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Abstract

In this note we give proofs of the following three algebraic facts which have applications in the theory of holonomy groups and homogeneous spaces: Any irreducibly acting connected subgroup $G \subset Gl(n, \mathbb{R})$ is closed. Moreover, if G admits an invariant bilinear form of Lorentzian signature, G is maximal, i.e. it is conjugated to $SO(1, n-1)_0$. We calculate the vector space of G-invariant symmetric bilinear forms, show that it is at most 3-dimensional, and determine the maximal stabilizers for each dimension. Finally, we give some applications and present some open problems.

MSC: 53C29; 53C30; 22E15; 20G05.

1 Background, results and applications

This article was motivated by three algebraic questions which are related to problems in holonomy theory of affine or semi-Riemannian manifolds and in the theory of homogeneous spaces.

Two of this questions are known for most experts in differential geometry but, besides special cases, general proofs are not easy to find in the literature. Thus, one goal of this paper is to supply such proofs. Indeed, we think that this results are not so well-known for experts in other areas of mathematics or physics. So, we hope this paper to be useful for non experts in differential geometry. We will give some applications in order to illustrate the use of it. We also state some open problems in the form of conjectures.

The three motivating problems are the following: Are holonomy groups closed? What are special holonomy groups of Lorentzian manifolds? And finally, how many G-invariant bilinear forms exist on a homogeneous space G/H?.

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Regarding the first question, the first thing to observe is that the the answer should not depend o the property of being a holonomy group, because due to a result of [HO65] any linear Lie group can be realised as a holonomy group of a linear connection (usually with non trivial torsion). Secondly one finds that the answer is 'yes' if one restricts it to holonomy groups of Riemannian manifolds, because any *irreducibly acting*, connected subgroup of $O(n)$ is closed (see for example [KN63, Appendix 5]), and thus, by the de Rham decomposition theorem [dR52] the holonomy group is a direct product of closed ones. But for a general connection the answer is 'no', there are a lot of examples of non-closed holonomy groups, see [HO65] for affine, *torsion-free* connections, [Wu67] for pseudo-Riemannian manifolds and [BI93] for Lorentzian manifolds. In all of these examples the holonomy group does not act irreducibly. So it arises the question if all irreducibly acting subgroups of $Gl(n, \mathbb{R})$ are closed? There is a note in [Bes87, Note 10.50, page 290] that the following theorem is true proved in [Wak71].

Theorem 1. *Any irreducibly acting connected Lie subgroup of $Gl(n, \mathbb{R})$ is closed in $Gl(n, \mathbb{R})$.*

In this note firstly we will prove this theorem (see Section 3) independently on the proof in [Wak71]. Regarding holonomy groups of linear connections this has the following consequence.

Corollary 1. *If the restricted holonomy group of a linear connection acts irreducibly, then it is closed. Furthermore, the restricted holonomy group of a semi-Riemannian manifold is closed if it acts completely reducibly.*

The second statement follows from the de Rham/Wu decomposition theorem [Wu64] and another theorem in [Wu67] (see below, last paragraph in section 3). Our proof of Theorem 1 uses a theorem of Yosida [Yos37] and Malcev [Mal45] and a very explicit description of the center of G. This description gives us two corollaries, the first of which will be useful in the proof of Theorem 2.

Corollary 2. *Let $G \subset O(p, q)$ a connected Lie subgroup of $O(p, q)$ which acts irreducibly. If G is not semisimple, then p and q are even and G is a subgroup of $U(p/2, q/2)$ with center $U(1)$.*

Applying this to the spin representations orthogonal groups we get:

Corollary 3. *Let $G \subset SO_0(p, q)$ be a connected Lie subgroup which acts irreducibly and $\tilde{G} \subset Spin(p, q)$ the pre-image of the covering $Spin(p, q) \to SO_0(p, q)$. If the spin representation of $\tilde{G}$ admits a trivial subrepresentation, then G is semisimple, or $G = U(1) \cdot G'$ with $G' \subsetneqq SU(p/2, q/2)$ and $(p+q)/2$ is even.*

Unfortunately, one cannot show that the existence of a trivial sub-representation of the spin representation implies semisimplicity of G as the following example shows: Let $G = U(1) \cdot Sp(p, q)$ with $p + q$ even. Then the intersection of the spaces of spinors which are annihilated by $U(1)$ and with the ones which are annihilated by $Sp(p, q)$ is a one-dimensional space (for details see [BK99]) .
Nevertheless, this corollary has applications to geometric problems. The first application is a well known fact. If the holonomy group of a semi-Riemannian manifold acts irreducibly and has a center, the manifold cannot admit parallel spinors. It was obtained by the classification of irreducible holonomy groups of semi-Riemannian manifolds with

parallel spinors ([Wan89] for Riemannian manifolds, and [BK99] for pseudo-Riemannian manifolds). But furthermore it gives results in any case where the holonomy group has a irreducibly acting component on which the existence of parallel spinors depends, as it is the case for indecomposable, non-irreducible Lorentzian manifolds, see [Lei02b].

Regarding the second problem which special Lorentzian holonomy groups might exist, one distinguishes between the irreducible and the indecomposable, non-irreducible case. While in the latter case there a several possibilities (for a classification see [BI93], [Lei02a], [Lei03a], [Lei03b], and [Gal05]), for the irreducible case the situation is very limited. The irreducible holonomy groups of semi-Riemannian manifolds were determined by M. Berger in [Ber55] and [Ber57]. In Riemannian and many other signatures the list depends essentially on the property of being a *holonomy group*, whereas in the Lorentzian case it turns out that *irreducibility* is sufficient to determine the group.

Theorem 2. *$SO(1,n)_0$ is the only connected Lie subgroup of $O(1,n)$ which acts irreducibly.*

The consequence for irreducible Lorentzian holonomy groups follows immediately.

Corollary 4. *If the restricted holonomy group of a Lorentzian manifold acts irreducibly, then it is equal to $SO(1,n)_0$.*

Here is an application to isotropy irreducibly Lorentzian homogeneous spaces.

Corollary 5. *Isotropy irreducibly Lorentzian homogeneous spaces have constant sectional curvature i.e. they are flat, de-Sitter or anti de-Sitter spaces.*

A direct and geometric proof of Theorem 2 was given in [DSO01]. An almost algebra-free proof which uses dynamical methods, can be found in [BZ04]. Finally, a purely algebraic proof was given in [BdlH04]. A nice and direct proof of Corollary 5, based on dynamical methods, can be found in [Zeg04]. In Section 4 we will give a short proof of Theorem 2 based on a theorem of Karpelevich [Kar53] and Mostow [Mos55] (i.e. Theorems 7 and 8). Theorem 2 also follows from Corollary 4.5.1 in [CheGre] which in turns depends upon Karpelevich-Mostow's theorem (cf. Lemma 4.4.3 in [CheGre] with Theorem 8).
We would like to thank the referee to call our attention about the paper by S.S. Chen and L. Greengerg [CheGre].

The result of the last section 5 is motivated by the geometric problem of describing the space of metrics or symplectic forms on a homogeneous space G/H which are invariant under G. Any G-invariant metric or symplectic form corresponds to a non-degenerate bilinear form on $\mathfrak{g}/\mathfrak{h}$ which is invariant under the linear isotropy representation $Ad_G(H) \subset Gl(\mathfrak{g}/\mathfrak{h})$. In our context $Ad_G(H)$ is assumed to act irreducibly. This is a special case of the following algebraic problem: Given an irreducibly acting Lie subgroup $G \subset Gl(n,\mathbb{R})$, what is the dimension of the space of G-invariant bilinear forms on $\mathbb{R}^n$. We prove the following statement.

Theorem 3. *Let G be an irreducibly acting subgroup of $Gl(n,\mathbb{R})$. The the space of G-invariant symmetric bilinear forms which are not of neutral signature (p,p) is at most one-dimensional. Moreover, the space of invariant symmetric bilinear forms is at most three-dimensional.*

We will describe all possible cases for the dimension of the space of (skew-) symmetric bilinear forms and determine the maximal subgroup which fixes these bilinear forms.

We should point out that many results in this paper rely on the classification of G-invariant endomorphisms for $G \subset Gl(n, \mathbb{R})$. This classification follows from Schur's lemma and the classification of associative division algebras by Frobenius, but we will give an elementary proof of it in Section 2.

2 The algebra of invariant endomorphisms

The results of this paper are mainly based on a description of the algebra of endomorphisms which are invariant under an irreducibly acting subgroup $G \subset Gl(n,\mathbb{R})$. If G is a Lie group and V and W two (real or complex) G-modules the algebra of invariant homomorphism is defined as

$$Hom_G(V,W) \ := \ \{X \in Hom(V,W) \mid A \circ X = X \circ A \text{ for all } A \in G\}.$$

Now, Schur's lemma says that $Hom_G(V,W) \subset Iso(V,W) \cup \{0\}$, and furthermore, if $V = W$ is complex, then $End_G(V) = \mathbb{C} \cdot Id$. In any case, it implies that $Hom_G(V,W)$ is a real associative division algebra, and thus by their classification of Frobenius (1878) it is isomorphic to the algebra of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or quaternions $\mathbb{H}$ (see e.g. [Pal68]). We are interested in the description of $End_G(V)$ where V is a real vector space, and in this section we will recall some facts about real irreducible representations which provide an elementary proof of this result.

Suppose that G is a real Lie group and V a real irreducible module. Then there are two cases which can occur for the complexified G-module $V^{\mathbb{C}}$. The first case is that $W := V^{\mathbb{C}}$ is still irreducible. In this case V or W is called *of real type*. One should remark that, if W is a complex irreducible G-module then its reellification $W_{\mathbb{R}}$ is a reducible G-module with invariant real subspace V if and only if W is the complexification of the real irreducible G-module V.
In the other and more complicated case, regarding the application of Schur's lemma, $V^{\mathbb{C}}$ is a reducible G-module. In this case $V^{\mathbb{C}}$ splits into two irreducible G-modules,

$$V^{\mathbb{C}} = W \oplus \overline{W}.$$

In fact, if W is an invariant complex subspace of $V^{\mathbb{C}}$ then $\overline{W}$, defined by the conjugation with respect to $V \subset V^{\mathbb{C}}$ is invariant too and the conjugate module. Furthermore, the spaces $W + \overline{W}$ and $W \cap \overline{W}$ are invariant and equal to their conjugation. Hence they are complexifications of real vector spaces, i.e. $W + \overline{W} = V_1^{\mathbb{C}}$ and $W \cap \overline{W} = V_2^{\mathbb{C}}$. Of course V_1 and V_2 are invariant subspaces of V and thus $V_1 = V$ and $V_2 = \{0\}$. The same argument ensures the irreducibility of W.
Now, since $W \cap \overline{W} = \{0\}$, the mapping $\psi : W_{\mathbb{R}} \ni v \mapsto \frac{1}{2}(v + \overline{v}) \in V$ is an isomorphism of real vector spaces yielding the identification

$$W_{\mathbb{R}} \ \overset{\psi}{\simeq} \ V \ \overset{\psi}{\simeq} \ \overline{W}_{\mathbb{R}} \tag{1}$$

of real G-modules. In this case V, respectively W, are called *of complex type*, and again we have that a complex module W has an irreducible reellification $V = W_{\mathbb{R}}$ if and only if $V^{\mathbb{C}} = W \oplus \overline{W}$ is reducible.
Now we are able to describe the algebra of invariant endomorphisms of a real irreducible G-module V.

Proposition 1. *Let G a Lie group and V a real irreducible G-module. Then $End_G(V)$ is isomorphic to one of the real algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

Proof. As above we consider two cases. Firstly assume that $V^{\mathbb{C}}$ is irreducible which ensures that $End_G(V^{\mathbb{C}}) = \mathbb{C} \cdot Id$ by Schur's lemma. Hence, if $A \in End_G(V)$, its complexification $A^{\mathbb{C}} \in End_G(V^{\mathbb{C}})$ is given by $A^{\mathbb{C}} = \lambda \cdot Id$ with $\lambda \in \mathbb{C}$. Since $A^{\mathbb{C}}$ leaves V

invariant and V is invariant under conjugation we get for $v = \overline{v} \in V$ that

$$\overline{\lambda}\, v \;=\; \overline{\lambda}\, \overline{v} \;=\; \overline{A^{\mathbb{C}} v} \;=\; A^{\mathbb{C}} v \;=\; \lambda v,$$

i.e. $\lambda \in \mathbb{R}$. $A = A^{\mathbb{C}}|_V$ gives that $End_G(V) = \mathbb{R} \cdot Id$.
For the second case we have to assume that $V^{\mathbb{C}}$ is reducible, i.e. by the above $V^{\mathbb{C}} = W \oplus \overline{W}$, $V \simeq W_{\mathbb{R}}$ and thus $End_G(V) = End_G(W_{\mathbb{R}})$. Now any real endomorphism on $W_{\mathbb{R}}$ decomposes uniquely into a complex linear and complex anti-linear part:

$$\begin{array}{ccccc} End(W_{\mathbb{R}}) & \simeq & End(W) & \oplus & Hom(W, \overline{W}) \\ A & = & \frac{1}{2}(A + iAi) & + & \frac{1}{2}(A - iAi). \end{array}$$

This decomposition descends to $End_G(W_{\mathbb{R}})$:

$$End_G(W_{\mathbb{R}}) \simeq End_G(W) \oplus Hom_G(W, \overline{W}).$$

Now Schur's lemma implies that $End_G(W) = \mathbb{C} \cdot Id$ and, since both, W and $\overline{W}$ are irreducible, that $Hom_G(W, \overline{W}) \subset Iso(W, \overline{W}) \cup \{0\}$.
If $Hom_G(W, \overline{W}) = \{0\}$ we get immediately

$$\mathbb{C} \cdot Id \;=\; End_G(W_{\mathbb{R}}) \;\simeq\; End_G(V) \;=\; span_{\mathbb{R}}\{Id, I\},$$

with the complex structure $I := \psi \circ (i \cdot Id) \circ \psi^{-1}$ where ψ is defined in (1).
Otherwise consider a non-zero $j \in Hom_G(W, \overline{W})$ which is an isomorphism by Schur's lemma. Then $j^2 \in End_G(W)$, hence $j^2 = \lambda \cdot Id$ with $0 \neq \lambda \in \mathbb{C}$. In fact, $\lambda \in \mathbb{R}$ since

$$\overline{\lambda}\, j(w) \;=\; j(\lambda\, w) \;=\; j\left(j^2(w)\right) \;=\; j^2\left(j(w)\right) \;=\; \lambda\, j(w)$$

for all $w \in W$. Finally $\lambda < 0$ since otherwise $W_{\mathbb{R}}$ would decompose into the G-invariant $\pm\sqrt{\lambda}$-eigenspaces of $j_{\mathbb{R}}$. Thus, we may assume $j^2 = -1$. For another $A \in Hom_G(W, \overline{W})$ we get $j \circ A \in End_G(W)$ and therefore $j \circ A = c \cdot Id$ for some $c \in \mathbb{C}$. On the other hand $j \circ (-\overline{c} j) = c \cdot Id$ and thus $A = -\overline{c} j$. Hence we obtain

$$End_G(W_{\mathbb{R}}) \;\simeq\; End_G(W) \oplus Hom_G(W, \overline{W}) \;=\; \mathbb{C} \cdot Id \oplus \mathbb{C} \cdot j,$$

which gives finally

$$End_G(V) = span_{\mathbb{R}}\{Id, I, J, I \circ J\} \simeq \mathbb{H},$$

with $I := \psi \circ i \circ \psi^{-1}$ and $J := \psi \circ j \circ \psi^{-1}$ anti-commuting complex structures. □

Corresponding to the structure of $End_G(V)$ the real irreducible G-module V is said to be of *real*, *complex* or *quaternionic* type. This corresponds to the convention to call a complex irreducible G-module W of *real type* if it is self-conjugated with respect to an anti-linear bijection J with $J^2 = Id$, of *quaternionic type* if it is self-conjugated with $J^2 = -Id$ and of *complex type* if it is not self-conjugated. Here is a useful consequence of the preceeding proposition.

Corollary 6. *For a real irreducible G-module V any $A \in End_G(V)$ is of the form $A = \alpha\, Id + \beta J$ with $\alpha, \beta \in \mathbb{R}$ and J a G-invariant complex structure (depending on A).*

Proof. Although this follows directly from Proposition 1 we will give another proof which will be useful later on. Applying the Schur-lemma we see that the minimal polynomial $\mu_A(x)$ of A is irreducible over $\mathbb{R}$ (cf. [KN63, Appendix 5, Lemma 1]). If $\mu_A(x) = x - \alpha$ is of degree one $0 = \mu_A(A) = A - \alpha \cdot Id$. Otherwise $\mu_A(x) = (x - \alpha)^2 + \beta^2$ is a polynomial of degree 2 with strictly positive quadratic supplement, since μ_A is irreducible. Thus $J := (A - \alpha \cdot Id)/\beta$ defines a complex structure on V. □

Finally in this section we describe the maximal representations of different types, i.e. any other irreducible representation occurs as a sub-representation of them.

Proposition 2. *Let $G \subset Gl(n,\mathbb{R})$ be an irreducibly acting subgroup. Then up to conjugation G is contained in one of the following subgroups $L \subset Gl(n,\mathbb{R})$:*

$End_G(\mathbb{R}^n)$	$L \subset Gl(n,\mathbb{R})$
$\mathbb{R}$	$Gl(n,\mathbb{R})$
$\mathbb{C}$	$Gl(n/2,\mathbb{C})$
$\mathbb{H}$	$Gl(n/4,\mathbb{H})$

Proof. We set $V := \mathbb{R}^n$ and $\mathbb{K} := End_G(V)$. Thus V becomes a *left* $\mathbb{K}$-vector space in a natural way. In order to make it a *right* $\mathbb{K}$-vector space we choose an anti-automorphism $\lambda \mapsto \overline{\lambda}$ of $\mathbb{K}$ (i.e. $\overline{\lambda+\mu} = \overline{\lambda}+\overline{\mu}$ and $\overline{\lambda\cdot\mu} = \overline{\mu}\cdot\overline{\lambda}$). Then $v\cdot\lambda := \overline{\lambda}(v)$ defines a right-multiplication on V with respect to the scalar field $\mathbb{K}$ (This is essential only in case of the non-commutative field $\mathbb{K}=\mathbb{H}$). The group $Gl(V,\mathbb{K})$ of $\mathbb{K}$-linear invertible maps from V into itself is by definition the centralizer of the homothety group $H_{\mathbb{K}} := \big\{\{v \mapsto v\lambda\} \,\big|\, \lambda \in \mathbb{K}^*\big\}$. By choosing a $\mathbb{K}$-basis $\{b_i\}_{i=1}^{n/d}$, where $d = \dim_{\mathbb{R}}\mathbb{K}$, we get $\mathbb{K}^{n/d} \simeq V$. Under this identification $Gl(V,\mathbb{K})$ corresponds to the group $Gl(n/d,\mathbb{K})$ of invertible $(n/d \times n/d)$-matrices acting on $\mathbb{K}^{n/d}$ from the left. By definition $H_{\mathbb{K}}$ is the centralizer of G and thus G is contained in $L := Gl(V,\mathbb{K})$. As explained this yields an inclusion $G \subset Gl(n/d,\mathbb{K})$. Conversely it is known that the centralizer of $Gl(n/d,\mathbb{K})$ equals $H_{\mathbb{K}}$, hence $End_L(\mathbb{R}^n) = H_{\mathbb{K}}$. Finally, the embedding $Gl(n/d,\mathbb{K}) \subset Gl(n,\mathbb{R})$ is obtained by associating to the $\mathbb{K}$-basis $\{b_i\}$ the real basis $\{b_i\lambda_k\}_{\substack{i=1,\dots,n/d\\ k=1,\dots,d}}$, where $\{\lambda_k\}_{k=1,\dots,d}$ is a basis of $\mathbb{K}$. □

Remark 1. In the proof of this proposition we see that if the action of a group $G \subset Gl(n,\mathbb{R})$ is defined by scalar multiplication from the right, the invariant endomorphism have to act from the left. Of course, this becomes only relevant in case of $End_G(\mathbb{R}^n) = \mathbb{H}$, and we can see this in the example of $G := Gl(1,\mathbb{H})$: It is

$$Gl(1,\mathbb{H}) \;=\; \{R_q : \mathbb{H} \to \mathbb{H} \mid q \in \mathbb{H}^* \text{ and } R_q(p) := p\cdot q\} \;=\; \mathbb{H}^*,$$

whereas

$$\begin{aligned} End_{Gl(1,\mathbb{H})}(\mathbb{R}^4) &= \{A \in Gl(4,\mathbb{R}) \mid A(R_q(p)) = R_q(A(p))\} \\ &= \{L_q \in Gl(4,\mathbb{R}) \mid L_q(p) := q\cdot p\} \\ &= \mathbb{H} \end{aligned}$$

since $L_q \circ R_p = R_p \circ L_q$ but $R_p \circ R_q \neq R_q \circ R_p$. This gives the seemingly paradoxical situation where both, the centraliser $Z_{Gl(4,\mathbb{R})}(G)$ and the group G itself are equal to $\mathbb{H}^*$, but its center $Z(G)$ which is the intersection of G with its centraliser is commutative and thus equal to $\mathbb{C}^*$.

3 Irreducibly acting, connected subgroups of $Gl(n, \mathbb{R})$

In this section we shall give a proof of Theorem 1 by using the results of the first section and a two general results from Lie theory. First we describe the identity component of the center of an irreducibly acting Lie subgroup of $Gl(n, \mathbb{R})$. We should remark that we mean 'Lie subgroup' always in the weaker sense of being a submanifold but not necessarily an immersion in order to make the statement of Theorem 1 non-trivial.

Proposition 3. *Let $G \subset Gl(n, \mathbb{R})$ be an irreducibly acting, connected Lie subgroup, $Z(G)$ its center and $Z(G)_0$ the identity component of the center. If $Z(G)_0$ is non-trivial, then $Z(G)_0$ is either*

(a) equal to $\mathbb{R}^+ Id$, or

(b) isomorphic to $\mathbb{C}^* = \mathbb{R}^+ \times S^1$, or

(c) isomorphic to a one-parameter subgroup of $\mathbb{C}^*$.

Cases (b) and (c) can only occur if n is even.

Proof. Let $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ be the Lie algebra of G and suppose that center $\mathfrak{z}$ of $\mathfrak{g}$ is non trivial. Considering the three cases of Proposition 1 we first assume that the the representation is of real type, i.e. that $End_G(\mathbb{R}^n) = \mathbb{R}Id$. Since $\mathfrak{z} \subset End_G(\mathbb{R}^n)$ we obtain in this case that $\mathfrak{z} = \mathbb{R}Id$ and therefore $Z(G)_0 = \exp(\mathfrak{z}) = \mathbb{R}^+ Id$.
Now suppose that $\mathbb{R}^n$ is a G-module of non-real type, i.e. $End_G(\mathbb{R}^{2n})$ isomorphic to $\mathbb{C}$ or $\mathbb{H}$. Again $\mathfrak{z}$ is an Abelian subalgebra of $End_G(\mathbb{R}^{2n})$. In case $End_G(\mathbb{R}^{2n}) \simeq \mathbb{H} \simeq \mathfrak{u}(2)$ any maximal Abelian subalgebra is isomorphic to $\mathbb{C}$. Hence $\mathfrak{z}$ is isomorphic to a subalgebra of $\mathbb{C} = span_{\mathbb{R}}(Id, J)$ where J is a complex structure on $\mathbb{R}^{2n}$. But $\exp tJ = (\cos t)Id + (\sin t)J$, i.e. $\exp(\mathbb{R}J) \simeq S^1$. But this implies that either isomorphic to $\mathbb{C}^*$, i.e.

$$Z(G)_0 \;=\; \mathbb{R}^+ Id \times \{(\cos t)Id + (\sin t)J \mid t \in \mathbb{R}\} \;\simeq\; \mathbb{R}^+ \times S^1 \;=\; \mathbb{C}^*,$$

or to a one-parameter subgroup of it, i.e.

$$\begin{aligned} Z(G)_0 &= \exp\left(\mathbb{R} \cdot (aId + bJ)\right) \\ &= \left\{ \left(\mathrm{e}^{at} \cdot Id\right) \circ \left((\cos bt)Id + (\sin bt)J\right) \mid t \in \mathbb{R} \right\}, \end{aligned}$$

for some real constants a and b. Of course if a or b are zero this is either $\mathbb{R}^+$ or S^1, if not this is a logarithmic spiral in $\mathbb{C}^*$. □

Proposition 3 will be the main ingredient in our proof of Theorem 1 but it implies also Corollaries 2 and 3 given in the introduction. But before we can prove these we have to recall that for a completely reducibly acting Lie subgroup $G \subset Gl(n, \mathbb{R})$ the center decides whether the Lie algebra is semisimple or not. This is due to a standard fact from the theory of Lie algebras, saying that a Lie algebra $\mathfrak{g}$ which admits a completely reducible representation is reductive. Hence $\mathfrak{g}$ admits a Lie algebra decomposition into its center and its derived Lie algebra,

$$\mathfrak{g} \;=\; \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}], \tag{2}$$

the derived Lie algebra being semisimple. A proof of this fact can be found in [Che47], see also [Bou71]. This means that the irreducibly acting, connected Lie subgroup in question is semisimple if the identity component of its center is trivial.

Remark 2. In this context we should remark that the center of a semisimple subgroup $G \subset Gl(n, \mathbb{R})$ is finite (see e.g. [Got48]): if G is semisimple, due to Weyl's theorem it acts completely reducibly, and furthermore its elements are of determinant 1, hence by Schur's lemma the center of G corresponds to the n_k-th roots of 1 where n_k are the dimensions of the irreducible subspaces.

For verifying Corollary 2 we assume that $G \subset O(p, q)$ is connected and acts irreducibly. If G is not semisimple, its Lie algebra $\mathfrak{g}$ has a non trivial center $\mathfrak{z}$, but the orthogonality of the representation implies that projection of the center on $\mathbb{R}Id$ is trivial. Hence the the representation is not of real type, i.e. $n = p + q$ is even, and $\mathfrak{z} = \mathbb{R}J$ where J is the complex structure which commutes with $\mathfrak{g}$. But on the other hand $J \in \mathfrak{so}(p, q)$, i.e. J is compatible with the inner product, which gives that p and q are even as well. Thus, by proposition 2,

$$\mathfrak{g} \subset \mathfrak{so}(p,q) \cap \mathfrak{gl}(n/2, \mathbb{C}) = \mathfrak{u}(p/2, q/2).$$

which is the statement of Corollary 2.
This also implies Corollary 3: If $\mathfrak{g}$ is not semisimple, then $\mathfrak{g} = \mathbb{R} \cdot J \oplus \mathfrak{g}'$ where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is semisimple. Hence, $\mathfrak{g}' = [\mathfrak{g}', \mathfrak{g}'] \subset \mathfrak{su}(r, s)$, with $\mathfrak{su}(r, s)$ defined by the complex structure J, $r = p/2$ and $s = q/2$. The complex structure J is given by

$$J = \sum_{k}^{r+s} \kappa_{2k} E_{2k-1\ 2k},$$

where $\langle e_i, e_j \rangle = \kappa_i \delta_{ij}$ is the corresponding orthonormal basis of . Let $u(\varepsilon_{r+s}, \ldots, \varepsilon_1)$ be the basis of the spinor module $\Delta_{(p,q)}$ as defined in [BK99]. This is an eigen basis for the spin representation of J:

$$Ju(\varepsilon_{r+s}, \ldots, \varepsilon_1) = i \left(\sum_{k}^{r+s} \kappa_{2k} \tau_{2k-1} \tau_{2k} \varepsilon_k \right) u(\varepsilon_{r+s}, \ldots, \varepsilon_1) = i \left(\sum_{k}^{r+s} \varepsilon_k \right) u(\varepsilon_{r+s}, \ldots, \varepsilon_1),$$

where $\tau_i = i$ if $\kappa_i = -1$ and 1 otherwise. If $V_{\mathbb{R}J}$ denotes the subspace in the spinor module which is annihilated by $\mathbb{R}J$ under its spin representation, then

$$V_{\mathbb{R}J} = \text{span} \left\{ u(\varepsilon_{r+s}, \ldots, \varepsilon_1) \mid \sum_{k}^{r+s} \varepsilon_i = 0 \right\},$$

which implies that J has zero eigen vectors only if $r + s$ is even. This is the statement of Corollary 3. The example given in the introduction of a non-semisimple irreducibly acting Lie algebra with trivial spin representation is given by $\mathfrak{g} = \mathbb{R}J \oplus \mathfrak{sp}(r, s)$ with $r + s$ even. Again in the notation of [BK99] it follows that

$$\varphi_{\frac{r+s}{2}} \sum_{\varepsilon_i = -1 \text{ for } \frac{r+s}{2} \text{ times}} u(\varepsilon_{r+s}, \varepsilon_{r+s}, \ldots, \varepsilon_1, \varepsilon_1)$$

is annihilated by $\mathfrak{sp}(r, s)$. But it is also annihilated by J:

$$J\varphi_{\frac{r+s}{2}} = 2 \sum_{\substack{\varepsilon_i = -1 \text{ for} \\ \frac{r+s}{2} \text{ many } \varepsilon_i\text{'s}}} \underbrace{(\varepsilon_1 + \ldots + \varepsilon_{r+s})}_{=0} u(\varepsilon_{r+s}, \varepsilon_{r+s}, \ldots, \varepsilon_1, \varepsilon_1).$$

Hence, $V_{\mathfrak{g}} = V_{\mathbb{R}J} \cap V_{\mathfrak{sp}(r,s)} = \mathbb{R}\varphi_{\frac{r+s}{2}}$ is one-dimensional.

Example. An example for an irreducible real representation of a Lie group with 2-dimensional center is the reellification of the representation of $S^1 \times CO(n,\mathbb{R})$ on $\mathbb{C}^n$. The Lie algebra consists of the matrices $\begin{pmatrix} A & aI_n \\ -aI_n & A \end{pmatrix} \in \mathfrak{gl}(2n,\mathbb{R})$ with $a \in \mathbb{R}$ and $A \in \mathfrak{co}(n,\mathbb{R})$, where I_n denotes the n-dimensional unit matrix. The center of the identity component of this group is $S^1 \times \mathbb{R}^+$, the semisimple part is $SO(n)$. In the same manner we can built an example where the center is a spiral in $\mathbb{C}^*$ by taking as Lie algebra

$$\mathfrak{g} \;:=\; \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle| A \in \mathfrak{so}(n) \right\} \;\oplus\; \mathbb{R} \cdot \begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} \;\subset\; \mathfrak{gl}(2n,\mathbb{R}),$$

and as group G the connected subgroup in $Gl(2n,\mathbb{R})$ with this Lie algebra. Both groups do not act orthogonally.

Now we can go ahead with the proof of Theorem 1. Let G be a connected, irreducibly acting Lie subgroup of $Gl(n,\mathbb{R})$, and $\mathfrak{g}$ be its Lie algebra. Our proof now relies on the following result of [Yos37] and [Mal45] (see also [Got48] where it is a corollary to a deeper result).

Theorem 4. *[Yos37], [Mal45],[Got48] A connected Lie subgroup of $Gl(n,\mathbb{R})$ is closed in $Gl(n,\mathbb{R})$ if and only if its radical is closed. In particular, if it is semisimple, it is closed.*

Recall that the radical of G is the connected Lie subgroup of G which corresponds to maximal solvable ideal in the Lie algebra $\mathfrak{g}$. Thus we have to show, that the radical of G is closed in $Gl(n,\mathbb{R})$. But by the remarks above, the Lie algebra of G is reductive, and thus the radical of G is equal to the identity component of its center, denoted by $Z(G)_0$. Now the closure $\overline{G}$ of G is still connected, acts irreducibly and has a reductive Lie algebra. By Theorem 4 the identity component of its center $Z\left(\overline{G}\right)_0$ is closed in $Gl(n,\mathbb{R})$. But $Z(G) \subset Z(\overline{G})$ because for $z \in Z(G)$ and $g = \lim g_n \in \overline{G}$ it is

$$z \cdot g \;=\; z \cdot \lim g_n \;=\; \lim(z \cdot g_n) \;=\; 0.$$

If we now assume that G is not closed we get by Theorem 4 that $Z(G)_0$ is not closed in $Gl(n,\mathbb{R})$, i.e.

$$Z(G)_0 \underset{\neq}{\subsetneq} \overline{Z(G)}_0 \subset Z(\overline{G})_0.$$

Now, since $\overline{G}$ is irreducible and connected, Proposition 3 leaves us only with the possibility that $Z(\overline{G})_0$ is isomorphic to $\mathbb{C}^*$ and $Z(G)_0$ is a one-parameter subgroup of $\mathbb{C}^*$. But these are closed in $\mathbb{C}^*$. This is a contradiction which completes the proof of Theorem 1.

Since holonomy groups are Lie subgroups of $Gl(n,\mathbb{R})$, the first point of Corollary 1 is a direct consequence of the Theorem 1. The second can be obtained by a theorem in [Wu67] which contains several results with different algebraic conditions for subgroups of the pseudo-orthogonal group, having consequences for holonomy groups.

Theorem 5. *[Wu67] The following subgroups of $Gl(p+q)$ are closed:*

1. *reductive, indecomposable subgroups of $O(p,q)$,*
2. *indecomposable subgroups of $O(p,q)$ if $p+q<6$,*
3. *holonomy groups of affine symmetric spaces.*

Here 'indecomposable' means 'no proper non-degenerate invariant subspace'. One should remark that the restriction to the dimension in the second point is sharp: In [Wu67] is constructed a 6-dimensional Kähler manifold whose reduced holonomy group is non-closed in $SO(4,2)$; also the Lorentzian examples in [BI93] are constructed in dimension 6. Also in [Wu67] is constructed an example of a symmetric space with solvable, non-Abelian holonomy group which shows that the third point does not follow from the first. Some of these examples are obtained by constructing subgroups containing a torus, which has non-closed 1-parameter subgroups. Our proof shows that such a situation can be excluded if the group acts irreducibly.
In order to obtain the second statement of Corollary 1, note that the first point of Theorem 5 implies that semi-Riemannian holonomy groups which act completely reducibly are closed: by the de Rham/Wu decomposition theorem [Wu64] any semi-Riemannian holonomy group is a product of indecomposably acting holonomy groups, but if the group is assumed to act completely reducibly it is reductive and hence closed by the first point of Theorem 5.
Since the dense line on the Clifford torus provides an example of a completely reducibly acting group which is not closed in $Gl(2,\mathbb{C})$, such a result cannot be true for holonomy groups of an arbitrary affine connection due to the result in [HO65], that any connected linear Lie group can be obtained as the holonomy group of an affine connection. It is not difficult to check that the connection of this example is not torsion free. But such a result might be true for torsion free connections.

Conjecture. *Let (M,∇) be an affine manifold where ∇ is a torsion-free connection. Assume that the restricted holonomy group $Hol^*(\nabla)_p$ acts completely reducible on T_pM. Then, $Hol^*(\nabla)_p$ is closed inside $GL(T_pM)$.*

4 Irreducibly acting, connected subgroups of $O(1,n)$

In this section we want to give a short proof of Theorem 2, that the only connected subgroup G of $O(1,n)$ which acts irreducibly on the Lorentzian space $\mathbb{R}^{1,n}$ is the connected component of the identity of $O(1,n)$ i.e. $G = SO(1,n)_0$. This statement was proven in [DSO01] where the main goal was to generalize to real hyperbolic space the following result about minimal homogeneous submanifolds i.e. orbits of isometry subgroups, in the Euclidean space.

Theorem 6. *[DS02] A (extrinsically) homogeneous minimal submanifold of the Euclidean space must be totally geodesic.*

It turns out that such result also holds in the real hyperbolic space (see [DSO01] for details). It is interesting to remark that further investigations of minimal homogeneous submanifolds were done in several directions [ADS03], [DS03]. In particular, the following conjecture was posed in [DS03].

Conjecture. *Let M be a Riemannian manifold that is either locally homogeneous or Einstein. Then, any minimal isometric immersion $f : M \to \mathbb{R}^n$ must be totally geodesic.*

Now, in order to prove Theorem 2 we assume that $G \subset O(1,n)$ acts irreducibly and is connected. By Corollary 2 it is semisimple and closed by Theorem 1. Our proof requires the following Karpelevich's theorem.

Theorem 7. *(Karpelevich [Kar53], Mostow [Mos55]) Let $M = Iso(M)/K$ be a Riemannian symmetric space of non-compact type. Then any connected and semisimple subgroup G of the full isometry group $Iso(M)$ has a totally geodesic orbit $G \cdot p \subset M$.*

The above theorem can also be stated in a purely algebraic way as follows.

Theorem 8. *Let $\mathfrak{g}'$ be a real semisimple Lie algebra of non compact type and let $\mathfrak{g} \subset \mathfrak{g}'$ be a semisimple Lie subalgebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for $\mathfrak{g}$. Then there exists a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ for $\mathfrak{g}'$ such that $\mathfrak{k} \subset \mathfrak{k}'$ and $\mathfrak{p} \subset \mathfrak{p}'$.*

The original proof of Theorem 7 is as a corollary of Theorem 8. The proof of Theorem 8 is not trivial and very algebraic (see [Mos55, Theorem 6] or [Oni04]). So should be nice to give a proof of Karpelevich's theorem by using geometric methods (or at least basic facts from Lie Theory e.g. Iwasawa theorem, etc). Indeed, should be nice to give a short and simpler proof when $\mathfrak{g}' = SL(n, \mathbb{R})$.

Remark 3. Notice that in Theorems 7 and 8 the hypothesis on being of "non compact type" can not be deleted. Any irreducible representation of a compact simple Lie group G gives an example. Indeed, the group $G \subset SO(n+1)$ acts on a sphere $S^n = SO(n+1)/SO(n)$ without totally geodesic orbits. Notice that if the orbit $G.x$ is totally geodesic then $G.x = S^n \bigcap V$, where V is a linear subspace of $\mathbb{R}^{n+1}$. So V is G-invariant giving a contradiction.

Now let us continue the proof of Theorem 2. Since $G \subset O(1,n)$ is semisimple Karpelevich's theorem applies to our situation. It implies that the action of G on the hyperbolic space $H^n = SO(1,n)/SO(n) \subset \mathbb{R}^{1,n}$ is transitive. Indeed, if the totally geodesic orbit $G \cdot p$ is not the whole hyperbolic space H^n then $G \cdot p$ is contained in a Lorentzian subspace $\mathbb{L}$ of $\mathbb{R}^{1,n}$. This is due to the fact that totally geodesic submanifolds of H^n are intersections $H^n \cap \mathbb{L}$ where $\mathbb{L}$ is a Lorentzian subspaces of $\mathbb{R}^{1,n}$. Thus, G can not act irreducibly as we had assumed.
Now, let K be a maximal connected compact subgroup of the semisimple group G. Then by Cartan's fixed point theorem K has a fixed point $p \in H^n$. Since $(G_p)_0$ is compact we get $K = (G_p)_0$. Thus, (G, K) is a symmetric pair such that $H^N = G/K$. Then, from the uniqueness of such symmetric pairs (see [Hel78, pp. 243]) we get $G = SO(1,n)_0$ and $K = SO(n)$. This proves Theorem 2.

Remark 4. Indeed, if $H^N = G/K$ with (G, K) a symmetric pair then the curvature tensor of H^N belong to the Lie algebra $\mathfrak{k}$ of the isotropy group K. So, $K = SO(n)$ and $G = SO(1,n)_0$. More in general, if $M = G/K$ where M is an irreducible Riemannian symmetric space of non-compact type and (G, K) is a symmetric pair then the curvature tensor R of M belong to $\mathfrak{k}$. Since R generate $\mathfrak{k}$ and $\mathfrak{k}$ generate $\mathfrak{g}$ it follows that $Iso(M)_0 = G$. This is simpler explanation of the "uniqueness" quoted above (see [Hel78, pp. 243])

A different, almost algebra-free proof of Theorem 2 which uses dynamical methods, can be found in [BZ04]. A purely algebraic proof was given in [BdlH04]. Theorem 2 also follows from Corollary 4.5.1 in [CheGre] which in turns depends upon Karpelevich-Mostow's theorem (cf. Lemma 4.4.3 in [CheGre] with Theorem 8).

Let M be a (locally) indecomposable Lorentzian manifold i.e. the restricted holonomy group Φ_p^* acts indecomposably on T_pM. In [DSO01] was proved that either Φ_p acts transitively on hyperbolic spaces $H_r := \{v \in T_pM : \langle v, v\rangle = -r^2\}$ or transitively on horospheres $Q_z \subset H_r$ centered at a point $z \in H_r(\infty)$. Observe that in the Riemannian case the non-transitivity (on a sphere) of the holonomy group implies the locally symmetry of the Riemannian manifold i.e. the Berger-Simons Theorem (see [Olm05] for a direct and geometric proof). It is interesting to note that the holonomy group of a non irreducible indecomposable Lorentzian symmetric space is abelian [CaWa70]. Thus, such holonomy group must act transitively on horospheres $Q_z \subset H_r$ centered at a point $z \in H_r(\infty)$, i.e. z is associated to the (unique) light-like direction invariant by Φ^*_p.

There are also non symmetric global (i.e. geodesically complete) examples of (non irreducible and indecomposables) homogeneous Lorentzian spaces such that the restricted holonomy group is abelian i.e. the so called *homogeneous plane waves* [BlLo]. Notice that such spaces had a rich family of totally geodesic Lorentzian surfaces trough each point. Namely, the exponential of the normal space to each horosphere $Q_z \subset T_pM$. Anyway, it should be desirable to know if some type of Berger-Simons theorem holds in the Lorenzian setting under suitable assumptions.
Finally, here is another application, i.e. Lemma 3.1 of [Sen06], see also [Hal91].

Proposition 4. *Let (M, g) be a simple connected Lorentzian manifold. Assume that there exist a non-zero covariantly constant symmetric tensor field $h_{\mu\nu}$ not proportional to the metric. Then, (M, g) is reducible, and further it is indecomposable only if there exists a unique (up to multiples) light-like parallel vector field.*

Indeed, if (M, g) is irreducible then the holonomy group Φ_p is $SO_0(n, 1)$. Thus, $h_{\mu\nu}$ being $SO(n, 1)_0$-invariant should be proportional to the metric g (See the first line in the table of Proposition 6 of the next section). The second part follows from the fact that if Φ_p leaves invariant two non proportional light-like vectors then Φ_p must leave invariant the two dimensional Lorentzian space generated by these vectors. Thus, (M, g) is decomposable. Notice that the "unicity" of a light-like parallel vector field does not implies indecomposibility of the Lorentzian manifold, e.g. a product between a Riemannian manifold and indecomposable Lorentzian manifold.

A modern exposition of Karpelevich's theory is [Oni04]. A generalization of Theorem 2 to arbitrary signatures (p, q) seems to be very difficult or impossible i.e. the classification of irreducible Lie subalgebras of $so(p, q)$. Anyway, should be interesting to know such classification for small signatures e.g. for $so(2, n)$ or $so(3, n)$.

5 Invariant bilinear forms of irreducible representations

As in the second section we consider an irreducibly acting subgroup $G \subset Gl(V)$ of a real vector space V and denote by $B_G(V)$ the vector space of G-invariant bilinear forms. If $B_G(V)$ is non-trivial it is intimately connected to $End_G(V)$. By Schur's lemma a non-zero $a \in B_G(V)$ is non-degenerate since its kernel is G-invariant and not equal to V. Thus

Riesz' theorem provides a one-to-one correspondence between $B_G(V)$ and $End_G(V)$ via

$$\begin{array}{ccc} End_G(V) & \stackrel{\sim}{\to} & B_G(V) \\ B & \mapsto & b = a(B(\cdot),\cdot). \end{array} \tag{3}$$

In particular this map endows $B_G(V)$ with the structure of an associative algebra and by Proposition 1 $B_G(V)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The unique decomposition of a bilinear form into symmetric and skew-symmetric parts applies also to G-invariant bilinear forms, since the (skew-)symmetrization of a G-invariant form inherits this property. Thus

$$B_G(V) = S_G(V) \oplus \Lambda_G(V). \tag{4}$$

This induces a decomposition of $End_G(V)$ into a-selfadjoint and a-skewadjoint operators,

$$End_G(V) = S^a_G(V) \oplus \Lambda^a_G(V). \tag{5}$$

If a is symmetric, $S_G(V)$ corresponds to $S^a_G(V)$ under (3) and if a is skew-symmetric, $S_G(V)$ corresponds to $\Lambda^a_G(V)$. The main question is what are the possible dimensions of $S_G(V)$ and what are the occurring signatures. A first answer gives the following statement.

Proposition 5. *Let $a, b \in B_G(V)$ be linear independent, $b = a(B(\cdot,\cdot))$ by (3) and $B = \alpha\, Id + \beta J$ according to Corollary 6.*

(i) If a and b are both symmetric (or skew-symmetric), $J \in S^a_G(V)$ and thus J is an anti-isometry with respect to both a and b. In particular $\mathrm{Sig}(a) = \mathrm{Sig}(b) = (n/2, n/2)$ where $n = \dim V$.

(ii) If a is symmetric and b is skew-symmetric, $B = \beta J \in \Lambda^a_G(V)$ and thus J is an isometry with respect to both a and b.

Proof. (i) Since B and Id are a-selfadjoint, the same holds for J. Using $[B, J] = 0$ we obtain

$$a(J(x), J(y)) = a(J^2(x), y) = -a(x, y), \text{ and}$$
$$b(J(x), J(y)) = a(B \circ J(x), J(y)) = -a(B(x), y) = -b(x, y).$$

(ii) Here B is a-skewadjoint. This implies for its minimal polynomial $\mu_B(x) = \mu_{-B}(x) = \mu_B(-x)$, hence $\mu_B(x) = x^2 + \beta^2$ (cf. Corollary 6), i.e. $B = \beta J$. The remaining part is analogous to (i). □

Next we determine all possible pairs $\big(\dim S_G(V), \dim \Lambda_G(V)\big)$ by describing their maximal representations analogous to Proposition 2 of Section 2. Recall that a representation is self-dual if and only if the space of non-degenerate invariant bilinear forms is non-trivial.

Proposition 6. *Let $\kappa : G \to Gl(n, \mathbb{R})$ be an irreducible self-dual representation on $\mathbb{R}^n$. Then up to conjugation $\kappa(G)$ is contained in one of the following subgroups $L \subset Gl(n, \mathbb{R})$*

with $p+q=n$:

$End_{\kappa(G)}(\mathbb{R}^n)$	$\dim S_{\kappa(G)}(\mathbb{R}^n)$	$\dim \Lambda_{\kappa(G)}(\mathbb{R}^n)$	$L \subset Gl(n,\mathbb{R})$
$\mathbb{R}$	1	0	$O(p,q)$
	0	1	$Sp(n/2,\mathbb{R})$
$\mathbb{C}$	2	0	$O(n/2,\mathbb{C})$, $(n\geq 4)$
	0	2	$Sp(n/4,\mathbb{C})$
	1	1	$U(p/2,q/2)$
$\mathbb{H}$	1	3	$Sp(p/4,q/4)$, $(n\geq 8)$
	3	1	$O^*(n/4)$, $(n\geq 8)$

Moreover we have the following isomorphisms[1]

$$\begin{aligned} O(n/2,\mathbb{C}) &\simeq O(n/2,n/2)\cap Gl(n/2,\mathbb{C}) \\ Sp(n/2,\mathbb{C}) &\simeq Sp(n,\mathbb{R})\cap Gl(n/2,\mathbb{C}) \\ U(p/2,q/2) &\simeq O(p,q)\cap Sp(n,\mathbb{R}) \\ Sp(p/4,q/4) &\simeq U(p/2,q/2)\cap Sp(n/4,\mathbb{C}) \\ O^*(n/4) &\simeq U(n/4,n/4)\cap O(n/2,\mathbb{C}). \end{aligned} \tag{6}$$

Remark 5. This proposition raises the question if there are proper subrepresentations of the different groups $L \subset Gl(n,\mathbb{R})$. The answer depends very much on the group L in question. To illustrate this, note that any compact simple Lie group admits irreducible representations in arbitrary high dimensions. All these representations are contained in $O(n)$ due to Weyl's trick. Considered the other way around $O(n)$ has in general a lot of irreducible subrepresentations. In contrast, there are no proper subgroups of $SO_0(1,n)$ which act irreducibly, see section 4.

Proof. For general considerations set $V := \mathbb{R}^n$; we will return at the end to $\mathbb{R}^n$ by choosing an appropriate basis. First note that (κ,V) is self-dual if and only if $B_{\kappa(G)}(V)\neq \{0\}$. In particular $End_{\kappa(G)}(V)\simeq B_{\kappa(G)}(V)$ according to (3) and we may distinguish between the various types of (κ,V). We determine in each case the maximal subgroup $L\subset Gl(V)$ fixing every element of $B_{\kappa(G)}(V)$ and thus $\kappa(G)\subset L$.

(κ, V) of real type: This is the simplest case, since $B_{\kappa(G)}(V)$ is 1-dimensional and thus spanned either by a symmetric or skew-symmetric bilinear form a (cf. (4)). In the symmetric case we can find a (pseudo-)orthonormal basis, i.e.

$$(a_{ij}) = \mathbb{I}_{p,q} := \begin{pmatrix} -\mathbb{I}_p & \\ & \mathbb{I}_q \end{pmatrix}.$$

[1] For details how the groups are embedded into $Gl(n,\mathbb{R})$ resp. $Gl(n,\mathbb{C})$ we refer to the proof.

Its isometry group is the (pseudo-)orthogonal group

$$O(p,q) := \{ A \in Gl(n,\mathbb{R}) \,|\, A^t \cdot \mathbb{I}_{p,q} \cdot A = \mathbb{I}_{p,q} \}.$$

In the skew-symmetric case we can find a symplectic basis, i.e.

$$(a_{ij}) = \mathbb{J}_{n/2} := \begin{pmatrix} & -\mathbb{I}_{n/2} \\ \mathbb{I}_{n/2} & \end{pmatrix}.$$

Its isometry group is the real symplectic group

$$Sp(n/2,\mathbb{R}) := \{ A \in Gl(n,\mathbb{R}) \,|\, A^t \cdot \mathbb{J}_{n/2} \cdot A = \mathbb{J}_{n/2} \}.$$

(κ, V) of complex type: First we fix a (skew-)symmetric $a \in B_G(V)$ and consider its bilinear extension $a_{\mathbb{C}}$ as well as its sesquilinear extension $\overline{a}_{\mathbb{C}}$ to the complexification $V_{\mathbb{C}}$:

$$\begin{aligned} a_{\mathbb{C}}(x+iy, u+iz) &:= a(x,u) - a(y,z) + i\big(a(x,z) + a(y,u)\big), \\ \overline{a}_{\mathbb{C}}(x+iy, u+iz) &:= a(x,u) + a(y,z) + i\big(a(x,z) - a(y,u)\big). \end{aligned}$$

$a_{\mathbb{C}}$ is (skew-)symmetric, $\overline{a}_{\mathbb{C}}$ is (skew-)Hermitian and they are linked by the formula $a_{\mathbb{C}}(\overline{v}, w) = \overline{a}_{\mathbb{C}}(v,w)$. From this follows

$$a_{\mathbb{C}}(\overline{v}, \overline{w}) = \overline{a_{\mathbb{C}}(v,w)} \qquad \text{and} \qquad \overline{a}_{\mathbb{C}}(\overline{v}, \overline{w}) = \overline{\overline{a}_{\mathbb{C}}(v,w)}. \tag{7}$$

Exactly one of them has to vanish on the $\kappa_{\mathbb{C}}$-irreducible subspace W. Indeed, if we suppose both to be non-zero and set $a_{\mathbb{C}}\big|_W = \overline{a}_{\mathbb{C}}\big|_W(J(\cdot),\cdot)$ one shows that $J \in \overline{End_\rho(W)}$, hence $J = 0$. On the other hand $a_{\mathbb{C}}\big|_{W\times W} = 0$ together with (7) implies $a_{\mathbb{C}}\big|_{\overline{W}\times\overline{W}} = 0$. Thus $a_{\mathbb{C}}\big|_{\overline{W}\times W} = \overline{a}_{\mathbb{C}}\big|_{W\times W}$ has to be non-degenerate and vice versa. Lets denote by $\widetilde{a}$ the non-vanishing form on W. Since $\widetilde{a}$ is ρ-invariant it induces the κ-invariant $\mathbb{C}$-valued $\mathbb{R}$-bilinear form $\psi_*\widetilde{a}$ on V via (1). In the following we will suppress the isomorphism ψ. Real and imaginary part of $\widetilde{a}$ are related by

$$Im\,(\widetilde{a})(x,y) = -Re\,(\widetilde{a})(x, I(y)).$$

In particular they are linear independent and thus $B_{\kappa(G)}(V)$ is spanned by these two forms. So the isometry group of $\widetilde{a}$ is isomorphic to the maximal subgroup of $L \subset Gl(V)$ which fixes any element of $B_{\kappa(G)}(V)$. Note that any element of L commutes with I and thus it is complex linear.
If $\widetilde{a}$ is symmetric, the same holds for its real and imaginary part and their signature has to be $(n/2, n/2)$ (cf. Proposition 5(i)). We can find a complex orthonormal basis, i.e. $(\widetilde{a}_{ij}) = \mathbb{I}_{n/2}$ and the isometry group is the complex orthogonal group

$$O(n/2,\mathbb{C}) = \{ A \in Gl(n/2,\mathbb{C}) \,|\, A^t \cdot A = \mathbb{I}_{n/2} \}.$$

If $\widetilde{a}$ is skew-symmetric, the same holds for its real and imaginary part. We can find a complex symplectic basis i.e. $(\widetilde{a}_{ij}) = \mathbb{J}_{n/4}$ and the isometry group is the complex symplectic group

$$Sp(n/2,\mathbb{C}) = \{ A \in Gl(n/2,\mathbb{C}) \,|\, A^t \cdot \mathbb{J}_{n/4} \cdot A = \mathbb{J}_{n/4} \}.$$

Finally $\widetilde{a}$ might be Hermitian (a complex skew-Hermitian form turns into a Hermitian one by multiplication with i). We can find a complex (pseudo-)orthonormal basis, i.e. $(\widetilde{a}_{ij}) = \mathbb{I}_{p/2,q/2}$ and the isometry group is the unitary group

$$U(p/2,q/2) := \left\{ A \in Gl(n/2,\mathbb{C}) \,\middle|\, \overline{A}^t \cdot \mathbb{I}_{p/2,q/2} \cdot A = \mathbb{I}_{p/2,q/2} \right\}.$$

In this case the real part is symmetric and has signature (p,q) and the imaginary part is skew-symmetric.
As mentioned in Proposition 2 the complex basis $\{b_i\}_{i=1}^{n/2}$ of V (with respect to I) induces the real basis $\{b_i, I(b_i)\}_{i=1}^{n/2}$. Thus $I = \mathbb{J}_{n/2}$ and we obtain the embedding

$$\begin{array}{rcl} Gl(n/2,\mathbb{C}) & \simeq & \left\{ C \in Gl(n,\mathbb{R}) \,\middle|\, C \circ \mathbb{J}_{n/2} = \mathbb{J}_{n/2} \circ C \right\} \\ A + iB & \mapsto & \begin{pmatrix} A & -B \\ B & A \end{pmatrix}. \end{array}$$

Now the real part of the symmetric form $\widetilde{a} = \mathbb{I}_{n/2}$ is given in the associated real basis by $Re(\widetilde{a}) = \begin{pmatrix} \mathbb{I}_{n/2} & \\ & -\mathbb{I}_{n/2} \end{pmatrix}$. Its isometry group is $O(n/2,n/2)$, thus we get the first identity of (6). Analogous, the real part of the symplectic form $\widetilde{a} = \mathbb{J}_{n/4}$ is given by $Re(\widetilde{a}) = \begin{pmatrix} \mathbb{J}_{n/4} & \\ & -\mathbb{J}_{n/4} \end{pmatrix}$. Its isometry group is conjugated to $O(n/2,n/2)$ which yields the second identity of (6). The real part of the Hermitian form $\widetilde{a} = \mathbb{I}_{p/2,q/2}$ is given by $Re(\widetilde{a}) = \begin{pmatrix} \mathbb{I}_{p/2,q/2} & \\ & \mathbb{I}_{p/2,q/2} \end{pmatrix}$. Its isometry group is conjugate to $O(p,q)$. Instead of taking the intersection with the centralizer of $\mathbb{J}_{n/2}$ as above we take here the isometry group of the imaginary part $Im(\widetilde{a}) = \mathbb{J}_{n/2}$ which is $Sp(n/2,\mathbb{R})$, hence the third identity of (6).

(κ, V) of quaternionic type: For representations of real or complex type all possible dimensions for the subspaces $S_{\kappa(G)}(V)$ and $\Lambda_{\kappa(G)}(V)$ occurred. This is no longer true in the quaternionic case.

Lemma 1. *If (κ, V) is self-dual and of quaternionic type then $S_{\kappa(G)}(V)$ and $\Lambda_{\kappa(G)}(V)$ are odd-dimensional, i.e. their dimension is 1 and 3. In particular, κ is both, orthogonal and symplectic. If the 1-dimensional subspace is spanned by $\{a\}$ then under the identification $End_{\kappa(G)}(V) \simeq \mathbb{H}$ the decomposition (5) is given by*

$$Re(\mathbb{H}) = S^a_{\kappa(G)}(V) \qquad Im(\mathbb{H}) = \Lambda^a_{\kappa(G)}(V).$$

Proof. Clearly $Re(\mathbb{H}) = \mathbb{R} \cdot Id \subset S^a_{\kappa(G)}(V)$. On the other hand $\Lambda^a_{\kappa(G)}(V) \subset Im(\mathbb{H})$ by Proposition 5, hence $Im(\mathbb{H}) = \left(S^a_{\kappa(G)}(V) \cap Im(\mathbb{H})\right) \oplus \Lambda^a_{\kappa(G)}(V)$. One of the subspaces has dimension greater or equal than two and is spanned by anti-commuting complex structures I, J. Irrespective of whether I, J are self- or skewadjoint with respect to a, their product is skewadjoint: $(I \circ J)^* = J^* \circ I^* = J \circ I = -I \circ J$. □

As in Proposition 2 we consider V as right $\mathbb{H}$-vector space via $x \cdot \lambda = \overline{\lambda}(x)$. Then an element $a \in B_{\kappa(G)}(V)$ as in the preceeding lemma yields the following quaternionic sesquilinear form on V:

$$a_{\mathbb{H}}(x,y) := a(x,y) + i \cdot a(xi,y) + j \cdot a(xj,y) + k \cdot a(xk,y).$$

Recall that one has to check $a_{\mathbb{H}}(x\lambda, y) = \overline{\lambda} a_{\mathbb{H}}(x,y)$ and $a_{\mathbb{H}}(x, y\lambda) = a_{\mathbb{H}}(x,y)\lambda$. Since multiplication (from the right) with imaginary quaternions is an a-skewadjoint operation according to Lemma 1, $a_{\mathbb{H}}$ is Hermitian if a is symmetric and skew-Hermitian otherwise.

By construction, $B_{\kappa(G)}(V)$ is spanned by the real and imaginary parts of $a_{\mathbb{H}}$, hence the group L which fixes all elements of $B_{\kappa(G)}(V)$ is the isometry group of $a_{\mathbb{H}}$ (again $\mathbb{H}$-linearity is ensured already by leaving $a_{\mathbb{H}}$ invariant).
Now, for any (skew-)Hermitian form one can find an orthogonal basis [Die71, Ch. I, §8]. In the Hermitian case the basis can be normed to the length ± 1. Thus the isometry group is the quaternionic unitary group

$$Sp(p/4, q/4) = \left\{ A \in Gl(n/4, \mathbb{H}) \,\middle|\, \overline{A}^t \cdot \mathbb{I}_{p/4,q/4} \cdot A = \mathbb{I}_{p/4,q/4} \right\}.$$

In the skew-Hermitian case the basis can be normed to the length i. Thus the isometry group is

$$O^*(n/4) = \left\{ A \in Gl(n/4, \mathbb{H}) \,\middle|\, \overline{A}^t \cdot i\mathbb{I}_{n/4} \cdot A = i\mathbb{I}_{n/4} \right\}.$$

The embedding $L \subset Gl(n, \mathbb{R})$ follows from the embedding $Gl(n/4, \mathbb{H}) \subset Gl(n, \mathbb{R})$ as in Proposition 2. In order to obtain the remaining identities of (6), we fix i as complex structure and thus represent any quaternionic matrix by two complex matrices: $A + iB + jC + kD = (A + iB) + (C + iD)j = U + Vj$. This yields the algebra isomorphism

$$\begin{array}{rcl} Gl(n/4, \mathbb{H}) & \simeq & \left\{ C \in Gl(n/2, \mathbb{C}) \,\middle|\, \overline{C} \circ \mathbb{J}_{n/4} = \mathbb{J}_{n/4} \circ C \right\} \\ U + Vj & \mapsto & \begin{pmatrix} U & -V \\ \overline{V} & \overline{U} \end{pmatrix}. \end{array}$$

Since under this identification the operation $C \mapsto \overline{C}^t$ is the same in $Gl(n/4, \mathbb{H})$ and $Gl(n/2, \mathbb{C})$ it is easily seen, that $Sp(p/4, q/4)$ is equal to the intersection of the isometry group $U(p/2, q/2)$ of the Hermitian form $\begin{pmatrix} \mathbb{I}_{p/4,q/4} & \\ & \mathbb{I}_{p/4,q/4} \end{pmatrix}$ with the isometry group $Sp(n/4, \mathbb{C})$ of the symplectic form $\begin{pmatrix} & -\mathbb{I}_{p/4,q/4} \\ \mathbb{I}_{p/4,q/4} & \end{pmatrix}$. Analogously we obtain $O^*(n/4)$ as intersection of the isometry group $U(n/4, n/4)$ of the skew-Hermitian form $\begin{pmatrix} i\mathbb{I}_{n/4} & \\ & -i\mathbb{I}_{n/4} \end{pmatrix}$ with the isometry group $O(n/2, \mathbb{C})$ of the symmetric form $\begin{pmatrix} & \mathbb{I}_{n/4} \\ \mathbb{I}_{n/4} & \end{pmatrix}$. This yields the remaining identities of (6).
We conclude the proof by showing that the maximal groups $L \subset Gl(n, \mathbb{R})$ are acting irreducibly on $\mathbb{R}^n$. One knows even more: For any subgroup $L \subset Gl(n/d, \mathbb{K}) \subset Gl(n, \mathbb{R})$ occuring in the list of the proposition its centralizer coincides with the corresponding homothety group $H_{\mathbb{K}}$:

$$End_L(\mathbb{R}^n) = H_{\mathbb{K}}, \quad L \subset Gl(\mathbb{K}).$$

For the symplectic groups this can be easily verified. For the unitary groups this is true beginning with $n/d \geq 2$ and for the orthogonal groups it is true for $n/d \geq 3$ (see [Die71, Ch. II, §3]). In this context the quaternionic groups $Sp(p/4, q/4)$ and $O^*(n/4)$ are comprehended as unitary groups. Since any homothety is invertible the above groups act irreducibly, otherwise the projection onto an invariant subspace would be an element of the centralizer which is certainly not invertible. It remains to discuss irreducibility in the excluded small dimensions. □

Remark 6. A quaternionic vector space does not admit any symmetric or skew-symmetric bilinear form. This is reflected in the fact that the space of symmetric or skew-symmetric bilinear forms is never 4-dimensional (cf. Lemma 1).

Remark 7. Changing the complex basis by the matrix $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & i\\ 1 & -i\end{smallmatrix}\right)$ we obtain the embedding $O^*(n/4) \subset Gl(n/2,\mathbb{C})$ as given in [Hel01, Ch.X,§2,1.]. There it is explained how $O^*(n/4)$ occurs as dual of the symmetric space $O(n/2)/U(n/4)$ which justifies the notation.

The considerations above can be generalized to non-irreducible representations of Lie groups or Lie algebras. This has been done in [MR93]. Of course the structure of the algebra $End_G(V)$ becomes more involved. On the other hand we may restrict our attention to special representations as e.g. the adjoint representation $Ad(G) \subset Gl(\mathfrak{g})$ of a Lie group G. To ask for an $Ad(G)$-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ becomes interesting from a geometrical point of view, since any such bilinear form induces a pseudo-Riemannian metric on G which is invariant under left- and right-multiplication. In particular G becomes a symmetric space. Hereafter we cite some results in this direction.
As shown above there are representations which are symplectic but not orthogonal. This fails for adjoint representations:

Proposition 7 ([MR93], Theorem 1.4)**.** *A Lie algebra* $\mathfrak{g}$ *admits an* $\mathfrak{ad}(\mathfrak{g})$*-invariant non-degenerate symmetric bilinear form if and only if it is self-dual.*

On the other hand it has been shown:

Proposition 8 ([MR93], Corollary 1.7)**.** *A Lie algebra* $\mathfrak{g}$ *admits an* $\mathfrak{ad}(\mathfrak{g})$*-invariant skew-symmetric bilinear form if and only if* $\mathrm{codim}_{\mathfrak{g}}[\mathfrak{g},\mathfrak{g}] \geq 2$.

In particular, for simple Lie algebras the adjoint representation is irreducible and $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Thus, they cannot be symplectic which excludes many cases of Proposition 6.

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