A NOTE ON THE EINSTEIN-HILBERT ACTION AND DIRAC OPERATORS ON $\mathbb{R}^n$

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Abstract. We prove an extension to $\mathbb{R}^n$, endowed with a suitable metric, of the relation between the Einstein-Hilbert action and the Dirac operator which holds on closed spin manifolds. By means of complex powers, we first define the regularised Wodzicki Residue for a class of operators globally defined on $\mathbb{R}^n$. The result is then obtained by using the properties of heat kernels and generalised Laplacians.

Introduction

In 1984 M. Wodzicki [40] introduced a trace on the algebra of classical pseudo-differential operators on a closed manifold M. A similar result was independently obtained by V. Guillemin [18], in order to give a *soft* proof of Weyl formula. Wodzicki residue became then a standard tool in Non-commutative Geometry. In 1988, A. Connes [10] proved that, for operator of order $-\dim(M)$, Dixmier Trace and Wodzicki residue are equivalent. Moreover, he conjectured that Wodzicki Residue could connect Dirac operators and Einstein-Hilbert actions on M. In 1995, D. Kastler [21], W. Kalau and M. Walze [19] proved this conjecture. Namely, let $D\!\!\!/$ be the classical Atiyah-Singer operator defined on a closed spin manifold $M = (M, g)$ of even dimension $n \geq 4$. Then

$$\operatorname{wres}(D\!\!\!/^{-n+2}) = -\frac{(n-2)\, 2^{[\frac{n}{2}]}}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}} \int_M \frac{1}{12} s(x)\, dx, \tag{0.1}$$

where $s(x)$ is the scalar curvature and dx the measure on M induced by the Riemannian metric g (see, e.g., [4] for an overview about Wodzicki Residue and Non-commutative Geometry). T. Ackermann [1] gave a proof of (0.1), using the relationship between heat trace and ζ-function and the properties of the second term in the asymptotic expansion of the heat trace of a generalised Laplacian. Y. Wang [37], [38], [39] suggested an extension of the result to a class of manifolds with boundary.

In the last years, the definition of Wodzicki residue has been extended to other settings: manifolds with boundary [14], manifolds with conical singularities [34], [15], SG-calculus on $\mathbb{R}^n$ [28], anisotropic operators on $\mathbb{R}^n$ [9], and manifolds with cylindrical ends [8]. In [26], in order to study critical metrics on $\mathbb{R}^n$, it has been introduced the regularised trace of operators, which leads to the definition of regularised ζ-function. Wodzicki residue has been used also by R. Ponge [31] to introduce *lower dimensional volumes*.

In this paper, we make use of SG-operators, a class of pseudodifferential operators on $\mathbb{R}^n$ whose symbols $a(x, \xi)$, for fixed $\mu, m \in \mathbb{R}$ and all $x, \xi \in \mathbb{R}^n$, satisfy the estimates

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1 + |x|)^{m-|\alpha|}(1 + |\xi|)^{\mu-|\beta|} \tag{0.2}$$

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for suitable constants $C_{\alpha,\beta} \geq 0$: the set of such symbols is denoted by $SG^{\mu,m}(\mathbb{R}^n)$. If one deals with operators acting on sections of a vector bundle, the symbols are matrices whose entries must satisfy the inequalities (0.2). The corresponding operators can be defined via the usual left-quantisation

$$Au(x) = \mathrm{Op}(a) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{ix\xi}\, a(x,\xi)\, \hat{u}(\xi)\, d\xi,$$

$u \in \mathcal{S}(\mathbb{R}^n)$, and the set $\{\mathrm{Op}(a) \mid a \in SG^{\mu,m}(\mathbb{R}^n)\}$ is denoted by $L^{\mu,m}(\mathbb{R}^n)$. Such operators form a graded algebra, that is $L^{\mu,m}(\mathbb{R}^n) \circ L^{\mu',m'}(\mathbb{R}^n) \subseteq L^{\mu+\mu',m+m'}(\mathbb{R}^n)$, map continuously $\mathcal{S}(\mathbb{R}^n)$ to itself and can be extended as continuous operators from $\mathcal{S}'(\mathbb{R}^n)$ to itself. $L^{-\infty}(\mathbb{R}^n) = \bigcap_{\mu,m\in\mathbb{R}} L^{\mu,m}(\mathbb{R}^n)$, the set of the so-called smoothing operators, coincides with the set of operators with kernel in $\mathcal{S}(\mathbb{R}^{2n})$: these constitute the residual elements of the SG-calculus. A main tool, in particular, is the subclass of SG-classical operators, whose symbols, belonging to $SG^{\mu,m}_{\mathrm{cl}}(\mathbb{R}^n) \subset SG^{\mu,m}(\mathbb{R}^n)$, admit a double asymptotic expansion in terms which are homogeneous w.r.t. the ξ-variable or the x-variable, respectively. More precisely, for $a \in SG^{\mu,m}_{\mathrm{cl}}(\mathbb{R}^n)$ we denote

(i) by $a_{\mu-j,\cdot}$ the terms of order $(\mu-j,m)$, homogeneous w.r.t. the ξ-variable, such that, for a fixed 0-excision function $\chi = \chi(\xi)$, $a \sim \sum_j \chi\, a_{\mu-j,\cdot} \mod SG^{-\infty,m}(\mathbb{R}^n)$;

(ii) by $a_{\cdot,m-k}$ the terms of order $(\mu, m-k)$, homogeneous w.r.t. the x-variable, such that, for a fixed 0-excision function $\omega = \omega(x)$, $a \sim \sum_k \omega\, a_{\cdot,m-k} \mod SG^{\mu,-\infty}(\mathbb{R}^n)$;

(iii) by $a_{\mu-j,m-k}$ the terms of order $(\mu-j,m-k)$, homogeneous w.r.t. the ξ-variable and the x-variable, such that, for the same 0-excision functions, $\chi\, a_{\mu-j,\cdot} \sim \sum_k \chi\,\omega\, a_{\mu-j,m-k} \mod SG^{\mu-j,-\infty}$ and $\omega\, a_{\cdot,m-k,} \sim \sum_j \chi\,\omega\, a_{\mu-j,m-k} \mod SG^{-\infty,m-k}$.

The set $\{\mathrm{Op}(a) \mid a \in SG^{\mu,m}_{\mathrm{cl}}(\mathbb{R}^n)\}$ is denoted by $L^{\mu,m}_{\mathrm{cl}}(\mathbb{R}^n)$: typical examples are differential operators with polynomial coefficients. For details about the SG-calculus and its properties, see, e.g., [7], [11], [13], [26], [29], [32], and the references quoted therein.

Here we prove a *regularised version* of (0.1) on $\mathbb{R}^n$, $n \geq 4$, endowed with a suitable Riemannian metric g and corresponding induced measure dx, see Sections 2 and 3 below. We consider a generalised positive Laplacian $\Delta = \nabla^*\nabla + \mathscr{K}$, defined on a Hermitian vector bundle E with connection ∇, where $\mathscr{K}$ is a symmetric endomorphism field. Moreover, in order to define and make use of its fractional powers, we assume that the spectrum $\sigma(\Delta)$ of Δ lies outside a sector of the complex plane, with at most the exception of the origin. Finally, we obtain

$$\widehat{\mathrm{wres}}(\Delta^{-\frac{n}{2}+1}) = \frac{(n-2)}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}} \fint \left[\frac{\mathrm{Rk}(E)}{6} s(x) - \mathrm{Trace}(\mathscr{K}_x)\right] dx, \tag{0.3}$$

where $\widehat{\mathrm{wres}}(\cdot)$ is a generalised Wodzicki residue, defined for SG-operators of order 0 w.r.t. the x-variable, and $\fint$ denotes the *finite part integral*[1].

The paper is organised as follows. Sections 1 and 2 are devoted to illustrate the definitions of the finite part integral of classical symbols, and of regularised trace and ζ-function for SG-operators, respectively. In Section 3 we give the proof of (0.3) under the assumption that, if the origin belongs to $\sigma(\Delta)$, it is an isolated point of $\sigma(\Delta)$, and, finally, we obtain a regularised version of (0.1).

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[1] A similar result was obtained in [7] by direct evaluation.

1. Finite part integral

The *finite-part integral*, introduced in [25], gives a meaning to the integral of a classical symbol a, and coincides with the usual integral when $a \in L^1(\mathbb{R}^n)$. dS denotes the usual measure on $|x| = 1$, induced by the Euclidean metric on $\mathbb{R}^n$, while, in this Section, dx denotes the standard Lebesgue measure on $\mathbb{R}^n$.

Definition 1.1. Let a be an element of the classical Hörmander symbol class $S^m_{\mathrm{cl}}(\mathbb{R}^n)$, that is,

(i) $a \in C^\infty(\mathbb{R}^n)$ and $\forall x \in \mathbb{R}^n$ $|D^\alpha a(x)| \leq C_\alpha (1 + |x|)^{m-|\alpha|}$;
(ii) a admits an asymptotic expansion in homogeneous terms a_{m-j} of order $m - j$: explicitly, for a fixed 0-excision function ω and all $N \in \mathbb{N}$,

$$a - \sum_{j=0}^{N-1} \omega\, a_{m-j} \in S^{m-N}(\mathbb{R}^n).$$

Then:

- if $m \in \mathbb{Z}$, set

$$\begin{aligned} ⨍ a(x)\,dx &:= \lim_{\rho\to\infty} \left[\int_{|x|\leq\rho} a(x)\,dx - \sum_{j=0}^{m-n} \int_{|x|\leq\rho} a_{m-j}(x)\,dx \right] \\ &= \lim_{\rho\to\infty} \left[\int_{|x|\leq\rho} a(x)\,dx - \sum_{j=0}^{m-n} \frac{\beta_j}{n+m-j} \rho^{n+m-j} - \beta_{n+m} \log\rho \right] \end{aligned}$$

where

$$\beta_j := \int_{|x|=1} a_{m-j}\,dS; \tag{1.4}$$

- if $m \notin \mathbb{Z}$, set

$$⨍ a(x)\,dx := \lim_{\rho\to\infty} \left[\int_{|x|\leq\rho} a(x)\,dx - \sum_{j=0}^{[m]-n-1} \int_{|x|\leq\rho} a_{m-j}(x)\,dx \right]. \tag{1.5}$$

From the above Definition it is clear that if $a \in L^1(\mathbb{R}^n)$ the finite part integral is equivalent to the standard integral. If $m \notin \mathbb{Z}$ the finite part integral coincides with the Kontsevich-Vishik density [22], [23].

Remark 1. *If one considers the radial compactification of $\mathbb{R}^n$ to $\mathbb{S}^n_+$, namely*

$$rc\colon \mathbb{R}^n \to S^n_+ : x = (x_1, \ldots, x_n) \mapsto y = \left[\frac{x_1}{(1+|x|^2)^{\frac12}}, \ldots, \frac{x_n}{(1+|x|^2)^{\frac12}}, \frac{1}{(1+|x|^2)^{\frac12}} \right],$$

and chooses y_{n+1} as boundary defining function on $\mathbb{S}^n_+$, its composition with rc coincides in the interior with $\frac{1}{(1+|x|^2)^{\frac12}}$, $x = rc^{-1}(y) \in \mathbb{R}^n$. Then

$$⨍ a(x)\,dx = {}^R\!\!\int_{\mathbb{S}^n_+} a(rc^{-1}(y))\,dS(y)$$

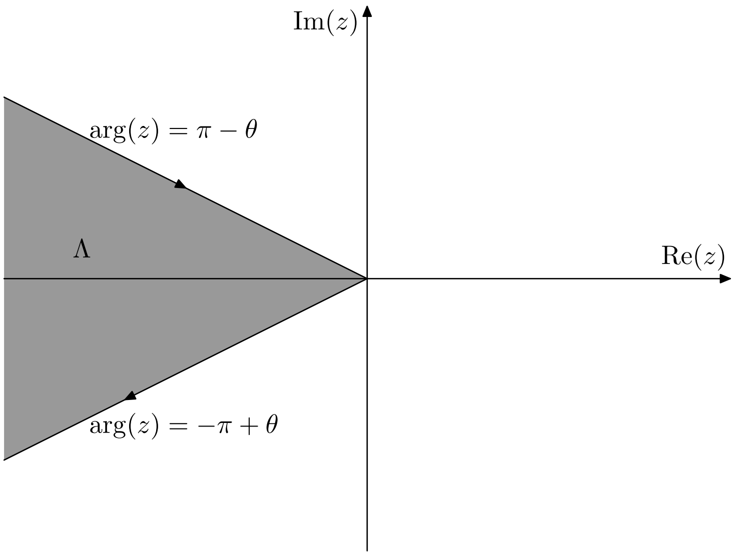
where the right hand side is defined as the term of order ϵ^0 in the asymptotic expansion of

$$\int_{\mathbb{S}^n_+ \cap \{y_{n+1} \geq \epsilon\}} a(rc^{-1}(y))\,dS(y), \quad \epsilon \searrow 0.$$

${}^R\!\!\int_{\mathbb{S}^n_+} f\,dS$ *is called* Renormalised integral, *see* [2] *and the references quoted therein for its precise definition, properties and applications.*

2. Regularised trace and regularised ζ-function

In the sequel we will often make reference to sectors of the complex plane with vertex at the origin, that is, subsets of $\mathbb{C}$ given by $\Lambda = \{z \in \mathbb{C} \mid -\pi + \theta \leq \arg(z) \leq \pi - \theta\}$, $0 < \theta < \pi$, as in the next picture.



The definition of Λ-elliptic operator is the standard one, here given for operators defined through matrix-valued symbols, whose spectrum we denote by $\sigma(a(x,\xi))$:

Definition 2.1. The operator $A \in L^{\mu,0}(\mathbb{R}^n)$ is Λ-elliptic if there exists a constant $R > 0$ such that

$$\sigma(a(x,\xi)) \cap \Lambda = \emptyset \quad \forall |\xi| \geq R, \quad \forall x \in \mathbb{R}^n \tag{2.6}$$

and

$$(a(x,\xi) - \lambda)^{-1} \in SG^{-\mu,0}(\mathbb{R}^n) \quad \forall |\xi| \geq R, \quad \forall x \in \mathbb{R}^n, \quad \forall \lambda \in \Lambda. \tag{2.7}$$

It is well known that, if an operator A is Λ-elliptic, we can build a *weak* parametrix $B(\lambda)$ such that

$$\begin{aligned} &B(\lambda) \circ (A - \lambda I) = \mathrm{Id} + R_1(\lambda), \\ &(A - \lambda I) \circ B(\lambda) = \mathrm{Id} + R_2(\lambda), \quad R_1, R_2 \in L^{-\infty,0}(\mathbb{R}^n). \end{aligned} \tag{2.8}$$

Moreover

$$\begin{aligned} &\lambda\, B(\lambda) \in L^{-\mu,0}(\mathbb{R}^n), \\ &\lambda^2 \left[(A - \lambda I)^{-1} - B(\lambda)\right] \in L^{-\infty,0}(\mathbb{R}^n), \quad \forall \lambda \in \Lambda \setminus \{0\}. \end{aligned} \tag{2.9}$$

From now on, $\mu > 0$ and A is considered as an unbounded operator with dense domain $D(A) = H^{\mu}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. To define the complex powers of a Λ-elliptic operator A, we assume that the following property holds for its spectrum $\sigma(A)$:

(A1) $\quad \sigma(A) \cap \{\Lambda \setminus \{0\}\} = \emptyset$ and the origin is at most an isolated point of $\sigma(A)$.

Proposition 2.1. *Let $A \in L^{\mu,0}(\mathbb{R}^n)$, $\mu > 0$, be a Λ-elliptic operator that satisfies Assumption (A1). The complex power A^z, $\Re\mathrm{e}(z) < 0$, can be defined as*

$$A^z := \frac{1}{2\pi i} \int_{\partial^+ \Lambda_\epsilon} \lambda^z (A - \lambda I)^{-1} d\lambda. \tag{2.10}$$

where $\Lambda_\epsilon = \Lambda \cup \{z \in \mathbb{C} \mid |z| \le \epsilon\}$, *with* $\epsilon > 0$ *chosen such that* $\sigma(A) \cap \{\Lambda_\epsilon \setminus \{0\}\} = \emptyset$ *and* $\partial^+\Lambda_\epsilon$ *is the (positively oriented) boundary of* Λ_ϵ.

Proof. By the definition of Λ-elliptic operator, we know that $(A - \lambda I)^{-1}$ exists for all $\lambda \in \Lambda \setminus \{0\}$. Moreover, by (2.9) we have that A is sectorial, so the integral (2.10) converges in $\mathcal{L}(L^2(\mathbb{R}^n))$. □

Remark 2. *The definition of* A^z *is then extended to arbitrary* $z \in \mathbb{C}$ *in the standard way, that is* $A^z := A^{z-j} \circ A^j$, *where* $j \in \mathbb{Z}_+$ *is chosen so that* $\Re\mathfrak{e}(z) - j < 0$, *see, e.g.,* [8], [26], [35].

Theorem 2.2. *Let* $A \in L^{\mu,0}(\mathbb{R}^n)$, $\mu > 0$, *be* Λ*-elliptic and satisfy Assumption* (A1). *Then,* $A^z \in L^{\mu z,0}(\mathbb{R}^n)$. *Moreover, if* A *is SG-classical then* A^z *is still SG-classical*[2].

Remark 3. *In order to define the symbol of* A^z, *the resolvent* $(A-\lambda I)^{-1}$ *can be approximated with the weak parametrix* $B(\lambda)$ *defined in* (2.8). *In this way, a symbol for* A^z *can be computed, modulo smoothing operators w.r.t. the* ξ*-variable.* A^z *can then be considered as an element of the algebra* $\mathscr{A}$ *given by*

$$\mathscr{A} := \bigcup_{\mu \in \mathbb{Z}} L^{\mu,0}(\mathbb{R}^n)/L^{-\infty,0}(\mathbb{R}^n). \tag{2.11}$$

The proof of the Theorem 2.2 has been given in [26].

From here on, dx will denote the measure induced on $\mathbb{R}^n$ by a smooth Riemannian metric $g = (g_{jk})$. In order to obtain a result similar to (0.1) we have to impose some condition on g, namely[3]

(A2) $\quad g$ is a matrix-valued SG-classical symbol of order $(0, 0)$.

If $A \in L^{\mu,m}(\mathbb{R}^n)$ is trace class, that is $\mu < -n, m < -n$, we can define its trace

$$TR(A) := \int K_A(x,x)\, dx,$$

where $K_A(x,x)$ is the kernel of A restricted to the diagonal. The concept of regularised trace, valid for classical SG-operators under less restrictive hypotheses on the order, has been introduced in [25], using the finite part integral defined in the previous Section:

Definition 2.2. Let $A \in L^{\mu,m}_{\mathrm{cl}}(\mathbb{R}^n)$ be such that $\mu < -n$. We define the regularised trace of A as

$$\widehat{TR}(A) := \fint K_A(x,x)\, dx. \tag{2.12}$$

Remark 4. *Note that the condition* $\mu < -n$ *implies that* $K_A(x,x)$ *is indeed a function and that the finite part integral* (2.12) *is well defined.*

Now, using the regularised integral, we can give the definition of regularised ζ-function:

Definition 2.3. Let $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, be a Λ-elliptic operator that satisfies (A1); then we define

$$\hat{\zeta}(A,z) := \widehat{TR}(A^{-z}) = \fint K_{A^{-z}}(x,x)\, dx, \quad \Re\mathfrak{e}(z) > \frac{n}{\mu}, \tag{2.13}$$

where $K_{A^{-z}}(x,x)$ is the kernel of the operator A^z.

[2] To define the class of symbols $SG^{z,w}(\mathbb{R}^n)$, $z, w \in \mathbb{C}$, we just have to substitute $\Re\mathfrak{e}(z)$ and $\Re\mathfrak{e}(w)$ in place of m, μ, respectively, in the estimates (0.2).

[3] In the $\flat$-calculus setting, this condition implies that the underlying metric is polyhomogeneous: this is used, for instance, in [3].

It is simple to prove that $\hat{\zeta}(A,z)$ is holomorphic for $\Re\mathfrak{e}(z) > \frac{n}{\mu}$, in view of the fact that the hypotheses imply that the kernel $K_{A^z}(x,x)$ is a function. As in the case treated in [35], we can look for meromorphic extensions of $\hat{\zeta}(A,z)$.

Theorem 2.3. *Let $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, be a SG-operator that admits complex powers. Then the function $\hat{\zeta}(A,z)$ can be extended as a meromorphic function with, at most, poles at the points $z_j = \frac{n-j}{\mu}$, $j \in \mathbb{N}$.*

Following the idea of M. Wodzicki [40], [20], we can now introduce a regularised version of non-commutative residue.

Definition 2.4. Let $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, be a Λ-elliptic operator that satisfies (A1). We define the regularised Wodzicki residue of A as

$$\widehat{\mathrm{wres}}(A) := \mu\, \mathrm{res}_{z=-1}\hat{\zeta}(A,z).$$

In the case $\mu \in \mathbb{N}$, using the explicit expression of the regularised integral and of the residues of $\hat{\zeta}(A,z)$, we get

$$\widehat{\mathrm{wres}}(A) = \frac{1}{(2\pi)^n}\lim_{\rho\to\infty}\left[\int_{|x|\le\rho}\int_{|\xi|=1} a_{-n,\cdot}(x,\xi)\,dS(\xi)dx - \sum_{j=0}^{\mu+n-1}\frac{\beta_j}{n-j}\rho^{n-j} - \beta_{\mu+n}\log\rho\right] \tag{2.14}$$

where

$$\beta_j = \int_{|x|=1}\int_{|\xi|=1} a_{n-j,\cdot}\, dS(\xi)\widetilde{dS}(x),$$

$\widetilde{dS}(x)$ the metric induced by g on $|x| = 1$. The case $\mu \notin \mathbb{Z}$ is not very interesting, since then $\widehat{\mathrm{wres}}(A)$ always vanishes, due to the fact that, in this case, the kernel $K_{A^{-z}}(x,x)$ has no poles at $z = -1$. The residue $\widehat{\mathrm{wres}}(\cdot)$ also vanishes on smoothing operators w.r.t. the ξ-variable, so it is well defined on the algebra $\mathscr{A}$. Incidentally, let us notice that the expression (2.14) is analogous to the functional $\mathrm{res}_\psi(A)$ defined by F. Nicola in [28], by means of holomorphic operator families.

3. A Kastler-Kalau-Walze type Theorem on $\mathbb{R}^n$

First, we restrict to the case of $\mathbb{R}^4$ and consider the classical Atiyah-Singer Dirac operator $\not{D}$ acting on the spinor bundle $\Sigma\mathbb{R}^4$. If the metric on $\mathbb{R}^4$ satisfies Assumption (A2), it is immediate to verify that $\not{D} \in L^{1,0}_{\mathrm{cl}}$. Let $\not{D}^{-2}$ denote a *weak* parametrix of the square of the Dirac operator, that is $\not{D}^2 \circ \not{D}^{-2} = I + R$, $R \in L^{-\infty,0}$. The calculus implies that $\not{D}^{-2} \in L^{-2,0}_{\mathrm{cl}}$. Via direct computation, following the idea of D. Kastler [21], it is possible to compute $a_{-4,\cdot}(x,\xi)$, the term of of order -4 in the asymptotic expansion w.r.t. the ξ-variable of the symbol of $\not{D}^{-2}$. Evaluating the integral on the sphere w.r.t. the ξ variable one gets

$$\int_{|\xi|=1} a_{-4,\cdot}(x,\xi)\,dS(\xi) = -\frac{1}{24\pi^2}s(x).$$

So we have that

$$\widehat{\mathrm{wres}}(\not{D}^{-2}) = -\frac{1}{24\pi^2}⨍ s(x)\,dx. \tag{3.15}$$

The proof of (3.15) is contained in [7]. Let us notice the slight abuse of notation in (3.15), due to the fact that, in general, $\not{D}^{-2}$ does not satisfy Assumption (A1): anyway, we can use (2.14) as a definition of $\widehat{\mathrm{wres}}(\not{D}^{-2})$ in this case.

In order to obtain a generalisation of (3.15) to higher dimensions and to more general operators, the direct approach seems to be rather cumbersome. For this reason, we follow an idea of T. Ackermann [1] and exploit the properties of the asymptotic expansion of the heat kernel of generalised Laplacians.

As explained in the previous Section, if $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, is Λ-elliptic and satisfies Assumption (A1), we can define the complex powers of A and the heat semigroup e^{-tA} as well:

$$e^{-tA} := \frac{i}{2\pi}\int_{\partial^+\Lambda_\epsilon} e^{-t\lambda}(A-\lambda I)^{-1}d\lambda.$$

In [25] it has been proved that e^{-tA} is a SG-operator belonging to $L^{-\infty,0}(\mathbb{R}^n)$, so we can also consider the regularised heat trace $\widehat{TR}(e^{-tA})$. There is a deep link between regularised heat trace and $\hat{\zeta}$-function:

Theorem 3.1. *Let $A \in L^{\mu,0}_{\mathrm{cl}}(\mathbb{R}^n)$, $\mu > 0$, be an operator that admits complex powers. Then, for suitable constants $c_{kl} = c_{kl}(A)$, the following two asymptotic expansions hold:*

$$\Gamma(z)\,\hat{\zeta}(A,z) \sim \sum_{k=0}^{\infty}\sum_{l=0}^{1} c_{kl}\left(z-\frac{n-k}{\mu}\right)^{-l+1} \tag{3.16}$$

$$\widehat{TR}(e^{-tA}) \sim \sum_{k=0}^{\infty}\sum_{l=0}^{1}(-1)^l c_{kl} t^{-\frac{n-k}{\mu}} \log^l t, \quad t \searrow 0. \tag{3.17}$$

Proof. The statement follows by adapting the arguments given in [25] to the present situation. A main role in the proof is played by an abstract theorem by G. Grubb and R. Seeley [17], connecting ζ-functions and heat traces. ◻

Let us now consider a generalised positive Laplacian $\Delta \in L^{-2,0}_{\mathrm{cl}}(E)$, where E is a Hermitian vector bundle on $\mathbb{R}^n$ with connection ∇, that is

$$\Delta = \nabla^*\nabla + \mathscr{K}, \quad \mathscr{K} \in C^\infty(\mathrm{End}(E)) \text{ symmetric endomorphism field.}$$

We require that Δ satisfies Assumption (A1): in this way, we can define $e^{-t\Delta}$ as above. In the case of closed manifolds, it is well known (see, e.g., [4]) that

$$K_{e^{-t\Delta}}(x,x) = k_t(x,x) \sim (4\pi t)^{-\frac{n}{2}}\left[1\cdot \mathrm{Id}_E + \left(\frac{1}{6}s(x)\mathrm{Id}_E - \mathscr{K}_x\right)t + O(t^2)\right], \quad t \searrow 0. \tag{3.18}$$

where $s(x)$ is the scalar curvature of the underlying manifold and the remainder term depends only on the connection and on the endomorphism field. The asymptotic expansion (3.18) also holds in the case of manifolds with cylindrical ends, since the computations are completely analogous and purely local, see [2]. The evaluation of the first term of the asymptotic expansion can be found in [24]: the expression of the second term then follows, using the properties of generalised Laplacians. In view of our hypotheses, the right hand side of (3.18) is a classical SG-symbol: then, we obtain

$$\begin{aligned}&\widehat{TR}(e^{-t\Delta}) \sim \\ &(4\pi t)^{-\frac{n}{2}}\left\{⨍ \mathrm{Rk}(E)\,dx + t ⨍\left[\frac{\mathrm{Rk}(E)}{6}s(x) - \mathrm{Trace}(\mathscr{K}_x)\right]dx + O(t^2)\right\},\ t \searrow 0.\end{aligned} \tag{3.19}$$

Since, trivially, when h is a meromorphic function with a simple pole in z_0, the function $\tilde{h}(z) = h(cz)$, $c \in \mathbb{R}$, is a meromorphic function with a simple pole in $\frac{z_0}{c}$ and

$$\mathrm{res}_{w=\frac{z_0}{c}}\tilde{h} = \frac{1}{c}\mathrm{res}_{z=z_0}h,$$

we also have that

$$\widehat{\mathrm{wres}}(\Delta^{-\frac{n}{2}+1}) = (2-n)\,\mathrm{res}_{z=-1}\hat{\zeta}(\Delta^{-\frac{n}{2}+1}, z) = 2\,\mathrm{res}_{z=\frac{n-2}{2}}\hat{\zeta}(\Delta, z) = 2\,\Gamma\left(\frac{n-2}{2}\right)^{-1} c_{2,0}(\Delta), \tag{3.20}$$

where $c_{2,0}(\Delta)$ is coefficient of the term of order $t^{-\frac{n-2}{2}}$ in the asymptotic expansion (3.17). Finally, by (3.19) and the properties of $\Gamma(z)$,

$$\widehat{\mathrm{wres}}(\Delta^{-\frac{n}{2}+1}) = \frac{n-2}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}} ⨍ \left[\frac{\mathrm{Rk}(E)}{6} s(x) - \mathrm{Trace}(\mathscr{K}_x)\right] dx. \tag{3.21}$$

Remark 5. *Assumption (A1) does not imply that Δ is invertible, since we allow the origin to be an isolated point of $\sigma(\Delta)$. In view of this, the operator $\Delta^{-\frac{n}{2}+1}$ has to be interpreted in the sense of the complex powers defined above.*

If we consider a generalised Laplacian Δ, then its principal homogeneous symbol is $g^{jk}(x)\xi_j\xi_k = |\xi|^2 > 0$, $\xi \neq 0$. Δ turns out to be always Λ-elliptic with respect to a suitable sector of the complex plane, while $\sigma(\Delta)$ can admit the origin as an accumulation point. For example, it is well known that the classical Atiyah-Singer Dirac operator on $\mathbb{R}^n$, endowed with the canonical Euclidean metric, has no point spectrum, but the essential spectrum is the whole real line. In this case Assumption (A1) of course fails to be true[4]. A simple example such that Assumption (A1) is satisfied can be built in the following way. Let us consider a general Dirac operator D, defined on a Clifford bundle E over $\mathbb{R}^n$: D^2 is then a generalised Laplacian and a non-negative operator. If we consider $D^2_\epsilon = D^2 + \epsilon I$, we obtain an invertible generalised Laplacian, that clearly satisfies (A1). If we consider the classical Atyiah-Singer Dirac operator $\not{D}$, formula (3.21) turns to

$$\widehat{\mathrm{wres}}((\not{D}^2_\epsilon)^{-\frac{n}{2}+1}) = \frac{(n-2)2^{[\frac{n}{2}]}}{\Gamma(\frac{n}{2})(4\pi)^{\frac{n}{2}}} \left(-\frac{1}{12} ⨍ s(x)\,dx - \epsilon ⨍ dx\right). \tag{3.22}$$

On the other hand, a natural example of a metric on $\mathbb{R}^n$ which can satisfy Assumption (A2) is an asymptotically flat one. In General Relativity, such an hypothesis on the metric is commonly assumed (e.g., in order to define the ADM-mass). Explicitly, we can consider a metric g such that, for a constant $\alpha > 0$,

$$g_{jk}(x) - \delta_{jk} = O(|x|^{-\alpha}) \text{ outside a compact set } K \subset \mathbb{R}^n.$$

Moreover, restricting ourself to $\mathbb{R}^4$, if $\alpha > 2$ the scalar curvature $s(x)$ is integrable: in this case, (3.15) becomes

$$\widehat{\mathrm{wres}}(\not{D}^{-2}) = -\frac{1}{24\pi^2} \int s(x)\,dx.$$

The method above can be used to treat also the case of manifolds with cylindrical ends, using the contents of [8]: one defines in this setting a regularised Wodzicki Residue and exploits its connection with the zeta function. The asymptotic expansion of the heat kernel as $t \searrow 0$ is locally defined, so, using suitable regularised integrals, see [8], the results can be generalised to those manifolds in this class which admit a spin structure. To keep this exposition at a reasonable length, we omit here the details.

[4]For further properties of the Dirac spectrum on open manifolds, the reader can refer, for instance, to the monograph by N. Ginoux [16].

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