i.e. strength and toughness) and of the notch opening angle, through ψ and λ.

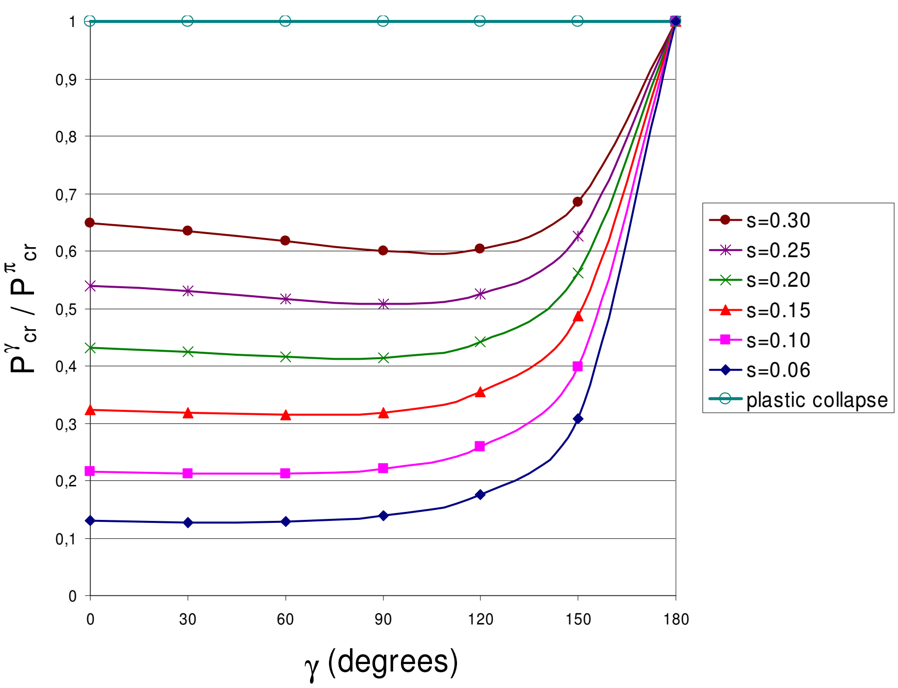

Figure 5. Ratio between critical loads at ω and π vs. the notch opening angle for different values of the brittleness number *s*.

In order to find the critical load for the V-notched specimen we still need the value of the generalized SIF K_I^*, since eqn (17) provides only its critical value. The generalized SIF contains the information about the geometry of the tested specimen.

Assuming a fixed slenderness value l/b (see fig. 3a), dimensional analysis arguments yield the following expression for the generalized SIF:

$$K_I^* = \frac{P^\omega\, l}{t\, b^{1+\lambda}}\; f^*\!\left(\frac{d}{b},\omega\right) \tag{18}$$

t being the specimen thickness. The shape function f^* varies along with the opening angle ω and the relative notch depth d/b. While $f^*\!\left(\frac{d}{b},\pi\right)$ can be easily found in the classic literature, for a generic angle ω the shape function f^* (i.e. the generalized SIF) must be computed *ad hoc*. A clever method for computing K_I^* (Sinclair et al. [17]) is to perform at first a finite element analysis of the TPB specimen; then to evaluate the so-called *H*-integral, i.e. a path independent integral which is a function of the real and of the complementary stress and displacement fields. It can be proven that the value of the *H*-integral coincides with K_I^*. Details of these computations will be given elsewhere.

At the critical condition K_I^* equals K_{Ic}^* in eqn (18). Taking the ratio between eqn (18) evaluated for a generic ω value and the same equation for $\omega = \pi$, yields:

$$\frac{P_{cr}^{\omega}}{P_{cr}^{\pi}} = \frac{f^*\!\left(\frac{d}{b},\pi\right)\quad \xi(\omega)}{f^*\!\left(\frac{d}{b},\omega\right)}\, s^{2(1-\lambda)} \tag{19}$$

where we used the result (17) and introduced the so-called brittleness number $s = K_{Ic} / (\sigma_u \sqrt{b})$ (Carpinteri [9, 18]). For a slenderness value equal to 4.22 and a relative notch depth equal to 0.1, eqn (19) is plotted vs. the notch opening angle in fig. 5 for different values of the brittleness number s. Note that the curves are not monotonically increasing, since they all show a minimum, which is more pronounced for high s numbers. It takes place at low opening angles for small s (brittle materials and/or large sizes) and at large angles for large s (ductile materials and/or small sizes). The result is particularly valuable since the minimum has been often detected experimentally (see, e.g., Carpinteri [13] and Seweryn [15]). Furthermore, observe that, if for small values of s the shape of the curve strictly resembles the variation of the stress singularity with the opening angle, for relatively large values of s this is no longer true.

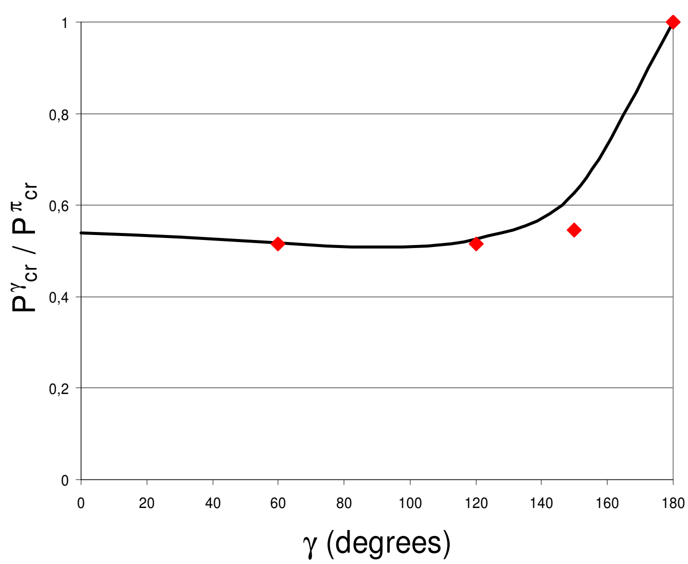

Figure 6. Ratio between critical loads at ω and π vs. the notch opening angle: theoretical result (s = 0.25, black line) and comparison with experimental data on polystyrene (red dots).

In order to have a preliminary check of the soundness of the present approach to V-notched structures, TPB tests were carried out at the Laboratory of the Department of Mechanical and Manufactural Engineering of the Trinity College of Dublin. The material used was Polystyrene. The geometrical dimensions were the following: length l = 76 mm, height h = 18 mm and thickness t = 3.7. The relative notch depth was 1/10 and the angle ω was set equal to 60°, 120°, 150° respectively. Also un-notched specimens were tested. For each geometry, five specimens were tested and the critical load recorded.

The strength value (70.4 MPa) was determined by using the result of the un-notched specimens; on the other hand, the toughness value (2.37 MPa $\sqrt{m}$) was obtained by fitting the 60° failure load value, since no cracked (ω = 0°) specimens were tested. These values yield a brittleness number s equal to 0.25. In fig. 6 the theoretical results are compared with the experimental data. As it can be seen, the fit is rather satisfactory. Particularly, the present criterion is able to explain the same failure load shown by the specimens with angle equal to 60° and 120°. This would not have been the case if the material were, for instance, PMMA. In that case s is equal to 0.06 since the material is much more brittle and the corresponding curve is the lowest one in fig. 6: then a monotonically increasing trend should have been expected.

Conclusions

A coupled fracture criterion has been addressed within the finite fracture mechanics framework. More in detail, we removed the hypothesis of a constant crack advancement; in this way it is possible to formulate a coupled FFM criterion that is consistent with both the energy and stress requirements.
Then we addressed two typical fracture mechanics problems: (i) the size effect upon strength of TPB cracked and uncracked beams and (ii) the strength variation by changing the opening angle in V-notched TPB specimens. For both the problems a comparison with experimental data was produced, leading to satisfactory fits.
It is worth observing that the present model is able to catch: (i) the concave-convex transition in the bi-logarithmic plot of strength vs. size when passing from cracked to uncracked specimens; (ii) the minimum in the critical load vs. notch angle plot, as usually shown by experiments.

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