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Published in compu.philica.com

Abstract

Entropy has a relevant role in several applications of information theory and in the image processing. Here, we discuss the Kaniadakis entropy for images. An example of bi-level image thresholding obtained by means of this entropy is also given. Keywords: Kaniadakis Entropy, Data Segmentation, Image processing, Thresholding.

Introduction

In the last twelve years several researches had been made on foundations and applications of the generalized statistical theory based on the κ-distribution of probabilities [1]. From this distribution, we can have an entropy, the κ-entropy, also known as the Kaniadakis entropy, named after Giorgio Kaniadakis, Politecnico di Torino, who proposed the generalized statistics [2]. Like the well-known Tsallis entropy [3], the κ-entropy becomes that given by Shannon in 1948, when its entropic index κ is going to zero. Shannon and Tsallis entropies are largely used for bi-level and multi-level thresholding in image processing [4-8]. It is therefore interesting to define Kaniadakis entropy for this purpose too.

Kaniadakis entropy

The Kaniadakis entropy, also known as κ-entropy [1,2], is given as:

$$S_\kappa = -\sum_i p_i \ln_{\{\kappa\}} p_i = -\frac{1}{2\kappa}\left\{\sum_i \left(p_i^{1+\kappa} - p_i^{1-\kappa}\right)\right\} \qquad (1)$$

In (1), we have the κ-logarithm of probabilities $\{p_i\}$. The index i is running from 1 to the total number of configurations. This entropy has a generalized additivity. Let us have two independent systems A and B, with κ-entropies S_κ^A and S_κ^B. In the limit $\kappa \to 0$, Kaniadakis entropy becomes Shannon entropy and therefore we expect finding the common Shannon additivity of entropies.

According to [9], in κ-calculus, the generalized sum of entropies is:

$$S_\kappa^{A\cup B} = S_\kappa^A I_\kappa^B + S_\kappa^B I_\kappa^A \quad where$$

$$S_\kappa^A = -\frac{1}{2\kappa}\left\{ \sum_{i,A} \left(p_i^{1+k} - p_i^{1-\kappa}\right)\right\}, \quad S_\kappa^B = -\frac{1}{2\kappa}\left\{ \sum_{j,B} \left(p_j^{1+k} - p_j^{1-\kappa}\right)\right\} \quad (2)$$

$$I_\kappa^A = \frac{1}{2}\left\{ \sum_{i,A} \left(p_i^{1+k} + p_i^{1-\kappa}\right)\right\}, \quad I_\kappa^B = \frac{1}{2}\left\{ \sum_{j,B} \left(p_j^{1+k} + p_j^{1-\kappa}\right)\right\}$$

In general, I_κ (let us call it "function I ") is given by:

$$I_\kappa = \frac{1}{2}\left\{\sum_i p_i^{1+\kappa} + \sum_i p_i^{1-\kappa}\right\} = -\kappa \frac{1}{2\kappa}\left\{\sum_i p_i^{1+\kappa} - \sum_i p_i^{1-\kappa}\right\} + \sum_i p_i^{1+\kappa}$$
$$= -\frac{1}{2}\sum_i p_i^{1+\kappa} + \frac{1}{2}\sum_i p_i^{1-\kappa} + \sum_i p_i^{1+\kappa} = \kappa S_\kappa + \sum_i p_i^{1+\kappa} = \kappa S_\kappa + \Pi_\kappa \quad (3)$$

Therefore, we have:

$$S_\kappa^{A\cup B} = S_\kappa^A I_\kappa^B + S_\kappa^B I_\kappa^A = S_\kappa^A \left(\kappa S_\kappa^B + \Pi_\kappa^B\right) + S_\kappa^B \left(\kappa S_\kappa^A + \Pi_\kappa^A\right)$$
$$= 2\kappa S_\kappa^A S_\kappa^B + S_\kappa^A \Pi_\kappa^B + S_\kappa^B \Pi_k^A \quad (4)$$

In the limit κ→0, Kaniadakis entropy becomes Shannon entropy, and therefore we must have the normal additivity:

$$S_\kappa^{A\cup B} = S^A \Pi_\kappa^B + S^B \Pi_\kappa^B = S^A + S^B \quad (5)$$
$$S^A, S^B \text{ are Shannon entropies.}$$

In the limit, Π_κ^A becomes $\sum_{i,A} p_i = 1$, and the same for *B*.

For entropy *S* and function *I*, the following relations are useful:

$$S_\kappa^{A\cup B} = S_\kappa^A I_\kappa^B + S_\kappa^B I_\kappa^A$$
$$I_\kappa^{A\cup B} = I_\kappa^A I_\kappa^B + \kappa^2 S_\kappa^A S_\kappa^B \quad (6)$$

In generalizing to three systems [10], we have for instance:

$$S_\kappa^{A\cup B\cup C} = S_\kappa^A I_\kappa^B I_\kappa^C + S_\kappa^B I_\kappa^C I_\kappa^A + S_\kappa^C I_\kappa^A I_\kappa^B + \kappa^2 S_\kappa^A S_\kappa^B S_\kappa^C \quad (7)$$

To evaluate the Kaniadakis entropy for gray-level images, we have simply to use the histogram of gray tones (see [4-8] for Shannon and Tsallis). In this manner, index *i* is indicating the gray level; usually, it is going from zero to 255. In the Figure 1 we are giving the behavior of S_κ, I_κ and ratio $R_\kappa = S_\kappa / I_\kappa$ as a function of the entropic index κ .

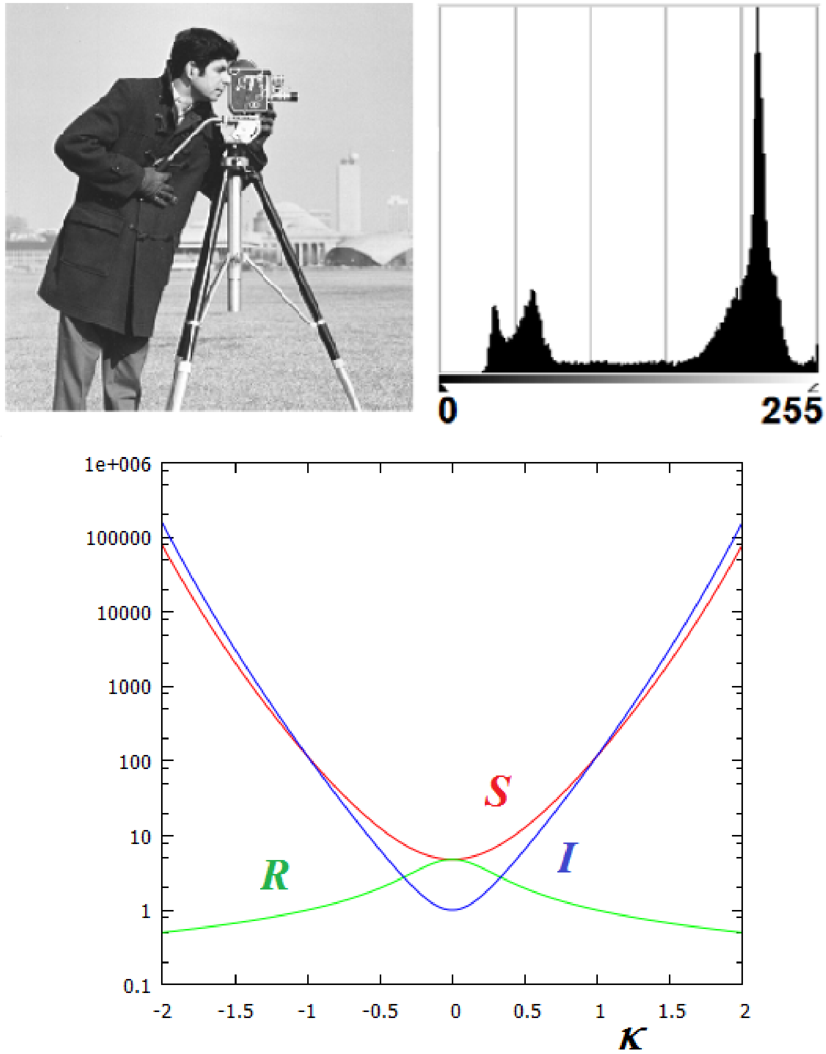

Figure 1: The image is Cameraman. On the right, its histogram. The plot gives the behavior of entropy *S*, function *I* and ratio *R* as a function of the entropic index. Note that functions are symmetric.

An example of bi-level thresholding

In Ref.11, the method for a bi-level thresholding using Kaniadakis entropy is discussed and the results compared to that given by Tsallis entropy (the two approaches compare positively). In a bi-level thresholding, the gray-level input image transforms into a bi-level black and white image, according to a given threshold. Pixels having a gray-tone above the threshold become white, those having a gray-tone below or equal the threshold become black. Of course, the output image depends on the value of the threshold.

Methods [5-7] for thresholding an image determine the best value of the threshold by maximizing Shannon or Tsallis entropy. Both Tsallis and Kaniadakis entropies have entropic indices that can give different results when the maximum entropy method is applied to an image. To show this, let us consider an example from a microscope image of a blood film. Figure 2 gives the original image and three bi-level images obtained with different values of κ entropic index. The image for this index equal to 0.01, is quite the result we can have using Shannon entropy. In this image we cannot see the red cells of blood, but, with an entropic index equal to 2, we have a quite different image. The use of a series of images obtained with different entropic indices is then important for segmentation. Let us conclude, remarking also that Kaniadakis entropy has the intuitive behavior of recovering the Shannon entropy, when its entropic index is going to zero.

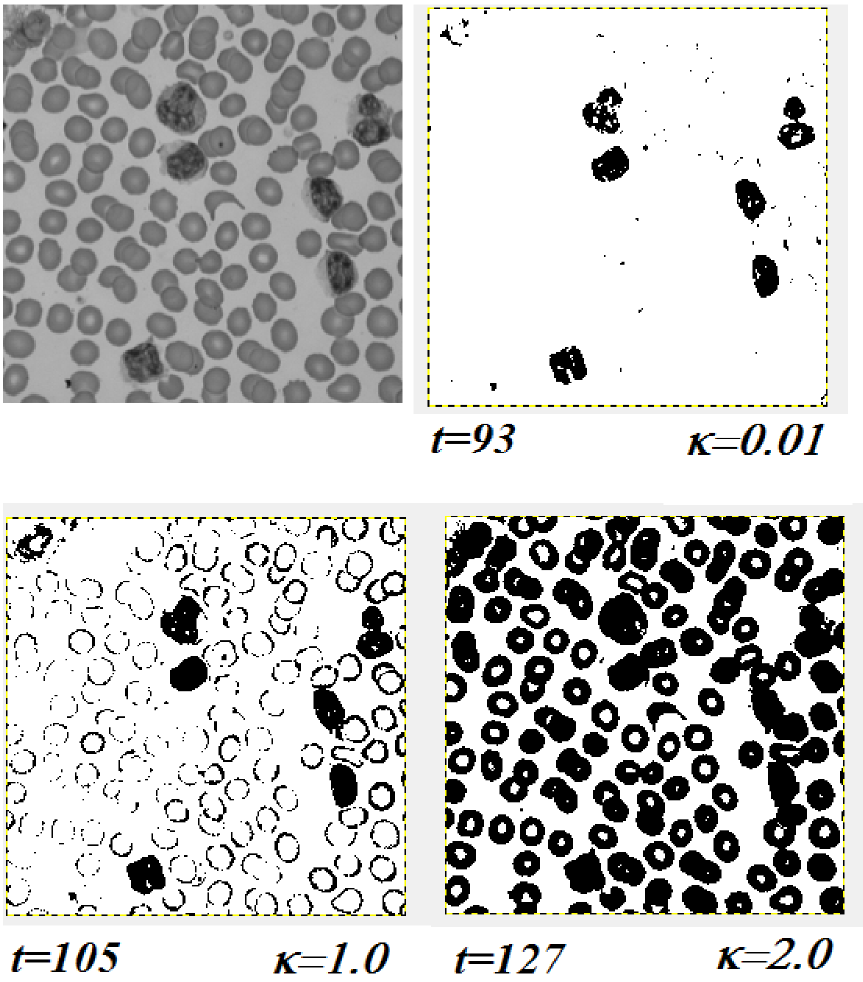

Figure 2: Bi-level thresholding obtained by means of Kaniadakis entropy, using three different values of the entropic index. For the index equal to 0.01, we have the Shannon limit. Note that, in this case, we cannot see the red cells of blood. These cells are well-defined in the image obtained when index is equal to 2.

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Information about this Article

Published on Saturday 28th February, 2015 at 11:29:41.

The full citation for this Article is:
Sparavigna, A. (2015). Kaniadakis Entropy and Images. *PHILICA.COM Article number 462*.