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On the Generalized Additivity of Kaniadakis Entropy

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Abstract: Since entropy has several applications in the information theory, such as, for example, in bi-level or multi-level thresholding of images, it is interesting to investigate the generalized additivity of Kaniadakis entropy for more than two systems. Here we consider the additivity for three, four and five systems, because we aim applying Kaniadakis entropy to such multi-level analyses.

Keywords: Kaniadakis Entropy, Data segmentation, Image processing, Thresholding

1. Introduction

As discussed in [1], in the last twelve years several researches had been made on foundations and applications of a generalized statistical theory, based on the κ-distribution of probabilities. This distribution provides an entropy, the κ-entropy, which is also known as the Kaniadakis entropy, named after Giorgio Kaniadakis, Politecnico di Torino, who proposed it and the κ-distribution [2]. Like the well-known Tsallis entropy [3], the κ-entropy is a generalization of that proposed by Shannon in 1948. In particular, we have that when its entropic index κ is going to zero, this entropy becomes the Shannon entropy.

Since Shannon and Tsallis entropies are largely used for bi-level and multi-level thresholding in image processing [4-7], it could be interesting to use Kaniadakis entropy for this purpose too. In a previous paper [8], we have discussed the bi-level thresholding by means of this entropy. Here, we are giving the method for the generalization of the additivity of κ-entropy to more than two systems. The application to image processing will addressed in a future paper.

2. Kaniadakis entropy

The Kaniadakis entropy, also known as κ-entropy [1,2],

$$S_\kappa = -\sum_i p_i \ln_{\{\kappa\}}(p_i) , \qquad (1)$$

has the remarkable property of having the same behavior of Shannon entropy, that is:

$$S_\kappa = \sum_i p_i \ln_{\{\kappa\}}\left(\frac{1}{p_i}\right) \qquad (2)$$

In (1) and (2), we have the κ-logarithm of probabilities $\{p_i\}$. The index i is running from 1 to the total number of configurations. The κ-logarithm is producing:

$$S_\kappa = -\frac{1}{2\kappa}\left\{\sum_i \left((p_i)^{1+k} - (p_i)^{1-k}\right)\right\} \qquad (3)$$

Let us have two independent systems A and B, with κ-entropies S_κ^A and S_κ^B. In the limit $\kappa \to 0$, Kaniadakis entropy becomes Shannon entropy and therefore we expect finding the common Shannon additivity of entropies. According to [10], in the κ-calculus, the generalized sum of entropies is:

$$S_\kappa^{A\cup B} = S_\kappa^A \mathfrak{I}_\kappa^B + S_\kappa^B \mathfrak{I}_\kappa^A \qquad (4)$$

In (4) we have:

$$\mathfrak{I}_\kappa^A = \frac{1}{2}\left\{\sum_{i,A}\left(p_i^{1+k} + p_i^{1-\kappa}\right)\right\}, \quad \mathfrak{I}_\kappa^B = \frac{1}{2}\left\{\sum_{i,B}\left(p_i^{1+k} + p_i^{1-\kappa}\right)\right\} \qquad (5)$$

In general, $\mathfrak{I}_\kappa$ is a function given by:

$$\mathfrak{I}_\kappa = \frac{1}{2}\left\{\sum_i p_i^{1+\kappa} + \sum_i p_i^{1-\kappa}\right\} = \kappa S_\kappa + \sum_i p_i^{1+\kappa} = \kappa S_\kappa + \Pi_\kappa \qquad (6)$$

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That is:

$$\Im_\kappa = \frac{1}{2}\left\{\sum_i p_i^{1+\kappa} + \sum_i p_i^{1-\kappa}\right\} = -\kappa\frac{1}{2\kappa}\left\{\sum_i p_i^{1+\kappa} - \sum_i p_i^{1-\kappa}\right\} + \sum_i p_i^{1+\kappa}$$
$$= -\frac{1}{2}\sum_i p_i^{1+\kappa} + \frac{1}{2}\sum_i p_i^{1-\kappa} + \sum_i p_i^{1+\kappa} = \Im_\kappa \tag{7}$$

Therefore, in (4) we have:

$$S_\kappa^{A\cup B} = S_\kappa^A \Im_\kappa^B + S_\kappa^B \Im_\kappa^A = S_\kappa^A\left(\kappa S_\kappa^B + \Pi_\kappa^B\right) + S_\kappa^B\left(\kappa S_\kappa^A + \Pi_\kappa^A\right) \tag{8}$$
$$S_\kappa^{A\cup B} = 2\kappa S_\kappa^A S_\kappa^B + S_\kappa^A \Pi_\kappa^B + S_\kappa^B \Pi_\kappa^A \tag{9}$$

As previously told, in the limit $\kappa \to 0$, Kaniadakis entropy becomes Shannon entropy, and therefore we must have the normal additivity:

$$S_\kappa^{A\cup B} = S^A \Pi_\kappa^B + S^B \Pi_\kappa^B = S^A + S^B \tag{10}$$

In (10), S^A, S^B are the Shannon entropies. In the limit, Π_κ^A becomes $\sum_{i,A} p_i = 1$, and the same for *B*.

Let us rewrite (9), to remark that it is symmetric when $\kappa \leftrightarrow -\kappa$:

$$S_\kappa^{A\cup B} = 2\kappa S_\kappa^A S_\kappa^B + S_\kappa^A \Pi_\kappa^B + S_\kappa^B \Pi_\kappa^A = 2\kappa S_\kappa^A S_\kappa^B + S_\kappa^A\left(\Im_\kappa^B - \kappa S_\kappa^B\right) + S_\kappa^B\left(\Im_\kappa^A - \kappa S_\kappa^A\right)$$
$$= 2\kappa S_\kappa^A S_\kappa^B + S_\kappa^A \Im_\kappa^B - \kappa S_\kappa^A S_\kappa^B + S_\kappa^B \Im_\kappa^A - \kappa S_\kappa^A S_\kappa^B = S_\kappa^A \Im_\kappa^B + S_\kappa^B \Im_\kappa^A \tag{11}$$

3. For three-levels

Therefore, if we have two independent systems *A* and *B*, the generalized additivity is:

$$S_\kappa^{A\cup B} = S_\kappa^A \Im_\kappa^B + S_\kappa^B \Im_\kappa^A \tag{12}$$

where $\Im_\kappa = \kappa S_\kappa + \Pi_\kappa$. To generalize to three systems, let us imagine having two systems *A* and *D*, where *D* is given by $B \cup C$. For the additivity (12):

$$S_\kappa^{A\cup B\cup C} = S_\kappa^A \Im_\kappa^{B\cup C} + S_\kappa^{B\cup C} \Im_\kappa^A =$$
$$= S_\kappa^A\left(\kappa S_\kappa^{B\cup C} + \Pi_\kappa^{B\cup C}\right) + \left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right)\Im_\kappa^A \tag{13}$$
$$= S_\kappa^A\left(\kappa\left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right) + \Pi_\kappa^{B\cup C}\right) + \left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right)\Im_\kappa^A$$

Let us use the property $\Pi_\kappa^{B\cup C} = \Pi_\kappa^B \Pi_\kappa^C$ ([9], pag.482):

$$S_\kappa^{A\cup B\cup C}$$
$$= S_\kappa^A\left(\kappa\left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right) + \Pi_\kappa^B \Pi_\kappa^C\right) + \left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right)\Im_\kappa^A \tag{14}$$
$$= \kappa S_\kappa^A S_\kappa^B \Im_\kappa^C + \kappa S_\kappa^A S_\kappa^C \Im_\kappa^B + S_\kappa^A \Pi_\kappa^B \Pi_\kappa^C + S_\kappa^B \Im_\kappa^C \Im_\kappa^A + S_\kappa^C \Im_\kappa^B \Im_\kappa^A$$

We have that:

$$S_\kappa^A \Pi_\kappa^B \Pi_\kappa^C = S_\kappa^A\left(\Im_\kappa^B - \kappa S_\kappa^B\right)\left(\Im_\kappa^C - \kappa S_\kappa^C\right)$$
$$= S_\kappa^A\left(\Im_\kappa^B \Im_\kappa^C + \kappa^2 S_\kappa^B S_\kappa^C - \kappa S_\kappa^B \Im_\kappa^C - \kappa S_\kappa^C \Im_\kappa^B\right) \tag{15}$$

And then:

$$S_\kappa^{A\cup B\cup C}$$
$$= \kappa S_\kappa^A S_\kappa^B \Im_\kappa^C + \kappa S_\kappa^A S_\kappa^C \Im_\kappa^B - \kappa S_\kappa^A S_\kappa^B \Im_\kappa^C - \kappa S_\kappa^A S_\kappa^C \Im_\kappa^B$$
$$+ S_\kappa^A \Im_\kappa^B \Im_\kappa^C + S_\kappa^B \Im_\kappa^C \Im_\kappa^A + S_\kappa^C \Im_\kappa^A \Im_\kappa^B + \kappa^2 S_\kappa^A S_\kappa^B S_\kappa^C \tag{16}$$
$$= S_\kappa^A \Im_\kappa^B \Im_\kappa^C + S_\kappa^B \Im_\kappa^C \Im_\kappa^A + S_\kappa^C \Im_\kappa^A \Im_\kappa^B + \kappa^2 S_\kappa^A S_\kappa^B S_\kappa^C$$

Note that, due to the symmetry $\kappa \leftrightarrow -\kappa$, in (16) we have only the term in κ^2. In the limit $\kappa \to 0$, we can easily see, as we did for two systems, that we have the Shannon additivity. Of course, we can generalize the calculation to several systems. To this purpose, let us note that, from (15) we can also have:

$$\begin{aligned}
\Im_\kappa^{B\cup C} &= \kappa S_\kappa^{B\cup C} + \Pi_\kappa^{B\cup C} = \kappa\left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right) + \Pi_\kappa^B \Pi_\kappa^C \\
&= \kappa\left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right) + \left(\Im_\kappa^B - \kappa S_\kappa^B\right)\left(\Im_\kappa^C - \kappa S_\kappa^C\right) \\
&= \kappa\left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right) - \kappa\left(S_\kappa^B \Im_\kappa^C + S_\kappa^C \Im_\kappa^B\right) + \Im_\kappa^B \Im_\kappa^C + \kappa^2 S_\kappa^B S_\kappa^C = \\
&= \Im_\kappa^B \Im_\kappa^C + \kappa^2 S_\kappa^B S_\kappa^C
\end{aligned} \tag{17}$$

4. Four levels

We start from:

$$S_\kappa^{A\cup B\cup C} = S_\kappa^A \Im_\kappa^B \Im_\kappa^C + S_\kappa^B \Im_\kappa^C \Im_\kappa^A + S_\kappa^C \Im_\kappa^A \Im_\kappa^B + \kappa^2 S_\kappa^A S_\kappa^B S_\kappa^C \tag{18}$$

Instead of *C*, we have $C \cup D$:

$$\begin{aligned}
S_\kappa^{A\cup B\cup C\cup D} &= S_\kappa^A \Im_\kappa^B \Im_\kappa^{C\cup D} + S_\kappa^B \Im_\kappa^{C\cup D} \Im_\kappa^A + S_\kappa^{C\cup D} \Im_\kappa^A \Im_\kappa^B + \kappa^2 S_\kappa^A S_\kappa^B S_\kappa^{C\cup D} \\
&= S_\kappa^A \Im_\kappa^B \left(\Im_\kappa^C \Im_\kappa^D + \kappa^2 S_\kappa^C S_\kappa^D\right) + S_\kappa^B \left(\Im_\kappa^C \Im_\kappa^D + \kappa^2 S_\kappa^C S_\kappa^D\right) \Im_\kappa^A \\
&+ \left(S_\kappa^C \Im_\kappa^D + S_\kappa^C \Im_\kappa^D\right) \Im_\kappa^A \Im_\kappa^B + \kappa^2 S_\kappa^A S_\kappa^B \left(S_\kappa^C \Im_\kappa^D + S_\kappa^C \Im_\kappa^D\right)
\end{aligned} \tag{19}$$

$$\begin{aligned}
&S_\kappa^{A\cup B\cup C\cup D} \\
&= S_\kappa^A \Im_\kappa^B \Im_\kappa^C \Im_\kappa^D + S_\kappa^B \Im_\kappa^C \Im_\kappa^D \Im_\kappa^A + S_\kappa^C \Im_\kappa^A \Im_\kappa^B \Im_\kappa^D + S_\kappa^D \Im_\kappa^A \Im_\kappa^B \Im_\kappa^C \\
&+ \kappa^2 \left(S_\kappa^A S_\kappa^B S_\kappa^C \Im_\kappa^D + S_\kappa^D S_\kappa^B S_\kappa^A \Im_\kappa^C + S_\kappa^C S_\kappa^D S_\kappa^A \Im_\kappa^B + S_\kappa^B S_\kappa^C S_\kappa^D \Im_\kappa^A\right)
\end{aligned} \tag{20}$$

5. Five levels

We start from (20), but instead of *D*, we have $D \cup E$:

$$\begin{aligned}
&S_\kappa^{A\cup B\cup C\cup D\cup E} \\
&= S_\kappa^A \Im_\kappa^B \Im_\kappa^C \Im_\kappa^{D\cup E} + S_\kappa^B \Im_\kappa^C \Im_\kappa^{D\cup E} \Im_\kappa^A + S_\kappa^C \Im_\kappa^A \Im_\kappa^B \Im_\kappa^{D\cup E} + S_\kappa^{D\cup E} \Im_\kappa^A \Im_\kappa^B \Im_\kappa^C \\
&+ \kappa^2 \left(S_\kappa^A S_\kappa^B S_\kappa^C \Im_\kappa^{D\cup E} + S_\kappa^{D\cup E} S_\kappa^B S_\kappa^A \Im_\kappa^C + S_\kappa^C S_\kappa^{D\cup E} S_\kappa^A \Im_\kappa^B + S_\kappa^B S_\kappa^C S_\kappa^{D\cup E} \Im_\kappa^A\right)
\end{aligned} \tag{21}$$

Let us consider that $\Im_\kappa^{D\cup E} = \Im_\kappa^E \Im_\kappa^D + \kappa^2 S_\kappa^D S_\kappa^E$. Then:

$$\begin{aligned}
&S_\kappa^{A\cup B\cup C\cup D\cup E} \\
&= S_\kappa^A \Im_\kappa^B \Im_\kappa^C \left(\Im_\kappa^E \Im_\kappa^D + \kappa^2 S_\kappa^D S_\kappa^E\right) + S_\kappa^B \Im_\kappa^C \left(\Im_\kappa^E \Im_\kappa^D + \kappa^2 S_\kappa^D S_\kappa^E\right) \Im_\kappa^A \\
&+ S_\kappa^C \Im_\kappa^A \Im_\kappa^B \left(\Im_\kappa^E \Im_\kappa^D + \kappa^2 S_\kappa^D S_\kappa^E\right) + \left(S_\kappa^E \Im_\kappa^D + S_\kappa^D \Im_\kappa^E\right) \Im_\kappa^A \Im_\kappa^B \Im_\kappa^C \\
&+ \kappa^2 \Big[S_\kappa^A S_\kappa^B S_\kappa^C \left(\Im_\kappa^E \Im_\kappa^D + \kappa^2 S_\kappa^D S_\kappa^E\right) \Im_\kappa^{D\cup E} + \left(S_\kappa^E \Im_\kappa^D + S_\kappa^D \Im_\kappa^E\right) S_\kappa^B S_\kappa^A \Im_\kappa^C \\
&+ S_\kappa^C \left(S_\kappa^E \Im_\kappa^D + S_\kappa^D \Im_\kappa^E\right) S_\kappa^A \Im_\kappa^B + S_\kappa^B S_\kappa^C \left(S_\kappa^E \Im_\kappa^D + S_\kappa^D \Im_\kappa^E\right) \Im_\kappa^A \Big]
\end{aligned} \tag{22}$$

$$\begin{aligned}
&S_\kappa^{A\cup B\cup C\cup D\cup E} \\
&= S_\kappa^A \Im_\kappa^B \Im_\kappa^C \Im_\kappa^D \Im_\kappa^E + S_\kappa^B \Im_\kappa^C \Im_\kappa^D \Im_\kappa^E \Im_\kappa^A + S_\kappa^C \Im_\kappa^D \Im_\kappa^E \Im_\kappa^A \Im_\kappa^B + S_\kappa^D \Im_\kappa^E \Im_\kappa^A \Im_\kappa^B \Im_\kappa^C \\
&+ S_\kappa^E \Im_\kappa^A \Im_\kappa^B \Im_\kappa^C \Im_\kappa^D + \kappa^2 \Big[S_\kappa^A S_\kappa^B S_\kappa^C \Im_\kappa^D \Im_\kappa^E + S_\kappa^A S_\kappa^B S_\kappa^D \Im_\kappa^E \Im_\kappa^C + S_\kappa^A S_\kappa^B S_\kappa^E \Im_\kappa^C \Im_\kappa^D \\
&+ S_\kappa^A S_\kappa^C S_\kappa^D \Im_\kappa^B \Im_\kappa^E + S_\kappa^A S_\kappa^C S_\kappa^E \Im_\kappa^B \Im_\kappa^D + S_\kappa^A S_\kappa^D S_\kappa^E \Im_\kappa^B \Im_\kappa^C + S_\kappa^B S_\kappa^C S_\kappa^E \Im_\kappa^A \Im_\kappa^D \\
&+ S_\kappa^B S_\kappa^C S_\kappa^D \Im_\kappa^A \Im_\kappa^E + S_\kappa^B S_\kappa^D S_\kappa^E \Im_\kappa^A \Im_\kappa^C + S_\kappa^C S_\kappa^D S_\kappa^E \Im_\kappa^A \Im_\kappa^B \Big] + \kappa^4 S_\kappa^A S_\kappa^B S_\kappa^C S_\kappa^D S_\kappa^E
\end{aligned} \tag{23}$$

Note that, we have, as usual, the symmetry $\kappa \leftrightarrow -\kappa$. There are 10 terms multiplied by κ^2 and one by κ^4. In the limit $\kappa \to 0$, we have the Shannon additivity.

In the Appendix, we give, for comparison, the generalized additions for Tsallis entropy [9].

6. Conclusion

In this paper, we have shown the method of calculation it is necessary to use when we want to apply Kaniadakis entropy to the multi-level thresholding methods, to substitute in them the Tsallis entropy. In a future paper, we will investigate the three-level thresholding of images with κ-entropy, and its comparison to Tsallis entropy.

Appendix on Tsallis entropy

Let us imagine having a set of probabilities $\{p_i\}$.For any real parameter q, the Tsallis entropy is defined as:

$$S_q(p_i) = \frac{1}{q-1}\left(1 - \sum_i p_i^q\right) \qquad \text{(A1)}$$

The link between Tsallis and Rényi entropy is given in [9]:

$$\bar{S}_q = \frac{1}{1-q}\ln\left[1 + (1-q)S_q\right] \qquad \text{(A2)}$$

Let us assume two independent systems *A* and *B* again. The Rényi entropy is additive:

$$\bar{S}_q^{A\cup B} = \bar{S}_q^A + \bar{S}_q^B \qquad \text{(A3)}$$

Using (A2) in (A3), we have:

$$(1-q)\bar{S}_q^{A\cup B} = \ln\left[1 + (1-q)S_q^A\right] + \ln\left[1 + (1-q)S_q^B\right] \qquad \text{(A4)}$$

From (A2) and (A4), we have:

$$\ln\left[1 + (1-q)S_q^{A\cup B}\right] = \ln\left[1 + (1-q)S_q^A\right] + \ln\left[1 + (1-q)S_q^B\right] \qquad \text{(A5)}$$

$$\begin{aligned}\left[1 + (1-q)S_q^{A\cup B}\right] &= \left[1 + (1-q)S_q^A\right]\left[1 + (1-q)S_q^B\right] \\ &= 1 + (1-q)S_q^A + (1-q)S_q^B + (1-q)^2 S_q^A S_q^B\end{aligned} \qquad \text{(A6)}$$

Then:

$$S_q^{A\cup B} = S_q^A + S_q^B + (1-q)S_q^A S_q^B \qquad \text{(A7)}$$

Let us imagine two systems, *A* and *D*, with $D = B \cup C$. From (A5):

$$(1-q)\bar{S}_q^{A\cup B\cup C} = \ln\left[1 + (1-q)S_q^A\right] + \ln\left[1 + (1-q)S_q^{B\cup C}\right] \qquad \text{(A8)}$$

Then:

$$\begin{aligned}\left[1 + (1-q)S_q^{A\cup B\cup C}\right] &= \left[1 + (1-q)S_q^A\right]\left[1 + (1-q)S_q^{B\cup C}\right] \\ &= 1 + (1-q)S_q^A + (1-q)S_q^{B\cup C} + (1-q)^2 S_q^A S_q^{B\cup C} \\ &= 1 + (1-q)S_q^A + (1-q)\left[S_q^B + S_q^C + (1-q)S_q^B S_q^C\right] \\ &+ (1-q)^2 S_q^A\left[S_q^B + S_q^C + (1-q)S_q^B S_q^C\right]\end{aligned} \qquad \text{(A9)}$$

And then:

$$S_q^{A\cup B\cup C} = S_q^A + S_q^B + S_q^C + (1-q)\left[S_q^A S_q^B + S_q^A S_q^C + S_q^B S_q^C\right] + (1-q)^2 S_q^A S_q^B S_q^C \qquad \text{(A10)}$$

And so on for more systems to add.

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