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Original
An explicit mechanical interpretation of Eringen non-local elasticity by means of fractional calculus / Cornetti, P., Carpinteri, A., Sapora, A.G., M., D.P., M., Z.. - ELETTRONICO. - (2009). (XIX congresso AIMETA Ancona 14-17 settembre 2009).

Availability:
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07 October 2026

An explicit mechanical interpretation of Eringen non-local elasticity by means of fractional calculus

Pietro Cornetti[1], Alberto Carpinteri[1], Alberto Sapora[1], Mario Di Paola[2], Massimiliano Zingales[2]

[1]*Department of Structural Engineering and Geotechnics, Politecnico di Torino, Italy*
E-mail: *alberto.carpinteri@polito.it, pietro.cornetti@polito.it, alberto.sapora@polito.it*

[2] *Department of Structural Engineering and Geotechnics, University of Palermo, Italy*
E-mail: *dipaola@diseg.unipa.it, zingales@diseg.unipa.it*

Keywords: Non-local elasticity, fractional calculus, point-spring models.

SUMMARY. If the attenuation function of strain is expressed as a power law, the formalism of fractional calculus may be used to handle Eringen non-local elastic model. Aim of the present paper is to provide a mechanical interpretation to this non-local fractional elastic model by showing that it is equivalent to a discrete, point-spring model. A one-dimensional geometry is considered; static, kinematic and constitutive equations as well as the proper boundary conditions are derived and discussed.

1 INTRODUCTION

While classical calculus treats integrals and derivatives of integer order, fractional calculus is the branch of mathematics dealing with the generalization of integrals and derivatives to any order. Note that, although the term fractional is, by tradition, universally accepted to define the subject, the order of differentiation may be given by any real value, i.e. not only a rational number.

The application of fractional calculus to applied sciences seems to be very attractive since, by varying the order of derivation, it is possible to describe the transition between completely different phenomena. In the last decades, many fractional differential equations have been proposed, solved and used to interpret experimental data that cannot be caught by usual differential models. They include relaxation equations, wave equations, diffusion equations [1,2].

For what concerns solid mechanics, most of the applications refer to rheological models, where the fractional derivative is taken with respect to the time variable (e.g. [3]). More recently, two research directions have been set that make use of fractional derivative with respect to the space variable. The former one [4] explores the connection between fractal sets and fractional calculus and applies the (local) fractional calculus formalism to address the problem of deformation and damage in solid mechanics. Fractal patterns often arise in heterogeneous materials and developing mathematical models able to catch the fractality of such phenomena is a matter of primary concern. The strength of such approach are the non-integer physical dimensions provided by the fractal geometry and fractional operators. This property has proven to be very effective in the description of the size-scale effects in solid mechanics. For a review, see [5].

The latter research direction aims to model non-local continua, i.e. solids characterized by non-local interactions [6,7]. The novelty is that internal forces are described by fractional derivatives [8]. One of the most remarkable achievements of this approach is that, by exploiting the Marchaud definition of fractional derivative, the fractional operators have a clear mechanical interpretation, i.e. springs connecting non-adjacent points of the body. The related stiffness decays along with the

distance among the material points. However, since only the integral part of the Marchaud derivative is retained in the equilibrium equation, the model developed by Di Paola and Zingales [8] does not coincide with the corresponding Eringen non-local fractional elastic model, whose mechanical interpretation is the subject of the present paper.

2 FRACTIONAL INTEGRALS AND DERIVATIVES

There are various definitions of fractional differintegral operators, not necessarily equivalent to each other. A complete list of these definitions can be found in the fractional calculus treatises, e.g. [9]. These definitions have different origins. The most frequently used definition of fractional integral of order β ($\beta \in \Re^+$) is due to Riemann-Liouville and is a straightforward generalization to non-integer values of Cauchy formula for repeated integrations:

$$I_{a+}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)}{(x-t)^{1-\beta}} \mathrm{d}t \tag{1}$$

From (1) it seems reasonable to define the fractional derivative of order β as the (integer) derivative of order n ($n \in \mathrm{N}$ and $n-1 < \beta < n$) of the fractional integral of order $(n-\beta)$. If $0 < \beta < 1$, we obtain:

$$D_{a+}^{\beta} f(x) = D[I_{a+}^{1-\beta} f(x)] = \frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}x} \int_a^x \frac{f(t)}{(x-t)^{\beta}} \mathrm{d}t \tag{2}$$

Eqn(2) represents the Riemann-Liouville definition of fractional derivative. However, it is also possible to define the fractional derivative as the fractional integral of order $(1-\beta)$ of the first derivative. In such a case we obtain the Caputo definition of fractional derivative, ${}_cD_{a+}^{\beta}$:

$${}_c D_{a+}^{\beta} f(x) = I_{a+}^{1-\beta}[D\, f(x)] = \frac{1}{\Gamma(1-\beta)} \int_a^x \frac{f'(t)}{(x-t)^{\beta}} \mathrm{d}t \tag{3}$$

which is valid for $0 < \beta < 1$. It is worth observing that the Riemann-Liouville derivative of a constant is not zero, whereas it is null the corresponding Caputo derivative. Since Caputo definition generalizes this well-known property of the derivatives of integer orders, Caputo fractional derivative is more useful for practical applications. The two definitions are related by the following identity:

$${}_c D_{a+}^{\beta} f(x) = D_{a+}^{\beta}[f(x) - f(a)] \tag{4}$$

By applying the formula of integration by parts to eqn(3), it is possible to give an alternative form to the Riemann-Liouville fractional derivative [9]:

$$D_{a+}^{\beta} f(x) = \frac{f(x)}{\Gamma(1-\beta)(x-a)^{\beta}} + \frac{\beta}{\Gamma(1-\beta)} \int_a^x \frac{f(x) - f(t)}{(x-t)^{1+\beta}} \mathrm{d}t \tag{5}$$

Eqn(5) is the Marchaud definition of fractional derivative ($0 < \beta < 1$). Since it coincides with the Riemann-Liouville definition (2) for a wide class of functions, we will indicate them with the same symbol.

Eqns(1-5) represent the so-called left (or forward) fractional integrals and derivatives. Analogously, it is possible to define the right (or backward) operators as:

$$I^{\beta}_{b-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \frac{f(t)}{(t-x)^{1-\beta}} \mathrm{d}t \tag{6}$$

$$D^{\beta}_{b-} f(x) = -D[I^{1-\beta}_{b-} f(x)] = -\frac{1}{\Gamma(1-\beta)} \frac{\mathrm{d}}{\mathrm{d}x} \int_x^b \frac{f(t)}{(t-x)^{\beta}} \mathrm{d}t \tag{7}$$

$${}_{c}D^{\beta}_{b-} f(x) = -I^{1-\beta}_{b-}[D\ f(x)] = -\frac{1}{\Gamma(1-\beta)} \int_x^b \frac{f'(t)}{(t-x)^{\beta}} \mathrm{d}t \tag{8}$$

$${}_{c}D^{\beta}_{b-} f(x) = D^{\beta}_{b-}[f(x) - f(b)] \tag{9}$$

$$D^{\beta}_{b-} f(x) = \frac{f(x)}{\Gamma(1-\beta)(b-x)^{\beta}} + \frac{\beta}{\Gamma(1-\beta)} \int_x^b \frac{f(x) - f(t)}{(t-x)^{1+\beta}} \mathrm{d}t \tag{10}$$

Summing up the Marchaud definitions (5) and (10) yields:

$$D^{\beta}_{a+} f(x) + D^{\beta}_{b-} f(x) = \frac{1}{\Gamma(1-\beta)} \left[\frac{f(x)}{(x-a)^{\beta}} + \frac{f(x)}{(b-x)^{\beta}} + \beta \int_a^b \frac{f(x) - f(t)}{|x-t|^{1+\beta}} \mathrm{d}t \right] \tag{11}$$

It is evident that the Marchaud definitions (5) and (10) hold only for $0 < \beta < 1$ (otherwise the integrals at the right-hand side diverge). However, it is possible to prove that eqn(11) holds true in the whole interval $0 < \beta < 2$. This non-trivial result (details will be given elsewhere) is of fundamental importance, since it will allow us to provide a mechanical interpretation to the Eringen non-local fractional model in the next section. For fractional calculus operators analogous to (11), but defined on infinite domains, see, e.g., [10].

3 ERINGEN NON-LOCAL FRACTIONAL MODEL

According to Eringen non-local elasticity, the stress at a given point depends on the strain in a neighborhood of that point by means of a convolution integral. This dependence is described by a proper attenuation function g, which decays along with the distance. In the case of a one-dimensional domain (i.e. a bar):

$$\sigma(x) = E \left[\varepsilon(x) + c_{\alpha} \int_a^b \varepsilon(t)\ g(x-t) \mathrm{d}t \right] \tag{12}$$

where σ is the stress, x the longitudinal coordinate, $x = a$ and $x = b$ the bar extreme coordinates, E the Young's modulus, ε the strain defined as the derivative of the longitudinal displacement u, and

c_α a material constant. The bar length is l ($l = b-a$). Note that the Eringen model [6] is sometime referred to as strong (or integral) non-locality, to distinguish it from the weak (or gradient) non-local elastic model, where the stress depends on the strain and its derivatives [7].

Let us now assume the following form for the attenuation function g:

$$g(\xi) = \frac{1}{\Gamma(1-\alpha)\,|\xi|^{\alpha}} \tag{13}$$

with $0 < \alpha < 1$.With the choice of eqn(13), the constitutive relationship becomes:

$$\sigma(x) = E\left[\varepsilon(x) + \frac{c_\alpha}{\Gamma(1-\alpha)}\int_a^b \frac{\varepsilon(t)}{|x-t|^{\alpha}}\,\mathrm{d}t\right] \tag{14}$$

By means of eqns(1) and (6), the formalism of fractional calculus can be introduced:

$$\sigma(x) = E\left[\varepsilon(x) + c_\alpha\left(I_{a+}^{1-\alpha}\varepsilon + I_{b-}^{1-\alpha}\varepsilon\right)\right] \tag{15}$$

Note that c_α has anomalous physical dimensions $[\mathrm{L}]^{\alpha-1}$. Before proceeding, it is worth highlighting the extreme cases. For $\alpha = 1$, function g acts as the Dirac function and the non-locality disappears; the additive term has the same local nature of the classical one ruled by Young's modulus:

$$\sigma(x) = E(1+2c_\alpha)\,\varepsilon(x) \tag{16}$$

For $\alpha = 0$, the fractional integrals in eqn(15) revert to their classical counterparts. Therefore:

$$\sigma(x) = E\left\{\varepsilon(x) + c_\alpha\left[u(b) - u(a)\right]\right\} \tag{17}$$

that is, the non-local bar acts as a classical elastic bar with a spring of stiffness EAc_α connecting the bar extremes (A being the bar cross section area).

Since $\varepsilon = \mathrm{d}u/\mathrm{d}x$, it is easy to recognize in eqn(15) the presence of the Caputo fractional derivatives of the displacement. Therefore, by exploiting eqns(3) and (8), the dependence of the stress upon the displacement becomes:

$$\sigma(x) = E\left[\frac{\mathrm{d}u}{\mathrm{d}x} + c_\alpha\left({}_{\mathrm{c}}D_{a+}^{\alpha}u - {}_{\mathrm{c}}D_{b-}^{\alpha}u\right)\right] \tag{18}$$

In order to get the equilibrium equation in terms of the displacement function $u(x)$, we simply need to substitute eqn(18) into the static equation $\mathrm{d}\sigma/\mathrm{d}x + f(x) = 0$, where $f(x)$ is the longitudinal force per unit volume. By means of eqns(4) and (9) and some more analytical manipulations, we get:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + c_\alpha\left\{D_{a+}^{1+\alpha}\left[u(x)-u(a)\right] + D_{b-}^{1+\alpha}\left[u(x)-u(b)\right]\right\} = -\frac{f(x)}{E} \tag{19}$$

Eqn(19) is a fractional differential equation. Note that, while the left fractional derivative coincides always with its integer order counterpart when the order of derivation is an integer number, the right fractional derivative coincides with the corresponding integer derivative only when the order of derivation is even; when the order of derivation is an odd number, it is equal to its opposite. Therefore, the term in curly brackets is equal to 2 $u''(x)$ when $\alpha = 1$, and vanishes when $\alpha = 0$.

4 EQUIVALENT POINT-SPRING MODEL

A useful interpretation of the governing equation (19) for the non-local elastic bar is sought. To this aim, it is convenient to express the sum of the Riemann-Liouville fractional derivatives in the Marchuad-like form (11) [8]. Note that this is possible since we proved that eqn(11) holds true also for orders of derivation between one and two, which is exactly the case in eqn(19) (while the order of fractional derivation was less than unity in [8]) . Hence, by exploiting the additive property of fractional derivatives and by letting $\beta = 1+\alpha$ in eqn(11), we get:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - c_\alpha \frac{\alpha}{\Gamma(1-\alpha)}\left[\frac{u(x)-u(a)}{(x-a)^{1+\alpha}} + \frac{u(x)-u(b)}{(b-x)^{1+\alpha}} + (1+\alpha)\int_a^b \frac{u(x)-u(t)}{|x-t|^{2+\alpha}}\mathrm{d}t\right] = -\frac{f(x)}{E} \tag{20}$$

where the gamma function property $\Gamma(1-\alpha) = -\alpha\,\Gamma(-\alpha)$ has been used. In this form it is evident that the first term at the left-hand side rules the local interactions, whereas the terms in the square brackets rule the non-local interactions by means of linear elastic springs and can be seen as an extra-force per unit of volume acting at the point of abscissa x. More in detail, the first two terms in the brackets refer to long-range interactions between the inner ($a < x < b$) and outer ($x = a$, $x = b$) points, whereas the integral term takes into account the interaction between two inner generic points. To make the concept even clearer, it is useful to write eqn(20) in discrete form. To this purpose, let us introduce a partition of the interval $[a, b]$ on the x axis made of $n-1$ ($n \in N$) intervals of length $\Delta x = l/(n-1)$. The generic point of the partition has the abscissa x_i, with $i = 1, \ldots, n$ and $x_1 = a$, $x_n = b$. Hence, for the inner points of the domain ($i = 2, \ldots, n-1$), the discrete form of eqn(20) reads:

$$\frac{-u_{i+1}+2u_i-u_{i-1}}{(\Delta x)^2} + \frac{c_\alpha \alpha}{\Gamma(1-\alpha)}\left[\frac{u_i-u_1}{(x_i-x_1)^{1+\alpha}} + \frac{u_i-u_n}{(x_n-x_i)^{1+\alpha}}\right] + \frac{c_\alpha \alpha(1+\alpha)}{\Gamma(1-\alpha)}\sum_{\substack{j=1 \\ j\neq i}}^{j=n} \frac{u_i-u_j}{|x_i-x_j|^{2+\alpha}}\Delta x = \frac{f_i}{E} \tag{21}$$

where $u_i \equiv u(x_i)$ and $f_i \equiv f(x_i)$. Multiplying both the sides of eqn(21) by $EA\Delta x$, eqn(21) may be re-written as:

$$k^l_{i,i+1}(u_i-u_{i+1}) + k^l_{i,i-1}(u_i-u_{i-1}) + k^{vs}_{i1}(u_i-u_1) + k^{vs}_{in}(u_i-u_n) + \sum_{\substack{j=1 \\ j\neq i}}^{j=n} k^{vv}_{ij}(u_i-u_j) = f_i A \Delta x \tag{22}$$

It is evident how the non-local fractional model is equivalent to a point-spring model where three kinds of springs appear: the local springs, ruling the local interactions, whose stiffness is k^l; the springs connecting the inner material points with the bar edges, ruling the volume-surface long-

range interactions, with stiffness k^{vs}; the springs connecting the inner material points each other, describing the non-local interactions between non-adjacent volumes, whose stiffness is k^{vv}. Provided that the indexes are never equal one to the other, the following expressions for the spring stiffnesses holds:

$$k^{l}_{i,i+1} = k^{l}_{i+1,i} = \frac{EA}{\Delta x} = k^{l} \tag{23}$$

$$k^{vs}_{i1} = k^{vs}_{1i} = EAc_{\alpha}\frac{\alpha}{\Gamma(1-\alpha)}\frac{\Delta x}{(x_i - x_1)^{1+\alpha}} \tag{24}$$

$$k^{vs}_{in} = k^{vs}_{ni} = EAc_{\alpha}\frac{\alpha}{\Gamma(1-\alpha)}\frac{\Delta x}{(x_n - x_i)^{1+\alpha}} \tag{25}$$

$$k^{vv}_{ij} = k^{vv}_{ji} = EAc_{\alpha}\frac{\alpha(1+\alpha)}{\Gamma(1-\alpha)}\frac{(\Delta x)^2}{\left|x_i - x_j\right|^{2+\alpha}} \tag{26}$$

The equivalent point-spring model is drawn in fig.1. Each point is connected to the adjacent points by two local springs, to the bar extremes by two volume-surface non-local springs and to all the other material points by the volume-volume non-local springs. Turning the attention to the whole bar, the number of the local springs is $n-1$, the number of the volume-surface springs is $2n-3$, the number of the volume-volume springs is $n(n-1)/2$.

5 PRINCIPLE OF VIRTUAL WORK AND BOUNDARY CONDITIONS

As the static and kinematic equations are the classical ones (i.e. only the constitutive equation is different), the Principle of Virtual Work reads:

$$\int_a^b f\ u\ dx + [\sigma\ u]_a^b = \int_a^b \sigma\ \varepsilon\ dx \tag{27}$$

where σ and f represent a statically equilibrated system of stresses and distributed forces, while u and ε are a kinematically admissible system of displacements and strains. The second term at the left-hand side represents the work of external forces at the bar extremes and it rules also the boundary conditions. They can be either kinematic or static:

$$u(a) = u_a \qquad \text{or} \qquad \sigma(a) = -\frac{F_a}{A}, \qquad \text{for } x = a \tag{28a}$$

$$u(b) = u_b \qquad \text{or} \qquad \sigma(b) = +\frac{F_b}{A}, \qquad \text{for } x = b \tag{28b}$$

F_a and F_b being respectively the forces acting at the left and right bar edges and positive if directed along x. To solve the governing eqn(20), the static boundary conditions must be given in terms of the displacement function by means of eqn(18). Since the left fractional Caputo derivative (eqn(3)) is zero in $x = a$ and the right fractional Caputo derivative (eqn(8)) is zero in $x = b$, the static boundary conditions reads, respectively:

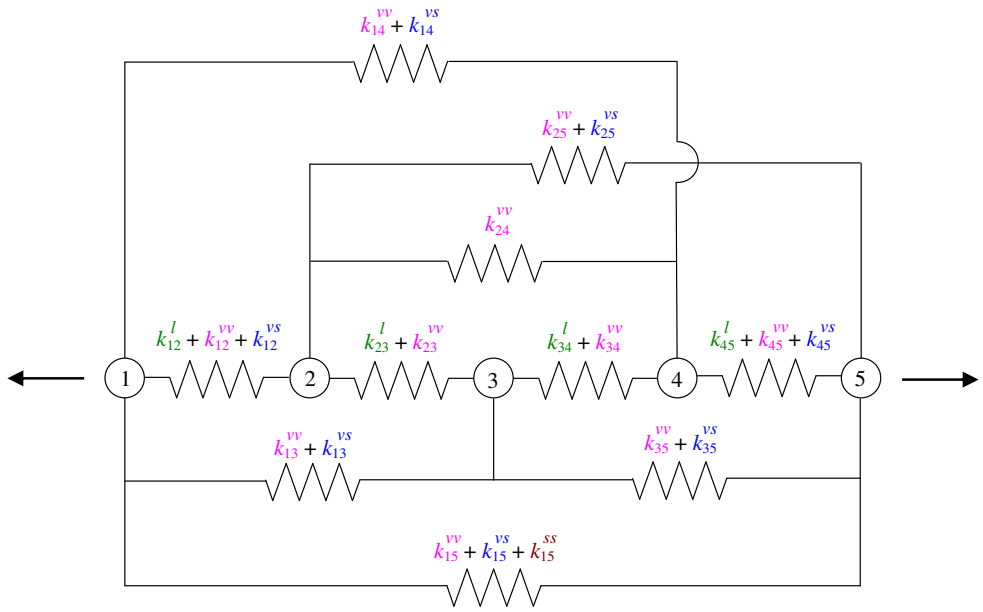

Figure 1: Pont-spring model equivalent to the non-local fractional elastic bar.

$$\sigma(a) = E\left[\frac{\mathrm{d}u}{\mathrm{d}x} - c_\alpha \left({}_cD_{b-}^{\alpha} u\right)\right]_{x=a} = -\frac{F_a}{A} \tag{29a}$$

$$\sigma(b) = E\left[\frac{\mathrm{d}u}{\mathrm{d}x} + c_\alpha \left({}_cD_{a+}^{\alpha} u\right)\right]_{x=b} = +\frac{F_b}{A} \tag{29b}$$

In other words, the static boundary conditions are expressed by combinations of the first derivative plus the right fractional derivative evaluated in the left extreme and of the first derivative plus the left fractional derivative evaluated in the right extreme. Of course, they are integral-type boundary conditions. This is evident by substituting, for instance, eqns(4) and (5) into eqn(29b), which now reads:

$$\sigma(b) = E\left\{\varepsilon(b) + \frac{c_\alpha}{\Gamma(1-\alpha)}\left[\frac{u(b)-u(a)}{(b-a)^\alpha} + \alpha\int_a^b \frac{u(b)-u(x)}{(b-x)^{1+\alpha}}\mathrm{d}x\right]\right\} = +\frac{F_b}{A} \tag{30}$$

Eqn(30) was derived analytically. We now wish to provide a physical meaning to this "strange" boundary condition. First of all, it should be observed that, while at the inner points of the bar the effect of the long-range (both volume-volume and volume-surface) interactions is equivalent to a force per unit volume, at the bar extremes the sum of the volume-surface interactions acts as a concentrated force. It means that, at $x = b$, the force F_b is partly carried by the local stress:

$$\sigma^{l}(b) = E\varepsilon(b) \tag{31}$$

and partly by the volume-surface interactions, whose sum reads, by recalling eqn(25):

$$\sigma^{vs}(b) = E\, c_\alpha \frac{\alpha}{\Gamma(1-\alpha)} \int_a^b \frac{u(b)-u(x)}{(b-x)^{1+\alpha}} \mathrm{d}x \tag{32}$$

Eqns(31) and (32) account for the first and third term in eqn(30). However the second term is still unjustified. In order to fill this discrepancy, we have to introduce a fourth set of springs, namely a unique spring, connecting the two bar extremes with stiffness:

$$k_{1n}^{ss} = k_{n1}^{ss} = \frac{EAc_\alpha}{\Gamma(1-\alpha)} \frac{1}{(x_n - x_1)^\alpha} \tag{33}$$

In such a way, a third stress contribution appears in $x = b$:

$$\sigma^{ss}(b) = \frac{E\, c_\alpha}{\Gamma(1-\alpha)} \frac{u(b)-u(a)}{(b-a)^\alpha} \tag{34}$$

and the boundary condition (30) is fully recovered physically: the force F_b is equilibrated by the sum of the three stress contributions (31,32,34). The same occurs at the left edge of the bar.

The subscript "ss" for the stiffness (33) is used since the spring connecting the bar edges can be seen as modeling the interactions between material points lying on the surface, which, in the simple one-dimensional model under examination, reduce to the two points $x = a,b$. The spring has been drawn in fig.1. Note that the presence of such a spring was implicitly embedded in the constitutive equation (15). However, since eqn(34) represents a constant contribution throughout the bar length, its presence was lost by derivation when inserting the constitutive relationship into the differential equilibrium equation, i.e. when passing from eqn(18) to eqn(19).

To summarize, the constitutive fractional relationship (15) for $0 < \alpha < 1$ is equivalent to a point-spring model with four sets of springs, one local (eqn(23)) and three non-local (24-26, 33). Note that their stiffnesses all decay with the distance, although the decaying velocity differs from one kind to the other. In the limit case $\alpha = 0$, the volume-volume and the volume-surface spring interactions ruled by eqns(24-26) vanish, and only the contribution (33) remains (together with the local springs): the non-local model corresponds to a classical (local) elastic bar in parallel with a spring of stiffness EAc_α. The related constitutive equation is correspondingly represented by eqn(17), whereas the governing equation (19) coincides with the classical $u'' = -f / E$. On the other hand, in the limit case $\alpha = 1$, since $\Gamma(0) = \infty$, the surface-surface (eqn(33)) and the volume-surface (eqns(24-25)) contributions disappear. For what concerns the interactions between inner material points (eqn(26)), only the interactions between adjacent material points are retained (the Gamma function tends to infinity, but the integral in eqn(20) diverges). Correspondingly, the additive term in eqn(15) has the same form as the classical (local) one, the model representing a bar with a stiffened (by a factor of $(1+2c_\alpha)$) Young's modulus (see eqn(16)), while the governing equation (19) becomes $u'' = -f / [E\,(1+2c_\alpha)]$.

6 NUMERICAL ANALYSIS

The equivalent discrete point-spring model allows one to solve directly the proposed one-dimensional non-local fractional equation. Let us introduce the following four stiffness matrices:

$$\mathbf{K}^{l}=\begin{pmatrix} k^{l} & -k^{l} & 0 & ... & 0 \\ -k^{l} & 2k^{l} & -k^{l} & ... & 0 \\ 0 & -k^{l} & 2k^{l} & ... & ... \\ ... & ... & ... & ... & -k^{l} \\ 0 & 0 & ... & -k^{l} & k^{l} \end{pmatrix} \qquad \mathbf{K}^{vv}=\begin{pmatrix} k_{11}^{vv} & -k_{12}^{vv} & -k_{13}^{vv} & ... & -k_{1n}^{vv} \\ -k_{12}^{vv} & k_{22}^{vv} & -k_{23}^{vv} & ... & -k_{2n}^{vv} \\ -k_{13}^{vv} & -k_{23}^{vv} & k_{33}^{vv} & ... & ... \\ ... & ... & ... & ... & ... \\ -k_{1n}^{vv} & -k_{2n}^{vv} & ... & ... & k_{nn}^{vv} \end{pmatrix} \tag{35}$$

$$\mathbf{K}^{vs}=\begin{pmatrix} k_{11}^{vs} & -k_{12}^{vs} & -k_{13}^{vs} & ... & -k_{1n}^{vs} \\ -k_{12}^{vs} & k_{22}^{vs} & 0 & 0 & -k_{2n}^{vs} \\ -k_{13}^{vs} & 0 & k_{33}^{vs} & 0 & ... \\ ... & 0 & 0 & ... & ... \\ -k_{1n}^{vs} & -k_{2n}^{vs} & ... & ... & k_{nn}^{vs} \end{pmatrix} \qquad \mathbf{K}^{ss}=\begin{pmatrix} k_{11}^{ss} & 0 & ... & 0 & -k_{1n}^{ss} \\ 0 & 0 & ... & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & ... & 0 & 0 \\ -k_{1n}^{ss} & 0 & ... & 0 & k_{nn}^{ss} \end{pmatrix}$$

whose non-diagonal terms are provided by eqns(23-26) and (33). The diagonal terms k_{ii} of each matrix is the opposite of the sum of all the other elements of the same row, i.e.:

$$k_{ii}=\sum_{j=1,\,j\neq i}^{j=n}k_{ij}, \qquad i=1,...,n \tag{36}$$

Note that all the four matrices are symmetrical, with positive elements on the diagonal and negative outside. More in detail, the local matrix $\mathbf{K}^{l}$ is tridiagonal; the non-local matrix $\mathbf{K}^{vv}$ ruling the long-range interactions between inner points is fully populated; the non-local matrix related to the inner-outer interactions $\mathbf{K}^{vs}$ has only border and diagonal elements different from zero; finally, the non-local matrix $\mathbf{K}^{ss}$ describing the interaction between the bar edges is empty except for the four corner elements. By introducing the displacement and the force vectors, $\mathbf{u}$ and $\mathbf{f}$, and the stiffness matrix of the structure as $\mathbf{K} = \mathbf{K}^{l} + \mathbf{K}^{vv} + \mathbf{K}^{vs} + \mathbf{K}^{ss}$, the solution is achieved by solving the following linear system:

$$\mathbf{K}\,\mathbf{u}=\mathbf{f} \tag{37}$$

A numerical example, referred to a clamped-loaded bar, is reported in fig.2.

Eventually, it is worth noting that the results here achieved yield also a straightforward generalization of the non-local model developed in [8] to fractional derivation orders comprised between 1 and 2, while in [8] only values between 0 and 1 were considered in the displacement equation. This finding is believed to be an important step, two being the order of derivation for which the non-local interactions degenerate in their local (classical) counterparts. From a numerical point of view, the solution of the Di Paola and Zingales' model [8] is obtained by retaining only the interaction between inner material points, so that the corresponding stiffness matrix becomes $\mathbf{K} = \mathbf{K}^{l} + \mathbf{K}^{vv}$. The solution is reported in fig.2 for the same values of the parameters used above. As expected, the model being less stiff (some springs are missing), the corresponding strain is approximately 50% higher.

7 CONCLUSIONS

In the present paper the possibility to apply fractional calculus to the study of non-local elastic

solids is investigated. It has been shown that, in the one-dimensional case, the non-local fractional bar is equivalent to a point-spring model with four sets of springs. It is argued that this springs may describe the long-range interactions between volume and surface elements of the solid, that, as well-known, play an important role at the smaller scales. However, further efforts are required to include the fractional approach in the field of (standard) non-local elastic solids [11].

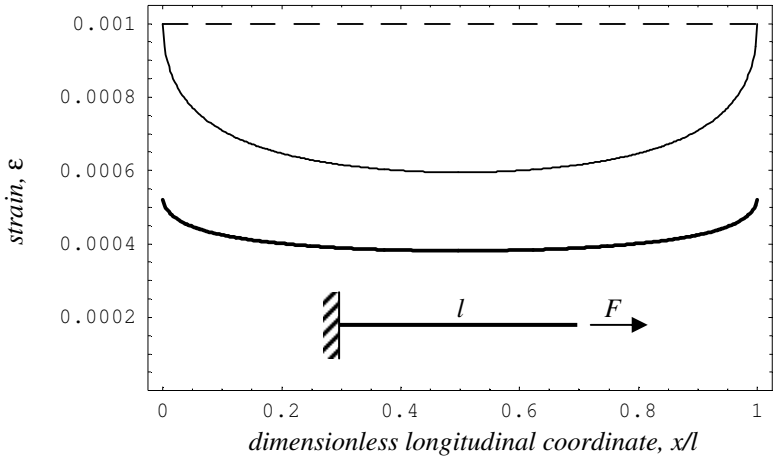

Figure 2: Strain for the non-local fractional elastic bar: clamped-loaded geometry. Thick line: present model. Thin line: Di Paola and Zingales' model [8]. Dashed line: local elastic bar. Values of the parameters: $F/EA = 10^{-3}$, $c_\alpha\, l^{1-\alpha} = 1$, $\alpha = 0.5$.

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