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Discrete kinetic and stochastic game theory for vehicular traffic: Modeling and mathematical problems

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Chapter 1

Introduction

Vehicular traffic is attracting a growing scientific interest because of its connections with other important problems, like, e.g., environmental pollution and cities congestion. Rational planning and management of vehicle fluxes are key topics in modern societies under both economical and social points of view, as the increasing number of projects aimed at monitoring the quality of the road traffic demonstrates. In spite of their importance, however, these issues cannot be effectively handled by simple experimental approaches. On the one hand, observation and data recording may provide useful information on the physics of traffic, highlighting some typical features like, e.g., clustering of the vehicles, the appearance of stop-and-go waves, the phase transition between the regimes of free and congested flow, the trend of the traffic in uniform flow conditions. The books by Kerner [32] and Leutzbach [37] extensively report about traffic phenomena, real traffic data, and their phenomenological interpretation. On the other hand, processing and organization of (usually huge amounts of) experimental measurements hardly allow to catch the real unsteady traffic dynamics, which definitely makes this approach scarcely predictive. Therefore vehicular traffic is not only an engineering matter but also a challenging mathematical problem.

The mathematical modeling of vehicular traffic requires, first of all, the choice of the scale of representation. The relevant literature offers many examples of models at any scale, from the microscopic to the macroscopic through the kinetic one. Each of them implies some technical approximations, and suffers therefore from related drawbacks, either analytical or computational.

The microscopic scale is like a magnifying glass focused on each single vehicle. The dynamics is described by a system of ordinary second order differential equations of the form

$$\ddot{x}_i = a_i[t, \{x_k\}_{k=1}^N, \{\dot{x}_k\}_{k=1}^N], \qquad i = 1, \dots, N, \tag{1.1}$$

where t is time, $x_i = x_i(t)$ the scalar position of the i-th vehicle along the road, $\dot{x}_i = \dot{x}_i(t)$ its velocity, and $N \in \mathbb{N}$ the total number of vehicles. The function a_i describes the acceleration of the i-th vehicle, which in principle might be influenced by the positions and velocities of all other vehicles simultaneously present on the road. As a matter of fact, each vehicle is commonly assumed to be influenced by its heading vehicle only (*follow-the-leader models*), so that the acceleration a_i depends at most on x_i, x_{i+1} and on $\dot{x}_i$, $\dot{x}_{i+1}$. As an example, one may have (see e.g., Aw *et al.* [3])

$$a_i[t, x_i, x_{i+1}, \dot{x}_i, \dot{x}_{i+1}] = C\frac{\dot{x}_{i+1} - \dot{x}_i}{(x_{i+1} - x_i)^{\gamma+1}}$$

for suitable constants $C > 0$, $\gamma \geq 0$. Further references on microscopic modeling of vehicular traffic are Gazis *et al.* [25], Helbing [28, 29], Kerner and Klenov [33], Treiber *et al.* [48, 49, 50], as well as the review paper by Hoogendoorn and Bovy [31], where cellular automaton and particle models are also considered. It is immediately seen that the size of system (1.1) rapidly increases with the number N of vehicles considered, which frequently makes the microscopic approach not competitive for computational purposes. Furthermore, from the analytical point of view it is often difficult to investigate the relevant global features of the system, also in connection with control and optimization problems.

On the other hand, both macroscopic and kinetic scales aim at describing the big picture without looking specifically at each single subject of the system, hence they are computationally more efficient: Few partial differential equations, that can be solved numerically in a feasible time, are normally involved, and the global characteristics of the system are readily accessible. Nevertheless, now the modeling is, in a sense, less accurate than in the microscopic case, due to the *continuum hypothesis* underlying both approaches, clearly not physically satisfied by cars along a road. The number of vehicles should be indeed large enough so that it makes sense to introduce the concepts of macroscopic density and kinetic distribution function, respectively, as pointwise continuous functions of space and, in the latter case, also velocity. Therefore, such a hypothesis must be accepted in the abstract as a technical approximation of the physics of the problem.

In the macroscopic approach, the flow of cars along a road is assimilated to the flow of fluid particles, for which suitable conservation or balance laws can be written. For this reason, macroscopic models are often called in this context *hydrodynamic models*. Cars are not followed individually, the point of view being rather that of the classical continuum mechanics. One looks at the evolution in time and space of some average quantities of interest, such as the mass density $n = n(t, x)$, the mean velocity $u = u(t, x)$, or the flux $q = nu$, referred to an infinitesimal reference volume individuated by a point in the geometrical one-dimensional space. In describing the system, the Eulerian point of view is normally adopted, meaning that the above dependent variables evaluated at (t, x) yield the evolution at time t of the vehicles flowing through the fixed in space position x. In other words, the spatial coordinate is not linked to any reference configuration but rather to the actual geometrical space, hence at different times the values of the state variables computed in the point x refer in general to different continuum particles flowing through x. Of course, models can in principle be restated in Lagrangian form, up to a proper change of variable.

The basic evolution equation translates the principle of conservation of the vehicles:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0. \tag{1.2}$$

According to this equation, the time variation in the amount of cars within any stretch of road comprised between two locations $x_1 < x_2$ is only due to the difference between the incoming flux at x_1 and the outgoing flux at x_2. Integration of Eq. (1.2) over the interval $[x_1, x_2]$ produces indeed

$$\frac{d}{dt}\int_{x_1}^{x_2} n(t,x)\,dx = q(t,x_1) - q(t,x_2),$$

where the integral at the left-hand side defines precisely the mass of cars contained in $[x_1, x_2]$ at time t.

It can be questioned that Eq. (1.2) does not give rise by itself to a self-consistent mathematical model, as it involves simultaneously two variables, n and u. To overcome this difficulty, one possibility is to devise suitable closures which allow to express the velocity u, or equivalently the flux q, as a function of the density n. This way Eq. (1.2) becomes a conservation law for the sole density n:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} f(n) = 0, \tag{1.3}$$

where $q = f(n)$ is the closure relation known in this context as the *fundamental diagram*. Fundamental diagrams are usually obtained by fitting some sets of experimental data measured in homogeneous uniform flow conditions (see e.g., Bonzani and Mussone [15]). Macroscopic models based on Eq. (1.3) are usually called *first order models*. Their most popular prototype is the celebrated Lighthill-Whitham-Richards (LWR) model (Lighthill and Whitham [39], Richards [45], Whitham [53]), in which the flux is taken to be

$$f(n) = n u_{\max}\left(1 - \frac{n}{n_{\max}}\right)$$

for positive constants $u_{\max}$, $n_{\max}$ denoting the maximum possible velocity and density, respectively.

A second possibility consists instead in joining Eq. (1.2) to an evolution equation for the flux q inspired by the momentum balance of a continuum:

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(qu) = A[n, u, Dn, Du], \tag{1.4}$$

where A is some material model for the generalized forces acting on the system and responsible for momentum variations. These forces express the macroscopic outcome of the microscopic interactions among the vehicles, and in most models are assumed to be determined by either the local density n or the local velocity u of the cars, as well as by their respective variations in time and/or space (generically denoted by the differential operator D in Eq. (1.4) above). Using $q = nu$ and taking Eq. (1.2) into account, Eq. (1.4) can be further manipulated to obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = a[n, u, Dn, Du],$$

where $a := A/n$ is a material model for the acceleration of the vehicles to be specified.

Finally, the resulting set of equations is

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial x}(nu) = 0 \\ \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = a[n, u, Dn, Du], \end{cases} \tag{1.5}$$

which is self-consistent in the unknowns n, u once the acceleration a has been conveniently designed. Mathematical models based on the system (1.5) are commonly termed in the literature *second order models*. Among the most popular ones we recall here the Payne-Whitham model (Payne [42], Whitham [53]) and the celebrated Aw-Rascle model [4], which corrects some severe drawbacks of the former (see Daganzo [19]) caused by a too straightforward application to cars of the analogy with fluid particles. For further details on macroscopic first and second order models the interested reader is referred to the main references listed in the above cited articles, as well as to the review paper by Bellomo *et al.* [7] and to the book by Garavello and Piccoli [23].

The kinetic scale constitutes an intermediate representation level between the microscopic and the macroscopic approaches. Indeed, despite the overall description of the system, it allows a microscopic modeling of the interactions among the vehicles, without the need for devising closure relations to get a self-consistent system of equations. As seen above, macroscopic models recover this missing information from experimental observations under steady flow conditions. In the kinetic approach, instead, fundamental diagrams are not assumed *a priori* but are mathematically studied as equilibrium solutions of the model itself, and possibly compared with the available experimental data. In addition, all interesting macroscopic quantities can be readily recovered *a posteriori* from the kinetic results via suitable average procedures, so that both levels of description are finally accessible.

In this thesis we are concerned with the mathematical modeling of vehicular traffic at the kinetic scale. In more detail, starting from the general structures proposed by Arlotti *et al.* [1, 2] and by Bellomo [5], we develop a discrete kinetic framework in which the velocity of the vehicles is not regarded as a continuous variable but can take a finite number of values only. Discrete kinetic models have historically been conceived in connection with the celebrated Boltzmann equation, primarily as mathematical tools to reduce the analytical complexity of the latter (see e.g., Bellomo and Gatignol Eds. [8], Gatignol [24]): The Boltzmann's integro-differential equation is converted into a set of partial differential equations in time and space, which share with the former some good mathematical properties being at the same time easier to deal with. In the present context, however, the discretization of the velocity plays a specific role in modeling the system rather than being simply a mathematical simplification, because it allows to relax the continuum hypothesis for the velocity variable and to include, at least partially, the strongly granular nature of the flow of cars in the kinetic theory of vehicular traffic.

The discrete velocity framework also gives rise to an interesting structure of the interaction terms of the kinetic equations, which are inspired to the stochastic game theory developed by Bertotti and Delitala [13]. Encounters among the vehicles are described in an essentially stochastic way, introducing the probability that a velocity transition occurs after an interaction between a car and another vehicle located in front of it. The set of all these transition probabilities constitutes the so-called *table of games*, a third order tensor which is furthermore assumed to depend on the solution of the problem itself via the macroscopic density of vehicles. This reflects the influence of the global conditions of traffic on the behavior of the vehicles: The probabilities of acceleration-and-overtaking or of braking-and-queuing change dramatically according to the crowding of the road. We remark that stochasticity is an essential ingredient in order to capture the real essence of the interactions among the vehicles. Velocity transitions are described without invoking any classical concept of force, since vehicles do not interact mechanically: They simply see each other and adjust their velocity according to the behavioral rules coded in the table of games. Finally, interactions are binary like in the classical kinetic theory, but not local: An interaction length is introduced, which defines a visibility zone for each vehicle. The final outcome on the speed of a specific vehicle is then determined by a weighted average of the interactions that the latter experiences with all vehicles comprised in its visibility zone.

The thesis is organized in five more chapters that follow this Introduction:

- Chapter 2 contains a detailed review of the main kinetic models of vehicular traffic available in the pertinent literature. The general structure of the equations and the main modeling assumptions are illustrated, then particular models are analyzed and discussed.

- Chapter 3 specifically reports about modeling by methods of the discrete kinetic theory. A general discrete velocity kinetic framework for vehicular traffic with binary nonlocal interactions is derived. Then, following some ideas on the discrete kinetic and stochastic game theory, suitable modeling guidelines are outlined, and finally a particular model is detailed.

- Chapter 4 deals with the spatially homogeneous problem, in which vehicles are supposed to be well mixed and their flow uniform in space. This problem allows to study the trend of the system toward the equilibrium, hence to draw fundamental diagrams and to compare theoretical predictions of the models with available experimental data. Qualitative analysis of the mathematical structures presented in Chapt. 3 is performed in terms of existence and uniqueness of solutions and equilibrium configurations of the system. Numerical results are also provided to compute the fundamental diagrams predicted by the specific model developed in Chapt. 3.

- Chapter 5 deals with the spatially inhomogeneous problem, which describes the evolution of the traffic accounting also for possible inhomogeneities in the spatial distribution of the vehicles. Well-posedness of both the initial value and the periodic initial-boundary value problem is addressed, with special emphasis on the possibility to extend local in time solutions to solutions defined for large times. Numerical simulations of some cases study (formation of a queue, bottleneck, clustering and stop-and-go waves), based on the model developed in Chapt. 3, are presented in order to test the ability of the model to reproduce some typical traffic effects documented in the specialized literature.

- Chapter 6 finally draws some conclusions about modeling and mathematical issues developed in the thesis, and briefly sketches further research perspectives in the field. These include: (i) The extension of the modeling framework to systems of active particles, with the aim of accounting for the presence of the drivers and their influence on the behavior of the vehicles; (ii) Some preliminary ideas toward the discretization of the whole state space, which should allow to include in the theory also the spatial granularity of the flow of vehicles.

Chapter 2

Review of kinetic models of vehicular traffic

2.1 The kinetic approach

The kinetic approach to the modeling of vehicular traffic is based on the choice of an intermediate representation scale between the macroscopic and the microscopic ones, technically called *mesoscopic*. Rather than looking at each single car of the system, like in the microscopic approach, a distribution function, usually denoted by

$$\mathcal{P} = \mathcal{P}(t, \mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(N)}), \tag{2.1}$$

is introduced over the microscopic mechanical states $\mathbf{w}^{(i)} = (x^{(i)}, v^{(i)})$, $i = 1, \dots, N$, of the N vehicles composing the system, $x^{(i)}$ standing for position and $v^{(i)}$ for velocity. In order to describe the unidirectional flow of cars along a stretch of road, these variables are usually supposed to be one-dimensional, with moreover $v^{(i)} \geq 0$ each i. In more detail, if $D_x \subseteq \mathbb{R}$, $D_v \subseteq \mathbb{R}_+$ denote the spatial domain and the velocity domain, respectively, and $D_{\mathbf{w}}^N = (D_x \times D_v)^N$ the state space, the assumption that cars are indistinguishable from one another allows in principle to describe the system by means of the marginal distribution f over the generic state $\mathbf{w} = (x, v)$ only:

$$f(t, x, v) = \int\limits_{D_{\mathbf{w}}^{N-1}} \mathcal{P}(t, x, v, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(N)})\, d\mathbf{w}^{(2)} \dots d\mathbf{w}^{(N)}.$$

The function f is technically termed the *one-particle distribution function*. However, since most of the times in kinetic theories and in kinetic models one deals with f rather than with $\mathcal{P}$, it is customary to rename f simply *distribution function*. The distribution function is such that the quantity

$$f(t, x, v)\, dx\, dv$$

represents a measure of the number of vehicles whose position at time t is comprised between x and $x + dx$, with a velocity comprised between v and $v + dv$. Consequently, under suitable integrability assumptions on f,

$$\mathcal{N}(t) = \int_{D_v}\int_{D_x} f(t,x,v)\,dx\,dv \tag{2.2}$$

gives the total amount of vehicles present in the system at time t. It is worth pointing out that f is in general not a probability density over (x,v), as the integral in Eq. (2.2) may not equal 1 at all times. However, if the total mass of cars is preserved then $\mathcal{N}(t)$ is constant in t, hence it is possible to rescale the distribution function so that it have a unit integral on the state space $D_x \times D_v$ for all times t of existence.

The knowledge of the distribution function enables one to recover the usual macroscopic variables of interest as local averages over the microscopic states of the vehicles. For instance, the number density n of cars at time t in the point x is given by

$$n(t,x) = \int_{D_v} f(t,x,v)\,dv, \tag{2.3}$$

while the flux q and the average velocity u are obtained from the first momentum of f with respect to v as

$$q(t,x) = \int_{D_v} vf(t,x,v)\,dv, \qquad u(t,x) = \frac{q(t,x)}{n(t,x)}. \tag{2.4}$$

Higher order momenta are related to other macroscopic variables, such as the average kinetic energy E and the variance of the velocity Θ:

$$E(t,x) = \frac{1}{2}\int_{D_v} v^2 f(t,x,v)\,dv, \qquad \Theta(t,x) = \frac{1}{n(t,x)}\int_{D_v} (v-u(t,x))^2 f(t,x,v)\,dv.$$

In particular, Θ is easily seen to be proportional to $E - \frac{1}{2}nu^2$, that is the internal energy of the system, therefore it is often linked, at a macroscopic level, to the traffic pressure, or the traffic temperature, responsible for the anticipation terms in the hydrodynamic equations of traffic (see e.g., Klar and Wegener [34] for further details).

Modeling of the system is obtained by stating an evolution equation in time and space for the distribution function f. Unlike the macroscopic approach, such an equation is written considering the influence of the microscopic interactions among the vehicles on the microscopic states, rather than referring to classical concepts of point and continuum mechanics, such as forces and stresses. It is useful to introduce a specific terminology to distinguish in an interaction the vehicle which is likely to change its state from the one which potentially causes such a change. The former is technically called the *candidate vehicle*, while the latter is called the *field vehicle*. Furthermore, a third kind of vehicle is often identified, namely the *test vehicle*, that is an ideal vehicle of the system whose microscopic state is targeted by a hypothetical observer. Although the test vehicle is not actually involved in the interactions, the results of the latter are evaluated in terms of its microscopic state, considering the number of candidate vehicles which get or lose the state of the test vehicle in the unit time.

Practically all kinetic models of vehicular traffic present in the literature describe the interactions by appealing to the following general guidelines:

1. Cars are regarded as points, their dimensions being negligible.
2. Interactions are binary, in the sense that those involving simultaneously more than two vehicles are disregarded.
3. Interactions modify by themselves only the velocity of the vehicles, not their positions. More specifically, in an interaction the sole velocity of the candidate vehicle may vary, that of the field vehicle remaining instead unchanged.
4. Vehicles are anisotropic particles which react mainly to frontal than to rear stimuli (see Daganzo [19]), therefore candidate vehicles only interact with field vehicles located ahead of them.
5. There exists a *probability of passing* P such that when a candidate vehicle encounters a slower field vehicle it may overtake it instantaneously with probability P without modifying its own velocity.
6. Interactions are conservative, in the sense that they preserve the total number of vehicles of the system.

The evolution equation relates the time variation of the number of vehicles in an arbitrarily fixed subset of the state space $D_x \times D_v$ to the state transitions caused by the interactions of the vehicles with each other. Therefore, in the absence of external actions on the system it reads

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = J[f], \tag{2.5}$$

where $J[f] = J[f](t, x, v)$ is an operator acting on the distribution function f, charged to describe the interactions and their effects on the states of the vehicles, and (x, v) is the microscopic state of the test vehicle. Following the classical terminology coming from the collisional kinetic theory of the gas dynamics (see e.g., Villani [51]), J is frequently termed the *collisional operator*, though in the present context this is a slight abuse of speech because interactions among vehicles are not like collisions among mechanical particles.

In view of the assumption 6 above, the operator J is required to satisfy

$$\int_{D_v} J[f](t, x, v)\, dv = 0, \qquad \forall\, x \in D_x,\, \forall\, t \geq 0$$

so that integrating Eq. (2.5) with respect to v and recalling Eqs. (2.3), (2.4) yields the macroscopic mass conservation equation (1.2).

Specific models are obtained from Eq. (2.5) by detailing the form of the collisional operator. In the next sections we will review some of the relevant contributions to the kinetic modeling of vehicular traffic available in the literature.

2.2 Prigogine model

In his pioneering work on the mathematical modeling of vehicular traffic by methods of the kinetic theory, Prigogine [43, 44] constructs the collisional operator through the contribution of two terms:

$$J[f] = J_r[f] + J_i[f_2]. \tag{2.6}$$

The first one, $J_r[f]$, is called *relaxation term* and models the tendency of each driver to adapt the state of her/his vehicle to a desired standard state. The latter is described by the *desired distribution function* $f_0 = f_0(t, x, v)$, which in essence corresponds to the driving program each driver aims at. In Prigogine's model, the function f_0 is expressed as

$$f_0(t, x, v) = n(t, x)\tilde{f}_0(v), \tag{2.7}$$

where $\tilde{f}_0$ is a prescribed probability distribution over the variable v, independent of x and t. The factorization (2.7) translates the hypothesis that the desired distribution of the velocity is, for each driver, unaffected by the local vehicle density. The relaxation term is then taken to be

$$J_r[f] = -\frac{f - f_0}{T},$$

where T is the relaxation time which depends on the probability of passing P as

$$T = \tau\frac{1 - P}{P} \qquad \text{for} \qquad P = 1 - \frac{n}{n_{\max}}. \tag{2.8}$$

In these formulas, τ is a positive parameter of the model and $n_{\max}$ denotes the maximum vehicle density locally allowed along the road according to the road capacity.

The second term appearing in the decomposition (2.6), $J_i[f_2]$, is called *interaction term* and is represented by an operator which models the interactions among candidate and field vehicles. As usual in the kinetic framework, it is further split into two more operators:

$$J_i[f_2] = G[f_2] - L[f_2].$$

The *gain operator* $G[f_2]$ accounts for the interactions of candidate vehicles having velocity $v_* > v$ with field vehicles with velocity v, which force the former to slow down to v if they cannot overtake the latter, causing this way a gain of cars with velocity state v:

$$G[f_2] = (1 - P)\int_v^{+\infty} (v_* - v) f_2(t, x, v_*, x, v)\, dv_*. \tag{2.9}$$

The *loss operator* $L[f_2]$ accounts instead for the interactions of candidate vehicles having velocity v with field vehicles with velocity $v_* < v$, which force the former to slow down to v_* if they cannot overtake the latter, thus giving rise to a loss of cars with velocity state v:

$$L[f_2] = (1 - P)\int_0^{v} (v - v_*) f_2(t, x, v, x, v_*)\, dv_*. \tag{2.10}$$

In Eq. (2.9) it is assumed $D_v = \mathbb{R}_+$, that is no upper limitation is imposed on the velocity of the cars.

The function f_2 appearing in Eqs. (2.9), (2.10) is the so-called *two-particles distribution function*, formally obtained from the distribution $\mathcal{P}$ given in Eq. (2.1) as

$$f_2(t,x,v,x_*,v_*) = \int_{D_{\mathbf{w}}^{N-2}} \mathcal{P}(t,\mathbf{w},\mathbf{w}_*,\mathbf{w}^{(3)},\ldots,\mathbf{w}^{(N)})\,d\mathbf{w}^{(3)}\ldots d\mathbf{w}^{(N)}$$

for $\mathbf{w} = (x,v)$ and $\mathbf{w}_* = (x_*,v_*)$. In other words, f_2 is the joint distribution function of candidate and field vehicles, such that $f_2(t,x,v,x_*,v_*)$ gives a measure of the joint probability to find a candidate vehicle in the state (x,v) and simultaneously a field vehicle in the state (x_*,v_*). Introducing the hypothesis of *vehicular chaos* (see e.g., Hoogendoorn and Bovy [31]), which states that vehicles are actually uncorrelated due to the mixing caused by overtaking, one can express f_2 in terms of f as

$$f_2(t,x,v,x_*,v_*) = f(t,x,v)f(t,x_*,v_*),$$

so that the interaction operator takes the form of a bilinear operator $J_i[f,f]$ acting on f:

$$J_i[f,f] = (1-P)f(t,x,v)\int_0^{+\infty}(v_*-v)f(t,x,v_*)\,dv_*$$

and Prigogine's model finally reads

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\frac{f(t,x,v)-f_0(t,x,v)}{T} + (1-P)f(t,x,v)\int_0^{+\infty}(v_*-v)f(t,x,v_*)\,dv_*. \quad (2.11)$$

Notice that the operators G and L in Eqs. (2.9), (2.10), and consequently also the final model (2.11), implicitly assume localized interactions ($x = x_*$) like in the classical Boltzmann collisional kinetic theory.

2.3 Paveri Fontana model

One of the main criticisms to Prigogine's model is that the desired speed distribution function $\tilde{f}_0$ in Eq. (2.7) is prescribed *a priori*, and is thus independent of the evolution of the system. In order to correct this drawback, Paveri Fontana [41] conceived a model in which the desired speed is taken into account as a further state variable w ranging in the same domain D_v of the true speed v. At the same time, a *generalized distribution function* $g = g(t,x,v,w)$ is introduced, such that

$$g(t,x,v,w)\,dx\,dv\,dw$$

gives a measure of the number of vehicles that at time t have a position comprised between x and $x+dx$, a true speed comprised between v and $v+dv$, and a desired speed comprised between w and $w+dw$. Notice that the distribution function f and the desired distribution function f_0 are readily recovered by integrating out w and v, respectively, from g:

$$f(t,x,v) = \int_{D_v} g(t,x,v,w)\,dw, \qquad f_0(t,x,w) = \int_{D_v} g(t,x,v,w)\,dv. \tag{2.12}$$

Using these relations, an evolution equation for g is straightforwardly derived from the corresponding equation (2.5) for f:

$$\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} = I[g], \tag{2.13}$$

where I is a new collisional operator acting on the generalized distribution function.

Analogously to Eq. (2.6), Paveri Fontana's choice consists in splitting I in a twofold contribution:

$$I[g] = I_r[g] + I_i[g,g],$$

where the first term is again a relaxation toward the desired speed having now the form

$$I_r[g] = -\frac{\partial}{\partial v}\left(\frac{w-v}{T}g\right), \tag{2.14}$$

while the second one describes the interactions among the vehicles. Specifically, in the same spirit as Prigogine's model (cf. Eqs. (2.9), (2.10)), a gain and loss term can be identified such that

$$I_i[g,g] = G[g,g] - L[g,g]$$

with specifically

$$G[g,g] = (1-P)\int_v^{+\infty} (v_* - v) f(t,x,v) g(t,x,v_*,w)\,dv_*, \tag{2.15}$$

$$L[g,g] = (1-P)\int_0^{v} (v - v_*) f(t,x,v_*) g(t,x,v,w)\,dv_*, \tag{2.16}$$

the probability of passing P being defined as

$$P = \left(1 - \frac{n}{n_{\max}}\right) H\left(1 - \frac{n}{n_c}\right), \tag{2.17}$$

where $H(\cdot)$ is the Heaviside function and $n_c \in (0, n_{\max})$ a critical density threshold above which overtaking is inhibited. Notice in particular that candidate vehicles are described via the generalized distribution function g in order to take their desired speed into account, while field vehicles are described by means of the distribution function f, in which the dependence on w has been integrated out, to focus rather on their actual speed. Moreover, Eqs. (2.15), (2.16) still assume localized interactions as confirmed by the fact that both f and g are evaluated at the same spatial position x.

Putting Eqs. (2.13)-(2.16) together we deduce the final form of Paveri Fontana's model:

$$\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} = -\frac{\partial}{\partial v}\left(\frac{w-v}{T}g\right) + (1-P)\left[f(t,x,v)\int_v^{+\infty}(v_*-v)g(t,x,v_*,w)\,dv_* \right.$$
$$\left. - g(t,x,v,w)\int_0^v (v-v_*)f(t,x,v_*)\,dv_*\right]. \tag{2.18}$$

Integrating Eq. (2.18) over w and using the first of Eqs. (2.12) gives an evolution equation for the distribution function f:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\frac{\partial}{\partial v}\left[\frac{1}{T}\int_0^{+\infty} wg(t,x,v,w)\,dw - \frac{vf(t,x,v)}{T}\right]$$
$$+ (1-P)f(t,x,v)\int_0^{+\infty}(v_*-v)f(t,x,v_*)\,dv_*.$$

This equation differs from the corresponding Eq. (2.11) of Prigogine's model only for the relaxation term at the right-hand side, which here maintains the explicit dependence on the generalized distribution function g.

Similarly, integration of Eq. (2.18) with respect to v, along with the second of Eqs. (2.12), yields the following evolution equation for the desired distribution function f_0:

$$\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial x}\left[\int_0^{+\infty} vg(t,x,v,w)\,dv\right] = 0,$$

which definitely confirms that f_0 depends now on the overall evolution of the system.

2.4 Enskog-like models

As noticed by some Authors (see e.g., Klar and Wegener [34]), the localized interactions framework used in classical kinetic models of traffic (like Prigogine's and Paveri Fontana's ones) prevents backward propagation of the perturbations in the negative x direction. This is essentially due to the fact that the flow of vehicles is unidirectional, given the positivity constraint on the velocity v. Indeed, integrating Eq. (2.5) along the characteristics, that is the lines $x - vt =$ constant in the time-space domain, gives

$$f(t,x,v) = f^0(x-vt,v) + \int_0^t J[f](s,x+v(s-t),v)\,ds,$$

where $f^0(x,v) = f(0,x,v)$ is the initial datum. Since $v \geq 0$ and $s \leq t$, one gets both $x - vt \leq x$ and $x + v(s-t) \leq x$, hence the value of f in x at time t depends only on its former values at points behind x, i.e., in the interval $(-\infty, x]$, and there is no way to extend such a domain of dependence to points located in the right interval $[x, +\infty)$. On the other hand, the localized interactions assumption is a heritage of the Boltzmann collisional kinetic theory, where however the aforementioned drawback is not present because the flow of gas particles need not be unidirectional.

To obviate this difficulty of the theory, Klar and coworkers suggest, in a series of papers [26, 27, 34, 35, 52] focusing among other things on this topic, see also the review by Klar and Wegener [36], to describe the collisional operator J of the kinetic equation (2.5) in terms of the two-particles distribution function as:

$$J[f_2] = G[f_2] - L[f_2] \tag{2.19}$$

where the gain operator is given by

$$G[f_2] = \sum_{j=1}^{M} \iint_{\Omega_j} |v_1 - v_2| \sigma_j(v;\, v_1, v_2, n) f_2(t, x, v_1, x + H_j(v_1, v_2), v_2)\, dv_1\, dv_2 \tag{2.20}$$

and the loss operator by

$$L[f_2] = \sum_{j=1}^{M} \int_{\omega_j} |v - v_2| f_2(t, x, v, x + H_j(v, v_2), v_2)\, dv_2.$$

Notice that the interacting pairs are not supposed to occupy the same spatial position. In particular, the field vehicle is assumed to be located in $x+H_j$, x being the position of the candidate vehicle. The thresholds $H_j > 0$, $j = 1, \ldots, M$, are introduced to delocalize the interactions, similarly to an Enskog-like kinetic setting, and to trigger different behaviors of the candidate vehicle (acceleration, deceleration) on the basis of both the distance separating it from its heading field cars and the specific velocities which it and the leaders are traveling at. Integration is performed over subsets of the velocity domain, $\Omega_j \subseteq D_v^2$, $\omega_j \subseteq D_v$, associated with each of the thresholds H_j.

In the simplest situation, two constant thresholds H_1, H_2 are considered to model, respectively, a deceleration of the candidate vehicle when the distance from the field vehicle falls below H_1, and an acceleration when it grows instead above H_2. Correspondingly, the deceleration domains in G and L are given by

$$\Omega_1 = \{(v_1, v_2) \in D_v^2 \,:\, v_1 > v_2\}, \qquad \omega_1 = \{v_2 \in D_v \,:\, v_2 > v\},$$

while the acceleration domains are

$$\Omega_2 = \{(v_1, v_2) \in D_v^2 \,:\, v_1 < v_2\}, \qquad \omega_2 = \{v_2 \in D_v \,:\, v_2 < v\}.$$

Specifically, in the case of a deceleration it is assumed that the post-interaction velocity v of the candidate vehicle either remains unchanged, thus equal to the initial velocity v_1, if an overtaking of the field vehicle is possible, or stochastically reduces to a certain fraction of the velocity v_2 of the field vehicle, if the candidate is forced to queue. This is expressed by the following form of the *transition probability* σ_1:

$$\sigma_1(v;\, v_1, v_2, n) = P\delta(v - v_1) + (1 - P)\frac{1}{(1-\beta)v_2}\chi_{[\beta v_2,\, v_2]}(v),$$

where P is the probability of passing defined like in Eq. (2.8), which carries the dependence of σ_1 on n, $\beta \in [0, 1]$ is a parameter, and χ_I is the indicator function of the set I.

Conversely, in the case of an acceleration the post-interaction velocity of the candidate vehicle is assumed to be uniformly distributed between the values v_1 and $v_1 + \alpha(v_{\max} - v_1)$, where α depends on n through P as $\alpha = \alpha_0 P$ for a suitable parameter $\alpha_0 > 0$, and $v_{\max}$ is the maximum allowed velocity on the road. Therefore, the transition probability associated with this second threshold takes the form

$$\sigma_2(v;\, v_1, v_2, n) = \frac{1}{\alpha(v_{\max} - v)}\chi_{[v_1,\, v_1 + \alpha(v_{\max} - v)]}(v).$$

In order to express the gain and loss operators in terms of the one-particle distribution function f, the hypothesis of vehicular chaos is invoked, which entails

$$f_2(t, x, v_1, x + H_j, v_2) = f(t, x, v_1)f(t, x + H_j, v_2)k(H_j, n(t, x)). \tag{2.21}$$

The function k appearing in this expression weights the interactions according to the distance of the interacting pairs and the local congestion of the road. Its presence in the collisional operator can be formally justified by referring to the Enskog kinetic equations of a dense gas. The interested reader might want to consult the book by Bellomo *et al.* [9] and the paper by Cercignani and Lampis [16] for further details on this part of the theory.

The final form of Klar and coworkers' model with two equal interaction thresholds ($H_1 = H_2 \equiv H$) writes:

$$\begin{aligned}\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = &\int_0^{+\infty}\!\!\int_0^{+\infty} |v_1 - v_2|\sigma(v;\, v_1, v_2, n)f(t, x, v_1)f(t, x + H, v_2)k(H, n)\, dv_1\, dv_2 \\ &- f(t, x, v)\int_0^{+\infty} |v - v_2| f(t, x + H, v_2)k(H, n)\, dv_2,\end{aligned} \tag{2.22}$$

where we have defined

$$\sigma(v;\, v_1, v_2, n) = \sum_{j=1}^{2} \sigma_j(v;\, v_1, v_2, n)\chi_{\Omega_j}(v_1, v_2).$$

In particular, it can be noticed that when considering just one interaction threshold for the deceleration, with furthermore $H = 0$, the collisional operator in (2.22) reduces to Prigogine's localized interaction operator for

$$\sigma(v;\, v_1, v_2, n) = P\delta(v_1 - v) + (1 - P)\delta(v_2 - v),$$

up to the weight function k.

We finally observe that the spirit in which the collisional operator is constructed in model (2.22) somehow differs from Prigogine's and Paveri Fontana's models in that interactions are now defined by detailing the short-range reactions of each driver to the dynamics of the neighboring vehicles rather than by interpreting her/his overall behavior.

2.5 Discrete velocity models

Recently new mathematical models of vehicular traffic, based on the discrete kinetic theory, have been proposed in the literature, with the aim of taking into account, still at a mesoscopic level of description, some aspects of the strong granular nature of the flow of vehicles. Indeed, as reported by several Authors (see e.g., Ben-Naim and Krapivsky [10, 11], Berthelin *et al.* [12]), cars along a road tend to cluster, which gives granularity in space, with a nearly constant speed within each cluster, which makes also the velocity a discretely distributed variable.

In this review we concisely report about the discrete velocity model by Coscia *et al.* [18], referring the reader to the next chapters for a thorough development and analysis of the model by Delitala and Tosin [22].

The main idea is to make the velocity variable v discrete by introducing in the domain D_v a grid $I_v = \{v_i\}_{i=1}^{2m-1}$ consisting of $2m-1$ points, $m \in \mathbb{N}$, of the form

$$0 = v_1 < v_2 < \cdots < v_m < \cdots < v_{2m-1} = 1,$$

and letting then $v \in I_v$, while time and space are left continuous. Each v_i is interpreted as a *velocity class*, encompassing a certain range of velocities v which are not individually distinguished. Correspondingly, the distribution function f is expressed as a linear combination of Dirac functions in the variable v, with coefficients depending on time and space:

$$f(t,x,v) = \sum_{i=1}^{2m-1} f_i(t,x)\delta(v-v_i), \tag{2.23}$$

where $f_i(t,x)$ gives the distribution of cars in the point x having at time t a velocity comprised in the i-th velocity class. Using this representation, the following expressions for the macroscopic variables of interest are easily derived from Eqs. (2.3)-(2.4):

$$n(t,x) = \sum_{i=1}^{2m-1} f_i(t,x), \qquad q(t,x) = \sum_{i=1}^{2m-1} v_i f_i(t,x), \qquad u(t,x) = \frac{\sum_{i=1}^{2m-1} v_i f_i(t,x)}{\sum_{i=1}^{2m-1} f_i(t,x)}.$$

One of the main features of model [18] is that the velocity grid I_v is conceived so as to have a variable step, which tends to zero for high vehicle concentrations (*adaptive velocity grid*). Specifically, the discrete velocities depend on the macroscopic vehicle density n as

$$v_i(n) = \frac{i-1}{m-1}V_e(n), \qquad i = 1, \ldots, 2m-1,$$

where $V_e(n)$ is an average equilibrium velocity, so that $v_1 = 0$, $v_{2m-1} = v_{\max} = 2V_e(n)$, and the grid is symmetric with respect to its central point $v_m = V_e(n)$. Since, by definition, $V_e(n) \to 0$ for $n \to n_{\max}$, we note that $v_i(n) \to 0$ each $i = 1, \ldots, 2m-1$ as the density locally approaches its maximum admissible value. The Authors of [18] recommend the use of the following equilibrium velocity:

$$V_e(n) = e^{-\alpha \frac{n}{n_{\max}-n}},$$

where $\alpha > 0$ is a constant related to road and environmental conditions ($\alpha = 0$ stands for good conditions and $\alpha = 1$ for bad conditions), and $n_{\max}$ denotes the road capacity. Alternatively, one can use the classical relation $V_e(n) = 1 - n/n_{\max}$ found in the Lighthill-Whitham-Richards first order macroscopic model (see Lighthill and Whitham [39], Whitham [53]).

The mathematical structure of the discrete kinetic equations to generate particular models relies on the localized Boltzmann-like collisional framework:

$$\frac{\partial f_i}{\partial t} + \frac{\partial}{\partial x}(v_i(n) f_i) = G_i[\mathbf{f}, \mathbf{f}] - f_i L_i[\mathbf{f}], \qquad i = 1, \ldots, 2m-1,$$

with $\mathbf{f} = (f_1, \ldots, f_{2m-1})$, for gain and loss operators inspired by a stochastic game theory approach (cf. Bertotti and Delitala [13]):

$$G_i[\mathbf{f}, \mathbf{f}] = \sum_{h,\,k=1}^{2m-1} \gamma |v_h - v_k| A^i_{hk} f_h f_k, \qquad L_i[\mathbf{f}] = \sum_{k=1}^{2m-1} \gamma |v_i - v_k| f_k,$$

where $\gamma > 0$ is a constant.

Acceleration and deceleration of the vehicles are described, like in Klar and coworkers' model, as short-range reactions of the drivers to the traffic dynamics in terms of velocity class transitions. Specifically, a *table of games* A^i_{hk} is introduced, which gives the probability that a candidate vehicle with velocity v_h adjusts its speed to v_i after an interaction with a field vehicle traveling at speed v_k. The following technical requirements are imposed:

$$A^i_{hk} \geq 0, \qquad \sum_{i=1}^{n} A^i_{hk} = 1, \qquad \forall\, i, h, k \in \{1, \ldots, 2m-1\},$$

in order for A^i_{hk} to represent a discrete probability distribution over the post-interaction velocity v_i of the candidate vehicle.

The table of games is assumed to be constant in time and space, and is constructed so as to allow interactions only among vehicles belonging to sufficiently close velocity classes ($|h-k| \leq 1$ in [18]). After an interaction, the candidate vehicle can only fall in a velocity class adjacent to its current one, thus the sole potentially non-zero coefficients A^i_{hk} are those for which $i = h-1$, $i = h$, and $i = h+1$. In addition, slow and fast cars, which are distinguished by comparing their velocity class to the central class v_m, are supposed to interact only with faster and slower cars, respectively, with a possible tendency to accelerate in the former case and to decelerate in the latter case. If $\epsilon_a, \epsilon_d \in [0,1]$ denote constant acceleration and deceleration probabilities, the table of games proposed by Coscia and coworkers reads as follows:

(i) for $h < m$

$$A^{h}_{h,\,h+1} = 1 - \epsilon_a, \qquad A^{h+1}_{h,\,h+1} = \epsilon_a, \qquad A^{i}_{hk} = 0 \quad \text{otherwise;}$$

(ii) for $h > m$

$$A^{h-1}_{h,\,h-1} = \epsilon_d, \qquad A^{h}_{h,\,h-1} = 1 - \epsilon_d, \qquad A^{i}_{hk} = 0 \quad \text{otherwise.}$$

A further particularization of these expressions is obtained by setting $\epsilon_a = \epsilon$, $\epsilon_d = \nu\epsilon$ for $0 \leq \nu \leq 1$, in order to identify a basic reaction probability $\epsilon \in [0, 1]$ and a measure ν of the relative strength of ϵ_a and ϵ_d.

Chapter 3

The discrete velocity model

3.1 Preliminaries

The flow of vehicles along a road is in many aspects a discrete process. First of all, the spatial distribution of the vehicles does not give rise to a continuous density, so that, as discussed in the Introduction, the continuum hypothesis applied to such a system is in principle only a modeling abstraction. However, since it offers several theoretical and practical advantages over the microscopic approach, it can ultimately be accepted in order to model the big picture. Of course, the age-old problem still remains of how one should understand the pointwise values of the vehicle density. It is perhaps a common experience that trying to give them a physical interpretation often results in obscure, ambiguous, and finally unconvincing explanations. Probably the best way to face the question is to honestly say that, strictly speaking, they are meaningless by themselves, because cars are clearly not point particles that can be tightly packed in a reference volume. What instead makes sense is to see the vehicle density on the whole as a measure of the macroscopic occupancy of the road, and as a tool to depict the spatial and temporal evolution of traffic waves. A second source of discreteness is represented by the velocity of the vehicles, which is not continuously distributed over the range of admissible values. Indeed, isolated vehicles maintain preferably a nearly constant speed, whose value strongly depends on the attitude of each driver. Packed vehicles tend instead to move in clusters featuring a specific average speed, which may substantially differ from one another. This is why it is more natural to speak roughly of slow and fast vehicles, rather than to distinguish meticulously between any two vehicles having an arbitrarily small difference in their speeds.

As anticipated in the Introduction, the goal of this thesis is to develop a mathematical framework of vehicular traffic modeling along a one-way road, focusing in particular on this second aspect of discreteness of the flow of vehicles. To this aim, we choose to work within the general setting of the discrete kinetic theory, because it is specifically devoted to model systems of interacting elements characterized by some discrete state variables. In addition, upon accepting the underlying continuum hypothesis in the sense discussed above, the kinetic approach allows an accurate microscopic-like modeling of the interactions among the vehicles along with a quick retrieval of all global, i.e. macroscopic, observable quantities of the system.

From the modeling point of view, the main independent one-dimensional variables needed to describe the system are the space coordinate x, the velocity v, and the time t, which we understand as dimensionless with respect to some characteristic values of length L, speed V, and consequently time $\tau = L/V$. This means that the physical, i.e. dimensional, space coordinate $\tilde{x}$, velocity $\tilde{v}$, and time $\tilde{t}$ are recovered as $\tilde{x} = Lx$, $\tilde{v} = Vv$, and $\tilde{t} = \tau t$, respectively. Concerning the above reference values, we observe that L may be taken for instance as the road length, so that the dimensionless spatial domain D_x coincides with the interval $[0, 1]$, although for theoretical purposes, like for instance the formal deduction of mathematical models and the investigation of their relevant qualitative properties, it is often convenient to think of D_x as unbounded, which is typically done by setting $D_x = \mathbb{R}$. Conversely, V may represent a certain characteristic maximum velocity attained by the vehicles, which implies that the dimensionless velocity domain D_v can be taken as the interval $[0, 1]$ as well. Actually, this choice may be questionable if V is understood as the typical mean velocity attained by the vehicles under light traffic conditions, for in this case the velocity domain should have more properly the form $[0, 1 + \mu]$, where $\mu > 0$ is a parameter accounting for the fact that isolated vehicles possibly travel at speeds higher than V (see e.g., Coscia *et al.* [18] and Delitala [21]). However, we anticipate that in a discrete velocity framework this detail is irrelevant, since μ is in general sufficiently small to allow to confuse $1 + \mu$ with 1 in a sense that we will make precise in a moment.

To make the velocity variable v discrete means technically to introduce a discrete grid with a finite number of points, say m points, $I_v = \{v_i\}_{i=1}^{m} \subset D_v$ and to let $v \in I_v$. This essentially amounts to partitioning the range of admissible speeds in m velocity classes, and to describing then velocity variations of the vehicles as state transitions of the latter from one class to another. In principle, the only requirement on m is that it be a nonnegative integer different from zero: $m \in \mathbb{N}$, $m > 0$, which will be always implicitly assumed in the sequel. In practice, however, it should be considered that the larger the number of classes is the smaller the range of velocities covered by each class is, hence an excessively high value of m results in a too fine, thus potentially meaningless from the physical point of view, velocity grid. Indeed, as previously mentioned, the spirit of the discrete kinetic approach in our project of traffic modeling is to render in the mathematical theory of traffic the actual impossibility to distinguish too carefully the speeds of the vehicles from one another due to their intrinsic discrete distribution.

We take the first velocity class to be $v_1 = 0$, which coincides with the left endpoint of the interval D_v. The other classes are then recovered as

$$v_{i+1} = v_i + (\Delta v)_i, \qquad i = 1, \ldots, m - 1,$$

where $(\Delta v)_i$ represents the amplitude of the i-th velocity class. A customary choice is to consider a uniformly spaced velocity grid over D_v, which implies a constant step

$$\Delta v = \frac{|D_v|}{m - 1},$$

$|D_v|$ being the length of the interval D_v. This way it results also

$$v_i = (i-1)\Delta v, \qquad i = 1, \dots, m,$$

with $v_m = |D_v|$. If one takes $D_v = [0, 1+\mu]$ with μ 'sufficiently small' then the resulting amplitude $\Delta v = \frac{1+\mu}{m-1}$ of the velocity classes little differs from the case $D_v = [0,1]$, which instead produces $\Delta v = \frac{1}{m-1}$. Consequently, one can choose to refer to the unit dimensionless velocity domain $D_v = [0,1]$, simply assuming that vehicles possibly traveling at speeds higher than 1 are included in the extreme velocity class $v_m = 1$.

In the sequel, whenever necessary, we will explicitly refer to a uniformly spaced velocity grid of the form

$$v_i = \frac{i-1}{m-1}, \qquad i = 1, \dots, m \tag{3.1}$$

with constant grid step

$$\Delta v = \frac{1}{m-1}.$$

The discreteness of the velocity variable entails a particular structure for the one-particle distribution function f, which can be formally rewritten as a linear combination of Dirac functions with coefficients depending on x and t:

$$f(t,x,v) = \sum_{i=1}^{m} f_i(t,x)\delta(v - v_i). \tag{3.2}$$

Since space and time are left continuous, each function f_i is defined over the set $\mathbb{R}_+ \times D_x$ and takes values in $\mathbb{R}_+$. The quantity $f_i(t,x)$ represents the density of vehicles belonging to the i-th velocity class which at time t are located in x. For the sake of convenience, in the sequel we will usually call it the *i-th distribution function* and, whenever needed, we will use the vector notation $\mathbf{f}(t,x) = (f_1(t,x), \dots, f_m(t,x))$.

Using the representation of the distribution function f given by Eq. (3.2), the following expressions for the classical macroscopic average quantities are easily derived:

(i) the vehicle density

$$n(t,x) = \sum_{i=1}^{m} f_i(t,x), \tag{3.3}$$

(ii) the vehicle flux

$$q(t,x) = \sum_{i=1}^{m} v_i f_i(t,x), \tag{3.4}$$

which should be understood in turn as dimensionless with respect to some characteristic values. If $\mathcal{N}$ denotes the reference value for the vehicle density n, so that the physical vehicle density is given by $\tilde{n} = \mathcal{N}n$, then the reference value Q for the vehicle flux has to be set coherently to $Q = V\mathcal{N}$. Notice that if each f_i is understood as dimensionless and linked to its dimensional counterpart $\tilde{f}_i$ by the relation

$$\tilde{f}_i(\tilde{t},\tilde{x}) = \mathcal{N} f_i(t,x), \qquad i = 1, \dots, m$$

as suggested by Eq. (3.3), then all these definitions apply straightforwardly and consistently. From the density n and the flux q, other important macroscopic quantities can be obtained, namely:

(iii) the average velocity

$$u(t,x) = \frac{q(t,x)}{n(t,x)} = \frac{\displaystyle\sum_{i=1}^{m} v_i f_i(t,x)}{\displaystyle\sum_{i=1}^{m} f_i(t,x)}, \tag{3.5}$$

rescaled with respect to the characteristic value V, and

(iv) the variance of the velocity

$$\Theta(t,x) = \frac{1}{n(t,x)} \sum_{i=1}^{m} (v_i - u(t,x))^2 f_i(t,x), \tag{3.6}$$

rescaled with respect to V^2. The quantity $n\Theta$ is sometimes termed the traffic pressure (see e.g., Aw and Rascle [4], Klar and Wegener [34, 52], Whitham [53]), especially in second order macroscopic traffic models.

3.2 Evolution equations

As usual in kinetic theories, an evolution equation for the distribution function is obtained from a balance principle which states that the time variation in the number of subjects of the system belonging to any subset of the state space is determined by the state transitions due to the interactions of the subjects themselves with each other. In particular, such a principle may imply the conservation of the mass, or of the total number of subjects, of the system, hence, as a conservation law, it is particularly suitable for vehicular traffic. For different systems, in which the totality of the interacting subjects need not be preserved, the same general idea still applies but without specific reference to the conservation of mass.

Confining the investigation to the standard kinetic Boltzmann-like framework of binary interactions, Arlotti and her coauthors discuss in [1, 2] a quite general structure of the kinetic equations, including the celebrated Boltzmann equation as a particular case, which enables one to address the modeling of various systems from applied sciences under a unified mathematical framework. In particular, Arlotti and coworkers deal with industrial applications, like the modeling of mixtures of dissipating gases, as well as biological applications in the fields of epidemiology and immune competition. The mathematical theory is then further developed and detailed in the book by Bellomo [5].

The general mathematical structure illustrated in [1, 2] refers to d-dimensional systems ($d = 1, 2, 3$ from the physical point of view) for which the (continuous) state variable, hereafter denoted by $\mathbf{w}$, is represented by the space position $\mathbf{x} \in D_{\mathbf{x}} \subseteq \mathbb{R}^d$, the velocity $\mathbf{v} \in D_{\mathbf{v}} \subseteq \mathbb{R}^d$, and the so-called *internal structure* $\mathbf{u} \in D_{\mathbf{u}}$, an l-dimensional vector quantity ($D_{\mathbf{u}} \subseteq \mathbb{R}^l$ with l any nonnegative integer) which characterizes the microscopic internal state of each element of the system. The evolution equation for the one-particle distribution function $f = f(t, \mathbf{w})$ is then written as

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{F}_{\mathbf{v}} \cdot \nabla_{\mathbf{v}} f + \nabla_{\mathbf{u}} \cdot (\mathbf{F}_{\mathbf{u}} f) = J[f, f], \tag{3.7}$$

where $\mathbf{F}_{\mathbf{v}} = \mathbf{F}_{\mathbf{v}}(t, \mathbf{x})$ and $\mathbf{F}_{\mathbf{u}} = \mathbf{F}_{\mathbf{u}}(t, \mathbf{u})$ model an external force field and the evolution of the internal state, respectively. The bilinear operator $J[f, f]$ at the right-hand side is the so-called *interaction operator*, which reads

$$\begin{aligned} J[f, f] = \int_{\mathcal{D}^3} [&\mathcal{F}(\mathbf{w}_1, \mathbf{w}_2;\, \mathbf{w}, \mathbf{w}_*) f(t, \mathbf{w}_1) f(t, \mathbf{w}_2) \\ &- \mathcal{F}(\mathbf{w}, \mathbf{w}_*;\, \mathbf{w}_1, \mathbf{w}_2) f(t, \mathbf{w}) f(t, \mathbf{w}_*)]\, d\mathbf{w}_1\, d\mathbf{w}_2\, d\mathbf{w}_* \end{aligned} \tag{3.8}$$

where $\mathcal{D}$ is the state space $D_{\mathbf{x}} \times D_{\mathbf{v}} \times D_{\mathbf{u}}$ and $\mathcal{F} : \mathcal{D}^4 \to \mathbb{R}_+$ is the transition distribution function, such that $\mathcal{F}(\mathbf{w}_1, \mathbf{w}_2;\, \mathbf{w}, \mathbf{w}_*)$ represents a measure of the probability that two subjects, with respective states $\mathbf{w}_1$, $\mathbf{w}_2$, change state into $\mathbf{w}$, $\mathbf{w}_*$ after interacting.

The structure depicted by Eqs. (3.7), (3.8) can be profitably used to derive formally our framework for traffic flow models, up to reformulating the equations in the discrete velocity context. In more detail, considering that in our specific case we deal with scalar space and velocity variables, $x \in D_x \subseteq \mathbb{R}$ and $v \in I_v \subset D_v \subseteq \mathbb{R}$, and that the microscopic internal structure $\mathbf{u}$ of the vehicles is not supposed to play a specific role, the state variable $\mathbf{w}$ is fully mechanical and is given by $\mathbf{w} = (x, v) \in D_x \times I_v$. Taking Eq. (3.2) into account and inserting it into Eq. (3.7) yields, after some technical calculations, the following system of equations for the distribution functions f_i:

$$\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial x} = G_i[\mathbf{f}, \mathbf{f}] - f_i L_i[\mathbf{f}], \qquad i = 1, \ldots, m, \tag{3.9}$$

where:

(i) $G_i[\mathbf{f}, \mathbf{f}]$ is the i-th *gain operator* defined as

$$G_i[\mathbf{f}, \mathbf{f}](t, x) = \sum_{j,h,k=1}^{m} \int_{D_x^3} \mathcal{F}_{hk}^{ij}(x_1, x_2;\, x, x_*) f_h(t, x_1) f_k(t, x_2)\, dx_1\, dx_2\, dx_*, \tag{3.10}$$

giving the amount of vehicles per unit time that, in consequence of binary collisions, get the test state (x, v_i);

(ii) $L_i[\mathbf{f}]$ is the i-th *loss operator* defined as

$$L_i[\mathbf{f}](t, x) = \sum_{j,h,k=1}^{m} \int_{D_x^3} \mathcal{F}_{ij}^{hk}(x, x_*;\, x_1, x_2) f_j(t, x_*)\, dx_1\, dx_2\, dx_*, \tag{3.11}$$

giving instead the amount of vehicles per unit time that, owing again to mutual interactions, lose the test state (x, v_i).

We observe that letting

$$J_i[\mathbf{f},\mathbf{f}] := G_i[\mathbf{f},\mathbf{f}] - f_i L_i[\mathbf{f}]$$

gives the discrete counterpart of the interaction operator defined by Eq. (3.8).

The transition distribution function $\mathcal{F}^{ij}_{hk} : D^4_x \to \mathbb{R}_+$, along with all possible permutations of the indexes i, j, h, k, is such that the quantity $\mathcal{F}^{ij}_{hk}(x_1, x_2;\, x, x_*)$ expresses a measure of the probability that the interaction between two vehicles, belonging to the h-th and to the k-th velocity class and located in x_1 and in x_2, respectively, leads the first one in x with a simultaneous transition to the i-th velocity class and the second one in x_* with a transition to the j-th velocity class. From the function $\mathcal{F}^{ij}_{hk}$ two other important quantities originate, which will be fundamental in structuring the mathematical model, namely:

(i) the *interaction rate*

$$\eta_{hk}(x_1, x_2) = \sum_{i,\,j=1}^{m} \int_{D^2_x} \mathcal{F}^{ij}_{hk}(x_1, x_2;\, x, x_*)\, dx\, dx_*, \tag{3.12}$$

which measures the frequency at which two vehicles located in x_1 and in x_2, and belonging to the h-th and to the k-th velocity class, respectively, may interact, and

(ii) the *table of games* (see Bertotti and Delitala [13])

$$A^i_{hk}(x_1, x_2;\, x) = \frac{1}{\eta_{hk}(x_1, x_2)} \sum_{j=1}^{m} \int_{D_x} \mathcal{F}^{ij}_{hk}(x_1, x_2;\, x, x_*)\, dx_*, \tag{3.13}$$

which yields the probability that a vehicle located in x_1 and belonging to the h-th velocity class moves to x and makes a transition to the i-th velocity class due to an interaction with a vehicle located in x_2 and belonging to the k-th velocity class.

Equations (3.9)-(3.11) constitute the general discrete velocity framework to generate particular models. We now introduce a preliminary characterization of the structure of the transition distribution function $\mathcal{F}^{ij}_{hk}$ in the case of the specific traffic model we are going to develop.

Consider two vehicles with states $\mathbf{w}_1 = (x_1, v_h)$, $\mathbf{w}_2 = (x_2, v_k)$, and assume that $x_1 \leq x_2$. Clearly, writing their speeds as v_h, v_k is shorthand to mean more properly that they belong to the h-th and k-th velocity class, respectively. Using the observation made by Daganzo [19] that a vehicle is essentially an anisotropic particle, in that it reacts mainly to frontal stimuli than to rear ones, we may generically suppose that the first vehicle, being located behind the other, is influenced by the presence of the second one on the road, but not the converse. As a consequence, we possibly expect a variation in the state $\mathbf{w}_1$, while $\mathbf{w}_2$ should remain basically unchanged after the interaction. The vehicle which is likely to change its current state due to an interaction is technically termed *candidate vehicle*, whereas any generic vehicle of the system potentially responsible for a change of state of the candidate vehicle is called *field vehicle*. The state $\mathbf{w} = (x, v_i)$, in which the candidate vehicle may fall, is said to be that of the *test vehicle*, the latter being understood as a hypothetical vehicle whose state is targeted by an ideal observer of the system. This discussion can be formally stated as follows:

Assumption 3.1. *After an interaction, neither the candidate nor the field vehicle changes position along the road. If $x, x_* \in D_x$ denote their respective post-interaction positions, this means that*

$$x = x_1, \quad x_* = x_2.$$

Assumption 3.2. *After an interaction, the field vehicle does not change velocity class (while, in principle, the candidate vehicle does). If $v_j \in I_v$ labels the post-interaction velocity class of the field vehicle, then*

$$v_j = v_k \quad \textit{or, equivalently,} \quad j = k.$$

Owing to Assumptions 3.1, 3.2 we detail the transition distribution function $\mathcal{F}_{hk}^{ij}$ as

$$\mathcal{F}_{hk}^{ij}(x_1, x_2; x, x_*) = \delta(x_1 - x)\delta(x_2 - x_*)\delta_{jk}\mathfrak{F}_{hk}^{i}(x_1, x_2), \tag{3.14}$$

where $\mathfrak{F}_{hk}^{i} : D_x^2 \to \mathbb{R}_+$ is charged to describe the actual stochastic mechanism leading the candidate vehicle to the i-th velocity class on the basis of its pre-interaction state and of that of the field vehicle. Notice, however, that in view of Assumption 3.1 we are henceforth allowed to speak generically of position of the candidate and the field vehicle without distinguishing between pre- and post-interaction position, and to use the symbols x_1, x and x_2, x_* interchangeably.

According to the above structure of $\mathcal{F}_{hk}^{ij}$, interactions are in general not local, indeed no term of the form (cf. e.g., Arlotti *et al.* [1])

$$\delta\left(\frac{x_1 + x - x_2 - x_*}{2}\right)$$

appears in Eq. (3.14). In other words, one may have $x_1 \neq x_2$, that is the candidate and the field vehicle need not be located in the same spatial position in order for the interaction to be triggered. This is meaningful in view of the observation that interactions among vehicles are structurally different from those among classical mechanical particles dealt with by the standard collisional kinetic theory. Indeed, vehicles need not be in contact to interact. More specifically, we may suppose that the state of the candidate vehicle be influenced by the presence of field vehicles within a finite distance ξ ahead of it along the road, that we call *interaction length* and understand as dimensionless with respect to the reference length L previously introduced. If x is the position of the candidate vehicle, the interaction length defines naturally an *interaction interval* $J_\xi(x) = [x, x + \xi]$ in which interactions are effective.

As a further general principle, it is reasonable to guess that the closer a field vehicle is to the candidate vehicle the more influential its presence is on the latter, so that finally interactions turn out to be more or less important, in terms of effect on the state of the candidate vehicle, according to the distance between the interacting vehicles.

We formalize the ideas above in the following

Assumption 3.3. *Interactions among a candidate vehicle located in $x \in D_x$ and the field vehicles are effective only within the interaction interval $J_\xi(x) = [x, x+\xi]$, and are weighted there on the basis of the distance separating the candidate vehicle from each of the field vehicles. Weighting is carried out by a* weight function $w : \mathbb{R} \to \mathbb{R}_+$ *such that:*

(i) $w(y) \geq 0$ *for all* $y \in D_x$,

(ii) $w(y) = 0$ *for all* $y \notin [0, \xi]$,

(iii) $\displaystyle\int_0^\xi w(y)\,dy = 1.$

In view of Assumption 3.3 the function $\mathcal{F}^i_{hk}$ specializes as

$$\mathcal{F}^i_{hk}(x_1, x_2) = a^i_{hk}(x_2)w(x_2 - x_1),$$

so that the transition distribution function $\mathcal{F}^{ij}_{hk}$ takes the form

$$\mathcal{F}^{ij}_{hk}(x_1, x_2;\, x, x_*) = \delta(x_1 - x)\delta(x_2 - x_*)\delta_{jk}a^i_{hk}(x_2)w(x_2 - x_1). \tag{3.15}$$

Here, $a^i_{hk} : D_x \to \mathbb{R}_+$ models in detail the stochastic dynamics of the velocity transitions operated by the candidate vehicle. Notice that it depends explicitly on the spatial position of the field vehicle only, the mutual position of the candidate and the field vehicle being already accounted for by the weight function w. Concerning this, we observe that, owing to Assumption 3.3-(ii), we have $w(x_2 - x_1) = 0$ for all $x_2 < x_1$ and all $x_2 > x_1 + \xi$, which guarantees that interactions are indeed effective only if the field vehicle precedes the candidate vehicle within the interaction interval of the latter.

Inserting the form of $\mathcal{F}^{ij}_{hk}$ given by Eq. (3.15) into Eqs. (3.10), (3.11) yields

$$G_i[\mathbf{f}, \mathbf{f}](t, x) = \sum_{h,\,k=1}^{m} \int_x^{x+\xi} a^i_{hk}(x_*)f_h(t, x)f_k(t, x_*)w(x_* - x)\,dx_*, \tag{3.16}$$

$$L_i[\mathbf{f}](t, x) = \sum_{j=1}^{m} \int_x^{x+\xi} \left(\sum_{h=1}^{m} a^h_{ij}(x_*)\right) f_j(t, x_*)w(x_* - x)\,dx_*, \tag{3.17}$$

whereas, according to Eqs. (3.12), (3.13), the interaction rate η_{hk} and the table of games A^i_{hk} take the form

$$\eta_{hk}(x_1, x_2) = \left(\sum_{i=1}^{m} a^i_{hk}(x_2)\right) w(x_2 - x_1),$$

$$A^i_{hk}(x_1, x_2;\, x) = \frac{a^i_{hk}(x_2)}{\sum_{i=1}^{m} a^i_{hk}(x_2)}\delta(x_1 - x).$$

Since a^i_{hk} depends on the space through x_2 only, from the latter expression we infer that for the specific modeling framework at hand the relevant entities in defining the table of games are

$$A^i_{hk}(x_2) := \frac{a^i_{hk}(x_2)}{\sum\limits_{i=1}^{m} a^i_{hk}(x_2)}, \tag{3.18}$$

which represent a discrete probability distribution over the velocity class i of the test vehicle, indeed from Eq. (3.18) we deduce

$$A^i_{hk}(x_2) \geq 0, \quad \sum_{i=1}^{m} A^i_{hk}(x_2) = 1, \qquad \forall\, h, k = 1, \ldots, m, \, \forall\, x_2 \in D_x. \tag{3.19}$$

Moreover, introducing the *unweighted interaction rate*

$$\bar{\eta}_{hk}(x_2) := \sum_{i=1}^{m} a^i_{hk}(x_2) \tag{3.20}$$

we also get

$$\eta_{hk}(x_1, x_2) = \bar{\eta}_{hk}(x_2) w(x_2 - x_1), \qquad a^i_{hk}(x_2) = \bar{\eta}_{hk}(x_2) A^i_{hk}(x_2). \tag{3.21}$$

Therefore, taking Eqs. (3.16), (3.17) into account, the set of discrete velocity kinetic evolution equations for the distribution functions $\{f_i\}_{i=1}^m$ finally becomes

$$\begin{aligned} \frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial x} = \sum_{h,\,k=1}^{m} \int\limits_{x}^{x+\xi} \bar{\eta}_{hk}(x_*) A^i_{hk}(x_*) f_h(t, x) f_k(t, x_*) w(x_* - x)\, dx_* \\ - f_i(t, x) \sum_{k=1}^{m} \int\limits_{x}^{x+\xi} \bar{\eta}_{ik}(x_*) f_k(t, x_*) w(x_* - x)\, dx_*, \end{aligned} \tag{3.22}$$

where, unlike Eq. (3.17), we have renamed k the (mute) index of the sum in the loss operator to avoid an unnecessary redundancy of indexes.

Equation (3.22) is actually a system of integro-differential equations with hyperbolic linear advection term at the left-hand side and nonlocal interaction operator at the right-hand side. Apart from Assumptions 3.1, 3.2, 3.3, the discrete velocity framework it depicts is still quite general, and can be specialized to originate particular models by acting on the specific forms of the table of games and the interaction rate, the latter detailed in both its unweighted and weighted component. We will deal with this topic in the next section.

3.3 Modeling microscopic interactions

According to the kinetic theory approach, the system of equations (3.22) calls for a microscopic analysis of the interactions among the vehicles, although it focuses on a global description of the traffic dynamics. Indeed, it allows an *a posteriori* retrieval of the usual macroscopic observable quantities via the set of distribution functions $\{f_i\}_{i=1}^m$ and the definitions (3.3)-(3.6).

In modeling the microscopic interactions, vehicles must be regarded as nonclassical particles, in spite of their fully mechanical state, because they interact, without colliding, in a nonlocal way. In particular, this implies that the relations usually found in the standard collisional kinetic theory (see e.g., Villani [51] and the main references listed therein) between pre- and post-collisional velocities need not apply in the present context: Interactions do not preserve in general either the momentum or the kinetic energy. Instead, they are described in an essentially probabilistic way via the table of games, without invoking any classical concept of force or law of point mechanics to account for velocity variations. Furthermore, the following implicit assumptions hold true:

(i) vehicles are regarded as points, their dimensions being negligible;

(ii) interactions involving simultaneously more than two vehicles are disregarded (binary interactions).

3.3.1 The table of games

The table of games A_{hk}^i models the stochastic dynamics of the velocity transitions of the vehicles, yielding the probability that the candidate vehicle changes its state class from h to that of the test vehicle i as a result of an interaction with a field vehicle lying in the state class k. In this subsection we present and discuss a possible form of the table of games, conceived on the basis of some elementary intuitions about the microscopic physics of traffic. Technical modifications and improvements, based either on specific theoretical modeling needs or on the particular physical setting at hand, are of course possible, some of which will be proposed and commented in Chapt. 5.

We suggest that a key role in the dynamics of the microscopic interactions should be played by the density n, regarded as an indicator of the macroscopic local conditions of traffic (cf. the discussion at the beginning of this chapter). In particular, we assume that it represents one of the main pieces of information used by a driver to decide how to adjust instantaneously the velocity of its vehicle, indeed phenomena like an acceleration, a deceleration or an overtaking are reasonably highly influenced by the free space locally available along the road. Observing that Eq. (3.18) allows in principle for a dependence of the table of games on the space variable via the position occupied by the field vehicle, we exploit this feature by devising a functional relationship linking A_{hk}^i to the density n computed for $x = x_2$ (or, equivalently, for $x = x_*$). Accordingly, in the sequel we will emphasize this by writing the table of games specifically as $A_{hk}^i[n]$. As the density n evolves in time, this makes the table of games depend also on the variable t, which is not explicitly provided for by Eq. (3.18). However, we notice that this fact does not invalidate by itself the formal derivation of the equations discussed in Sect. 3.2, requiring simply to allow a dependence on t of the transition distribution function $\mathcal{F}$ in Eq. (3.8), as well as of $\mathcal{F}_{hk}^{ij}$ in Eqs. (3.10), (3.11) when reducing to the discrete velocity framework.

Another important factor strongly affecting the flux of vehicles is represented by the road conditions. Bumpy roads make people drive more carefully, keeping slow speeds and avoiding accelerating and overtaking, while smooth roads usually offer more opportunities of maneuver. We duly incorporate this aspect in the table of games via a phenomenological parameter $\alpha \in [0, 1]$, whose lower and higher values are related to bad and good road conditions, respectively.

Let us focus on a candidate vehicle belonging to the velocity class h which interacts with a field vehicle belonging to the velocity class k, and let i be the velocity class of the test vehicle, $i, h, k \in \{1, \ldots, m\}$. We address separately the three cases $h < k$, $h > k$, and $h = k$, for in principle different behavioral rules apply in each of these situations.

In conceiving the explicit dependence of the table of games on the dimensionless vehicle density, one has to take carefully into account the admissible range of values of n resulting from the performed nondimensionalization, so as to fulfil the requirements expressed by Eq. (3.19). This amounts in essence to specifying the physical sense given to the reference value $\mathcal{N}$: The particular form of the elements $A^i_{hk}[n]$ may vary if one gives $\mathcal{N}$ a different physical interpretation, the formal structure of the whole model remaining however unchanged.

One of the most common choices is to identify $\mathcal{N}$ with the maximum density $n_{\max}$ allowed on the road according to the road capacity, whence it follows $0 \leq n \leq 1$. The form of the table of games we propose is specifically based on this kind of nondimensionalization, therefore it assumes *a priori* that the dimensionless vehicle density is bounded between 0 and 1. Consequently, it is meaningful as long as the resulting model is able to provide solutions that actually keep such a constraint. It is worth pointing out that this latter property cannot be assumed in turn *a priori* but requires to be proved by an appropriate qualitative analysis of the mathematical model itself. However, for the sake of modeling we temporarily refrain from dealing with this issue, postponing its discussion to the subsequent Chapts. 4 and 5.

Interaction with a faster vehicle ($h < k$)

When $h < k$, the candidate vehicle is encountering a faster field vehicle. The result of this interaction can be modeled according to a *follow-the-leader* strategy, which implies that the candidate vehicle either maintains its current speed or possibly accelerates, depending on the available surrounding free space. We set then (Fig. 3.1a)

$$A^i_{hk}[n] = \begin{cases} 1 - \alpha(1-n) & \text{if } i = h \\ \alpha(1-n) & \text{if } i = h+1 \\ 0 & \text{otherwise} \end{cases} \qquad (h, k = 1, \ldots, m). \tag{3.23}$$

Note that when $\alpha = 0$ (worst road conditions) the candidate vehicle simply keeps its current speed, and does not accelerate in any case. Conversely, when $\alpha = 1$ (best road conditions) the result of the interaction is essentially dictated by the local traffic congestion.

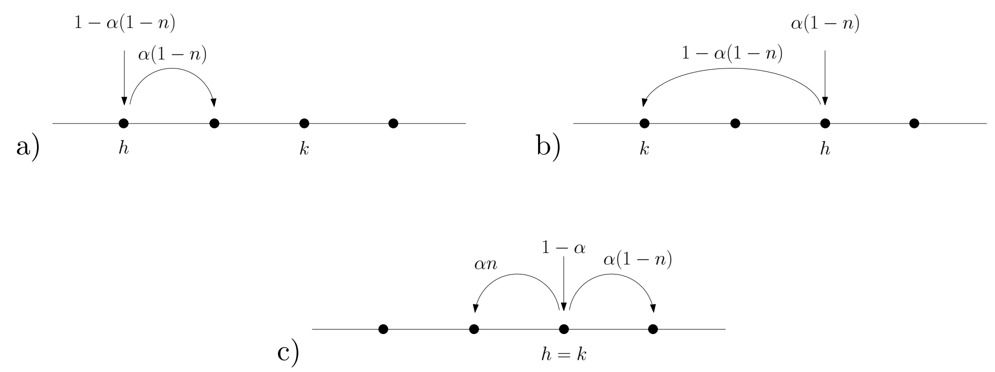

Figure 3.1. Pictorial representation of the interaction rules coded in the table of games. In the diagrams, h is the velocity class of the candidate vehicle, k that of the field vehicle. The arrows indicate the possible changes of class of the former due to an encounter with the latter, and are labeled with the probabilities associated to each transition.

Interaction with a slower vehicle ($h > k$)

When $h > k$, the candidate vehicle interacts with a slower field vehicle. In this case, we assume it does not accelerate. Rather, either it is forced to queue, reducing its speed to that of the leading vehicle, or it maintains its current speed, because it has enough free space to overtake. Consequently, we define (Fig. 3.1b)

$$A^i_{hk}[n] = \begin{cases} 1 - \alpha(1-n) & \text{if } i = k \\ \alpha(1-n) & \text{if } i = h \\ 0 & \text{otherwise} \end{cases} \qquad (h, k = 1, \ldots, m). \tag{3.24}$$

Note that this choice amounts to defining a *probability of passing* (cf. Eqs. (2.8), (2.17)) $P_\alpha = P_\alpha[n]$ depending on the local traffic and parameterized by the road conditions:

$$P_\alpha[n] = \alpha(1-n).$$

The emptier the road is the closer to α this probability becomes, and, if the road conditions allow, the candidate vehicle is more likely to overtake the leading field vehicle without the need for slowing down.

Interaction with an equally fast vehicle ($h = k$)

When $h = k$, the candidate vehicle and the field vehicle are traveling at the same speed. In this case, the result of the interaction has a higher degree of randomness than in the previous cases: The physical situation does not suggest any *a priori* more probable upshot. Therefore, we use as guideline the phenomenological idea of the spread of the velocity: The two vehicles are unlikely to strictly preserve their speed during the motion, for this would imply they do not interact, behaving as if they were alone along the road. Thus, we distribute the effect of the interaction over four possible outcomes (Fig. 3.1c):

$$A^i_{hh}[n] = \begin{cases} \alpha n & \text{if } i = h-1 \\ 1-\alpha & \text{if } i = h \\ \alpha(1-n) & \text{if } i = h+1 \\ 0 & \text{otherwise} \end{cases} \qquad (h = 2, \dots, m-1). \tag{3.25}$$

In this definition, α plays the role of a tuning parameter that regulates the mutual relevance of the outcomes. If $\alpha = 0$, then $A^h_{hh}[n] = 1$ each $h \in \{2, \dots, m-1\}$ and each n, so that one obtains the trivial interaction that does not cause any velocity transition. Conversely, if $\alpha = 1$ the interaction between two equally fast vehicles results in a full spread of the velocity, since $A^h_{hh}[n] = 0$ each n, and consequently none of them is allowed to maintain its current speed.

Note that the form of $A^i_{hh}[n]$ given by Eq. (3.25) applies only if $h \neq 1, m$. In contrast, a technical modification is needed at the boundary of the velocity grid, since when $h = 1$ or $h = m$ the candidate vehicle cannot brake or accelerate respectively, due to the lack of further lower or higher velocity classes. In these cases, we merge the deceleration or the acceleration into the tendency to preserve the current speed:

$$A^i_{11}[n] = \begin{cases} 1-\alpha(1-n) & \text{if } i = 1 \\ \alpha(1-n) & \text{if } i = 2 \\ 0 & \text{otherwise,} \end{cases} \qquad A^i_{mm}[n] = \begin{cases} \alpha n & \text{if } i = m-1 \\ 1-\alpha n & \text{if } i = m \\ 0 & \text{otherwise.} \end{cases} \tag{3.26}$$

3.3.2 The interaction rate

The interaction rate η_{hk} is a measure of the frequency at which vehicles in the traffic flow interact. According to Eq. (3.21), it can be split into a first component, the unweighted interaction rate, which accounts for the frequency of interaction between candidate and field vehicles regardless of their relative location along the road, and a second component, the weight function, which instead takes into account the distance between the interacting pairs in order to define the effectiveness of the interactions.

From its definition (cf. Eq. (3.20)), we see that the unweighted interaction rate $\bar{\eta}_{hk}$ may depend in general not only on the respective velocity classes of the interacting vehicles, like in the Boltzmann-like collisional kinetic theory, but also on the spatial position of the field vehicle along the road. In modeling this term of the equations it is worth stressing once again that interactions among vehicles are strongly different from mechanical collisions among classical particles. As a consequence, the standard forms of the interaction rate, deduced under momentum and energy conservation principles and involving then the relative velocity of the interacting pairs, are not suitable for the system at hand and require to be revised on the basis of different guidelines, resorting to the human-like behavioral nature of the vehicles.

Specifically, we choose not to emphasize the link between the interaction rate and the relative velocity of the interacting vehicles, assuming that η_{hk}, hence also $\bar{\eta}_{hk}$, is independent of the specific velocity classes h, k. Furthermore, borrowing some ideas from

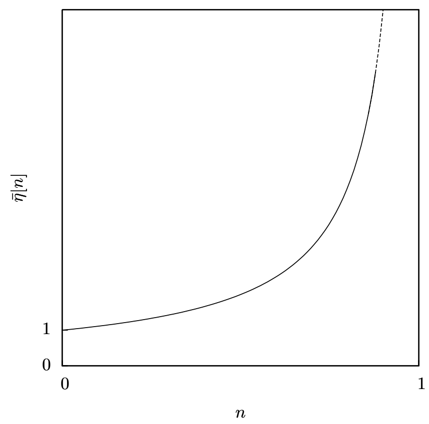

Figure 3.2. The unweighted interaction rate $\bar{\eta}$ as a function of the macroscopic density n on the basis of Eq. (3.27)

the Enskog kinetic theory of dense gases (see e.g., Bellomo *et al.* [9]) and coherently with the considerations previously introduced about the table of games, we formalize the dependence of the unweighted interaction rate on the position of the field vehicle via the vehicle density n, meaning that the frequency of the interactions experienced by the candidate vehicle is affected by the macroscopic conditions of traffic in the stretch of road ahead of it. Owing to all of this, in the sequel we will explicitly write $\bar{\eta}[n]$ instead of $\bar{\eta}_{hk}$.

The discussion above suggests that one can reinterpret the unweighted interaction rate $\bar{\eta}[n]$ in terms of reactiveness of the vehicles (or possibly of their drivers) to the traffic conditions along the road. In view of this, higher or lower rates of interaction should be understood as corresponding higher or lower levels of awareness of the traffic evolution from the vehicles, determined in particular by the local traffic congestion. Therefore, a form of $\bar{\eta}[n]$ which closely recall that of the analogous term in the kinetic equations for a dense Enskog gas may apply also in this case (see Fig. 3.2):

$$\bar{\eta}[n] \simeq \frac{1}{1-n}, \tag{3.27}$$

where we have implicitly assumed a nondimensionalization of n with respect to the reference value $\mathcal{N} = n_{\max}$. Accordingly, the fact that this function is monotonically increasing for $n \in [0, 1)$ implies that the local interaction rate becomes higher and higher as the vehicle density increases toward its limit threshold fixed by the road capacity. Or, rephrasing in other words, that the reactiveness of the vehicles rises as the traffic conditions on the road become more and more serious, requiring more and more carefulness. We observe that for $n = 1$, denoting the maximum possible occupancy of the road, the unweighted interaction rate given by Eq. (3.27) blows up, meaning an infinite reactiveness of the vehicles. Clearly, this is just a mathematical abstraction which schematizes the physical reality, hence it is questionable from the modeling point of view. Notice, however, that for very high values of road occupancy the physics itself of the system is likely to change dramatically. Indeed, vehicles cannot actually withstand arbitrarily large packing levels, as demonstrated by the fact that bump-to-bump configuration is never reached in real traffic. Refraining from dealing with such an issue in this work, we nevertheless observe that if one allows density values only strictly less than the maximum possible threshold then one gets from Eq. (3.27) a physically and mathematically consistent rate of interaction, which does not blow up for $n < 1$.

Concerning the weight function w, Assumption 3.3 fixes all its relevant properties in order for the integrals appearing in the gain and loss operators of the kinetic equations to represent an average of the interactions experienced by the candidate vehicle with the field vehicles within the interaction interval $J_\xi(x)$. As for the specific form of w, different choices can be made according to different possible criteria to evaluate the relevance of the interactions. For merely theoretical purposes, we note here that setting $w(y) = \delta(y)$, the Dirac function centered at the origin, allows a formal *a posteriori* retrieval of the local interactions framework with, in addition, Assumption 3.3 satisfied (possibly in the sense of distributions). Nevertheless, local interactions are hardly compatible with the physical problem at hand, therefore more realistic weight functions fitting in the nonlocal interactions setting need to be sought.

Probably the simplest form of such a w is obtained from the following piecewise constant function:

$$w(y) = \frac{1}{\xi}\chi_{[0,\,\xi]}(y), \tag{3.28}$$

where $\chi_{[0,\,\xi]}$ denotes the indicator function of the interval $[0, \xi]$. Introducing this term in Eq. (3.22) gives a uniform average of the interactions in the interaction interval $J_\xi(x)$. More sophisticated weight functions may be proposed, assuming for instance that the relevance of the interactions decreases with the distance separating the interacting pairs (cf. e.g., Delitala [21]). This amounts to requiring explicitly that w satisfies the further condition of being monotonically decreasing in $[0, \xi]$:

$$w(y_2) \leq w(y_1), \qquad \forall\, y_1, y_2 \in [0, \xi],\ y_1 \leq y_2.$$

As an example, one might consider the family of functions

$$w(y) = ae^{-by}\chi_{[0,\,\xi]}(y)$$

for suitable choices of the constants $a, b > 0$ that guarantee the fulfilment of the conditions expressed by Assumption 3.3.

3.4 Concluding remarks

The form of the table of games and of the interaction rate proposed in the previous Sect. 3.3 deserves several comments from both theoretical and technical points of view.

As a major issue, we observe that the dependence of the table of games and the interaction rate on the vehicle density n introduces a new nonlinearity in the gain and loss operators $G_i[\mathbf{f}, \mathbf{f}]$ and $L_i[\mathbf{f}]$, initially not accounted for by the general discrete velocity framework depicted by Eqs. (3.9)-(3.11). The transition distribution function $\mathcal{F}^{ij}_{hk}$ is indeed originally not supposed to depend in turn on the distribution function $\mathbf{f}$. In other words, strictly holding to Eqs. (3.10), (3.11), the whole dependence of the gain and loss operators on the functions $\{f_i\}_{i=1}^m$ should be expressed by the quadratic term $f_h f_k$ for the former and the linear term f_j for the latter, without any further influence from other terms. However, as the transition distribution function $\mathcal{F}_{hk}^{ij}$ generates the table of games (cf. Eq. (3.13)) and the interaction rate (cf. Eq. (3.12)), this would force a stochastic dynamics of the interactions independent of the spatial and temporal evolution of the local traffic conditions on the road, which seems unrealistic considering the nonclassical nature of the vehicles. Therefore, the choice of envisaging a dependence of the table of games and the interaction rate on the distribution function $\mathbf{f}$ through n, although at present unjustified from the point of view of the classical theoretical framework of reference, is motivated by the necessity to address the modeling of nonclassical subjects featuring a specific ability to adapt their behavior to the instantaneous evolution of the system.

As a minor issue, since for this specific model the interaction rate does not depend on the velocity classes of the interacting vehicles, the i-th loss operator is formally independent of the index i and can be technically specialized as

$$L_i[\mathbf{f}](t,x) = L[\mathbf{f}](t,x) = \int_x^{x+\xi} \bar{\eta}[n](t,x_*)n(t,x_*)w(x_*-x)\,dx_*,$$

so that one can finally consider the evolution equations (3.22) in the equivalent form

$$\begin{aligned}\frac{\partial f_i}{\partial t} + v_i\frac{\partial f_i}{\partial x} = \sum_{h,\,k=1}^{m} \int_x^{x+\xi} \bar{\eta}[n](t,x_*)A_{hk}^i[n](t,x_*)f_h(t,x)f_k(t,x_*)w(x_*-x)\,dx_* \\ - f_i(t,x)\int_x^{x+\xi} \bar{\eta}[n](t,x_*)n(t,x_*)w(x_*-x)\,dx_* \qquad (3.29)\end{aligned}$$

for $i = 1,\dots,m$. However, for notational purposes (see Chapt. 5) and in view of a usual trend in kinetic theories, we will maintain in the sequel the notation L_i for the loss operators of our equations.

Summing the right-hand sides of the above equations on i and taking Eq. (3.19) into account yields

$$\sum_{i=1}^{m} (G_i[\mathbf{f},\mathbf{f}] - f_i L_i[\mathbf{f}]) = 0,$$

which implies

$$\frac{\partial n}{\partial t} + \frac{\partial q}{\partial x} = 0, \qquad (3.30)$$

that is the conservation of mass for the model described by Eq. (3.29).

Chapter 4

The spatially homogeneous problem

4.1 Mathematical setting

In this chapter we address the spatially homogeneous problem, in which the distribution functions $\{f_i\}_{i=1}^m$ are assumed to be independent of the spatial variable x:

$$f_i = f_i(t) : \mathbb{R}_+ \to \mathbb{R}_+, \qquad i = 1, \dots, m.$$

As a consequence, it results

$$\frac{\partial f_i}{\partial x} = 0, \qquad \forall\, i = 1, \dots, m,$$

so that the system of equations (3.29) reduces to

$$\frac{df_i}{dt} = \bar{\eta}[n] \left(\sum_{h,\,k=1}^{m} A^i_{hk}[n] f_h f_k - n f_i \right), \qquad i = 1, \dots, m, \tag{4.1}$$

where we have specifically used the fact that, owing to Assumption 3.3, the weight function w satisfies

$$\int\limits_{x}^{x+\xi} w(x_* - x)\, dx_* = 1,$$

as it can be immediately seen by performing the change of variable $y = x_* - x$ in the above integral.

The mathematical formalization of the spatially homogeneous problem consists in the system of ordinary differential equations (4.1) in the unknowns f_i, supplemented by a suitable set of initial conditions

$$f_i(0) = \varphi_i \in \mathbb{R}_+, \qquad i = 1, \dots, m. \tag{4.2}$$

Like for the distribution functions f_i, we similarly introduce the shorthand vector notation

$$\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_m) \in \mathbb{R}^m$$

to denote at once the m scalar initial conditions of system (4.1).

The spatially homogeneous problem is a good benchmark to test the reliability of the theoretical predictions of the model with respect to the available experimental data, since it provides some information on the trend of the system toward the equilibrium, the so-called *fundamental diagrams*, that can be duly compared with the measurements performed under uniform flow conditions (see e.g., Kerner [32]). Qualitative and computational investigation of fundamental diagrams will be the object of the second part of the analysis presented in this chapter (Sect. 4.3), after establishing the well-posedness of the spatially homogeneous problem.

In order to address the qualitative analysis of Problem (4.1)-(4.2), we introduce, for $T > 0$, the linear space

$$X_T = C([0, T]; \mathbb{R}^m)$$

of the vector-valued continuous functions $\mathbf{u} = \mathbf{u}(t) : [0, T] \to \mathbb{R}^m$, $\mathbf{u}(t) = (u_1(t), \ldots, u_m(t))$, endowed with the ∞-norm

$$\|\mathbf{u}\|_\infty = \sup_{t \in [0,\, T]} \|\mathbf{u}(t)\|_1,$$

where, for every fixed $t \in [0, T]$, the quantity $\|\mathbf{u}(t)\|_1$ denotes the 1-norm of the vector $\mathbf{u}(t)$ in $\mathbb{R}^m$:

$$\|\mathbf{u}(t)\|_1 = \sum_{i=1}^{m} |u_i(t)|, \qquad t \in [0, T].$$

We observe that $(X_T, \|\cdot\|_\infty)$ is a real Banach space. Furthermore, if $\mathbf{u}(t) \geq 0$ then the 1-norm of the vector $\mathbf{u}(t)$ yields the 'mass' $n_{\mathbf{u}}(t)$ of the function $\mathbf{u}$ at time t:

$$\|\mathbf{u}(t)\|_1 = \sum_{i=1}^{m} u_i(t) =: n_{\mathbf{u}}(t).$$

The 1-norm is expected to take advantage more directly of the conservation of mass featured by the system at hand (cf. Sect. 3.4). However, we point out that setting in $\mathbb{R}^m$ a different p-norm (see Delitala and Tosin [22]) does not affect substantially the results we are going to state, up to some technicalities in the proof of the theorems, because in $\mathbb{R}^m$ all norms are equivalent.

Regarding the table of games $A^i_{hk}[n]$ and the unweighted interaction rate $\bar{\eta}[n]$ appearing in Eq. (4.1), we assume that, besides the general modeling properties stated in Chapt. 3, they technically satisfy the following requirements:

Assumption 4.1. *We assume that the dependence of the table of games on the vehicle density is such that*

$$0 \leq A^i_{hk}[n] \leq 1$$

whenever $0 \leq n \leq 1$.

Assumption 4.2. *We assume that for all nonnegative* $K < 1$ *there exists* $C_{\bar{\eta},K} > 0$ *such that*

$$0 < \bar{\eta}[n] \leq C_{\bar{\eta},K}$$

whenever $0 \leq n \leq K$.

Note that the table of games and the unweighted interaction rate proposed in Sect. 3.3 agree with these hypotheses. However, Assumptions 4.1, 4.2 enable one to conceive different stochastic dynamics of the interactions, and to generate then alternative models, for which the qualitative results we are going to establish still hold.

4.2 Well-posedness

Well-posedness of the spatially homogeneous problem means global in time existence and uniqueness of a solution $\mathbf{f} = \mathbf{f}(t)$ to the Cauchy problem (4.1)-(4.2), featuring in addition the following properties:

(i) nonnegativity, i.e., $f_i(t) \geq 0$ for all $t > 0$ and all $i = 1, \ldots, m$;

(ii) uniform in time boundedness, i.e.,

$$\sup_{t>0} \max_{i=1,\ldots,m} f_i(t) < +\infty,$$

related to the confinement of the vehicle density $n(t)$ within a certain maximum threshold linked to the road capacity (cf. Subsect. 3.3.1).

To achieve these results we go through two classical steps: First we prove existence and uniqueness of a local in time solution $\mathbf{f} \in X_{T^*}$ for a certain $T^* > 0$, featuring all the desired characteristics, then extend it to a global solution defined for all $t > 0$.

4.2.1 Local existence

We agree to denote by n_0 the initial mass of the system as fixed by the initial condition $\boldsymbol{\varphi} \geq 0$:

$$n_0 := \|\boldsymbol{\varphi}\|_1 = \sum_{i=1}^{m} \varphi_i.$$

Existence and uniqueness of a local in time solution to Problem (4.1)-(4.2) are now obtained.

Theorem 4.3. *Let Assumptions 4.1, 4.2 hold, and let in addition* $0 \leq n_0 < 1$. *Then there exists* $T^* > 0$ *such that Problem* (4.1)-(4.2) *admits a unique nonnegative local solution* $\mathbf{f} \in X_{T^*}$ *satisfying the a priori estimate*

$$\|\mathbf{f}(t)\|_1 = n_0, \qquad \forall\, t \in (0, T^*]. \tag{4.3}$$

Proof. Let us preliminarily introduce, for any $\mathbf{u} \in X_T$, the antiderivative $N_{\mathbf{u}}(t)$ vanishing for $t = 0$ of the function $\bar{\eta}[n_{\mathbf{u}}](t) n_{\mathbf{u}}(t)$:

$$N_{\mathbf{u}}(t) = \int_0^t \bar{\eta}[n_{\mathbf{u}}](s) n_{\mathbf{u}}(s)\, ds;$$

we agree to drop in the sequel the subscript $\mathbf{u}$ from $n_{\mathbf{u}}$ and $N_{\mathbf{u}}$ whenever referring to these quantities computed for $\mathbf{u} = \mathbf{f}$, i.e., the expected solution to Problem (4.1)-(4.2).

After multiplying both sides of Eq. (4.1) by $e^{N(t)}$ and integrating over $[0, t]$, $0 \le t \le T$, the system (4.1) is formally rewritten as follows:

$$f_i(t) = e^{-N(t)} \varphi_i + \int_0^t e^{N(s)-N(t)} \bar{\eta}[n](s) \sum_{h,\,k=1}^{m} A^i_{hk}[n](s) f_h(s) f_k(s)\, ds. \tag{4.4}$$

We now consider the subset of X_T

$$\mathcal{B}_T = \{\mathbf{u} \in X_T \,:\, \mathbf{u}(t) \ge 0,\, n_{\mathbf{u}}(t) = n_0 \text{ for all } t \in [0, T]\}$$

and the operator S defined componentwise on X_T as

$$(S\mathbf{u})_i(t) = e^{-N_{\mathbf{u}}(t)} \varphi_i + \int_0^t e^{N_{\mathbf{u}}(s)-N_{\mathbf{u}}(t)} \bar{\eta}[n_{\mathbf{u}}](s) \sum_{h,\,k=1}^{m} A^i_{hk}[n_{\mathbf{u}}](s) u_h(s) u_k(s)\, ds$$

for $\mathbf{u} = (u_1, \ldots, u_m) \in X_T$.

From Eq. (4.4) we deduce that any possible solution in $\mathcal{B}_T$ to Problem (4.1)-(4.2) corresponds to a fixed point of S on $\mathcal{B}_T$. In particular, if $n_0 = 0$ then $\boldsymbol{\varphi} = 0$, and the set $\mathcal{B}_T$ reduces to $\{0\}$. In this case, the function $\mathbf{f} = 0$ is trivially the unique fixed point of S on $\mathcal{B}_T$, i.e., the unique local in time solution to Problem (4.1)-(4.2) corresponding to zero initial conditions, and the proof is completed. Therefore we hereafter assume $n_0 > 0$.

Let us point out the following:

(i) For all $\mathbf{u} \in \mathcal{B}_T$ we have $N_{\mathbf{u}}(t) = \bar{\eta}[n_0] n_0 t$, where $0 < \bar{\eta}[n_0] \le C_{\bar{\eta},n_0}$ is well-defined (i.e., not infinite) in view of Assumption 4.2 because $n_0 < 1$, and constant. As a consequence, the operator S on $\mathcal{B}_T$ takes the form

$$(S\mathbf{u})_i(t) = e^{-\bar{\eta}[n_0] n_0 t} \varphi_i + \bar{\eta}[n_0] \int_0^t e^{\bar{\eta}[n_0] n_0 (s-t)} \sum_{h,\,k=1}^{m} A^i_{hk}[n_0] u_h(s) u_k(s)\, ds. \tag{4.5}$$

In addition, the coefficients $A^i_{hk}[n_0]$ of the table of games are constant and bounded between 0 and 1 owing to Assumption 4.1.

(ii) S maps $\mathcal{B}_T$ into itself. Indeed, given $\mathbf{u} \in \mathcal{B}_T$ it is immediate to check from Eq. (4.5) that $S\mathbf{u} \in X_T$, due to the well-known continuity properties of the integral and of the exponential function, and that $(S\mathbf{u})(t) \geq 0$ for all $t \geq 0$, due to the nonnegativity of the initial condition $\boldsymbol{\varphi}$. Moreover, by a direct computation we obtain

$$\begin{aligned}\sum_{i=1}^{m}(S\mathbf{u})_i(t) &= n_0 e^{-\bar{\eta}[n_0]n_0 t} + \bar{\eta}[n_0]\int\limits_0^t e^{\bar{\eta}[n_0]n_0(s-t)} \sum_{h,\,k=1}^{m} u_h(s)u_k(s)\,ds \\ &= n_0 e^{-\bar{\eta}[n_0]n_0 t} + n_0\left(1 - e^{-\bar{\eta}[n_0]n_0 t}\right) = n_0,\end{aligned}$$

which confirms that $S(\mathcal{B}_T) \subseteq \mathcal{B}_T$.

(iii) The set $\mathcal{B}_T$ is closed in X_T, each $T > 0$. Indeed, let $\{\mathbf{u}^{[k]}\}_{k\in\mathbb{N}} \subseteq \mathcal{B}_T$ be a sequence converging to some $\bar{\mathbf{u}} \in X_T$, i.e., $\|\bar{\mathbf{u}} - \mathbf{u}^{[k]}\|_\infty \to 0$ when $k \to \infty$. Then $\mathbf{u}^{[k]}(t)$ converges pointwise to $\bar{\mathbf{u}}(t)$ on $[0, T]$, i.e., $\|\bar{\mathbf{u}}(t) - \mathbf{u}^{[k]}(t)\|_1 \to 0$ when $k \to \infty$ for all $t \in [0, T]$, which in turn implies $|\bar{u}_i(t) - u_i^{[k]}(t)| \to 0$ as $k \to \infty$ for all $i = 1, \ldots, m$ and all $t \in [0, T]$. Since $u_i^{[k]}(t) \geq 0$, this says that $\bar{u}_i(t) \geq 0$. Moreover:

$$\left|\sum_{i=1}^{m} \bar{u}_i(t) - n_0\right| = \left|\sum_{i=1}^{m} \bar{u}_i(t) - \sum_{i=1}^{m} u_i^{[k]}(t)\right| \leq \|\bar{\mathbf{u}}(t) - \mathbf{u}^{[k]}(t)\|_1,$$

whence, considering that the right-hand side of the above inequality can be made arbitrarily small for large enough k, we conclude $\sum_{i=1}^{m} \bar{u}_i(t) = n_0$ for all $t \in [0, T]$, and finally $\bar{\mathbf{u}} \in \mathcal{B}_T$.

In view of the previous points (i)-(iii), it is sufficient to prove that, for a suitable choice of T, the mapping $S : \mathcal{B}_T \to \mathcal{B}_T$ is a contraction to obtain from Banach Fixed Point Theorem the existence and uniqueness of a local in time solution to Problem (4.1)-(4.2). To this end, we introduce the operator $A : \mathcal{B}_T \to X_T$ defined componentwise by

$$(A\mathbf{u})_i(t) = e^{\bar{\eta}[n_0]n_0 t} \sum_{h,\,k=1}^{m} A_{hk}^i[n_0]u_h(t)u_k(t).$$

Therefore, one has

$$(S\mathbf{u})(t) = e^{-\bar{\eta}[n_0]n_0 t}\left(\boldsymbol{\varphi} + \bar{\eta}[n_0]\int\limits_0^t (A\mathbf{u})(s)\,ds\right),$$

whence, for $\mathbf{u}, \mathbf{v} \in \mathcal{B}_T$,

$$\|(S\mathbf{u})(t) - (S\mathbf{v})(t)\|_1 \leq \bar{\eta}[n_0]e^{-\bar{\eta}[n_0]n_0 t}\int\limits_0^t \|(A\mathbf{u})(s) - (A\mathbf{v})(s)\|_1\,ds,$$

which, recalling Assumption 4.2 and taking the supremum over $t \in [0, T]$ of both sides, entails

$$\|S\mathbf{u} - S\mathbf{v}\|_\infty \leq C_{\bar{\eta},n_0} T \|A\mathbf{u} - A\mathbf{v}\|_\infty. \tag{4.6}$$

If we can show that A is Lipschitz on $\mathcal{B}_T$ (though not necessarily a contraction), Eq. (4.6) will provide suitable conditions on T that make S a contraction on $\mathcal{B}_T$. For this we compute:

$$\begin{aligned}\|(A\mathbf{u})(t)-(A\mathbf{v})(t)\|_1 &= e^{\bar{\eta}[n_0]n_0 t}\sum_{i=1}^{m}\left|\sum_{h,\,k=1}^{m}A^i_{hk}[n_0]\left[u_h(t)u_k(t)-v_h(t)v_k(t)\right]\right| \\ &\leq e^{\bar{\eta}[n_0]n_0 t}\sum_{h,\,k=1}^{m}|u_h(t)u_k(t)-v_h(t)v_k(t)|;\end{aligned}$$

adding and subtracting $u_h(t)v_k(t)$ yields

$$\begin{aligned}&\leq e^{\bar{\eta}[n_0]n_0 t}\left(\sum_{h=1}^{m}|u_h(t)|\sum_{k=1}^{m}|u_k(t)-v_k(t)|\right.\\ &\quad\left.+\sum_{k=1}^{m}|v_k(t)|\sum_{h=1}^{m}|u_h(t)-v_h(t)|\right)\\ &= e^{\bar{\eta}[n_0]n_0 t}(n_{\mathbf{u}}(t)+n_{\mathbf{v}}(t))\|\mathbf{u}(t)-\mathbf{v}(t)\|_1\\ &= 2n_0 e^{\bar{\eta}[n_0]n_0 t}\|\mathbf{u}(t)-\mathbf{v}(t)\|_1,\end{aligned}$$

thus

$$\|A\mathbf{u}-A\mathbf{v}\|_\infty \leq 2n_0 e^{\bar{\eta}[n_0]n_0 T}\|\mathbf{u}-\mathbf{v}\|_\infty.$$

Owing to Eq. (4.6) we deduce

$$\|S\mathbf{u}-S\mathbf{v}\|_\infty \leq 2n_0 C_{\bar{\eta},n_0}Te^{\bar{\eta}[n_0]n_0 T}\|\mathbf{u}-\mathbf{v}\|_\infty,$$

therefore S is a contraction on $\mathcal{B}_T$ provided $T>0$ is chosen in such a way that

$$Te^{\bar{\eta}[n_0]n_0 T} < \frac{1}{2n_0 C_{\bar{\eta},n_0}}.$$

As the function $t \mapsto te^{\bar{\eta}[n_0]n_0 t}$ is continuous and strictly increasing for $t \geq 0$, vanishes for $t=0$, and tends to $+\infty$ for $t \to +\infty$, there exists at least a positive value of T, say $T=T^*$, satisfying the above inequality. Thus we conclude that Problem (4.1)-(4.2) admits a unique local in time solution $\mathbf{f} \in \mathcal{B}_{T^*}$ such that

$$\|\mathbf{f}(t)\|_1 = n(t) = n_0, \qquad \forall\, t \in (0,T^*],$$

which completes the proof. □

The *a priori* estimate (4.3) has its physical counterpart in the conservation of the vehicle mass fulfilled by the system, as it can be further checked by summing Eq. (4.1) over i: one gets

$$\frac{d}{dt}\sum_{i=1}^{m}f_i(t)=0,$$

which states precisely that for the local in time solution $\mathbf{f}$ the macroscopic mass n is constant in t. We observe moreover that this *a priori* estimate can be easily converted into an estimate for the local solution $\mathbf{f}$ in X_{T^*}, in fact:

$$\|\mathbf{f}\|_\infty = \sup_{t\in[0,\,T^*]} \|\mathbf{f}(t)\|_1 = n_0.$$

Before ending this subsection, we want to stress the importance of the hypotheses of Theorem 4.3. The boundedness of the initial density n_0 between 0 and 1 is crucial to have a table of games $A^i_{hk}[n]$ consistent with the probabilistic interpretation discussed in Sect. 3.3, hence for the Cauchy problem (4.1)-(4.2) to make sense with respect to the general framework within which it is conceived. This issue is not present in other works sharing with the present one a similar framework, like e.g., Bertotti and Delitala [13] or Coscia *et al.* [18], because there the table of games is not assumed to depend on the solution $\mathbf{f}$ itself of the problem.

4.2.2 Global existence

The existence of a local in time solution $\mathbf{f}(t)$ to Problem (4.1)-(4.2) and the *a priori* estimate (4.3) allow to extend $\mathbf{f}(t)$ on the whole positive real axis $\mathbb{R}_+$. In fact, we can prove:

Theorem 4.4. *Under the same hypotheses of Theorem 4.3, one can take $T^* = +\infty$, that is Problem* (4.1)-(4.2) *admits a unique nonnegative global solution* $\mathbf{f} \in C(\mathbb{R}_+; \mathbb{R}^m)$ *satisfying*

$$\|\mathbf{f}(t)\|_1 = n_0, \qquad \forall\, t > 0.$$

Proof. It suffices to apply the same reasoning developed in the proof of Theorem 4.3 on the interval $(T^*, 2T^*]$, taking $\mathbf{f}(T^*)$ as new initial condition. Since $f_i(T^*) \geq 0$ for all $i = 1, \ldots, m$, and moreover

$$\sum_{i=1}^{m} f_i(T^*) = n_0 \in [0, 1),$$

we are in the same hypotheses of Theorem 4.3, hence we conclude on the existence and uniqueness of a local in time continuous solution on $(T^*, 2T^*]$ satisfying

$$\|\mathbf{f}(t)\|_1 = n_0, \qquad \forall\, t \in (T^*, 2T^*].$$

Therefore, the local solution $\mathbf{f} \in \mathcal{B}_{T^*}$ to Problem (4.1)-(4.2) given by Theorem 4.3 can be uniquely extended to a local solution $\mathbf{f} \in \mathcal{B}_{2T^*}$. Iterating this reasoning on all intervals of the form $(kT^*, (k+1)T^*]$, $k \in \mathbb{N}$, we construct the global solution on $\mathbb{R}_+$ and we are done. □

The *a priori* estimate on the global solution implies its uniform in time boundedness, in fact, observing that $f_i(t) \leq \|\mathbf{f}(t)\|_1$ for all $t > 0$ and all $i = 1, \ldots, m$ due to the nonnegativity of each f_i, we have

$$\sup_{t>0} \max_{i=1,\,\ldots,\,m} f_i(t) \leq \sup_{t>0} \|\mathbf{f}(t)\|_1 = n_0 < +\infty. \tag{4.7}$$

As far as the regularity of the solution is concerned, we observe that, strictly speaking, Theorem 4.4 defines $\mathbf{f}$ as a mild solution of Problem (4.1)-(4.2), indeed it asserts the continuity in time of $\mathbf{f}$ but not its differentiability. Actually, this is a consequence of the fact that both local and global well-posedness of the spatially homogeneous problem have been addressed by Theorems 4.3 and 4.4 on the basis of Eq. (4.4), which, being a reformulation of the Cauchy problem (4.1)-(4.2) in an integral, thus weak, form, only requires the continuity of the functions f_i. However, classical (and even more) regularity of $\mathbf{f}$ can be straightforwardly recovered *a posteriori* as follows:

Corollary 4.5. *The solution* $\mathbf{f}$ *to Problem* (4.1)-(4.2) *is of class* C^∞ *on* $\mathbb{R}_+$.

Proof. The continuity of $\mathbf{f}$ on $\mathbb{R}_+$ implies that the right-hand side of Eq. (4.1) is continuous in $t \geq 0$ for each $i = 1, \dots, m$, that is $f_i' \in C(\mathbb{R}_+)$ each i. Differentiating once Eq. (4.1) with respect to t gives, by a similar reasoning, $f_i'' \in C(\mathbb{R}_+)$ each i. Proceeding inductively this way finally yields the thesis. □

4.3 Fundamental diagrams

Fundamental diagrams show the dependence of some macroscopic quantities of interest, like e.g., the vehicle flux q (cf. Eq. (3.4)) or the average velocity u (cf. Eq. (3.5)), on the density n at the equilibrium, i.e., under uniform homogeneous flow conditions. Technically, the name 'fundamental diagram' should be used specifically for the diagram of q vs. n, while the diagram of u vs. n should be called more properly *velocity diagram*. However, for the sake of simplicity, in the present context we will use the term fundamental diagram to refer to any diagram relating a certain macroscopic quantity to the vehicle density at the equilibrium.

The use of fundamental diagrams is especially popular in first order hydrodynamic traffic models, where they are employed to devise a phenomenological closure $q = q(n)$ (or, alternatively, $u = u(n)$, considering that $q = nu$) of the macroscopic mass conservation equation (1.2), see also Eq. (1.3). Some of them, trying to mimic the experimentally computed fundamental diagrams (examples of which are reported in the Introduction of the book by Kerner [32] for several US highways), can be found in the works by Bonzani [14], Bonzani and Mussone [15], and Whitham [53], as well as in the review paper by Bellomo and Coscia [6].

Kinetic models do not rely instead on this procedure to get a self-consistent system of evolution equations, therefore in this case fundamental diagrams can be studied *a posteriori* as a by-product of the model itself. In particular, this can be done by means of the spatially homogeneous problem, because the possible equilibria of system (4.1) identify precisely the points of the fundamental diagrams corresponding to a certain initially fixed macroscopic density n_0.

One consequence of Theorem 4.4 is that the set

$$B = \{\mathbf{x} \in \mathbb{R}^m \,:\, \mathbf{x} \geq 0,\, \|\mathbf{x}\|_1 = n_0 \in [0, 1)\} \tag{4.8}$$

is positively invariant for the system (4.1): If $\mathbf{f}(0) = \boldsymbol{\varphi} \in B$, then $\mathbf{f}(t) \in B$ for all $t > 0$. On the other hand, we have already discussed in Subsect. 4.2.1 the necessity to choose the initial condition (4.2) in B, so that this subset of the state space $\mathbb{R}^m$ turns out to be the only one relevant for our purposes.

According to the discussion above, in order to plot fundamental diagrams we need to investigate and characterize the equilibria of system (4.1), that is the points $\mathbf{x} \in B$ such that if $\boldsymbol{\varphi} = \mathbf{x}$ then $\mathbf{f}(t) = \mathbf{x}$ for all $t > 0$. More specifically, we are preliminarily interested in their existence and uniqueness. We anticipate that we will be much more exhaustive on the first issue, while for the second one we will essentially limit ourselves to the analysis of particular cases, simple enough to make the analytical investigation affordable but at the same time quite representative of the main features of the system.

We begin by stating a result that says that equilibria actually exist for our system.

Theorem 4.6. *The system* (4.1) *has at least one equilibrium point in the set* B *defined by Eq.* (4.8)*.*

Proof. The equilibria of the system (4.1) on B are given by

$$\sum_{h,\,k=1}^{m} A^i_{hk}[n_0]x_h x_k - n_0 x_i = 0, \qquad i = 1,\,\dots,m. \tag{4.9}$$

Notice in particular that, due to $n_0 < 1$ and to Assumption 4.2 on the unweighted interaction rate, the term $\bar{\eta}[n_0]$ is a constant to be hidden in the time scale.

If $n_0 = 0$ then the set B reduces to the point $\mathbf{x} = 0$, which is actually an equilibrium. Otherwise, if $0 < n_0 < 1$ then any solution to Eq. (4.9) can be viewed as a fixed point of the operator $S : B \to \mathbb{R}^m$ defined componentwise as

$$(S\mathbf{x})_i = \frac{1}{n_0}\sum_{h,\,k=1}^{m} A^i_{hk}[n_0]x_h x_k, \qquad i = 1,\,\dots,m.$$

Looking more closely at the set B, we discover the following characteristics:

(i) B is bounded, indeed it is contained into the (closed) ball of $\mathbb{R}^m$ with radius n_0 and center at the origin.

(ii) B is closed in $\mathbb{R}^m$. Let $\{\mathbf{x}^{[k]}\}_{k\in\mathbb{N}} \subseteq B$ be a sequence converging to some $\bar{\mathbf{x}} \in \mathbb{R}^m$, that is $\|\bar{\mathbf{x}} - \mathbf{x}^{[k]}\|_1 \to 0$ when $k \to \infty$. To see that $\bar{\mathbf{x}} \in B$ we first observe that $|\bar{x}_i - x^{[k]}_i| \to 0$ when $k \to \infty$ for all $i = 1,\dots,m$, which, owing to $x^{[k]}_i \geq 0$, implies $\bar{x}_i \geq 0$. Furthermore,

$$\big|\|\bar{\mathbf{x}}\|_1 - n_0\big| = \left|\sum_{i=1}^{m}\bar{x}_i - \sum_{i=1}^{m}x^{[k]}_i\right| \leq \|\bar{\mathbf{x}} - \mathbf{x}^{[k]}\|_1,$$

whence $\|\bar{\mathbf{x}}\|_1 = n_0$ as the right-hand side of the above inequality can be made arbitrarily small for sufficiently high k.

Since $B \subseteq \mathbb{R}^m$, properties (i) and (ii) say that B is compact. In addition:

(iii) B is convex. Let $\mathbf{x}, \mathbf{y} \in B$ and let us define $\mathbf{z}_\lambda \in \mathbb{R}^m$ as a convex linear combination of $\mathbf{x}$ and $\mathbf{y}$ via a parameter $\lambda \in [0,1]$, that is $\mathbf{z}_\lambda = \lambda\mathbf{x} + (1-\lambda)\mathbf{y}$. To see that $\mathbf{z}_\lambda \in B$ for all $\lambda \in [0,1]$ we first check that $\mathbf{z}_\lambda \geq 0$, then, owing to the nonnegativity of $\mathbf{x}, \mathbf{y}$, we further obtain $\|\mathbf{z}_\lambda\|_1 = \lambda\|\mathbf{x}\|_1 + (1-\lambda)\|\mathbf{y}\|_1 = n_0$.

As for the operator S, we instead observe that it maps B into itself: Using the properties of the table of games expressed by Eq. (3.19) we immediately get the nonnegativity of $S\mathbf{x}$ and also $\sum_{i=1}^m (S\mathbf{x})_i = \frac{1}{n_0} n_0^2 = n_0$, thus $S(B) \subseteq B$. In addition, for $\mathbf{x}, \mathbf{y} \in B$ it results

$$\|S\mathbf{x} - S\mathbf{y}\|_1 = \sum_{i=1}^{m} |(S\mathbf{x})_i - (S\mathbf{y})_i| \leq \frac{1}{n_0} \sum_{h,\,k=1}^{m} |x_h x_k - y_h y_k|$$

whence, adding and subtracting $x_h y_k$,

$$\leq \frac{1}{n_0}(\|\mathbf{x}\|_1 + \|\mathbf{y}\|_1)\|\mathbf{x} - \mathbf{y}\|_1 = 2\|\mathbf{x} - \mathbf{y}\|_1,$$

which implies the continuity (actually, the Lipschitz continuity) of S on B.

In view of the above reasoning we are in a position to apply Brouwer Fixed Point Theorem, which guarantees existence of fixed points of S in B, and thus leads us to the thesis. □

As it often happens when using a fixed point technique, Theorem 4.6 gives existence but not uniqueness of equilibria in B. As a matter of fact, uniqueness cannot be ensured for all possible choices of the coefficients of the table of games, for neither hypotheses (3.19) nor Assumption 4.1 is in general sufficient to exclude multiple equilibria.

As a simple example, assume that the table of games is such that

$$A_{hh}^h[n_0] = 1, \qquad \forall\, h = 1, \dots, m,$$

and, by consequence,

$$A_{hh}^i[n_0] = 0, \qquad \forall\, i \neq h.$$

For instance, this may be obtained from the structure presented in Subsect. 3.3.1 by setting $\alpha = 0$ in Eqs. (3.23)-(3.26). Consider then the point $\mathbf{x}^{i_0} \in B$ whose components are given by

$$x_i^{i_0} = \begin{cases} n_0 & \text{if } i = i_0 \\ 0 & \text{otherwise} \end{cases}$$

for a certain fixed $i_0 \in \{1, \dots, m\}$ and for $0 < n_0 < 1$. Referring to the operator S defined in the proof of Theorem 4.6, it is easy to compute $(S\mathbf{x}^{i_0})_i = A_{i_0,i_0}^i[n_0]n_0$ for all $i = 1, \dots, m$, hence finally

$$(S\mathbf{x}^{i_0})_i = \begin{cases} n_0 & \text{if } i = i_0 \\ 0 & \text{otherwise,} \end{cases}$$

that is $S\mathbf{x}^{i_0} = \mathbf{x}^{i_0}$. This means that any trivial distribution $\mathbf{x}^{i_0}$ in which all vehicles are traveling at the same speed in a unique cluster is a fixed point of S for all $i_0 = 1, \ldots, m$, hence in this case the system has at least m admissible equilibria.

Conversely, if there exists $i_0 \in \{1, \ldots, m\}$ such that $A^{i_0}_{i_0,i_0}[n_0] < 1$ then the corresponding trivial distribution is no longer an equilibrium point, in fact in this case we have $(S\mathbf{x}^{i_0})_{i_0} = A^{i_0}_{i_0,i_0}[n_0]n_0 < n_0 = x^{i_0}_{i_0}$. In particular, if one chooses $\alpha > 0$ in Eqs. (3.23)-(3.26) then

$$A^h_{hh}[n_0] < 1, \qquad \forall\, h = 1, \ldots, m,$$

thus none of the previous points $\mathbf{x}^{i_0}$ is a possible equilibrium for the system. In other words, in this case vehicles do not tend to concentrate in a cluster of constant velocity but undergo the speed spread discussed in Subsect. 3.3.1.

Apart from these general considerations and in view of the difficulty to address the full m-dimensional case, we now analyze in detail the equilibrium configurations of the system in the relatively simple cases $m = 2$, $m = 3$, with specific reference to the table of games presented in Chapt. 3. Although somehow limiting from the modeling point of view, these cases study give however a quite good idea of the complexity of the dynamics of the equilibria, highlighting at the same time some key issues they can be sensibly affected by.

4.3.1 The case study $m = 2$

For $m = 2$ the velocity grid is constituted by two points only, $v_1 = 0$ and $v_2 = 1$, therefore each vehicle is characterized by a binary state: It is either staying or moving, without any differentiation between fast and slow vehicles.

Given a density $n_0 \in (0, 1)$, the set B where we look for equilibria is

$$B = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = n_0\}.$$

Due to the conservation of mass fulfilled by the system, with the substitution $x_2 = n_0 - x_1$ in Eq. (4.9) the problem reduces to find the solutions to the following single equation

$$(A^1_{11} - A^1_{12} - A^1_{21} + A^1_{22})x_1^2 - (1 - A^1_{12} - A^1_{21} + 2A^1_{22})n_0 x_1 + A^1_{22} n_0^2 = 0$$

in the unknown $x_1 \in [0, n_0]$, which, according to the definition of the coefficients $A^i_{hk}[n_0]$ discussed in Subsect. 3.3.1, further specializes to

$$(\alpha - 1)x_1^2 - (2\alpha - 1)n_0 x_1 + \alpha n_0^3 = 0. \tag{4.10}$$

If $\alpha = 0$, whence $A^1_{11}[n_0] = A^2_{22}[n_0] = 1$, the solutions to this equation are $x_1 = 0$, $x_1 = n_0$, corresponding to the trivial equilibrium distributions $(0, n_0)$ and $(n_0, 0)$ already predicted by the general considerations developed above.

For $\alpha > 0$, it is instead convenient to introduce the function $F : \mathbb{R}^2 \to \mathbb{R}$ defined by the left-hand side of Eq. (4.10), namely

$$F(x_1, \alpha) = (\alpha - 1)x_1^2 - (2\alpha - 1)n_0 x_1 + \alpha n_0^3,$$

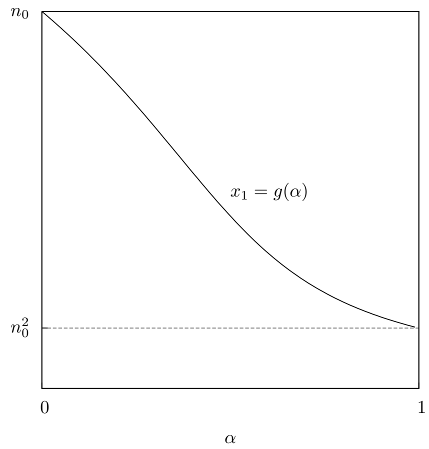

Figure 4.1. The equilibrium point x_1 as a function of α in the case $m = 2$.

which is such that

$$\frac{\partial F}{\partial x_1}(x_1, \alpha) = 2(\alpha - 1)x_1 - (2\alpha - 1)n_0.$$

Since $(\partial_{x_1} F)(0,0) = -(\partial_{x_1} F)(n_0, 0) = n_0$, and recalling that we are working under the hypothesis $n_0 \neq 0$, we infer that in suitable neighborhoods of the points $(x_1, \alpha) = (0, 0)$, $(x_1, \alpha) = (n_0, 0)$ Eq. (4.10), which can be read in terms of F as $F(x_1, \alpha) = 0$, defines implicitly x_1 as a function of α. In detail, there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that about $\alpha = 0$ one has $x_1 = f(\alpha)$ with $f(0) = 0$ and $x_1 = g(\alpha)$ with $g(0) = n_0$, respectively. The Implicit Function Theorem enables one to compute the derivatives of f and of g for $\alpha = 0$:

$$f'(0) = -\frac{(\partial_\alpha F)(0,0)}{(\partial_{x_1} F)(0,0)} = -n_0^2 < 0, \qquad g'(0) = -\frac{(\partial_\alpha F)(n_0, 0)}{(\partial_{x_1} F)(n_0, 0)} = -n_0(1 - n_0) < 0.$$

Since they are both negative, we see that for sufficiently small α the point $x_1 = 0$ moves toward negative values (indeed $f(\alpha) = f'(0)\alpha + o(\alpha)$ for $\alpha \to 0^+$), hence it leaves the interval $[0, n_0]$ and the corresponding equilibrium disappears, while the point $x_1 = n_0$ moves in the interior of $[0, n_0]$ (in fact $g(\alpha) = n_0 + g'(0)\alpha + o(\alpha)$ for $\alpha \to 0^+$) and still defines an admissible equilibrium.

In this case it is easy to compute explicitly the function g by solving the second order polynomial equation (4.10):

$$g(\alpha) = \begin{cases} n_0 \dfrac{2\alpha - 1 - \sqrt{1 + 4\alpha(\alpha - 1)(1 - n_0)}}{2(\alpha - 1)} & \text{if } 0 \leq \alpha < 1, \\ n_0^2 & \text{if } \alpha = 1; \end{cases}$$

the graph of g depicted in Fig. 4.1 confirms the theoretical prediction obtained above and shows that the (right) neighborhood of $\alpha = 0$ in which they hold actually coincides with

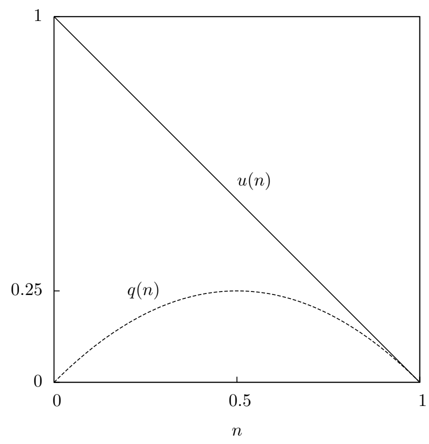

Figure 4.2. The classical fundamental diagrams with linear velocity and parabolic flux profiles, extensively used by many Authors in hydrodynamic first order traffic models, are obtained here as equilibria of our discrete kinetic model with two velocities only.

the whole interval $[0,1]$, since the point $x_1 = g(\alpha)$ remains in the admissible domain $[0, n_0]$ for all $\alpha \in [0,1]$.

Note that letting $\alpha = 1$ yields $x_1 = n_0^2$ and consequently $x_2 = n_0 - n_0^2$, therefore the average velocity u and the macroscopic flux q at the equilibrium are expressed in terms of the vehicle density n as

$$u(n) = 1 - n, \qquad q(n) = n(1 - n).$$

These functions, plotted in Fig. 4.2, correspond to the linear velocity and parabolic flux profiles often cited as prototypes of closure relations of the mass conservation equation in first order hydrodynamic traffic models. As a consequence, we can look at those models under a new point of view, arguing that they presuppose a quite poor dynamics of the microscopic interactions among the vehicles however hidden by average macroscopic modeling.

4.3.2 The case study $m = 3$

For $m = 3$ the velocity grid consists of three points, which, according to Eq. (3.1), are $v_1 = 0$, $v_2 = \frac{1}{2}$, $v_3 = 1$. The dynamics of the interactions is now richer than in the previous case, since three state classes are represented, with possible differentiation between fast and slow vehicles.

Given a density $n_0 \in (0,1)$, the set B becomes specifically

$$B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 = n_0\}.$$

Furthermore, after the substitution $x_3 = n_0 - x_1 - x_2$, Eq. (4.9) originates the following

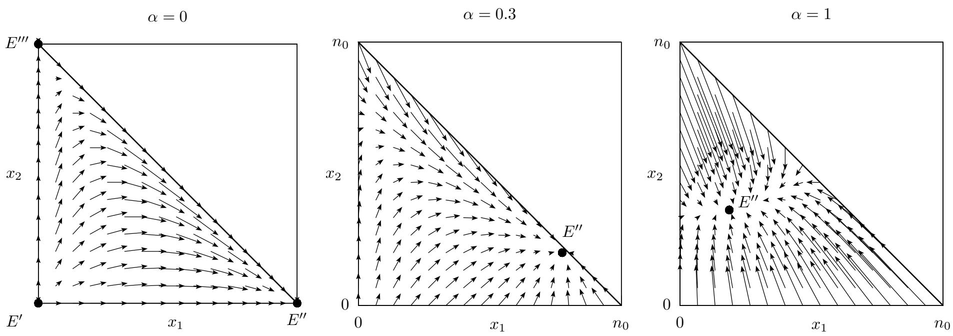

Figure 4.3. The dependence of the equilibrium point (x_1, x_2) on α in the case $m = 3$. The arrows represent the vector field $F = (F_1, F_2)$ as a function of (x_1, x_2).

system of two equations in the unknowns $x_1, x_2 \in [0, n_0]$:

$$\left\{ \begin{array}{rcl} [\alpha(1-n_0)-1]x_1^2 + \alpha n_0 x_2^2 + [1-\alpha(1-n_0)]n_0 x_1 & = & 0, \\ \alpha n_0 x_1^2 + [\alpha(3-n_0)-2]x_1 x_2 + [\alpha(1-n_0)-1]x_2^2 & & \\ +\alpha n_0(1-3n_0)x_1 + (1-\alpha)n_0 x_2 + \alpha n_0^3 & = & 0. \end{array} \right. \tag{4.11}$$

By the same previous technique, we try to characterize how the parameter α influences the equilibria of the system.

For $\alpha = 0$ the following three solutions to Eq. (4.11) are found:

$$E' = (0,0), \quad E'' = (n_0, 0), \quad E''' = (0, n_0),$$

which constitute the vertexes of the triangle obtained by projecting $B \subset \mathbb{R}^3$ onto the plane $x_3 = 0$ (see Fig. 4.3).

In order to treat the case $\alpha > 0$, we introduce the vector-valued function $F : \mathbb{R}^3 \to \mathbb{R}^2$, $F = (F_1, F_2)$, defined by

$$\begin{aligned} F_1(x_1, x_2, \alpha) &= [\alpha(1-n_0)-1]x_1^2 + \alpha n_0 x_2^2 + [1-\alpha(1-n_0)]n_0 x_1, \\ F_2(x_1, x_2, \alpha) &= \alpha n_0 x_1^2 + [\alpha(3-n_0)-2]x_1 x_2 + [\alpha(1-n_0)-1]x_2^2 \\ &\quad + \alpha n_0(1-3n_0)x_1 + (1-\alpha)n_0 x_2 + \alpha n_0^3, \end{aligned}$$

and compute its Jacobian matrix with respect to x_1, x_2 in E', E'', E''' for $\alpha = 0$:

$$(D_{(x_1,x_2)}F) = (E', 0) = -(D_{(x_1,x_2)}F)(E'', 0) = \begin{pmatrix} n_0 & 0 \\ 0 & n_0 \end{pmatrix},$$

$$(D_{(x_1,x_2)}F)(E''', 0) = \begin{pmatrix} n_0 & 0 \\ -2n_0 & -n_0 \end{pmatrix}.$$

Since for $n_0 \neq 0$ none of these three matrices is singular, the system (4.11), which can be read in terms of the function F as $F(x_1, x_2, \alpha) = 0$, defines implicitly (x_1, x_2) as a function of α in a neighborhood of $\alpha = 0$ for each of the points E', E'', E'''. More specifically, there exist vector-valued functions $f, g, h : \mathbb{R} \to \mathbb{R}^2$ such that $(x_1, x_2) = f(\alpha), g(\alpha), h(\alpha)$, respectively, with moreover $f(0) = E'$, $g(0) = E''$, $h(0) = E'''$. As before, we compute now the derivatives $f'(0)$, $g'(0)$, $h'(0)$ in order to discover how the equilibria E', E'', E''' are perturbed by α:

$$f'(0) = -[(D_{(x_1,x_2)}F)(E',0)]^{-1}\frac{\partial F}{\partial \alpha}(E',0) = -(0, n_0^2),$$

$$g'(0) = -[(D_{(x_1,x_2)}F)(E'',0)]^{-1}\frac{\partial F}{\partial \alpha}(E'',0) = n_0(1-n_0)(-1,1),$$

$$h'(0) = -[(D_{(x_1,x_2)}F)(E''',0)]^{-1}\frac{\partial F}{\partial \alpha}(E''',0) = n_0(-n_0,1).$$

For $n_0 \neq 0$ both vectors $f'(0)$ and $h'(0)$ point outward the triangle $E'E''E'''$; recalling

$$f(\alpha) = E' + f'(0)\alpha + o(\alpha), \quad h(\alpha) = E''' + h'(0)\alpha + o(\alpha) \qquad (\alpha \to 0^+),$$

we see that that for $\alpha > 0$ sufficiently small the equilibria E', E''' disappear. Conversely, the vector $g'(0)$ points inward the triangle (more precisely, along the direction $E''E'''$), therefore E'' moves toward a still admissible and unique (at least for small α) equilibrium.

Figure 4.3 shows how E'' moves toward the interior of the triangle $E'E''E'''$ and confirms the theoretical results obtained in this subsection, implying at the same time that for all $0 < \alpha \leq 1$ the point E'' remains indeed the unique stable equilibrium of the system.

4.3.3 Computational analysis and the phase transition

Numerical simulations of Eq. (4.1) have been carried out to obtain the fundamental diagrams relating the average velocity u, the macroscopic flux q, and also the variance of the velocity Θ (cf. Eq. (3.6)) to the vehicle density n at the equilibrium. Time integration has been performed by a standard fourth-order Runge-Kutta scheme, using a uniform velocity grid with $m = 6$ velocity classes ($v_1 = 0, \ldots, v_6 = 1$).

Figure 4.4 shows the numerical results for three different values of the parameter α, namely $\alpha = 0.3$, $\alpha = 0.6$, $\alpha = 1$, corresponding to different road conditions. Considering in particular the cases of intermediate and good roads ($\alpha = 0.6$ and $\alpha = 1$, respectively), we observe that for low density the flux q exhibits an almost linear behavior, which is in agreement with the experimental observations reported by Kerner [32] under free flow conditions. Its subsequent strongly nonlinear decrease to zero suggests a critical change in the characteristics of the traffic for high density, that we interpret as the well-known *phase transition* between the free and congested flow regimes, experimentally described by Kerner himself and mathematically studied by some Authors, like e.g., Colombo [17], in the framework of macroscopic hyperbolic models.

As a further confirmation of the ability of the model to capture such phase transition, we note that the maximum of the variance of the velocity Θ is located precisely in correspondence of the density value for which the change in the flux behavior occurs. In addition, the average velocity u rapidly switches from a nearly constant value for low

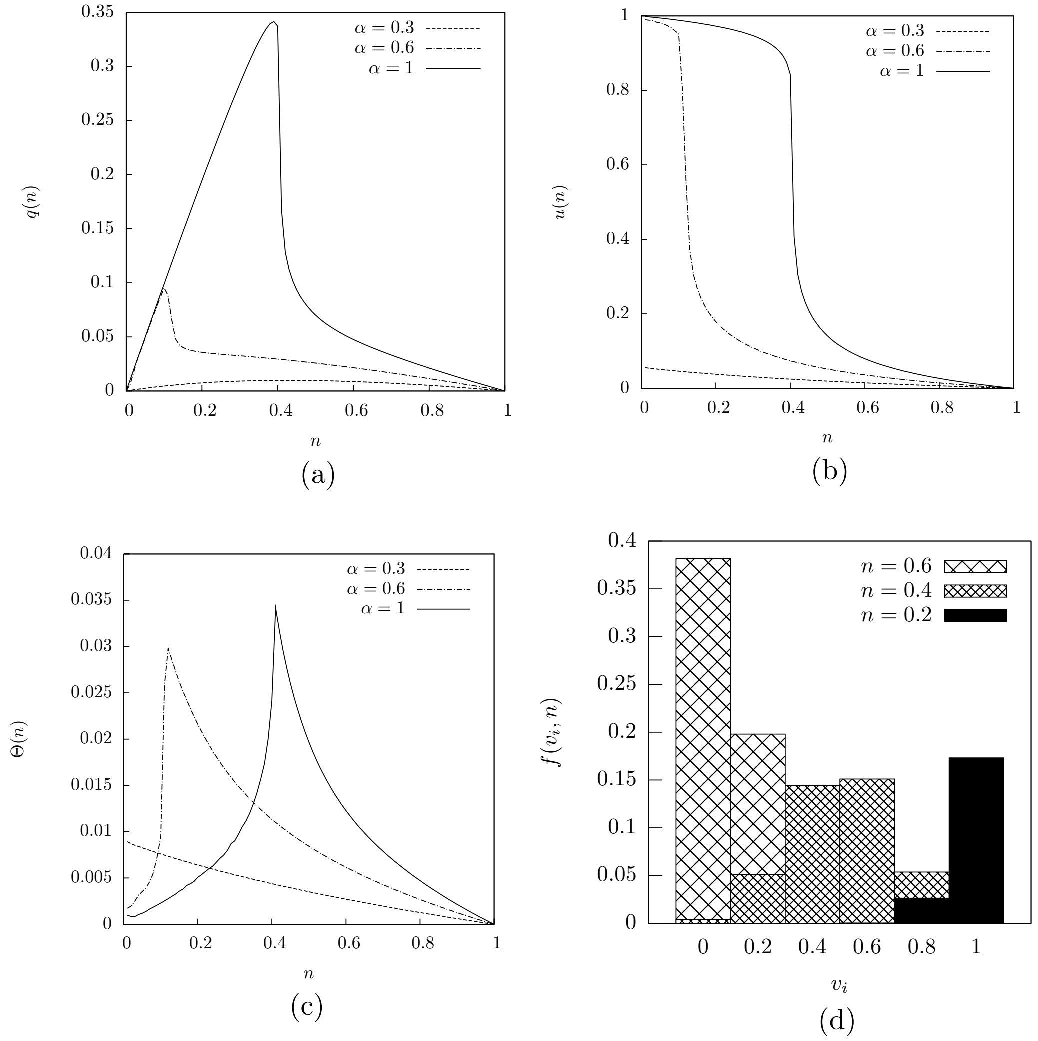

Figure 4.4. Fundamental diagrams for the macroscopic flux (a), the average velocity (b) and the velocity variance (c) as functions of the macroscopic density, obtained under various road conditions ($\alpha = 0.3$, $\alpha = 0.6$, $\alpha = 1$, respectively). In (d) the equilibrium distribution of the velocity is shown for three possible values of the density ($n = 0.2$, $n = 0.4$, $n = 0.6$) and for $\alpha = 1$.

density, close to the maximum one allowed on the road, to values near to zero once the above mentioned critical density threshold has been overcome.

Finally, we note that when the quality of the road decreases the phase transition is correspondingly anticipated at lower values of n.

It is worth stressing that this experimentally observed feature is reproduced by our discrete velocity kinetic model as a result of the evolution of the system. In other words, it is not postulated *a priori* as a modeling assumption, like for instance in the above-cited paper by Colombo [17], but is described by the model itself on the basis of more general principles, thanks to a detailed analysis of the microscopic interactions among the vehicles.

In Fig. 4.4(d) the equilibrium distribution of the velocity is shown for three possible values of the vehicle density, $n = 0.2$, $n = 0.4$, $n = 0.6$, and for $\alpha = 1$, the results for different α being in principle similar for suitable corresponding densities. We observe, as expected, the concentration of the vehicles in the extreme velocity classes for low and high n, respectively, and instead their central distribution for intermediate density.

Chapter 5

The spatially inhomogeneous problem

5.1 Mild formulation of the problem

This chapter deals with the spatially inhomogeneous problem, which describes the spatial and temporal evolution of the traffic subjected to suitable initial and boundary conditions. The distribution functions f_i feature now a full dependence on both variables t and x, which are assumed to range in suitable intervals $[0, T]$ and $D_x \subseteq \mathbb{R}$, respectively, $T > 0$ denoting the final time (possibly $+\infty$):

$$f_i = f_i(t, x) : [0, T] \times D_x \to \mathbb{R}_+, \qquad i = 1, \ldots, m.$$

The spatially inhomogeneous problem consists of the system of equations

$$\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial x} = G_i[\mathbf{f}, \mathbf{f}] - f_i L_i[\mathbf{f}], \qquad i = 1, \ldots, m, \tag{5.1}$$

supplemented by a set of initial conditions

$$f_i(0, x) = \varphi_i(x), \qquad i = 1, \ldots, m \tag{5.2}$$

and of boundary values conveniently prescribed at the proper endpoints of the interval D_x (or possibly at infinity if D_x is unbounded). For the sake of convenience, we recall here the definitions of the i-th gain and loss operators as they have been modeled in Chapt. 3:

$$\begin{cases} G_i[\mathbf{f}, \mathbf{f}](t, x) = \displaystyle\sum_{h,k=1}^{m} \int_x^{x+\xi} \bar{\eta}[n](t, x_*) A_{hk}^i[n](t, x_*) f_h(t, x) f_k(t, x_*) w(x_* - x)\, dx_* \\ L_i[\mathbf{f}](t, x) = \displaystyle\sum_{k=1}^{m} \int_x^{x+\xi} \bar{\eta}[n](t, x_*) f_k(t, x_*) w(x_* - x)\, dx_*. \end{cases} \tag{5.3}$$

Equation (5.1) is an integro-differential system of equations with linear hyperbolic advection part, whose characteristic curves satisfy in the space-time domain the ordinary differential equations

$$\frac{dx}{dt} = v_i, \qquad i = 1, \ldots, m.$$

Hence, there exist m families of characteristics constituted by the lines $x - v_i t =$ constant. In particular, the equation of the characteristic of the i-th family issuing from the point $x = x_0$ at the initial time $t = 0$ is $x - v_i t = x_0$. Since $v_i \geq 0$ for all $i = 1, \ldots, m$, the characteristics advect the data rightward (except at most for the characteristics of the first family, which do not advect data as $v_1 = 0$). Therefore, the inflow boundary of the spatial domain is the left endpoint of the interval D_x, where boundary values possibly have to be prescribed.

Following the approach introduced by Nishida and Mimura [40], and adopted among others by Tartar [46] and Toscani [47], we hereafter denote by $\hat{f}_i(t, x)$ the restriction of the i-th distribution function to the characteristic of its own family issuing from the point x at $t = 0$, that is

$$\hat{f}_i(t, x) := f_i(t, x + v_i t). \tag{5.4}$$

Bearing this in mind, we will henceforth use the hat over whatsoever function to indicate the restriction of that function to a proper family of characteristics of the system at hand. Notice in particular that when the i-th distribution function is restricted to the characteristic of the j-th family, that is when one considers a quantity like $f_i(t, x + v_j t)$, then Eq. (5.4) generalizes to

$$f_i(t, x + v_j t) = \hat{f}_i(t, x + (v_j - v_i)t). \tag{5.5}$$

The main example of this use, that will be extensively referenced in the sequel, concerns the gain and loss operators (5.3):

$$\begin{aligned}
\hat{G}_i[\mathbf{f}, \mathbf{f}](t, x) &:= G_i[\mathbf{f}, \mathbf{f}](t, x + v_i t) \\
&= \sum_{h,\, k=1}^{m} \int_{x+v_i t}^{x+v_i t+\xi} \bar{\eta}[n](t, x_*) A^i_{hk}[n](t, x_*) f_h(t, x + v_i t) \\
&\quad \times f_k(t, x_*) w(x_* - x - v_i t)\, dx_* \\
&= \sum_{h,\, k=1}^{m} \int_{x}^{x+\xi} \bar{\eta}[n](t, x_* + v_i t) A^i_{hk}[n](t, x_* + v_i t) f_h(t, x + v_i t) \\
&\quad \times f_k(t, x_* + v_i t) w(x_* - x)\, dx_* \\
&= \sum_{h,\, k=1}^{m} \int_{x}^{x+\xi} \bar{\eta}[n](t, x_* + v_i t) A^i_{hk}[n](t, x_* + v_i t) \hat{f}_h(t, x + (v_i - v_h)t) \\
&\quad \times \hat{f}_k(t, x_* + (v_i - v_k)t) w(x_* - x)\, dx_*,
\end{aligned} \tag{5.6}$$

and analogously

$$\hat{L}_i[\mathbf{f}](t,x) := L_i[\mathbf{f}](t,x+v_it)$$
$$= \sum_{k=1}^{m}\int_{x}^{x+\xi} \bar{\eta}[n](t,x_*+v_it)\hat{f}_k(t,x_*+(v_i-v_k)t)w(x_*-x)\,dx_*. \qquad (5.7)$$

In addition, we observe that the macroscopic vehicle density n can be expressed in terms of the restrictions $\hat{f}_i$ as

$$n(t,x) = \sum_{i=1}^{m}\hat{f}_i(t,x-v_it). \qquad (5.8)$$

Remark. As a matter of fact, the restrictions of the gain and loss operators to the characteristics should be denoted more properly by $\widehat{G_i[\mathbf{f},\mathbf{f}]}$ and $\widehat{L_i[\mathbf{f}]}$, respectively, in strict accordance with the meaning of the hat introduced above. However, in order not to make the notation dull reading, we allow ourselves to use henceforth the corresponding lighter writings $\hat{G}_i[\mathbf{f},\mathbf{f}]$, $\hat{L}_i[\mathbf{f}]$ as defined by Eqs. (5.6), (5.7).

From Eq. (5.4) we deduce

$$\frac{\partial\hat{f}_i}{\partial t}(t,x) = \frac{\partial f_i}{\partial t}(t,x+v_it) + v_i\frac{\partial f_i}{\partial x}(t,x+v_it)$$

whence, taking Eqs. (5.6), (5.7) into account, we rewrite Eq. (5.1) along the characteristics as

$$\frac{\partial\hat{f}_i}{\partial t}(t,x) = \hat{G}_i[\mathbf{f},\mathbf{f}](t,x) - \hat{f}_i(t,x)\hat{L}_i[\mathbf{f}](t,x), \qquad i=1,\,\dots,m. \qquad (5.9)$$

Furthermore, we anticipate that for the subsequent treatment it will be customary to consider the following transformation of the functions f_i, $\hat{f}_i$:

$$\phi_i(t,x) = f_i(t,x)e^{\lambda t}, \qquad \hat{\phi}_i(t,x) = \hat{f}_i(t,x)e^{\lambda t}, \qquad (5.10)$$

where $\lambda > 0$ is a parameter to be chosen conveniently. Denoting $\boldsymbol{\phi} = (\phi_1,\,\dots,\phi_m)$, from Eqs. (5.6), (5.7) it is readily computed that

$$\hat{G}_i[\mathbf{f},\mathbf{f}](t,x) = \hat{G}_i[\boldsymbol{\phi},\boldsymbol{\phi}](t,x)e^{-2\lambda t}, \qquad \hat{L}_i[\mathbf{f}](t,x) = \hat{L}_i[\boldsymbol{\phi}](t,x)e^{-\lambda t}. \qquad (5.11)$$

In addition, differentiating the second of Eqs. (5.10) with respect to t we discover

$$\frac{\partial\hat{f}_i}{\partial t} = e^{-\lambda t}\left(\frac{\partial\hat{\phi}_i}{\partial t} - \lambda\hat{\phi}_i\right)$$

so that system (5.9) can be reformulated in terms of the $\hat{\phi}_i$'s as

$$\frac{\partial\hat{\phi}_i}{\partial t} = \hat{G}_i[\boldsymbol{\phi},\boldsymbol{\phi}]e^{-\lambda t} + \hat{\phi}_i\left(\lambda - \hat{L}_i[\boldsymbol{\phi}]e^{-\lambda t}\right), \qquad i=1,\,\dots,m. \qquad (5.12)$$

Notice that the initial conditions on the f_i's remain unchanged when passing to the functions $\hat{\phi}_i$, indeed $\hat{\phi}_i(0,x) = \phi_i(0,x) = f_i(0,x)$. Therefore, system (5.12) is coupled to the same initial conditions given by Eq. (5.2):

$$\hat{\phi}_i(0,x) = \varphi_i(x), \qquad i = 1, \ldots, m.$$

Integrating Eq. (5.12) up to a time instant $t \leq T$ yields the mild formulation of the spatially inhomogeneous problem that we will refer to in the sequel:

$$\hat{\phi}_i(t,x) = \varphi_i(x) + \int_0^t \left\{ \hat{G}_i[\boldsymbol{\phi}, \boldsymbol{\phi}](s,x)e^{-\lambda s} + \hat{\phi}_i(s,x)\left(\lambda - \hat{L}_i[\boldsymbol{\phi}](s,x)e^{-\lambda s}\right) \right\} ds. \tag{5.13}$$

It is understood that solving (5.13) for $\hat{\boldsymbol{\phi}} = (\hat{\phi}_1, \ldots, \hat{\phi}_m)$ is equivalent to solving (5.1) for $\mathbf{f} = (f_1, \ldots, f_m)$ in the sense that from any $\hat{\boldsymbol{\phi}}$ one can uniquely recover a mild solution $\mathbf{f}$ to the original problem via the relations (5.10).

5.2 The initial value problem

In addressing the well-posedness of the spatially inhomogeneous problem we focus first of all on the case in which the spatial domain D_x coincides with the whole real axis $\mathbb{R}$. In other words, we consider the Cauchy problem generated by the system of equations (5.1) joined to the set of initial conditions (5.2), which we specifically consider in the mild form given by Eq. (5.13), for $x \in \mathbb{R}$.

Let us introduce, for $T > 0$, the Banach space

$$X_T = C([0,T];\, (L^1(\mathbb{R}))^m)$$

of the vector-valued functions $\mathbf{u} = \mathbf{u}(t,x) : [0,T] \times \mathbb{R} \to \mathbb{R}^m$ such that for all fixed $t \in [0,T]$ the function $x \mapsto (\mathbf{u}(t))(x) = \mathbf{u}(t,x)$ belongs to $(L^1(\mathbb{R}))^m$, i.e.,

$$\|\mathbf{u}(t)\|_1 = \sum_{i=1}^m \|u_i(t)\|_1 = \sum_{i=1}^m \int_{\mathbb{R}} |u_i(t,x)|\, dx < +\infty, \quad \forall\, t \in [0,T], \tag{5.14}$$

with moreover the mapping $t \mapsto \|\mathbf{u}(t)\|_1$ continuous on $[0,T]$. We take the quantity

$$\|\mathbf{u}\|_{X_T} := \sup_{t \in [0,\,T]} \|\mathbf{u}(t)\|_1 \tag{5.15}$$

as a norm on X_T.

In dealing with the mild formulation (5.13), it is necessary to translate some desired features of the functions of X_T, among which we look for the solution $\boldsymbol{\phi}$ of the problem, to their corresponding restrictions to the characteristics. For this, we make the following preliminary observations:

(i) Given $\mathbf{u} \in X_T$, by a simple change of variable in Eq. (5.14) we discover $\|\hat{\mathbf{u}}\|_{X_T} = \|\mathbf{u}\|_{X_T}$, therefore we conclude that $\mathbf{u} \in X_T$ if and only if $\hat{\mathbf{u}} \in X_T$.

(ii) We clearly have $\mathbf{u}(t,x) \geq 0$ almost everywhere in $\mathbb{R}$ and for all $t \in [0,T]$ if and only if the analogous property holds for the restriction $\hat{\mathbf{u}}$ too, i.e., if and only if $\hat{\mathbf{u}}(t,x) \geq 0$ for almost every $x \in \mathbb{R}$ and for all $t \in [0,T]$.

(iii) In order for the vehicle density n to stay below a certain threshold $K \in [0,1]$ almost everywhere in $\mathbb{R}$ and for all $t \in (0,T]$, the solution $\mathbf{f}$ of the spatially inhomogeneous problem should formally satisfy, owing to Eqs. (5.8), (5.10),

$$n(t,x) = \sum_{i=1}^{m} \hat{f}_i(t, x - v_i t) \leq K,$$

or equivalently

$$n(t,x) = e^{-\lambda t} \sum_{i=1}^{m} \hat{\phi}_i(t, x - v_i t) \leq K, \tag{5.16}$$

almost everywhere in $\mathbb{R}$ and for all $t \in (0,T]$. However, we anticipate that this condition alone turns out to be insufficient for our purposes, and has rather to be reinforced as

$$\sum_{i=1}^{m} \hat{f}_i(t, x - v_i \tau) \leq K,$$

that is

$$\sum_{i=1}^{m} \hat{\phi}_i(t, x - v_i \tau) \leq K e^{\lambda t},$$

for almost every $x \in \mathbb{R}$ and for all $t \in (0,T], \tau \geq 0$. The reason for this stronger constraint relies in the specific structure of the domains of dependence of the points $(x,t) \in \mathbb{R} \times [0,T]$ as determined by the m families of characteristics of the problem. Its utility will be evident when inspecting the proof of Theorem 5.6 below. For the moment, we simply observe that setting $\tau = t$ in the above expressions allows to recover, as a particular case, the desired condition

$$n(t,x) \leq K, \quad \text{a.e. in } \mathbb{R},\, \forall\, t \in (0,T].$$

A mild solution to Problem (5.1)-(5.2) is a continuously integrable nonnegative vector-valued function $\mathbf{f} = \mathbf{f}(t,x)$ defined over the set $(0,T] \times \mathbb{R}$ for a certain final time $T > 0$, which has the additional property that the macroscopic density n is uniformly bounded in time below the maximum threshold $n = 1$. In particular, in view of the discontinuity of the unweighted interaction rate $\bar{\eta}[n]$ at $n = 1$, we require, like in the spatially homogeneous problem (cf. Chapt. 4), that n be bounded below a maximum threshold $K < 1$. Owing to Eq. (5.10) and to the above reasoning, in considering the mild reformulation (5.13) of the problem we seek a solution $\boldsymbol{\phi} = \boldsymbol{\phi}(t,x) : (0,T] \times \mathbb{R} \to \mathbb{R}^m$ such that

(i) $\hat{\boldsymbol{\phi}} \in X_T$,

(ii) $\hat{\boldsymbol{\phi}}(t,x) \geq 0$ for almost every $x \in \mathbb{R}$ and all $t \in (0,T]$,

(iii) $\sum_{i=1}^{m} \hat{\phi}_i(t, x - v_i\tau) \leq Ke^{\lambda t}$ for almost every $x \in \mathbb{R}$ and all $t \in (0, T], \tau \geq 0$.

Before going into the technical details of the theorems, few considerations on the unweighted interaction rate, the table of games, and the weight function are in order. In particular, concerning $\bar{\eta}[n]$ and $A^i_{hk}[n]$ we still require the validity of the Assumptions 4.1, 4.2 formulated on the occasion of the spatially homogeneous problem, but we complement them with the following further hypothesis:

Assumption 5.1. *We assume that both $\bar{\eta}[n]$ and $A^i_{hk}[n]$ (all $i, h, k = 1, \ldots, m$) are Lipschitz continuous functions of the macroscopic density n, i.e., that there exist constants $L_\eta, L_A > 0$ such that*

$$|\bar{\eta}[n_2] - \bar{\eta}[n_1]| \leq L_\eta |n_2 - n_1|, \qquad |A^i_{hk}[n_2] - A^i_{hk}[n_1]| \leq L_A |n_2 - n_1| \tag{5.17}$$

for all $n_1, n_2 \in [0, K]$ and all $i, h, k = 1, \ldots, m$.

We observe in particular that the Lipschitz condition on the table of games expressed by Eq. (5.17) amounts to a uniform Lipschitz continuity with respect to the indexes i, h, k. If each of the functions $A^i_{hk}[n]$ is Lipschitz continuous with respect to n with Lipschitz constant $L^{ihk}_A > 0$, then Eq. (5.17) holds with

$$L_A := \max_{i, h, k=1, \ldots, m} L^{ihk}_A,$$

which is independent of the indexes i, h, k.

As for the weight function, we simply recall here Assumption 3.3, which can be equivalently restated by saying that $w = w(y) : \mathbb{R} \to \mathbb{R}$ is a nonnegative function supported in the interval $[0, \xi]$ and with unit integral over $\mathbb{R}$. In this context, we simply add the following boundedness property:

Assumption 5.2. *We assume that w is bounded in the interval $[0, \xi]$, i.e., that there exists a constant $C_w > 0$ such that*

$$0 \leq w(y) \leq C_w, \quad \forall\, y \in [0, \xi].$$

Notice that Assumption 5.2 allows for a wide choice of weight functions w to generate specific mathematical models (in other words, it is not a too restrictive hypothesis). Weight functions are indeed required to feature good boundedness properties within the interaction interval $[0, \xi]$ only, being set to zero by default outside it. In particular, no continuity of w is required on $\mathbb{R}$.

5.2.1 Local existence

Local in time existence and uniqueness of a solution to the Cauchy problem (5.1)-(5.2) for $x \in \mathbb{R}$, $t \in [0, T]$ are now obtained through a series of successive intermediate steps.

As previously anticipated, we refer to the mild formulation (5.13) of the problem, introducing for this the operator S defined componentwise ($i = 1, \ldots, m$) on X_T by its restriction to the characteristics as

$$(\widehat{S\mathbf{u}})_i(t,x) = \varphi_i(x) + \int_0^t \left\{ \hat{G}_i[\mathbf{u},\mathbf{u}](s,x)e^{-\lambda s} + \hat{u}_i(s,x)\left(\lambda - \hat{L}_i[\mathbf{u}](s,x)e^{-\lambda s}\right)\right\} ds. \quad (5.18)$$

In addition, motivated by the previous discussions, we look for the solution in the following subset of X_T:

$$\mathcal{B}_T = \left\{ \mathbf{u} \in X_T \,:\, \hat{\mathbf{u}}(t,x) \geq 0, \sum_{i=1}^m \hat{u}_i(t, x - v_i\tau) \leq Ke^{\lambda t}, \right.$$
$$\left. \|\mathbf{u}(t)\|_1 \leq K_1 \text{ a.e. in } \mathbb{R}, \forall\, t \in [0,T], \tau \geq 0 \right\},$$

K, K_1 being fixed nonnegative constants with in particular $K < 1$.

Finally, we assume that the initial data are such that $\varphi_i \in L^1(\mathbb{R})$, $\varphi_i(x) \geq 0$ a.e. in $\mathbb{R}$ for all $i = 1, \ldots, m$, and that they further satisfy

$$\sum_{i=1}^m \varphi_i(x - v_i t) \leq K_0, \quad \text{for a.e. } x \in \mathbb{R}, \forall\, t \geq 0 \quad (5.19)$$

for a suitable nonnegative constant $K_0 < K$, with in addition $\|\boldsymbol{\varphi}\|_1 < K_1$.

Remark. In what follows we will constantly switch, according to the convenience of the moment, between a function and its restriction to the characteristics. This is possible, and is indeed useful, due to the previously discussed one-to-one correspondence linking any $\mathbf{u}$ to its 'restricted' counterpart $\hat{\mathbf{u}}$.

Lemma 5.3. *There exists $T_1 > 0$ such that S maps $\mathcal{B}_{T_1}$ into itself.*

Proof. 1. To prove $S\mathbf{u} \in X_T$ for $\mathbf{u} \in X_T$, we check the equivalent property $\widehat{S\mathbf{u}} \in X_T$. For this we compute preliminarily

$$\|(\widehat{S\mathbf{u}})_i(t)\|_1 \leq \|\varphi_i\|_1 + \int_0^t \left\{ \|\hat{G}_i[\mathbf{u},\mathbf{u}](s)\|_1 e^{-\lambda s} + \lambda \|\hat{u}_i(s)\|_1 \right.$$
$$\left. + \|\hat{L}_i[\mathbf{u}](s)\hat{u}_i(s)\|_1 e^{-\lambda s} \right\} ds \quad (5.20)$$

and then we estimate, in view of Assumptions 4.2 on $\bar{\eta}$ and 5.2 on w:

$$\begin{aligned}\|\hat{G}_i[\mathbf{u},\mathbf{u}](s)\|_1 = \int_{\mathbb{R}} \Bigg| \sum_{h,\,k=1}^{m} \int_{x}^{x+\xi} & \bar{\eta}[n_{\mathbf{u}}](s, x_* + v_i s) A^i_{hk}[n_{\mathbf{u}}](s, x_* + v_i s) \\ & \times \hat{u}_h(s, x + (v_i - v_h)s)\hat{u}_k(s, x_* + (v_i - v_h)s) w(x_* - x)\, dx_* \Bigg| \, dx \\ \leq C_{\bar{\eta},K} C_w & \sum_{h,\,k=1}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} A^i_{hk}[n_{\mathbf{u}}](s, x_* + v_i s)\hat{u}_h(s, x + (v_i - v_h)s) \\ & \times \hat{u}_k(s, x_* + (v_i - v_k)s)\, dx_*\, dx \\ = C_{\bar{\eta},K} C_w & \sum_{h,\,k=1}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} A^i_{hk}[n_{\mathbf{u}}](s, x_*)\hat{u}_h(s, x - v_h s) \\ & \times \hat{u}_k(s, x_* - v_k s)\, dx_*\, dx\end{aligned}$$

where, in the same spirit as the spatially homogeneous problem and according to Eq. (5.16), we have set

$$n_{\mathbf{u}}(s, x_*) := e^{-\lambda s} \sum_{i=1}^{m} \hat{u}_i(s, x_* - v_i s).$$

Notice in particular that $n_{\mathbf{u}} \leq K$ for $\mathbf{u} \in \mathcal{B}_T$, hence Assumption 4.2 on the unweighted encounter rate applies. Summing over i and using Eq. (3.19) we get

$$\sum_{i=1}^{m} \|\hat{G}_i[\mathbf{u},\mathbf{u}](s)\|_1 \leq \sum_{h,\,k=1}^{m} C_{\bar{\eta},K} C_w \|\hat{u}_h(s)\|_1 \|\hat{u}_k(s)\|_1 = C_{\bar{\eta},K} C_w \|\mathbf{u}(s)\|_1^2. \tag{5.21}$$

Analogous computations on the loss operator yield

$$\begin{aligned}\|\hat{L}_i[\mathbf{u}](s)\hat{u}_i(s)\|_1 = \int_{\mathbb{R}} \Bigg| \sum_{k=1}^{m} \int_{x}^{x+\xi} & \bar{\eta}[n_{\mathbf{u}}](s, x_* + v_i s)\hat{u}_k(s, x_* + (v_i - v_k)s) \\ & \times w(x_* - x)\, dx_* \Bigg| \hat{u}_i(s, x)\, dx \\ \leq C_{\bar{\eta},K} C_w & \sum_{k=1}^{m} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{u}_k(s, x_* + (v_i - v_k)s)\hat{u}_i(s, x)\, dx_*\, dx \\ = C_{\bar{\eta},K} C_w & \sum_{k=1}^{m} \|\hat{u}_k(s)\|_1 \|\hat{u}_i(s)\|_1 = C_{\bar{\eta},K} C_w \|\mathbf{u}(s)\|_1 \|\hat{u}_i(s)\|_1,\end{aligned}$$

whence

$$\sum_{i=1}^{m} \|\hat{L}_i[\mathbf{u}](s)\hat{u}_i(s)\|_1 \leq C_{\bar{\eta},K} C_w \|\mathbf{u}(s)\|_1^2. \tag{5.22}$$

Collecting together Eqs. (5.21), (5.22) into Eq. (5.20) gives

$$\begin{aligned}\|(S\mathbf{u})(t)\|_1 = \sum_{i=1}^{m} \|(\widehat{S\mathbf{u}})_i(t)\|_1 &\le \|\boldsymbol{\varphi}\|_1 + \int_0^t \Big\{ C_{\bar\eta,K} C_w \|\mathbf{u}(s)\|_1^2 e^{-\lambda s} + \lambda \|\mathbf{u}(s)\|_1 \\ &\quad + C_{\bar\eta,K} C_w \|\mathbf{u}(s)\|_1^2 e^{-\lambda s} \Big\}\, ds \\ &\le \|\boldsymbol{\varphi}\|_1 + 2tC_{\bar\eta,K} C_w \|\mathbf{u}\|_{X_T}^2 + \lambda t \|\mathbf{u}\|_{X_T}\end{aligned}$$

and finally, taking the supremum over $t \in [0,T]$,

$$\|S\mathbf{u}\|_{X_T} \le \|\boldsymbol{\varphi}\|_1 + 2TC_{\bar\eta,K} C_w \|\mathbf{u}\|_{X_T}^2 + \lambda T \|\mathbf{u}\|_{X_T}.$$

This technically says that $S\mathbf{u} \in L^\infty(0,T;\,(L^1(\mathbb{R}))^m)$. To get actually the continuity of the mapping $t \mapsto \|(S\mathbf{u})(t)\|_1$ we have to accept for the moment the fact that, as we will prove independently in the next point 2, it is possible to choose $\lambda > 0$ such that

$$\lambda - \hat{L}_i[\mathbf{u}](s,x) e^{-\lambda s} \ge 0 \tag{5.23}$$

almost everywhere in $\mathbb{R}$, for all $s \ge 0$, and for all $i = 1, \ldots, m$. With such a choice of λ, we infer from Eq. (5.18) that $(\widehat{S\mathbf{u}})_i(t,x)$ is the sum of nonnegative terms, thus for $t_1, t_2 \in [0,T]$, $t_1 \le t_2$, we can write

$$\|(S\mathbf{u})_i(t_2)\|_1 - \|(S\mathbf{u})_i(t_1)\|_1 = \int_{\mathbb{R}} \Big\{ (\widehat{S\mathbf{u}})_i(t_2,x) - (\widehat{S\mathbf{u}})_i(t_1,x) \Big\}\, dx.$$

In particular

$$\begin{aligned}(\widehat{S\mathbf{u}})_i(t_2,x) - (\widehat{S\mathbf{u}})_i(t_1,x) = \int_{t_1}^{t_2} \Big\{ &\hat{G}_i[\mathbf{u},\mathbf{u}](s,x) e^{-\lambda s} \\ &+ \hat{u}_i(s,x) \Big(\lambda - \hat{L}_i[\mathbf{u}](s,x) e^{-\lambda s} \Big) \Big\}\, ds,\end{aligned}$$

so that, owing to the nonnegativity of both the gain and loss operators for $\mathbf{u} \in \mathcal{B}_T$, we obtain the following estimate

$$\|(S\mathbf{u})_i(t_2)\|_1 - \|(S\mathbf{u})_i(t_1)\|_1 \le \int_{t_1}^{t_2} \Big\{ \|\hat{G}_i[\mathbf{u},\mathbf{u}](s,x)\|_1 e^{-\lambda s} + \lambda \|\hat{u}_i(s)\|_1 \Big\}\, ds.$$

Summing over i and using Eq. (5.21) we further discover

$$\|(S\mathbf{u})(t_2)\|_1 - \|(S\mathbf{u})(t_1)\|_1 \le \int_{t_1}^{t_2} \Big\{ C_{\bar\eta,K} C_w \|\mathbf{u}(s)\|_1^2 e^{-\lambda s} + \lambda \|\mathbf{u}(s)\|_1 \Big\}\, ds$$

and finally

$$\left|\|(S\mathbf{u})(t_2)\|_1 - \|(S\mathbf{u})(t_1)\|_1\right| \le (t_2 - t_1)\left\{C_{\bar\eta,K}C_w\|\mathbf{u}\|^2_{X_T} + \lambda\|\mathbf{u}\|_{X_T}\right\},$$

which shows that $\|(S\mathbf{u})(t_2)\|_1 \to \|(S\mathbf{u})(t_1)\|_1$ when $t_2 \to t_1$ and gives therefore the required continuity, i.e., $S\mathbf{u} \in X_T$.

2. We now check that $(\widehat{S\mathbf{u}})(t,x) \ge 0$ for a.e. $x \in \mathbb{R}$ and for all $t \in [0,T]$. Since $\hat{G}_i[\mathbf{u},\mathbf{u}](t,x) \ge 0$ for $\mathbf{u} \in \mathcal{B}_T$, while $\varphi_i(x) \ge 0$ by assumption, the nonnegativity of $(S\mathbf{u})_i(t,x)$ each $i = 1, \ldots, m$ strictly depends on the possibility to find $\lambda > 0$ such that the inequality (5.23) holds. For this we observe that $\mathbf{u} \in \mathcal{B}_T$ implies

$$\sum_{k=1}^{m} \hat{u}_k(s, x_* + (v_i - v_k)s) = \sum_{k=1}^{m} \hat{u}_k(s, (x_* + v_i s) - v_k s) \le K e^{\lambda s}$$

almost everywhere in $\mathbb{R}$ and for all $s \in [0,T]$, therefore from the expression of $\hat{L}_i[\mathbf{u}]$ resulting from Eq. (5.7) we get

$$\begin{aligned}\hat{L}_i[\mathbf{u}](s,x)e^{-\lambda s} &\le K \int_x^{x+\xi} \bar\eta[n_{\mathbf{u}}](s, x_* + v_i s) w(x_* - x)\, dx_* \\ &\le K C_{\bar\eta,K} \int_x^{x+\xi} w(x_* - x)\, dx_* = K C_{\bar\eta,K}.\end{aligned}$$

The nonnegativity of $(\widehat{S\mathbf{u}})(t,x)$ is then achieved by fixing $\lambda \ge K C_{\bar\eta,K}$.

3. To show that $\sum_{i=1}^{m} (\widehat{S\mathbf{u}})_i(t, x - v_i\tau) \le K e^{\lambda t}$ for a.e. $x \in \mathbb{R}$ and for all $t \in [0,T]$, $\tau \ge 0$ we first notice that, due to the nonnegativity of $\hat{L}_i[\mathbf{u}](s,x)$ for $\mathbf{u} \in \mathcal{B}_T$, it results

$$\begin{aligned}(\widehat{S\mathbf{u}})_i(t, x - v_i\tau) \le \varphi_i(x - v_i\tau) + \int_0^t \Big\{ &\hat{G}_i[\mathbf{u},\mathbf{u}](s, x - v_i\tau)e^{-\lambda s} \\ &+ \lambda \hat{u}_i(s, x - v_i\tau) \Big\}\, ds;\end{aligned}$$

in addition, recalling Assumption 4.1 on the table of games, we have

$$\begin{aligned}\hat{G}_i[\mathbf{u},\mathbf{u}](s, x - v_i\tau)e^{-\lambda s} &\le C_{\bar\eta,K} e^{-\lambda s} \int_{x - v_i\tau}^{x - v_i\tau + \xi} \sum_{h,\,k=1}^{m} \hat{u}_h(s, (x - v_i(\tau - s)) - v_h s) \\ &\quad \times \hat{u}_k(s, (x_* + v_i s) - v_k s) w(x_* - x + v_i\tau)\, dx_* \\ &\le K^2 C_{\bar\eta,K} e^{\lambda s} \int_{x - v_i\tau}^{x - v_i\tau + \xi} w(x_* - x + v_i\tau)\, dx_* = K^2 C_{\bar\eta,K} e^{\lambda s},\end{aligned}$$

whence

$$(\widehat{S\mathbf{u}})_i(t,x-v_i\tau) \le \varphi(x-v_i\tau) + \int\limits_0^t \left\{K^2 C_{\bar{\eta},K} e^{\lambda s} + \lambda \hat{u}_i(s,x-v_i\tau)\right\} ds$$

and finally

$$\sum_{i=1}^{m} (\widehat{S\mathbf{u}})_i(t,x-v_i\tau) \le K_0 + K\left(1 + \frac{mKC_{\bar{\eta},K}}{\lambda}\right)\left(e^{\lambda t} - 1\right).$$

The request on the left-hand side is certainly fulfilled if

$$K_0 + \frac{mK^2 C_{\bar{\eta},K}}{\lambda}\left(e^{\lambda t} - 1\right) - K \le 0, \qquad \forall\, t \in [0,T].$$

In particular, owing to the monotonicity of the exponential function, it is sufficient that this relation holds for $t = T$, which entails

$$T \le \frac{1}{\lambda} \log\left(1 + \frac{\lambda(K-K_0)}{mK^2 C_{\bar{\eta},K}}\right) > 0. \tag{5.24}$$

4. Finally we verify that the L^1-norm of $(S\mathbf{u})(t)$ remains bounded from above by K_1. Exploiting the calculations performed at the previous point 1 we easily deduce

$$\begin{aligned}\|(S\mathbf{u})(t)\|_1 &\le \|\boldsymbol{\varphi}\|_1 + \int\limits_0^t \left\{2C_{\bar{\eta},K} C_w \|\mathbf{u}(s)\|_1^2 e^{-\lambda s} + \lambda \|\mathbf{u}(s)\|_1\right\} ds \\ &\le \|\boldsymbol{\varphi}\|_1 + (2C_{\bar{\eta},K} C_w K_1^2 + \lambda K_1)t,\end{aligned}$$

hence we get the required bound on the left-hand side for all $t \in [0,T]$ provided the final time T is chosen in such a way that

$$T \le \frac{K_1 - \|\boldsymbol{\varphi}\|_1}{2C_{\bar{\eta},K} C_w K_1^2 + \lambda K_1} > 0. \tag{5.25}$$

In view of the results so far obtained, we conclude that choosing any $T_1 > 0$ satisfying simultaneously the inequalities (5.24), (5.25), that is such that

$$T_1 \le \min\{\text{Eq. (5.24), Eq. (5.25)}\},$$

yields $S(\mathcal{B}_{T_1}) \subseteq \mathcal{B}_{T_1}$ and thus the thesis. □

Lemma 5.4. *There exists $T_2 > 0$ such that S is a contraction on $\mathcal{B}_{T_2}$.*

Proof. Proving this lemma amounts in essence to showing that for a suitable choice of the final time T the operator S is Lipschitz continuous on $\mathcal{B}_T$ with Lipschitz constant strictly less than 1. To this end we take $\mathbf{u}, \mathbf{v} \in \mathcal{B}_T$ and we compute

$$
\begin{aligned}
(\widehat{S\mathbf{u}})_i(t,x) - (\widehat{S\mathbf{v}})_i(t,x) &= \int_0^t \left\{\hat{G}_i[\mathbf{u},\mathbf{u}](s,x) - \hat{G}_i[\mathbf{v},\mathbf{v}](s,x)\right\} e^{-\lambda s}\,ds \\
&\quad + \lambda \int_0^t \{\hat{u}_i(s,x) - \hat{v}_i(s,x)\}\,ds \\
&\quad + \int_0^t \left\{\hat{L}_i[\mathbf{v}](s,x)\hat{v}_i(s,x) - \hat{L}_i[\mathbf{u}](s,x)\hat{u}_i(s,x)\right\} e^{-\lambda s}\,ds,
\end{aligned}
$$

whence

$$
\begin{aligned}
\|(S\mathbf{u})_i(t) - (S\mathbf{v})_i(t)\|_1 &= \|(\widehat{S\mathbf{u}})_i(t) - (\widehat{S\mathbf{v}})_i(t)\|_1 \\
&\leq \int_0^t \|\hat{G}_i[\mathbf{u},\mathbf{u}](s,x) - \hat{G}_i[\mathbf{v},\mathbf{v}](s,x)\|_1 e^{-\lambda s}\,ds \\
&\quad + \lambda \int_0^t \|\hat{u}_i(s,x) - \hat{v}_i(s,x)\|_1\,ds \\
&\quad + \int_0^t \|\hat{L}_i[\mathbf{v}](s,x)\hat{v}_i(s,x) - \hat{L}_i[\mathbf{u}](s,x)\hat{u}_i(s,x)\|_1 e^{-\lambda s}\,ds. \qquad (5.26)
\end{aligned}
$$

In more detail, we have

$$
\begin{aligned}
\|\hat{G}_i[\mathbf{u},\mathbf{u}](s,x) - \hat{G}_i[\mathbf{v},\mathbf{v}](s,x)\|_1 = \int_{\mathbb{R}} \Bigg| \sum_{h,\,k=1}^{m} \int_x^{x+\xi} &\big\{(\bar{\eta}[n_{\mathbf{u}}] - \bar{\eta}[n_{\mathbf{v}}])A^i_{hk}[n_{\mathbf{u}}]\hat{u}_h\hat{u}_k \\
&+ \bar{\eta}[n_{\mathbf{v}}](A^i_{hk}[n_{\mathbf{u}}] - A^i_{hk}[n_{\mathbf{v}}])\hat{u}_h\hat{u}_k \\
&+ \bar{\eta}[n_{\mathbf{v}}]A^i_{hk}[n_{\mathbf{v}}](\hat{u}_h - \hat{v}_h)\hat{u}_k \\
&+ \bar{\eta}[n_{\mathbf{v}}]A^i_{hk}[n_{\mathbf{v}}](\hat{u}_k - \hat{v}_k)\hat{v}_h\big\}\,w\,dx_* \Bigg|\,dx,
\end{aligned}
$$

where we have omitted the variables of the functions under the integrals for brevity. Using the nonnegativity of $\mathbf{u}, \mathbf{v} \in \mathcal{B}_T$, and consequently that of the the table of games and of the unweighted interaction rate (Assumptions 4.1, 4.2) we obtain

$$\begin{aligned}\|\hat{G}_i[\mathbf{u},\mathbf{u}](s,x)-\hat{G}_i[\mathbf{v},\mathbf{v}](s,x)\|_1 \leq &\sum_{h,\,k=1}^{m}\int_{\mathbb{R}}\int_{\mathbb{R}} |\bar{\eta}[n_{\mathbf{u}}]-\bar{\eta}[n_{\mathbf{v}}]|A^i_{hk}[n_{\mathbf{u}}]\hat{u}_h\hat{u}_k w\,dx_*\,dx \\ &+\sum_{h,\,k=1}^{m}\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{\eta}[n_{\mathbf{v}}]|A^i_{hk}[n_{\mathbf{u}}]-A^i_{hk}[n_{\mathbf{v}}]|\hat{u}_h\hat{u}_k w\,dx_*\,dx \\ &+\sum_{h,\,k=1}^{m}\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{\eta}[n_{\mathbf{v}}]A^i_{hk}[n_{\mathbf{v}}]|\hat{u}_h-\hat{v}_h|\hat{u}_k w\,dx_*\,dx \\ &+\sum_{h,\,k=1}^{m}\int_{\mathbb{R}}\int_{\mathbb{R}} \bar{\eta}[n_{\mathbf{v}}]A^i_{hk}[n_{\mathbf{v}}]|\hat{u}_k-\hat{v}_k|\hat{v}_h w\,dx_*\,dx.\end{aligned}$$

The Lipschitz continuity of the unweighted interaction rate and of the table of games with respect to n given by Assumption 5.1 implies

$$|\bar{\eta}[n_{\mathbf{u}}]-\bar{\eta}[n_{\mathbf{v}}]| \leq L_{\bar{\eta}}|n_{\mathbf{u}}-n_{\mathbf{v}}| \leq L_{\bar{\eta}}\sum_{j=1}^{m}|\hat{u}_j-\hat{v}_j|e^{-\lambda s}$$

as well as

$$|A^i_{hk}[n_{\mathbf{u}}]-A^i_{hk}[n_{\mathbf{v}}]| \leq L_A|n_{\mathbf{u}}-n_{\mathbf{v}}| \leq L_A\sum_{j=1}^{m}|\hat{u}_j-\hat{v}_j|e^{-\lambda s},$$

therefore we deduce

$$\begin{aligned}\|\hat{G}_i[\mathbf{u},\mathbf{u}](s,x)-\hat{G}_i[\mathbf{v},\mathbf{v}](s,x)\|_1 &\leq (L_{\bar{\eta}}C_wK+L_AC_{\bar{\eta},K}C_wK)\|\mathbf{u}(s)\|_1\|\mathbf{u}(s)-\mathbf{v}(s)\|_1 \\ &\quad+C_{\bar{\eta},K}C_w(\|\mathbf{u}(s)\|_1+\|\mathbf{v}(s)\|_1)\|\mathbf{u}(s)-\mathbf{v}(s)\|_1 \\ &\leq C'\|\mathbf{u}(s)-\mathbf{v}(s)\|_1\end{aligned}$$

where $C'>0$ is a cumulative constant deduced when bounding from above with K_1 the L^1-norms of $\mathbf{u}(s)$ and $\mathbf{v}(s)$.

Analogous calculations yield

$$\begin{aligned}\|\hat{L}_i[\mathbf{v}](s)\hat{v}_i(s)-\hat{L}_i[\mathbf{u}](s)\hat{u}_i(s)\|_1 &\leq (L_{\bar{\eta}}+C_{\bar{\eta},K})C_w\|v_i(s)\|_1\|\mathbf{v}(s)-\mathbf{u}(s)\|_1 \\ &\quad+C_{\bar{\eta},K}C_w\|\mathbf{u}(s)\|_1\|v_i(s)-u_i(s)\|_1,\end{aligned}$$

so that Eq. (5.26) finally gives, after summing over the index $i=1,\ldots,m$,

$$\|(S\mathbf{u})(t)-(S\mathbf{v})(t)\|_1 \leq (mC'+C'')\int_0^t\|\mathbf{u}(s)-\mathbf{v}(s)\|_1e^{-\lambda s}\,ds+\lambda\int_0^t\|\mathbf{u}(s)-\mathbf{v}(s)\|_1\,ds$$

for $C''=(L_{\bar{\eta}}+2C_{\bar{\eta},K})C_w$. Consequently

$$\|S\mathbf{u}-S\mathbf{v}\|_{X_T} \leq (mC'+C''+\lambda)T\|\mathbf{u}-\mathbf{v}\|_{X_T},$$

which shows that if T_2 is chosen such that

$$T_2 < \frac{1}{mC' + C'' + \lambda}$$

then S is a contraction on $\mathcal{B}_{T_2}$ as desired. □

Lemma 5.5. *The set $\mathcal{B}_T$ is closed in X_T each $T \geq 0$.*

Proof. Let $\{\mathbf{u}_n\}_{n\geq 1} \subset \mathcal{B}_T$ be a sequence converging to $\bar{\mathbf{u}} \in X_T$, that is

$$\lim_{n\to\infty} \|\mathbf{u}_n - \bar{\mathbf{u}}\|_{X_T} = 0. \tag{5.27}$$

The closure of $\mathcal{B}_T$ in X_T corresponds to the fact that actually $\bar{\mathbf{u}} \in \mathcal{B}_T$ as well. Owing to the definition of the norm in X_T (cf. Eq. (5.15)), Eq. (5.27) implies

$$\lim_{n\to\infty} \|\mathbf{u}_n(t) - \bar{\mathbf{u}}(t)\|_1 = 0, \qquad \forall\, t \in [0,T], \tag{5.28}$$

that is for every fixed $t \in [0,T]$ the sequence $\{\mathbf{u}_n(t)\}_{n\geq 1}$ converges to $\bar{\mathbf{u}}(t)$ in $\left(L^1(\mathbb{R})\right)^m$. Upon passing to a subsequence if necessary, we can hence assume that the functions $\mathbf{u}_n(t,x)$ converge pointwise to $\bar{\mathbf{u}}(t,x)$ for almost every $x \in \mathbb{R}$, namely

$$\lim_{n\to\infty} \mathbf{u}_n(t,x) = \bar{\mathbf{u}}(t,x), \qquad \text{for a.e. } x \in \mathbb{R}, \forall\, t \in [0,T]. \tag{5.29}$$

On this basis we check that $\bar{\mathbf{u}}$ satisfies all the requirements needed to be an element of the set $\mathcal{B}_T$.

1. Equation (5.29) immediately implies $\hat{\bar{\mathbf{u}}}(t,x) \geq 0$ for a.e. $x \in \mathbb{R}$ and all $t \in [0,T]$, as the same is true for $\hat{\mathbf{u}}_n(t,x)$ each $n \geq 1$ by assumption.

2. Let $t \in [0,T], \tau \geq 0$, then

$$\begin{aligned}\sum_{i=1}^m \hat{\bar{u}}_i(t, x - v_i\tau) &= \sum_{i=1}^m \lim_{n\to\infty} (\hat{u}_i)_n(t, x - v_i\tau) \\ &= \lim_{n\to\infty} \sum_{i=1}^m (\hat{u}_i)_n(t, x - v_i\tau) \leq Ke^{\lambda t} \qquad \text{for a.e. } x \in \mathbb{R}.\end{aligned}$$

3. Equation (5.28) implies $\|\bar{\mathbf{u}}(t)\|_1 = \lim_{n\to\infty} \|\mathbf{u}_n(t)\|_1$ all t, hence we get $\|\bar{\mathbf{u}}(t)\|_1 \leq K_1$ for all $t \in [0,T]$. □

We are now in a position to prove our local existence theorem for the initial value spatially inhomogeneous problem. For the sake of definiteness, we recall in the statement of the theorem all hypotheses that bring to the result.

Theorem 5.6. *Let Assumptions 4.1, 4.2, 5.1, 5.2 on the unweighted interaction rate, the table of games, and the weight function hold. Fix moreover two constants* $0 \leq K_0 < K < 1$ *and assume that the initial data are such that* $\varphi_i \in L^1(\mathbb{R})$, $\varphi_i(x) \geq 0$ *for a.e.* $x \in \mathbb{R}$ *for all* $i = 1, \ldots, m$, *with in addition*

$$\sum_{i=1}^{m} \varphi_i(x - v_i t) \leq K_0, \qquad \text{for a.e. } x \in \mathbb{R}, \, \forall\, t \in [0, T].$$

Then there exists $T^* > 0$ *such that Problem* (5.1)-(5.2) *admits a unique nonnegative solution* $\mathbf{f} \in X_{T^*}$ *such that*

$$n(t, x) \leq K, \qquad \text{for a.e. } x \in \mathbb{R}, \, \forall\, t \in (0, T^*].$$

Proof. Let us choose $T^* \leq \min\{T_1, T_2\}$, where $T_1, T_2 > 0$ are determined by Lemmas 5.3 and 5.4, respectively. Then the operator S maps $\mathcal{B}_{T^*}$ into itself (Lemma 5.3), and is moreover a contraction on $\mathcal{B}_{T^*}$ (Lemma 5.4). In addition, owing to Lemma 5.5, $\mathcal{B}_{T^*}$ is closed in X_{T^*}, therefore we can apply Banach Fixed Point Theorem and conclude on the existence and uniqueness of a fixed point $\boldsymbol{\phi} \in \mathcal{B}_{T^*}$ of S. Specifically, since $\boldsymbol{\phi}$ satisfies by definition $S\boldsymbol{\phi} = \boldsymbol{\phi}$, from Eq. (5.18) we see that it solves Eq. (5.13) on $\mathbb{R} \times (0, T^*]$, hence it generates, via the transformation (5.10), the unique solution $\mathbf{f} \in X_{T^*}$ to Problem (5.1)-(5.2).

It is plain that $\mathbf{f} \geq 0$, indeed $\boldsymbol{\phi} \in \mathcal{B}_{T^*}$ implies in particular $\boldsymbol{\phi} \geq 0$. Furthermore we deduce $\sum_{i=1}^{m} \hat{\phi}_i(t, x - v_i\tau) \leq Ke^{\lambda t}$ for a.e. $x \in \mathbb{R}$ and all $t \in (0, T^*], \tau \geq 0$, whence setting $\tau = t$ and recalling Eq. (5.16) entails the desired estimate on $n(t, x)$. □

5.2.2 Further remarks on the local solution

Theorem 5.6 states that if the initial data $\{\varphi_i\}_{i=1}^{m}$ satisfy some basic requirements of boundedness, nonnegativity, and integrability then the initial value spatially inhomogeneous problem admits a unique solution $\mathbf{f} = \mathbf{f}(t, x) = (f_1(t, x), \ldots, f_m(t, x))$ defined for $t \in (0, T^*]$, $x \in \mathbb{R}$, which features some nice mathematical properties making it meaningful from the physical point of view. In particular, it is possible to control the macroscopic density of vehicles so that it never exceeds the maximum density allowed along the road as determined by the road capacity. This is expressed by a bound of the form

$$n(t, x) \leq K, \qquad \text{a.e. in } \mathbb{R}, \, \forall\, t \in (0, T^*]$$

which, given the nonnegativity of the local solution, implies a uniform in time and space boundedness estimate for the f_i's:

$$\max_{t \in (0, T^*]} \operatorname*{ess\,sup}_{x \in \mathbb{R}} f_i(t, x) \leq K, \qquad i = 1, \ldots, m,$$

to be compared with the analogous estimate (4.7) for the spatially homogeneous problem.

Actually, it can be observed that if the initial datum $\boldsymbol{\varphi} \in (L^1(\mathbb{R}))^m$ is further required to fulfil

$$\lim_{|x| \to \infty} \boldsymbol{\varphi}(x) = 0, \tag{5.30}$$

then the solution $\mathbf{f}(t) \in (L^1(\mathbb{R}))^m$ provided by Theorem 5.6 satisfies the analogous property

$$\lim_{|x|\to\infty} \mathbf{f}(t,x) = 0, \qquad \forall\, t \in (0,T^*].$$

This means that, under condition (5.30), the solution to the Cauchy problem (5.1)-(5.2) belongs more precisely to the subset $\mathcal{B}^0_{T^*} \subset \mathcal{B}_{T^*}$ of the functions which, at all times, are infinitesimal at infinity. This is technically proved by applying Theorem 5.6 in $\mathcal{B}^0_{T^*}$ after showing, via slight modifications of Lemmas 5.3, 5.5, that for each $T > 0$ the operator S maps $\mathcal{B}^0_T$ into itself and that, in addition, $\mathcal{B}^0_T$ is closed in X_T.

Therefore we discover in particular

$$\lim_{|x|\to\infty} q(t,x) = \lim_{|x|\to\infty} \sum_{i=1}^{m} v_i f_i(t,x) = 0, \qquad \forall\, t \in (0,T^*].$$

As a consequence, integration over $x \in \mathbb{R}$ of the macroscopic mass conservation equation (3.30) yields

$$\frac{d}{dt} \int_{\mathbb{R}} n(t,x)\, dx = 0,$$

that is

$$\int_{\mathbb{R}} \sum_{i=1}^{m} f_i(t,x)\, dx = \int_{\mathbb{R}} \sum_{i=1}^{m} \varphi_i(x)\, dx, \qquad \forall\, t \in (0,T^*]$$

or, in terms of norms,

$$\|\mathbf{f}(t)\|_1 = \|\boldsymbol{\varphi}\|_1, \qquad \forall\, t \in (0,T^*]. \tag{5.31}$$

In other words, the L^1-norm of the solution $\mathbf{f}(t)$ is preserved, and in particular it equals that of the initial datum. Equation (5.31) can be viewed as the analogous, in the spatially inhomogeneous case, of the *a priori* estimate (4.3) valid on the solution to the spatially homogeneous problem. In addition, it demonstrates that the condition $\|\mathbf{u}(t)\|_1 \leq K_1$ imposed on the functions of $\mathcal{B}_{T^*}$ is not restrictive, as it is automatically satisfied by the solution $\mathbf{f}$ for all possible choices of the constant $K_1 > \|\boldsymbol{\varphi}\|_1$. Its utility is rather merely technical, being due to the necessity to have $\mathcal{B}_{T^*}$ bounded in X_{T^*} as needed in Lemma 5.4 to prove the Lipschitz continuity of the operator S.

Another interesting property of the local solution $\mathbf{f}$ concerns a more precise lower bound that can be obtained by suitably manipulating Eq. (5.9). The nonnegativity of $\mathbf{f}$ entails $\hat{G}_i[\mathbf{f},\mathbf{f}](t,x) \geq 0$, hence

$$\frac{\partial \hat{f}_i}{\partial t}(t,x) + \hat{f}_i(t,x)\hat{L}_i[\mathbf{f}](t,x) \geq 0, \qquad \text{a.e. in } \mathbb{R},\ \forall\, t \in (0,T^*]. \tag{5.32}$$

If we let

$$\hat{\Lambda}_i[\mathbf{f}](t,x) := \int_0^t \hat{L}_i[\mathbf{f}](s,x)\, ds$$

denote the formal antiderivative of $\hat{L}_i[\mathbf{f}](s,x)$ with respect to s vanishing for $s = 0$, from Eq. (5.32) we deduce

$$\frac{\partial}{\partial t}\left(e^{\hat{\Lambda}_i[\mathbf{f}](t,x)}\hat{f}_i(t,x)\right) \geq 0,$$

thus, after integration in time,

$$\hat{f}_i(t,x) \geq \varphi_i(x)e^{-\hat{\Lambda}_i[\mathbf{f}](t,x)}, \qquad \text{a.e. in } \mathbb{R},\ \forall\, t \in (0,T^*].$$

However, recalling from Eq. (5.11) that $\hat{L}_i[\mathbf{f}](s,x) = \hat{L}_i[\boldsymbol{\phi}](s,x)e^{-\lambda s}$ and inspecting the proof of Lemma 5.3 we get $\hat{L}_i[\mathbf{f}](s,x) \leq KC_{\bar{\eta},K}$, whence $\hat{\Lambda}_i[\mathbf{f}](t,x) \leq KC_{\bar{\eta},K}t$ and consequently

$$\hat{f}_i(t,x) \geq \varphi_i(x)e^{-KC_{\bar{\eta},K}t}, \qquad \text{a.e. in } \mathbb{R},\ \forall\, t \in (0,T^*].$$

Coming back to the solution $\mathbf{f}$ this rewrites as

$$f_i(t,x) \geq \varphi_i(x - v_i t)e^{-KC_{\bar{\eta},K}t}, \qquad \text{a.e. in } \mathbb{R},\ \forall\, t \in (0,T^*], \tag{5.33}$$

which shows in particular that at time t the amount of vehicles of the i-th velocity class located in the spatial position x is bounded from below by their amount at time $t = 0$ in the corresponding initial position $x_0 = x - v_i t$. Notice the presence, in the estimate (5.33), of an exponential damping factor decaying in time at a rate fixed by the maximum macroscopic density K and by the maximum value $C_{\bar{\eta},K}$ attained by the unweighted interaction rate $\bar{\eta}$ for that density. Since, as discussed in Chapt. 3, Subsect. 3.3.2, the unweighted interaction rate may be thought of as a measure of the reactiveness of the drivers, Eq. (5.33) suggests that the prompter the latter are the higher the number of possible velocity class transitions is with respect to the initial configuration.

We end this section by briefly commenting on the assumption (5.19) on the initial datum. If no interaction terms were present in the kinetic equations, namely if the solution $\mathbf{f}$ were simply given by the advection of $\boldsymbol{\varphi}$ along the characteristics, this condition would produce $n(t,x) \leq K_0$ for a.e. $x \in \mathbb{R}$ and for all $t \in (0,T^*]$. In presence of interactions, the control (5.19) on the initial datum is still necessary to obtain the uniform in time and space boundedness of the macroscopic density of cars, although in this case the advection of $\boldsymbol{\varphi}$ along the characteristics is no longer the only factor which plays a role in shaping the solution (see the next Subsect. 5.2.3 for further discussions on this issue). A straightforward way to choose the initial data $\{\varphi_i\}_{i=1}^m$ so that Eq. (5.19) holds is

$$\varphi_i(x) \leq \frac{K_0}{m}, \qquad \text{for a.e. } x \in \mathbb{R},\ \forall\, i = 1,\,\dots,m.$$

In particular, if some of the φ_i's, say $m_0 < m$ of them, are zero then the remaining non-zero initial data can be increased accordingly as

$$\varphi_i(x) \leq \frac{K_0}{m - m_0}, \qquad \text{for a.e. } x \in \mathbb{R}.$$

5.2.3 On the existence of the solution for large times

Existence of a global in time solution to Problem (5.1)-(5.2) may be proved starting from the local solution $\mathbf{f}$ by showing that the results of Theorem 5.6 extend to the whole real axis, i.e., that one can formally take $T^* = +\infty$ (cf. the spatially homogeneous case, Theorems 4.3, 4.4). For this, a basic requirement is that the local solution $\mathbf{f}$ features at the final time $t = T^*$ all properties assumed on the initial datum, so that, with the further aid of proper *a priori* estimates ensuring boundedness in time, one can first repeat the reasoning of local existence on the interval $(T^*, 2T^*]$ taking $\mathbf{f}(T^*, x)$ as new initial condition, and then use it inductively to uniquely extend $\mathbf{f}$ to a solution defined for all $t > 0$.

For the local solution of the spatially homogeneous problem at hand, however, this procedure fails because Theorem 5.6 does not guarantee the same uniform in space bound on the initial datum $\boldsymbol{\varphi}$ and on the solution $\mathbf{f}$. Indeed, starting from the assumption

$$\sum_{i=1}^{m} \varphi_i(x - v_i t) \leq K_0, \qquad \text{a.e. in } \mathbb{R},\, \forall\, t \geq 0$$

one gets (see Lemma 5.3)

$$\sum_{i=1}^{m} f_i(t, x - v_i(\tau - t)) \leq K, \qquad \text{a.e. in } \mathbb{R},\, \forall\, t \in (0, T^*], \tau \geq 0$$

so that choosing $\mathbf{f}(T^*, x)$ as new initial condition and renaming $\tau - T^* =: \tau'$ leads to

$$\sum_{i=1}^{m} f_i(T^*, x - v_i \tau') \leq K, \qquad \text{a.e. in } \mathbb{R},\, \forall\, \tau' \geq 0.$$

Unfortunately, since $K > K_0$, this is not the same condition holding on $\boldsymbol{\varphi}$. On the other hand, Lemma 5.3 clearly shows that it is impossible to assume $K = K_0$, for this would force $T_1 = 0$ (cf. Eq. (5.24)) and consequently also $T^* = 0$, i.e., no local solution in practice. This necessary non-zero gap between K_0 and K can be explained considering that, even if the total mass of cars (i.e., the integral of the vehicle density $n(t, x)$ over $x \in \mathbb{R}$) is conserved, one cannot exclude local mass generation along the characteristics caused by the state transitions of the vehicles among the different velocity classes. In other words, the local contributions of the gain and loss operators may not balance, as it has to be expected since the final solution is actually not simply the advection of the initial datum along the characteristics.

Such an issue prevents from using the local solution to obtain a global in time solution to the spatially inhomogeneous problem via the reasoning explained above. As a matter of fact, at present we are unable to fix directly this inconvenience. Therefore, here we propose to follow a different strategy, namely to slightly modify the mathematical structure of the problem so as to obtain a theoretical setting which enables one to successfully look at least for a solution existing for arbitrarily large times T, possibly also $T = +\infty$. Nevertheless, we immediately point out that this operation will not be of zero cost, indeed the gain

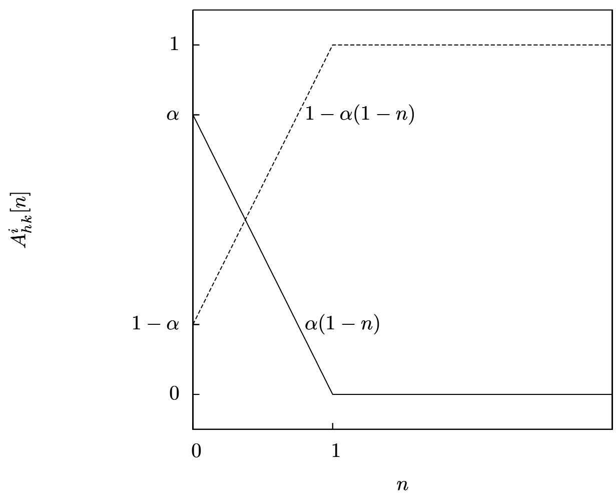

Figure 5.1. A possible form of the coefficients of the table of games fulfilling the new requirement given by Eq. (5.34).

of such a solution will have as counterpart the loss of control on the uniform in time boundedness of the macroscopic density n.

The basic idea is to remove the explicit requirement of upper boundedness of the vehicle density n from the set of assumptions of the problem. In doing so, we clearly have to mind that the table of games and the unweighted interaction rate still fulfil the fundamental properties dictated by the general modeling framework we have presented in Chapt. 3.

In particular, the coefficients $A^i_{hk}[n]$ must remain a discrete probability distribution for the transitions among the velocity classes. The lack of an explicit upper bound on n, however, may lead in principle to a violation of the condition $0 \leq A^i_{hk}[n] \leq 1$, therefore we force this by restating Assumption 4.1 as follows:

$$0 \leq A^i_{hk}[n] \leq 1, \qquad \forall\, n \geq 0. \tag{5.34}$$

We incidentally observe that the table of games proposed in Chapt. 3, Subsect. 3.3.1, does not comply with this new assumption, indeed for $n > 1$ some of the coefficients $A^i_{hk}[n]$ become negative (e.g., $\alpha(1-n)$) and, correspondingly, some others greater than 1 (e.g., $1-\alpha(1-n)$). However, this can be readily fixed by suitably modifying the analytical expression of the mappings $n \mapsto A^i_{hk}[n]$. A constant table of games, that is a table of games whose coefficients A^i_{hk} do not depend either on n or on x, like the one used by Coscia *et al.* [18], is the prototype of the admissible tables of games in the present context. Another possibility is to use the indicator function

$$\chi_{[0,\,1]}(n) = \begin{cases} 1 & \text{if } n \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

to transform the above expressions in $\alpha(1-n)\chi_{[0,\,1]}(n)$, $1-\alpha(1-n)\chi_{[0,\,1]}(n)$, respectively (see Fig. 5.1). A more refined choice might involve mollified versions of these functions.

This way one mimics Eqs. (3.23)-(3.26) for $0 \leq n \leq 1$ while fulfilling at the same time Eq. (5.34) for $n > 1$. This preserves also the property $\sum_{i=1}^{m} A^i_{hk}[n] = 1$ for all $h, k = 1, \ldots, m$ (cf. Eq. (3.19)) and the Lipschitz continuity of the mapping $n \mapsto A^i_{hk}[n]$ (cf. Assumption 5.1) for all $n \geq 0$.

Concerning the unweighted interaction rate, since we are dealing with unconstrained positive densities n we can no longer allow for a blow up of the mapping $n \mapsto \bar{\eta}[n]$ when $n \to 1^-$, hence we are going to assume that Assumption 4.2 holds for every $n \geq 0$, i.e., that there exists a constant $C_{\bar{\eta}} > 0$ such that

$$0 < \bar{\eta}[n] \leq C_{\bar{\eta}}, \qquad \forall\, n \geq 0. \tag{5.35}$$

As for Assumption 5.1, we simply update it by requiring that the mapping $n \mapsto \bar{\eta}[n]$ be Lipschitz continuous on the whole real positive axis. It is plain that this prevents from using directly the unweighted interaction rate given by Eq. (3.27). In particular, from the modeling point of view the global boundedness of $\bar{\eta}[n]$ stated by Eq. (5.35) amounts to assuming a finite reactiveness of the drivers at all densities, with a maximum value fixed by $C_{\bar{\eta}}$.

In this modified setting, we need not have the boundedness of the initial datum $\boldsymbol{\varphi}$ anymore, i.e., the hypothesis (5.19) drops, and we can first seek a local solution in the new set

$$\tilde{\mathcal{B}}_T = \{\mathbf{u} \in X_T \,:\, \hat{\mathbf{u}}(t,x) \geq 0, \|\mathbf{u}(t)\|_1 \leq K_1 \text{ a.e. in } \mathbb{R}, \forall\, t \in [0,T]\} \supseteq \mathcal{B}_T.$$

By inspecting the proofs of Lemmas 5.3, 5.4, 5.5 one sees that the same conclusions hold when one replaces the original set $\mathcal{B}_T$ with $\tilde{\mathcal{B}}_T$. In particular, for suitable choices $\tilde{T}_1, \tilde{T}_2$ of the final time T, the operator S introduced in Eq. (5.18) maps $\tilde{\mathcal{B}}_{\tilde{T}_1}$ into itself and is a contraction on $\tilde{\mathcal{B}}_{\tilde{T}_2}$, while $\tilde{\mathcal{B}}_T$ turns out to be closed in X_T for all $T \geq 0$. Thus, picking up a final time $0 < \tilde{T}^* \leq \min\{\tilde{T}_1, \tilde{T}_2\}$ and using the transformation (5.10) we get again existence and uniqueness of a nonnegative local in time solution $\mathbf{f} \in X_{T^*}$ to Problem (5.1)-(5.2), satisfying further the *a priori* estimate (5.31) and the lower bound (5.33).

The advantage is that now this local solution can be extended for large times via the reasoning illustrated at the beginning of this subsection, because $\mathbf{f}(T^*, x)$ shares with the initial datum $\boldsymbol{\varphi}$ all necessary properties for that machinery to be booted. Nevertheless, the drawback of this approach is that we are unable to predict the boundedness of the macroscopic density of cars within reasonable values, say $n \leq 1$ or at most $n \leq 1 + \nu$ for moderately small $\nu > 0$, which may be tolerated if one imagines to perform the nondimensionalization of n with respect to a characteristic average value rather than to the maximum road capacity (see e.g., Hilliges and Weidlich [30]). Actually, for bounded initial conditions we can at least guarantee, in view of Theorem 5.6, that $n(t)$ is uniformly bounded from above in space by a constant $C_t > 0$ possibly depending on t, i.e., that $n(t,x) \leq C_t$ for a.e. $x \in \mathbb{R}$. However, the open problem still remains that C_t grows in principle unboundedly with t.

5.3 The periodic initial-boundary value problem

The periodic initial-boundary value problem is a particular application of the spatially inhomogeneous problem, which simulates the evolution of the traffic on a road whose ends coincide, for instance a ring.

From the experimental point of view, this is probably the mainly used setting to perform real measurements in inhomogeneous flow conditions, see e.g., Kerner [32]. Indeed, the limited spatial extension of the ring and its simultaneous virtually unlimited length (in the sense that it allows vehicles to cover in principle whatever distance) make it suitable for data recording, while permitting at the same time the observation of a wide variety of traffic phenomena like, e.g, stop-and-go waves or clustering.

This kind of problem is mathematically described by the system of equations (5.1) on a bounded spatial domain, say in dimensionless form $D_x = [0, 1]$, joined to the initial condition (5.2) for $x \in (0, 1)$ and to periodic boundary conditions of the form

$$f_i(t, 0) = f_i(t, 1), \qquad \forall\, t \in (0, T], \;\; i = 1, \dots, m. \tag{5.36}$$

By consequence, the solution $\mathbf{f}$ is expected to be periodic in space with unit period.

Let us introduce the space $(L^1_{\#}(\mathbb{R}))^m$ consisting of all vector-valued 1-periodic functions $\mathbf{u} : \mathbb{R} \to \mathbb{R}^m$, $\mathbf{u}(x) = \mathbf{u}(x + 1)$ for a.e. $x \in \mathbb{R}$, which are integrable on $[0, 1]$, along with the norm

$$\|\mathbf{u}\|_{1,\#} := \sum_{i=1}^{m} \int_0^1 |u_i(x)|\, dx.$$

It is immediate to notice that this space can be identified with $(L^1(0, 1))^m$, with however the further property that its functions are integrable on *every* interval of $\mathbb{R}$ of unit length. In fact, a simple computation shows that for all $a \in \mathbb{R}$ it results

$$\sum_{i=1}^{m} \int_a^{a+1} |u_i(x)|\, dx = \sum_{i=1}^{m} \left[\int_0^1 |u_i(x)|\, dx - \int_0^a |u_i(x)|\, dx + \int_1^{a+1} |u_i(x)|\, dx \right] = \|\mathbf{u}\|_{1,\#},$$

since the last two integrals in the above expression cancel out mutually due to the periodicity of each u_i, $i = 1, \dots, m$.

In Eq. (5.1), the interaction length ξ must obviously be assumed lower than the length of the ring, thus $\xi < 1$. As a result, under suitable nonnegativity properties of the functions, the integrals over $[x, x + \xi]$ appearing in the gain and loss terms of the kinetic equations can be bounded from above by the corresponding integrals over $[0, 1]$. In addition, the formal extension of the functions to $\mathbb{R}$ by periodicity makes it possible to use the mild formulation (5.13) of the problem with the same transformation (5.10), so that finally all results proved for the initial value problem apply straightforwardly also to this case. Specifically, after introducing the Banach space $X_T = C([0, T];\, (L^1_{\#}(\mathbb{R}))^m)$ endowed with the norm

$$\|\mathbf{u}\|_{X_T} := \sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{1,\#} = \sup_{t \in [0, T]} \sum_{i=1}^{m} \int_0^1 |u_i(t, x)|\, dx,$$

together with its subset

$$\mathcal{B}_T = \Bigg\{ \mathbf{u} \in X_T \,:\, \hat{\mathbf{u}}(t,x) \geq 0,\, \sum_{i=1}^{m} \hat{u}_i(t, x - v_i\tau) \leq Ke^{\lambda t},$$

$$\|\mathbf{u}(t)\|_{1,\#} \leq K_1 \text{ a.e. in } [0,1],\, \forall\, t \in [0,T], \tau \geq 0 \Bigg\},$$

one can show, under the same assumptions on the initial condition given for the initial value problem, and by slightly adapting the proofs of Lemmas 5.3, 5.4, 5.5, and of Theorem 5.6, that Problem (5.1)-(5.2)-(5.36) admits a unique nonnegative local solution $\mathbf{f} \in X_{T^*}$ defined up to a final time $T^* > 0$, such that

$$n(t,x) \leq K, \qquad \text{for a.e. } x \in (0,1),\, \forall\, t \in (0,T^*].$$

This solution satisfies in addition both the *a priori* estimate (5.31) and the lower bound (5.33). In particular, the former is now entailed by the periodicity of $\mathbf{f}$, which implies $q(t,0) = q(t,1)$ for all $0 < t \leq T^*$.

Concerning the regularity of $\mathbf{f}$, we observe the initial datum has to be consistent with the boundary conditions, i.e., $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}(1)$. Indeed, if this is not the case then a jump across the boundary of the spatial domain arises, which propagates along the characteristics making the solution discontinuous.

Finally, regarding the possibility to extend the local solution to a global in time solution there is at present no improvement here with respect to the issues discussed in Subsect. 5.2.3 about the initial value problem.

5.4 Computational analysis of some cases study

We end this chapter devoted to the study of the spatially inhomogeneous problem by performing a numerical analysis of some representative cases well documented in the specialized literature (see e.g., Klar and Wegener [34]), in order to test the ability of the model to reproduce some typical characteristics of the inhomogeneous flow of vehicles.

In all problems discussed below we use a uniform velocity grid I_v constituted by six velocity classes, with in particular $v_1 = 0$ and $v_6 = 1$ (cf. Eq. (3.1)). The hyperbolic system (5.1) has been integrated by a high resolution conservative method based on the slope-limiters technique, in which the first order upwind flux is combined with a correction term given by the Superbee limiter (see LeVeque [38] for further details). We choose the interaction length to be $\xi = 0.05$ and use the weight function given by Eq. (3.28). The table of games and the unweighted interaction rate are as described in Chapt. 3, Subsects. 3.3.1, 3.3.2, for different values of the road parameter α to be specified from time to time.

5.4.1 Formation of a queue

We intend to simulate the formation of a queue due to the accumulation of some incoming vehicles behind a pre-existing group of motionless vehicles.

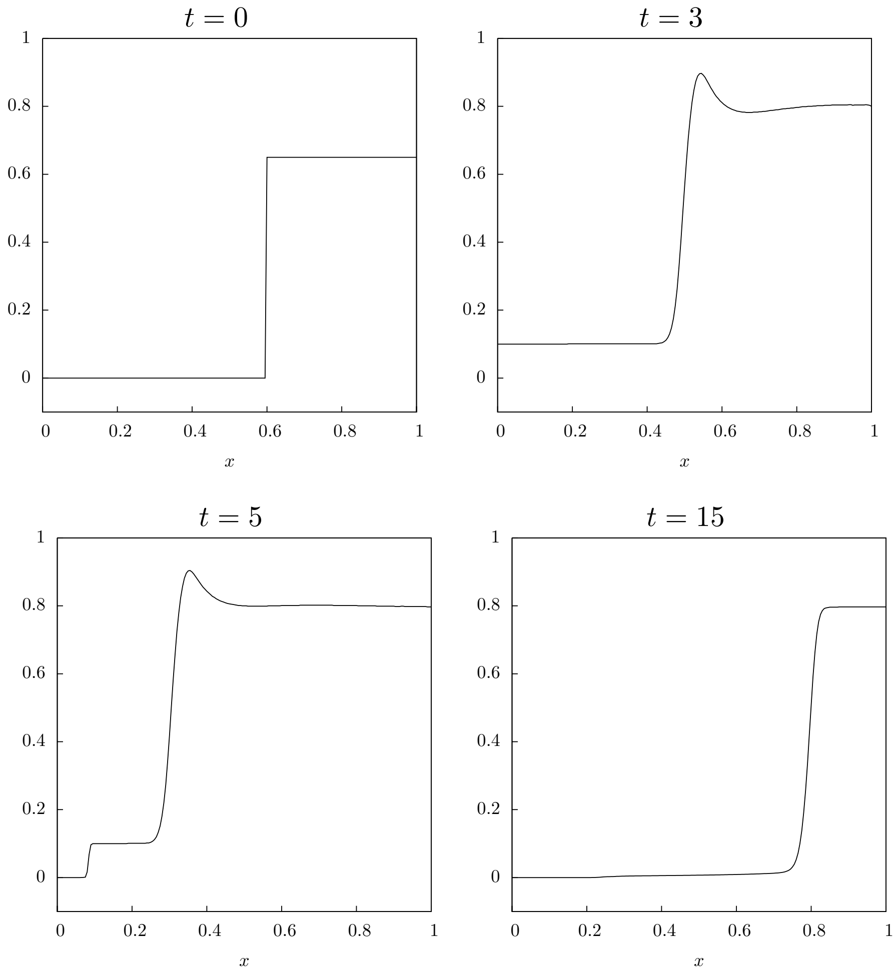

Figure 5.2. Evolution of the density of cars in a queue at different times. An initial plateau of motionless vehicles ($t = 0$) is reached by an incoming flux of vehicles ($t = 3$) entering the domain from the left boundary $x = 0$. This makes the local density increase, and gives rise to a density wave propagating backward until new vehicles stop entering the domain ($t = 5$). At that time the emptying of the queue begins as a consequence of the flow of vehicles from the right ($t = 15$).

As initial condition we set all distribution functions f_i to zero but the one corresponding to the first velocity class $i = 1$, which is instead assumed constant in a suitable stretch behind the outflow boundary $x = 1$ of the spatial domain. Hence, f_1 determines for $t = 0$ the initial profile of the density n, which exhibits a plateau representing the above-mentioned pre-existing queue. At the inflow boundary $x = 0$ we imagine a group of incoming vehicles entering the domain with a certain positive velocity, that we choose as the maximum possible according to our velocity grid.

Figure 5.2 shows the result of the simulation in terms of the macroscopic density n. Note in particular the expected enlargement of the plateau due to a backward propagation of the queue toward the inflow boundary, with a nearly constant maximum value of the density located in the rear part of the group of vehicles. A fundamental contribution in reproducing correctly these features is given by both the dislocation of the interactions over the whole amplitude of the visibility zone and the increment in the interaction rate for growing density. Indeed, it is well known that without the former there is no way to obtain backward propagation of the information (see e.g., Klar and Wegener [34]), that is the initial plateau would not grow longitudinally along the road, its rear front remaining fixed at the same initial location for all $t > 0$. On the other hand, without the latter the density may achieve locally values greater than 1. This risk is especially high in the rear part of the queue, at the attack point between the slow queued vehicles and the fast incoming ones, where many incoming vehicles are required to slow down due to the little space left to overtake. The effect of the interaction rate, which grows to infinity for n close to 1, is precisely that of inducing numerous transitions of velocity class by the vehicles, avoiding then the superposition of different density waves which may locally sum to more than 1.

At time $t = 5$ the left boundary condition switches to zero for all distribution functions f_i, that is no vehicle is entering the domain anymore from then on. As a consequence, a slow flow of vehicles from the right boundary begins, with a progressive emptying of the queue.

5.4.2 The bottleneck

Next we want to study the effect on the traffic of a variation in the maximum density allowed along the road. This situation may arise for instance as a consequence of a reduction in the number of lanes available to the vehicles, or more in general because of a narrowing of the roadway, and is usually referred to as a bottleneck.

In our model we obtain this effect by simply rescaling the nondimensional density n by a variable maximum value, which depends on x as shown by the dashed line in Fig. 5.3. This corresponds to performing a nondimensionalization of the physical density of cars with respect to a characteristic maximum value given by a function of x: $\mathcal{N} = n_{\max} = n_{\max}(x)$. In particular, we use a bottleneck density profile which is close to 1 at the inflow boundary and decreases to 0.4 at the outflow boundary, causing a reduction of approximately 60% in the road capacity.

Initial and boundary conditions for this problem are similar to those prescribed in Subsect. 5.4.1, with a group of slow vehicles inside the bottleneck and an incoming group of fast vehicles at the left boundary. Once again, formation and backward propagation of a queue are observed, with a density profile which closely follows that of the bottleneck. We stress that such a result is mainly due to the action of the interaction rate, which regulates the number of vehicles undergoing velocity jumps on the basis of the maximum density locally allowed.

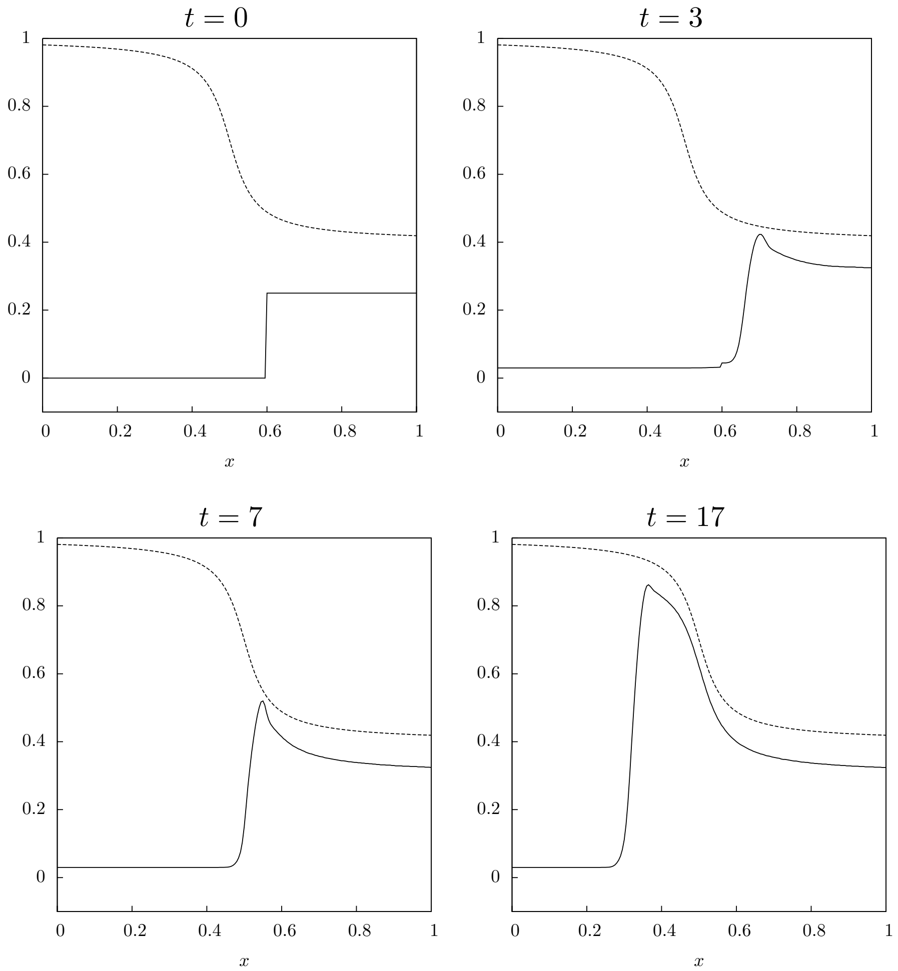

Figure 5.3. Formation and evolution of a queue in a bottleneck, that is a variation in the maximum density allowed along the road due to a reduction in the road capacity. Dashed line represents the bottleneck density profile, obtained in this simulation using an arc tangent-like function.

5.4.3 Merging of two clusters and stop-and-go waves

Finally we consider the case of a ring-road where overtaking is forbidden, which can be formally obtained by setting $\alpha = 0$ in the table of games (3.23)-(3.26). This may be due, for instance, to very bad road conditions or to some sort of special limitations imposed on the traffic. The result is that vehicles simply tend to maintain their current speed until they reach other slower vehicles. Then, they are forced to slow down to the velocity of the leaders and to queue, no matter how much free space is actually available on the road.

The simulation (see Fig. 5.4) starts with two clusters of vehicles at the same density ($n = 0.5$) traveling at different speeds. In particular, the front cluster is slower than the rear one, the difference of speed between them being of one class in the grid I_v. When the

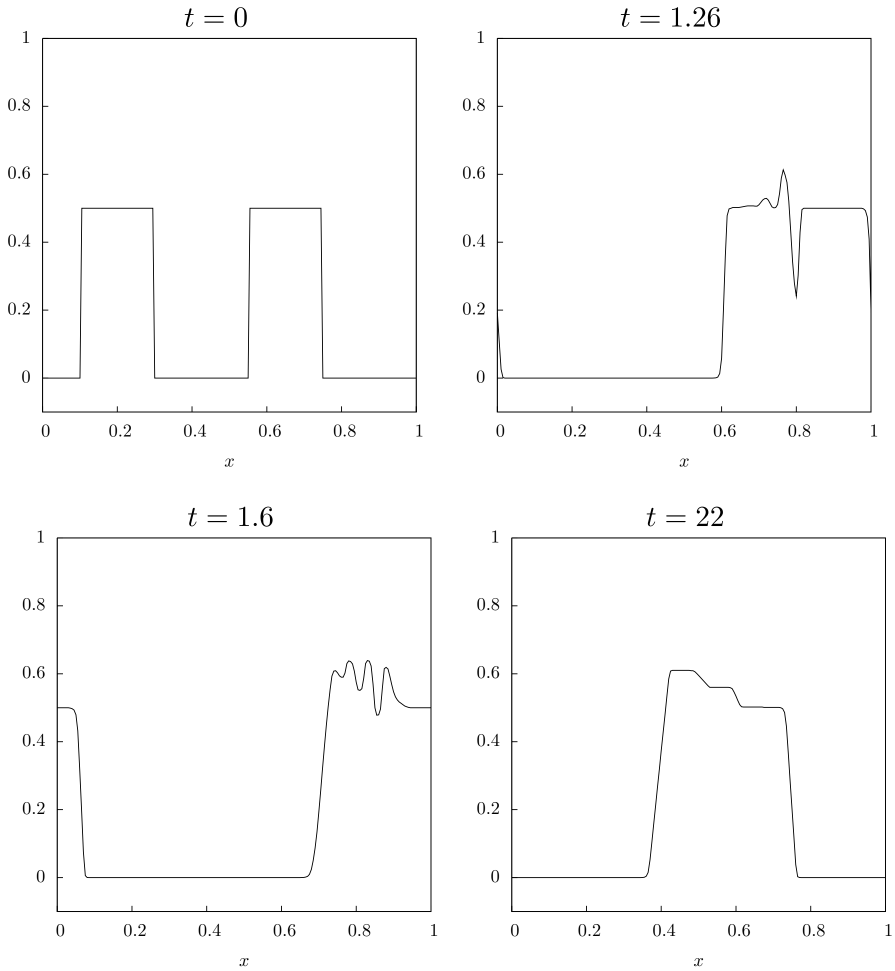

Figure 5.4. Merging of two clusters of vehicles, whose the faster pursues the slower, along a ring-road where overtaking is forbidden. Immediately after the encounter ($t = 1.26$), a perturbation arises in the density distribution of the rear cluster, due to a progressive slowdown of the vehicles. This causes a packing of the two clusters and a simultaneous appearance of some stop-and-go-like waves ($t = 1.6$). These finally disappear when the velocity in the rear cluster becomes uniform again, and equal to that of the front cluster ($t = 22$).

second cluster reaches the first, they merge into a unique group of vehicles which keeps the velocity of the slower one.

It is interesting to note the formation of stop-and-go-like waves (cf. Günther *et al.* [27]) in the rear cluster as a reaction to the velocity transitions occurring immediately after the encounter with the front one. Such waves, that finally smooth out as the velocity within the rear cluster becomes uniform, are observed thanks to the dislocation of the interactions, which makes the leading vehicles of the rear cluster realize first the need for slowing down while approaching the front cluster.

Chapter 6

Conclusions and research perspectives

In this thesis we have systematically developed a discrete velocity framework for one-dimensional kinetic modeling of vehicular traffic, using as guidelines the kinetic structures formalized by Arlotti *et al.* [1, 2] and by Bellomo [5], as well as some general ideas about discrete kinetic and stochastic game theory proposed by Bertotti and Delitala [13]. The main features of this modeling framework include:

(i) *Discreteness of the velocity variable*, which is allowed to range in a finite set of values only. Vehicles are thus grouped in *velocity classes*, that roughly express their tendency to travel slowly or fast without distinguishing between any two arbitrarily close velocities. Rather than being a pure mathematical technicality aimed at reducing the analytical and computational complexity of the problem (like in the traditional discrete kinetic theory associated with the classical Boltzmann equation, see e.g., Bellomo and Gatignol Eds. [8] and Gatignol [24]), in this context the discretization of the velocity is conceived so as to account for one of the main sources of granularity of the flow of cars along a road.

(ii) *Microscopic modeling* of the interactions among the vehicles, despite the global, i.e., macroscopic, point of view on the evolution of the system. Interactions are modeled by detailing the short-range reactions of the drivers to the traffic conditions ahead instead of trying to interpret their overall behavior. On the other hand, the latter is accessible *a posteriori*, as by-product of the integration of the equations of the model.

(iii) *Stochastic interactions* among the vehicles responsible for velocity variations. Non-classical interactions experienced by cars are modeled without invoking any concept of force, proper of point mechanics, but appealing instead to a stochastic game theory, which defines suitable transition probabilities among the velocity classes (*table of games*). These probabilities are determined on the basis of the current speeds of the interacting pairs, and possibly also of the local congestion and quality of the road.

(iv) *Nonlocal interactions.* Vehicles are assumed to interact in pairs over a *visibility area*, which extends from each car up to a certain characteristic *visibility length* along the road. Within the visibility area, interactions are weighted by the distance separating the interacting pairs, and occur with more or less frequency according to the reactiveness of the drivers, which is possibly influenced in turn by the local congestion of the road.

Within this framework, we have also developed a specific traffic model by detailing the microscopic interactions of the cars in terms of table of games and interaction rate. The model has shown ability to describe specific traffic flow phenomena in both the spatially homogeneous and the spatially inhomogeneous problem.

In the first case, existence, uniqueness, and *a priori* boundedness of a physically relevant solution have been proved. Moreover, the dependence of the equilibrium configurations on a phenomenological parameter linked to the quality of the road has been investigated, at least in some relatively simple but representative cases. Numerical simulations provide fundamental diagrams and equilibrium velocity distributions highly close to the corresponding available experimental data. In more detail, the fundamental diagrams of the macroscopic flux and the average velocity exhibit, at least for good road conditions, a strong variation in their behaviors when passing from low to high density, which can be interpreted as the ability of the model to capture the phase transition between free and congested flow. Specifically, the almost linear behavior of the flux for low density, with a corresponding nearly constant average velocity close to the maximum allowed value, and its subsequent nonlinear decrease to zero, with a simultaneous steep decrease of the velocity toward low values, are in good agreement with the experimental observations performed in free and congested uniform flow conditions.

The spatially inhomogeneous problem has been addressed from the qualitative point of view by investigating existence and uniqueness of admissible solutions, especially in connection with the possibility to control the maximum value attained by the density of cars. Indeed, confining the density n below the (dimensionless) road capacity, say $n_{\max} = 1$, is important not only from the obvious physical point of view but possibly also for mathematical reasons of consistency of the models generated by the framework. Since the macroscopic density, which varies in time and space, is in general supposed to affect the table of games, the necessity arises to guarantee that the latter actually represents a discrete distribution probability over the velocity variable at all times and all spatial locations. Depending on the specific analytical expressions of the coefficients of the table of games as functions of n, this requirement may be violated if the density locally becomes negative or exceeds its maximum allowed value $n_{\max}$. For instance, the table of games proposed in Chapt. 3 contains expressions like αn, $\alpha(1-n)$, which are conceived under the physically sharable assumption of a (dimensionless) density of cars bounded between 0 and 1. However, the mathematical model need not keep automatically this constraint. On the other hand, should n become negative or take values greater than 1, the table of games itself would lose its mathematical consistency, becoming incompatible with the aforementioned probabilistic setting. Notice that in the spatially homogeneous case this risk is ruled out by Theorems 4.3, 4.4, which guarantee the global well-posedness of the problem under a suitable choice of the initial data. Conversely, in the spatially inhomogeneous case local in time results have been proved to hold for the Cauchy problem and for the periodic initial-boundary value problem (cf. Theorem 5.6 and Sect. 5.3), nevertheless at present they cannot be straightforwardly extended to global in time results. However, by slightly stiffening the structural assumptions on the table of games and, at the same time, relaxing the constraint of uniform upper boundedness of the density n, it has been possible to achieve existence and uniqueness results for arbitrarily large times, which may constitute a cue toward deeper analysis and suitable improvements of the framework.

Numerical simulations of the spatially inhomogeneous problem have been carried out, by addressing three representative cases study which highlight the ability of the model to reproduce correctly some interesting features of traffic. In particular, we recall here the merging of two clusters of vehicles with concomitant appearance of stop-and-go waves (Chapt. 5, Subsect. 5.4.3), and the formation and backward propagation of a queue, possibly in presence of a bottleneck (Chapt. 5, Subsects. 5.4.1, 5.4.2), with a vehicle density profile which in this second case closely follows that of the bottleneck, and which in both cases never overcomes the maximum value fixed by the road capacity. If the dislocation of the interactions is undoubtedly responsible for backward density waves in spite of the unidirectional motion of the vehicles, the interaction rate introduced in the interaction terms of the equations favors instead the boundedness of the density. Since the rate of the interactions among the candidate vehicle and the field vehicles strongly increases when the macroscopic density approaches the upper limit threshold, a larger number of velocity transitions toward lower velocity classes is induced in such a case, thus preventing the superposition of different density waves which may locally sum to more than the threshold.

The modeling framework developed in this thesis accounts for a fully mechanical microscopic state of the vehicles, which is described by their position x and their (discrete) velocity v_i. On the other hand, it can be argued that cars are actually not completely mechanical subjects, for the presence of the drivers is likely to affect their behavior in an essentially unconventional way.

The specialized literature offers some examples of models trying to explicitly account for the action of the drivers on the evolution of the traffic. For instance, in the context of first order hydrodynamic models, De Angelis [20] introduces the concept of *fictitious density* n^*, that is the density of cars along the road as perceived by the drivers, in order to express the personal feelings of the latter about the local conditions of traffic, and to model accordingly their instantaneous reactions. Specifically, the fictitious density is linked to the actual density n and to its spatial gradient $\partial_x n$ in such a way that when n is increasing (respectively, decreasing) n^* increases (respectively, decreases) in turn but more quickly than n, while for n approaching the limit threshold 1 also n^* tends to saturate to 1. In other words, the sensitivity of the drivers should consist in feeling a fictitious congestion state of the road higher or lower than the real one, and by consequence in anticipating certain behaviors that a purely mechanical agent would exhibit only in presence of the appropriate external conditions. The dependence of n^* on n, $\partial_x n$ proposed by De Angelis is

$$n^*[n, \partial_x n] = n + \gamma(1-n)\frac{\partial n}{\partial x}, \tag{6.1}$$

where $\gamma > 0$ is a parameter. It can be observed that for increasing actual density, that is $\partial_x n > 0$, the fictitious density is larger than n, thus suggesting a more careful attitude of the system car-driver with respect to the purely mechanical case, while for decreasing actual density, namely $\partial_x n < 0$, the fictitious density is lower than n, which implies a more aggressive behavior induced by the presence of the driver.

Particular mathematical models are obtained by plugging Eq. (6.1) into the expression of the velocity diagram used in first order macroscopic models. This amounts essentially to devising a closure relation of the mass conservation equation (1.2) of the form $u = u(n^*)$, so that the evolution equation finally reads

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu(n^*)) = 0.$$

At the kinetic level of representation, no model has been so far introduced in the literature taking the action of the driver explicitly into account. As a matter of fact, one may observe that the stochastic game theory framework acknowledges some nonclassical behavioral components in the evolution of traffic, nevertheless these are by themselves only implicit hints about a more complex structure of the system, which calls instead for a suitable modeling via proper mathematical tools.

Enhancements of the discrete velocity kinetic framework presented in this thesis can be conceived in order to effectively introduce in the theory the action of the drivers. This can be done by referring to the generalized kinetic methods for systems of *active particles* (Bellomo [5]), that is particles whose microscopic state includes, besides the standard mechanical variables, also an additional internal variable named *activity*, a scalar or a vector quantity depending on the specific application.

In the case of vehicular traffic, the activity may be represented by a scalar dimensionless variable u, ranging in the domain $D_u = [0, 1]$, which stands for the driving ability, or alternatively for the personality, of each driver. Low values of u close to 0 denote timid and inexperienced drivers, whereas high values close to 1 identify aggressive and experienced drivers. In more detail, we observe that it may be convenient to think of the activity as a discretely distributed variable, for this exempts from giving it a so fine definition to catch the difference between any two arbitrarily close values of u. Indeed, the driving ability being a subjective, hence hardly quantifiable, concept, what really matters is simply to make a rough distinction between bad and good drivers. Therefore, one can introduce a grid $I_u = \{u_j\}_{j=1}^p$ in D_u, which classifies the drivers in *activity classes* according to their driving skills, and let consequently $u \in I_u$. As for the discretization of the velocity, the number p of activity classes has to be kept in principle small enough for the above mentioned rough distinction among the different abilities of drivers to make sense, coherently with the general modeling ideas underlying the discrete kinetic theory of traffic.

In this enlarged framework, the distribution function f depends also on u in such a way that the quantity

$$f(t, x, v, u)\, dx\, dv\, du$$

gives a measure of the number of cars that at time t are located between x and $x+dx$, with a velocity comprised between v and $v+dv$, and an activity state between u and $u+du$. In particular, in view of the discretization of both v and u, the following representation of f can be introduced:

$$f(t,x,v,u) = \sum_{i=1}^{m}\sum_{j=1}^{p} f_{ij}(t,x)\delta(v-v_i)\delta(u-u_j),$$

where $f_{ij}(t,x)$ gives the distribution at time t and in the point x of the vehicles with velocity and activity comprised in the i-th velocity class and in the j-th activity class, respectively. Summing over the index j allows formally to recover the distribution function f_i of the vehicles in the i-th velocity class:

$$f_i(t,x) = \sum_{j=1}^{p} f_{ij}(t,x), \tag{6.2}$$

whereas summing over the index i gives the distribution of the vehicles in the j-th activity class:

$$P_j(t,x) = \sum_{i=1}^{m} f_{ij}(t,x).$$

The macroscopic variables of interest are instead recovered by summing out both indexes i, j or, equivalently, by applying the definitions (3.3)-(3.6) to Eq. (6.2).

Suitable evolution equations in time and space for the collection of distribution functions $\{f_{ij}(t,x)\}_{i,j}$ can be derived, following an analogous procedure to that illustrated in Chapt. 3. Notice that the internal structure u of the vehicles has to be explicitly accounted for as a component of the microscopic state, which is now represented by the triple $\mathbf{w} = (x, v_i, u_j)$. In particular, the interaction terms, and specifically the table of games, should now depend also on the activities u_j of the drivers, so as to introduce their effect on the dynamics of the system.

A second research thread may involve the discretization of the whole state space, namely not only of the velocity and possibly the activity but also of the space variable, in order to include in the kinetic theory of vehicular traffic the spatial granularity of the flow of cars. Some hints and ideas toward this topic are provided in the already cited book by Bellomo [5], however research in this area is still at a very embryonic stage. Here we simply remark that not only modeling issues but also nontrivial technical difficulties have to be dealt with, like, among others, the proper way to render in a discrete space context the spatial derivatives appearing in the kinetic equations.

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