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MORSE INDEX AND LINEAR STABILITY OF RELATIVE EQUILIBRIA IN SINGULAR MECHANICAL SYSTEMS

DOCTORAL THESIS

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To my family:
Roberto, Silvia, Federica and Zippo

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1 INTRODUCTION

This dissertation consists of two papers [BJP14a; BJP14b] written in collaboration with Alessandro Portaluri and Vivina Barutello. We have focussed on the study of the linear stability of some particular periodic orbits in planar singular mechanical systems with symmetry, and we have achieved the results that are going to be presented using quite advanced mathematical techniques. These involve some homotopy invariants, such as the spectral flow, and some index theory, namely a theorem stating the equality between the Morse index of an orbit seen as a critical point of a Lagrange action functional and the Maslov index of the fundamental solution of the associated Hamiltonian system. Albeit we started with an analysis of a generalised n-body problem, that is, an n-body problem with a more general potential, what we have found holds in a more general situation, as we are going to illustrate.

Simple mechanical systems are a special class of Hamiltonian systems in which the Hamiltonian function can be written as the sum of the potential and kinetic energies. The search for special orbits, such as equilibria and periodic orbits, and the understanding of their stability properties are amongst the major subjects in the whole theory of Dynamical Systems. In 1970, in one of his famous papers [Sma70], S. Smale, following the ideas sketched out by E. Routh in [Rou77], examined the stability of relative equilibria of simple mechanical systems with symmetries. For a general system of this kind, a *relative equilibrium* is a dynamical fixed point (i.e. an equilibrium point) in the *reduced* phase space obtained by quotienting the original phase space by the symmetry group. Thus, generally speaking, relative equilibria are the analogue of fixed points for systems without symmetry (whence their great importance), yet they can also be viewed as one-parameter group orbits. Of course, the larger the symmetry group is, the richer the supply of relative equilibria becomes. For a system of particles in the plane described in the coordinates of the centre of mass, subject to the action of the rotation group $SO(2)$ — like the one that we examine here — relative equilibria are solutions in which the whole system rotates with constant angular velocity around the barycentre. For this reason they are also called *dynamical motions in steady rotation*.

Given a relative equilibrium, it is natural to investigate its stability properties in order to understand the dynamical behaviour of the orbits nearby. Two of the main methods used to study the *stability* of relative equilibria are the *Energy-Casimir method* and the *Energy-*

Momentum method; however, even when applicable, they do not give any information about *instability* without further investigation. One of the few feasible methods to study the matter of stability is to show that the Hamiltonian $\mathscr{H}$, or some other integral, has a maximum or minimum at a critical point: if the maximum or minimum is isolated then $\mathscr{H}$ is a Lyapunov function and the equilibrium point is stable. Unfortunately, in the n-body context, it is easy to see [Moe94, p. 86] that this approach never works in the case of relative equilibria, and for this reason it is hopeless to try to prove their stability (or instability). Instead of that, we concentrate here on the notions of *linear* and *spectral stability* (see Section 2.1 for their definition): we linearise the Hamiltonian system around a relative equilibrium and analyse its features. This involves the computation of the spectrum of a Hamiltonian matrix, which is symmetric with respect to both axes in the complex plane. A direct consequence of this fact is that relative equilibria are never asymptotically stable.

In studying symmetric systems of particles it is usual to introduce the so-called *augmented potential* $\mathcal{U}_\Xi$, which is equal to the potential of the system plus a term coming from the centrifugal forces (see [Mar92] and references therein). The reason is that relative equilibria are precisely the critical points of this modified potential [see Sma70].

Our main result on this topic reads as follows (see Theorem 2.21 and Section 2.4 for a more precise statement, further details and the proof).

Theorem. *Let $\bar{x}$ be a critical point of the augmented potential and assume that it has even nullity. If its Morse index is odd, then the relative equilibrium corresponding to $\bar{x}$ is spectrally unstable.*

An immediate consequence is the following.

Corollary. *Let $\bar{x}$ be a critical point of the augmented potential. If its Morse index or its nullity are odd then the corresponding relative equilibrium is linearly unstable.*

The main tool that we use in the proof of this theorem is the *spectral flow* (in the very elementary case of Hermitian matrices), a well-known integer-valued homotopy invariant of paths of self-adjoint Fredholm operators introduced by M. F. Atiyah, V. K. Patodi and I. M. Singer in [APS76]. In finite-dimensional situations it is just the difference of the Morse index at the endpoints (see Section 2.3 for its definition and Section A.1 for its main properties). Up to perturbation, non-degeneracy and transversality conditions, this invariant can be computed in terms of the so-called *crossing forms*, which, intuitively speaking, count in an appropriate way the net number of eigenvalues crossing the value 0 transversally. In our setting this need not be the case; however, A. Portaluri developed in other papers (see for instance [GPP04a], or Section A.2 for a short description) a non-perturbative analysis of non-transversal intersections. The reason behind the choice of a non-perturbative technique lies in the fact that, in general, perturbative methods preserve global invariants but completely destroy the local information concerning the single intersection. By means of this theory, based on what has been termed *partial signatures*, we have been able to prove Theorem 2.21.

The main applications of our result (see Chapter 3) are directed towards singular α-homogeneous and logarithmic potentials of the form

$$U_\alpha(q) := \sum_{\substack{i,j=1\\ i<j}}^{n} \frac{m_i m_j}{|q_i - q_j|^\alpha}, \quad \alpha \in (0,2), \tag{1.1a}$$

$$U_{\log}(q) := \sum_{\substack{i,j=1\\ i<j}}^{n} m_i m_j \log \frac{1}{|q_i - q_j|}, \tag{1.1b}$$

although also some other interesting classes can be reformulated in our framework, such as the Lennard-Jones interaction potential. Our theorem offers indeed a unifying viewpoint of all these quite different situations, since the property that it unravels descends only from the rotational invariance of the mechanical system. All of these potentials are extensively studied in literature: the α-homogeneous ones are the natural generalisation of the gravitational attraction ($\alpha = 1$) and they are employed in different atomic models, whilst the logarithmic potential naturally arises when looking from a dynamical point of view at the stationary helicoidal solutions of the n-vortex filaments model, which is popular and useful in Fluid Mechanics. See [RT95; Ven02; DPS06] and references therein for the homogeneous cases, [PP13] for the logarithmic one and [BFT08] for a general overview.

To detect a relative equilibrium in a *generalised* n*-body problem* (or n*-body-type problem*) means to determine a periodically moving *planar central configuration* of the bodies which solves Newton's equations

$$m\ddot{q}_i = \frac{\partial U}{\partial q_i}, \qquad i = 1, \dots, n, \tag{1.2}$$

(where U is one of the above potentials) and in which the attractive force is perfectly balanced by the centrifugal one. This is currently the only way known to obtain exact solutions, albeit finding central configurations amounts to solving a system of highly nonlinear algebraic equations and is therefore very hard (cf. [Moe94] for the Newtonian case and [FP08] for the α-homogeneous one).

Being invariant under the symmetry group of Euclidean transformations and admitting linear momentum, angular momentum and energy as first integrals, n-body-type problems are highly degenerate. This in particular yields Jacobians with nullity 8 (cf. [Mey99;

MHO09] for the gravitational force), but only in an inertial reference frame: indeed, if we move (as we do) to a suitable uniformly rotating coordinate system (so that the relative equilibrium becomes an effective equilibrium) six out of the eight eigenvalues produced by the first integrals depend on the angular velocity. This is not surprising at all, since linear stability properties strongly depend on the choice of the frame of the observer. For this reason, studying the case $\alpha = 1$, R. Moeckel in [Moe94] defined the linear and spectral stability by ruling out all the eigenvalues linked to this kind of degeneracy. In the same context, K. R. Meyer and D. S. Schmidt concluded in [MS05] a deep study of the linearised equations: in particular, they introduced a suitable system of symplectic coordinates in which the matrices are block-diagonal, with one block representing the translational invariance of the problem and another one carrying the symmetries induced by dilations and rotations. These two submatrices generate the eight eigenvalues responsible of degeneracy, whilst a third (and last) block contains all the information about stability, in the sense mentioned above. We observe that an analogous decomposition holds also for the potentials that we examine (see Section 3.2).

In this picture, it is worth mentioning a conjecture on linear stability stated by Moeckel [see ACS12, Problem 16], which we report here:

Moeckel's Conjecture. *In the planar Newtonian n-body problem, the central configuration associated with a linearly stable relative equilibrium is a non-degenerate minimum of the potential function restricted to the* shape sphere *(i.e. the* $\mathrm{SO}(2)$*-quotient of the ellipsoid of inertia).*

This claim is still unproved; however, X. Hu and S. Sun have made some progress. More precisely, they showed in [HS09b] that if the Morse index or the nullity of a central configuration (viewed as a critical point of the potential restricted to the shape sphere) are odd, then the corresponding relative equilibrium is linearly unstable. Therefore the central configurations giving rise to linearly stable relative equilibria should correspond to a critical point with even Morse index and nullity. The main result in [HS09b] is the first attempt towards the understanding of the relationship (if there is any) between two dynamics: the gradient flow on the shape sphere and Hamilton's equations in the phase space.

The contribution of the present work in this setting is twofold:

1) We provide a complete and detailed proof of the result on linear instability proved in [HS09b] and we extend it to a very general class of interaction potentials by using spectral flow techniques.

2) We prove a result on spectral instability by means of the theory of partial signatures previously developed in [GPP04a]. Note that Corollary 3.8 is actually the main result in [HS09b] (written there in the gravitational setting only).

Moeckel's Conjecture can thus be adapted to the class of potentials that we study; accordingly, we reformulate it as follows:

Conjecture. *In planar* $SO(2)$*-symmetric mechanical systems, a critical point of the augmented potential associated with a linearly stable relative equilibrium, is a non-degenerate minimum.*

We cast some light on this question with Theorem 2.21 and with Theorem 3.6 in the special case of generalised n-body problems.

Furthermore, following G. E. Roberts' approach in [Rob99], we are able to give a sufficient condition for spectral instability of a relative equilibrium (at least in the α-homogeneous case) in terms of the potential evaluated at a central configuration. It is in fact rather foreseeable that the linear stability depends also on the homogeneity parameter α (see Subsection 3.4 and cf. Corollary 3.10 for a precise statement).

Theorem. *Let* $\bar{x}$ *be a central configuration. If the following inequality holds*

$$\sum_{\substack{i,j=1\\ i<j}}^{n} \frac{m_i + m_j}{|\bar{x}_i - \bar{x}_j|^{\alpha+2}} > \frac{2n+\alpha-4}{\alpha} U_\alpha(\bar{x})$$

then the arising relative equilibrium is linearly unstable.

We point out that no sufficient condition for detecting the linear or spectral stability has been found thus far. This question is in fact addressed in [BJP14b], where we are trying to establish in a precise way the stability properties of the relative equilibria by using some symplectic and variational techniques, mainly based on the Maslov index, index theorems and topological invariants.

We consider to this end a planar 3-body-type problem governed by a singular potential function U of one of the forms (1.1). Newton's equations for this problem are given by (1.2) and we seek solutions that satisfy periodic boundary conditions. By taking into account the conservation law of the centre of mass we see that the *configuration space* is 4-dimensional and is given by

$$\widehat{X} := \left\{ q \in \mathbb{R}^6 \;\middle|\; \sum_{i=1}^{3} m_i q_i = 0,\; q_i \neq q_j \;\forall\, i \neq j \right\}.$$

Let (q, v) be an element of the tangent bundle $T\widehat{X}$, so that $q \in \widehat{X}$ and $v \in T_q\widehat{X}$. The *Lagrangian function* $\mathscr{L} \in \mathscr{C}^\infty(T\widehat{X}, \mathbb{R})$ is given by

$$\mathscr{L}(q, v) := \frac{1}{2} \sum_{i=1}^{3} m_i |v_i|^2 + U(q), \tag{1.3}$$

Let $W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X})$ be the Sobolev space of L^2-loops with weak L^2-derivatives and define on it the *Lagrangian action functional* $\mathbb{A}$ as

$$\mathbb{A}(\gamma) := \int_0^{2\pi} \mathscr{L}\big(\gamma(t), \dot{\gamma}(t)\big)\, dt, \tag{1.4}$$

which is smooth on its domain, since it consists of collisionless loops. Its critical points in this space are the 2π-periodic (classical) solutions of Equations (1.2).

The first solutions of the classical ($U = U_\alpha$ with $\alpha = 1$) planar 3-body problem have been shown in 1772 by J.-L. Lagrange [Lag72]: for any choice of the three masses there exists a family of periodic motions during which the bodies are always arranged in an equilateral triangle that rotates around its barycentre, changing its size but not its shape; moreover, each particle describes a Keplerian conic. In the special case where the trajectory of each body around the centre of mass is a circle swept with some appropriate angular frequency, Lagrange's triangular solution is an example of relative equilibrium, called *Lagrange circular orbit*. We observe that this kind of circular motion is maintained also in the case of the more general potentials defined in (1.1).

Given a periodic solution of (1.2), it is natural to investigate its stability properties in order to understand the dynamical behaviour of the orbits nearby. Our main concern is the linear stability of these circular Lagrangian solutions. It turns out that it depends on two parameters: the *mass parameter*

$$\beta := 27\,\frac{m_1 m_2 + m_2 m_3 + m_1 m_3}{(m_1 + m_2 + m_3)^2} \in (0, 9]$$

and the homogeneity parameter $\alpha \in [0, 2)$. Note that we now include the value $\alpha = 0$ because it will be shown that this corresponds to the logarithmic case. These two parameters define a family of Lagrangian circular solution, which we denote by $\gamma_{\alpha,\beta}$.

In order to investigate the linear stability of this family we need to reformulate the Newtonian problem (1.2) in Hamiltonian language. A 2π-periodic solution of this autonomous Hamiltonian system is *spectrally stable* if the spectrum of the monodromy matrix of the corresponding linearised system is contained in the unit circle of the complex plane; it is *linearly stable* if in addition such matrix is diagonalisable.

When analysing $\gamma_{\alpha,\beta}$ we face a very degenerate situation because of the invariance of n-body-type problems under the symmetry group of Euclidean transformations and the presence of first integrals. It is possible, however, through an ingenious change of coordinates originally found by Meyer and Schmidt and here modified, to factorise the contributions of these constants of motion and split the phase space into a direct sum of invariant 4-dimensional symplectic subspaces: $T^*X = E_1 \oplus E_2 \oplus E_3$ [see Moe94; MS05; BJP14a]. It turns

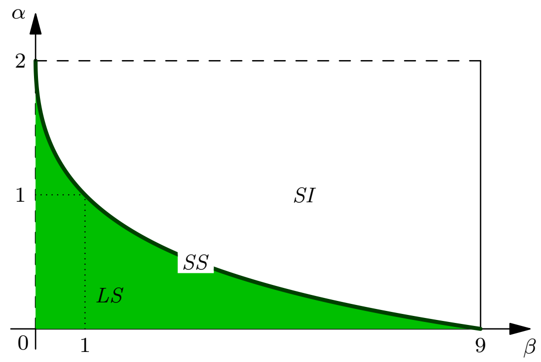
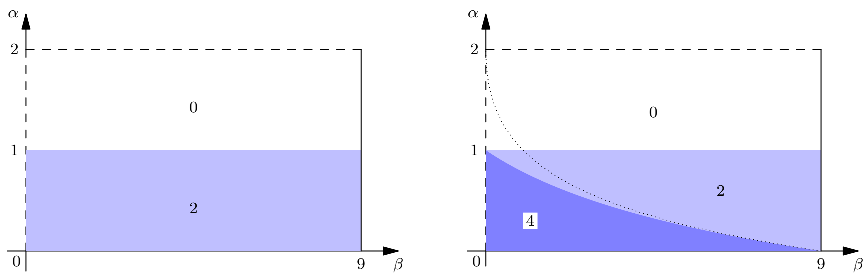


Figure 1.1. Stability regions: spectral instability (*SI*), spectral stability (*SS*) and linear stability (*LS*). The curve drawn is the *stability curve* $\beta = 9\left(\frac{\alpha-2}{\alpha+2}\right)^2$, which marks the transition between stability and instability. For a fixed value of α the motion becomes linearly stable if β is small enough, i.e. if there is a dominant mass. For a fixed value of β, linear stability is achieved if α is small enough, i.e. if the potential is sufficiently weak.

out that the degeneracy is confined in E_1 and partly in E_2, whereas E_3 contains the essence of the dynamics. More precisely, the subspace E_1 corresponds to the four integrals of motion of the center of mass, whilst E_2 includes the conservation of the angular momentum. Furthermore, the restriction of the Hamiltonian to the symplectic invariant subspace E_2 of the phase space agrees with the Hamiltonian of a generalised Kepler problem (i.e. a Kepler problem with potential U of the form (1.1)). We note that the eigenvalues of the monodromy matrix restricted to E_2 are $1, 1, e^{\pm 2\pi i\sqrt{2-\alpha}}$; hence, for any $\alpha \in (0, 2)$, the circular solutions of the Kepler-type problem (corresponding to the line $\beta = 0$ in Figure 1.1) are spectrally stable and $\alpha = 2$ is the boundary of their stability region (which is also called in the literature *elliptic region*). The portrait of the stability properties of $\gamma_{\alpha,\beta}$, which takes into account the essence of the dynamics, taking place on E_3, is depicted in Figure 1.1, where one can neatly distinguish three regions: that of spectral instability (*SI*), that of linear stability (*LS*) and the curve of spectral stability (*SS*) that separates them.

A very deep and intriguing question is the relation between the linear stability of a periodic solution or of a closed geodesic and the Morse index of its iterations [Bot56]: a famous result by H. Poincaré states that every closed minimising geodesic on a Riemannian surface is unstable. Motivated by this question we computed the Morse index of the Lagrangian circular orbit in the free loop space of $\widehat{X}$. Very few results are known about this topic; a classical one is due to W. B. Gordon [Gor77], who proved that the minimisers of the Lagrangian action functional for the Kepler problem on the subspace of $W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}^2 \setminus \{0\})$ of loops with winding number ± 1 with respect to the origin are the ellipses. S. Zhang and Q. Zhou [ZZ01] and A. Venturelli [Ven01] proved in 2001 that the Lagrangian equilat-



(a) Values of $i_{Morse}(\gamma_{\alpha,0})$. On the line $\alpha = 1$ it is equal to 0.

(b) Values of $i_{Morse}(\gamma_{\alpha,\beta})$. For a fixed α, the Morse index is a monotone decreasing function of β that attains the minimum value in correspondence of equal masses. The dotted curve is the stability curve.

Figure 1.2. Values of the Morse index of the generalised Kepler problem (a) and of the Lagrange circular solution (b).

eral triangle solutions of the 3-body problem are minimisers of the corresponding action functional with $\alpha = 1$. However, M. Ramos and S. Terracini showed in [RT95] a sort of double variational characterisation of the set of all periodic solutions of the α-homogeneous Kepler problem; this can give a heuristic explanation of the degeneracy occurring at $\alpha = 1$. In [Ven02] Venturelli proved that for $\alpha \in (1, 2)$ and winding numbers ± 1 the minimisers are precisely the circular solutions, whilst for $\alpha \in (0, 1)$ the minima are attained by the ejection-collision solutions. He left, however, completely open the problem of computing the Morse index of the circular solutions in the case $\alpha \in (0, 1)$.

Our first main result concerns the computation of the Morse index of the circular solution of Kepler-type problems (we write $\gamma_{\alpha,0}$ for the Keplerian trajectory, in view of the formal correspondence with the case $\beta = 0$). As already observed, this means to compute the Morse index of the restriction of $\gamma_{\alpha,\beta}$ to the subspace E_2 (see Figure 1.2a). Note that this quantity does not depend on β; however, we represent its values in the plane (β, α) in order to relate them more clearly with the restriction of the system to E_3: the Morse index of the original problem is indeed given by the sum of the indices of the restrictions and it is easy to visualise this by superposing the graphs.

Theorem. *The Morse index of the circular solution ($\gamma_{\alpha,0}$) of the generalised Kepler problem is*

$$i_{Morse}(\gamma_{\alpha,0}) = \begin{cases} 0 & \textit{if } \alpha \in [1, 2) \\ 2 & \textit{if } \alpha \in [0, 1). \end{cases}$$

We can then go further by computing the Morse index of any k-th iterate $\gamma^k_{\alpha,0}$ of $\gamma_{\alpha,0}$ for $k \in \mathbb{N}$, $k > 1$; this is made possible by the

ω-index theory and the Bott-Long iteration formula. What we obtain is that $i_{\mathrm{Morse}}(\gamma^k_{\alpha,0})$ is a piecewise constant and non-increasing function of α for every fixed $k > 1$. In particular, for any fixed k, there exists an interval $\left(2-\frac{1}{k^2}, 2\right)$ on which $i_{\mathrm{Morse}}(\gamma^k_{\alpha,0}) = 0$. On the other hand, for any fixed value of α, the quantity $i_{\mathrm{Morse}}(\gamma^k_{\alpha,0})$ diverges to $+\infty$ as $k \to +\infty$. Let us observe that $\alpha_k \coloneqq 2 - \frac{1}{k^2}$ tends to the value 2 as k diverges: this means that the jumps of the Morse index tend to the boundary of the stability region for the Kepler-type problem. See Figure 1.3 for some examples.

As for the Morse index of the family of circular Lagrangian solutions of the planar 3-body-type problem, an interesting result is due to Venturelli [Ven02, Theorem 3.1.7, p. 25], who proved that for $\alpha = 1$ the minimisers of the Lagrangian action functional among the loops under a homological constraint are circular orbits. Moreover, he showed that for equal masses ($\beta = 9$) and $\alpha \in (1, 2)$ the periodic solution $\gamma_{\alpha,9}$ is a strict local minimiser, whereas for $\alpha \in [0, 1)$ it is a saddle. The problem of determining the Morse index of the circular Lagrangian orbit for different masses and for any parameter $\alpha \in (0, 2)$ has been left unsolved until now.

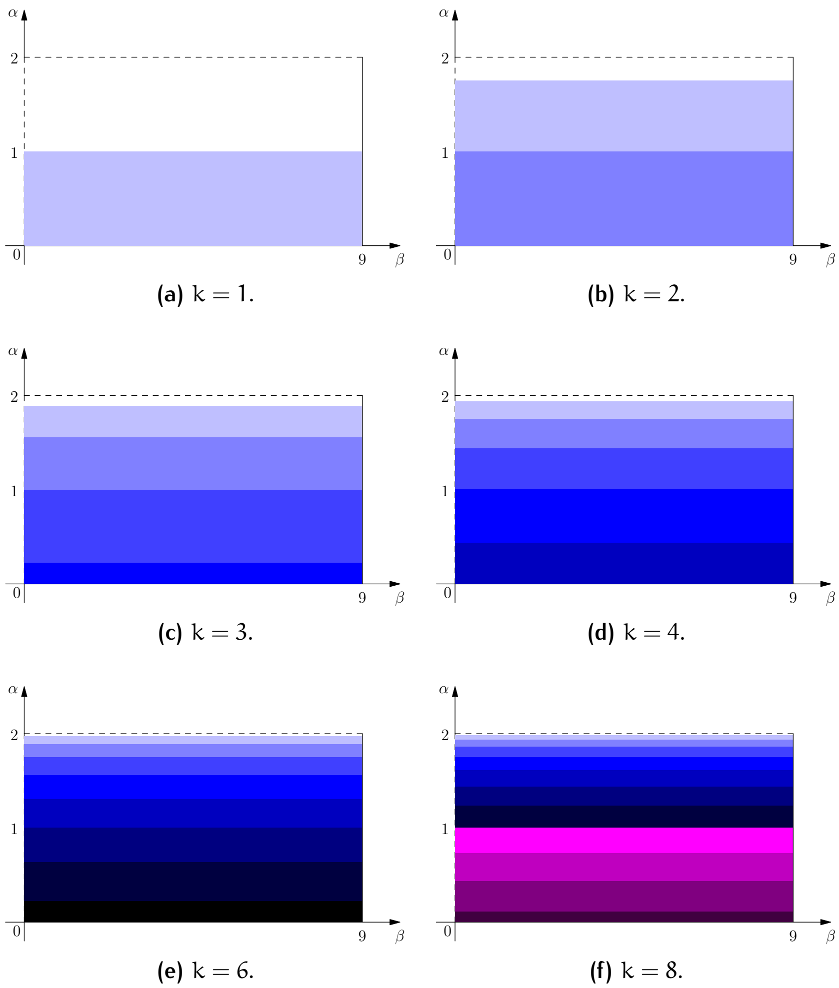
Theorem. *The Morse index of the Lagrangian circular solution $\gamma_{\alpha,\beta}$ is given by*

$$i_{\mathrm{Morse}}(\gamma_{\alpha,\beta}) = \begin{cases} 0 & \textit{if } \alpha \in [1,2) \\ 2 & \text{if } \beta \geqslant \dfrac{36(1-\alpha)}{(\alpha+2)^2} \text{ and } \alpha \in [0,1) \\ 4 & \text{if } 0 < \beta < \dfrac{36(1-\alpha)}{(\alpha+2)^2}. \end{cases}$$

The result is depicted in Figure 1.2b.

In the particular case of $\alpha = 1$ we recover the results proved by X. Hu and S. Sun in [HS10, Formulæ (55)–(56)]: they compute the Morse index of the Lagrangian elliptic orbits of the classical 3-body problem taking as parameters the eccentricity of the orbit and β.

As for the generalised Kepler problem, we are able to determine, via ω-index theory, any $i_{\mathrm{Morse}}(\gamma^k_{\alpha,\beta})$ for all $k \geqslant 2$. It is worth noting that $\alpha = 2$ is the limit of some values α_k that are the points where the Morse index of the k-th iteration of the circular Keplerian solution jumps. Moreover this limit value (which coincides with the lower bound of the strong force condition) is the boundary of the spectral stability region. The very same behaviour appears also in the restriction to the symplectic invariant subspace E_3, although the curves of the (β, α)-plane over which the Morse index of all the k-iterations jumps are no longer straight lines. As we show at the end of Subsection 5.4.2, the boundary of the stability region is the enveloping curve of a two-parameter family of curves representing the jumps in

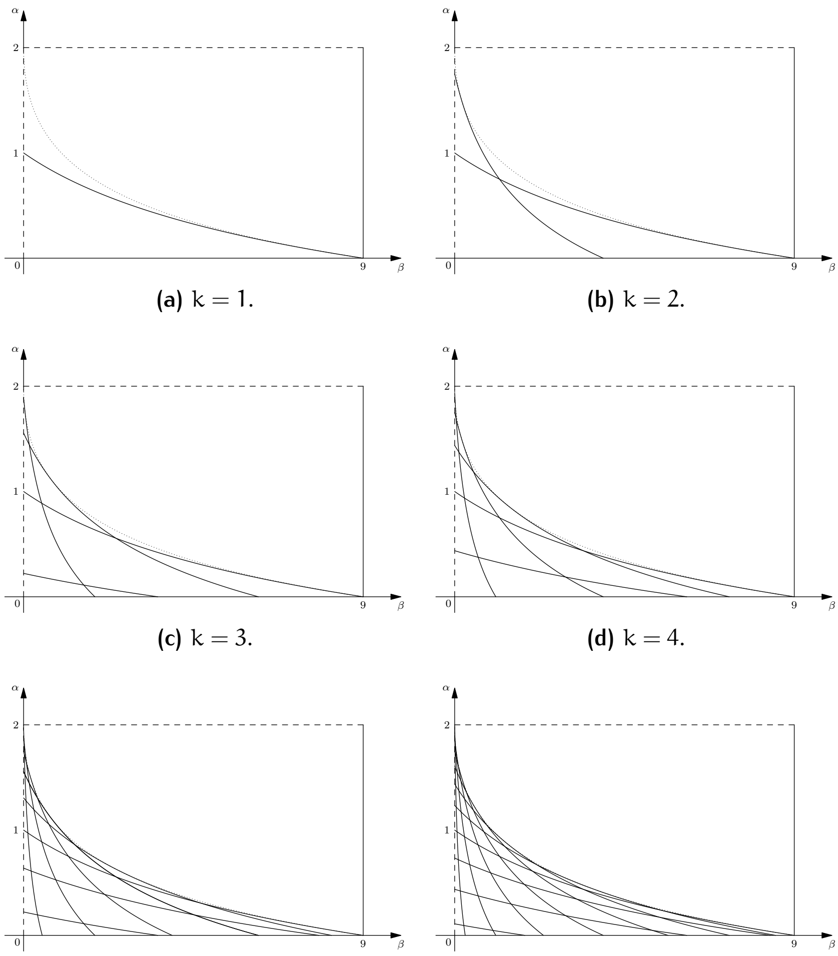


(a) $k = 1$. **(b)** $k = 2$.

(c) $k = 3$. **(d)** $k = 4$.

(e) $k = 6$. **(f)** $k = 8$.

Figure 1.3. Values of the Morse index of the k-th iteration of the Kepler circular orbit $\gamma^k_{\alpha,0}$ for some values of k. The white upper band in each subfigure represents the value 0; going downwards and passing through the lower boundary of each band increases the Morse index by 2. As k increases there is an accumulation of bands at the value $\alpha = 2$.



(a) $k = 1$. (b) $k = 2$.

(c) $k = 3$. (d) $k = 4$.

(e) $k = 6$. (f) $k = 8$.

Figure 1.4. Curves $\{f_{k,l}\}$ on which the Maslov index of the k-th iterate of the problem restricted to E_3 jumps (the values of k taken into account are shown below each subfigure). The dotted line is the stability curve, which is approximated more and more accurately as k increases.

the Morse index of the iterations of the solution. It seems then quite plausible to conjecture that the points at which a transition of stability occurs could be locally approximated, in a suitable sense, by curves along which there is a change in the Morse index of all the iterates.

Let us now compare our result with some other important contributions on the subject. Being every relative equilibrium a zero-average loop solution, our theorem and [Ven02, Theorem 3.1.7, p. 25] seem to be in striking contrast with the main theorem by A. Chenciner and N. Desolneux in [CD98], where they proved that $\gamma_{\alpha,\beta}$, for $\alpha \in (0, +\infty)$, are global minima of the action functional defined on the space of $W^{1,2}$-loops with zero average (and fixed centre of mass). However, although for $\alpha \in [0, 1)$ we show that the Morse index is strictly positive, there is no contradiction because we do not restrict ourselves to the zero-average $W^{1,2}$-loop space. One might observe that the domain of the functional analysed by Chenciner and Desolneux includes collisions and ours does not, but this is not at all influential on the question: even taking into account those singularities the Morse index would not be affected, being it a local function and being relative equilibria always collisionless by definition. The main result in [CD98] has been recently generalised in [BT04], where V. Barutello and S. Terracini proved that for every $\alpha \in (0, +\infty)$ the absolute minimum among simple choreographies is attained on a relative equilibrium motion associated with the regular n-gon. We observe that in imposing the choreographic symmetry constraint the authors require as well that the masses be equal, so that the symmetry may act transitively on the bodies' labels. This corresponds in our setting to fixing $\beta = 9$. Their result [BT04, Theorem 1] entails that the circular Lagrange solution is an absolute minimum of the action functional on the $W^{1,2}$-choreographies. However, by our theorem we have that for $\alpha \in [0, 1)$ the Morse index is 2. We observe as above that this is not in contrast with our result since we are computing the Morse index in a strictly larger space.

The main tool we used to demonstrate these results is an index theory, namely a Morse index theorem that relates the Morse index of a critical point of the Lagrangian action functional and the Maslov index of the fundamental solution associated with the corresponding Hamiltonian system: the problem of computing the former index is then translated into the computation of the latter. The aforementioned theorem become in this way the key ingredient that allows to switch to this symplectic invariant, the Maslov index: in order to compute it we avail ourselves of some canonical transformations that involve a symplectic change of coordinates. Such new coordinates provide two useful advantages: first, the linearised Hamiltonian system becomes autonomous; second, the reduced phase space is split into two symplectic 4-dimensional subspaces E_2 and E_3 which are invariant under the phase flow. As a consequence, the Maslov index is obtained as the sum of the Maslov indices of the restrictions of the fundamental solution to these subspaces. Although some formulæ for the computation of the Maslov index exist for non-degenerate situations (involving for instance the Krein signature), we point out that E_2 gives rise to a really degenerate setting. We overcome all of these problems by using different notions of Maslov index available in the literature, all of which differ by the contribution at the endpoints and by their homotopy properties. In order to overcome the degeneracy on E_2 we used the axiomatic definition given by S. E. Cappell, R. Lee and E. Y. Miller in their well-known paper [CLM94], while to manage the degeneracy represented by the boundary of the stability region on E_3 we mainly employ the Maslov index introduced by Y. Long. In Chapter 4 we recall the puzzle of all these indices trying to point out their main properties as well as the intertwining relations between them. Due to the low dimension, in all of our computation a big role is played by the geometry of $\mathrm{Sp}(2)$. To this set is dedicated Section A.4, where we fix our notation and recall some well-known facts scattered in the literature. As already observed, a key result is represented by the Morse index theorem stating the relation between the Morse index of the essentially positive Fredholm quadratic forms associated with the second variation and the Maslov index of the periodic solution. Section A.5 is devoted to clarifying the functional-analytical setting.

2 LINEAR STABILITY IN SIMPLE SINGULAR PLANAR MECHANICAL SYSTEMS WITH SYMMETRY

Contents

In this chapter we present the notions of linear and spectral stability for autonomous Hamiltonian systems and outline the setting of the problem, introducing the concept of relative equilibria in a rather general way. Then we proceed with some auxiliary results and set up the machinery and the general tools needed for the proof of the first main theorem mentioned in the Introduction.

2.1 DEFINITION OF LINEAR AND SPECTRAL STABILITY FOR AUTONOMOUS HAMILTONIAN SYSTEMS

We recall here some basic definitions and well-known facts about the linear stability of autonomous Hamiltonian systems, starting with the definition of the symplectic group and its Lie algebra. We refer to [Abb01] for more details.

The *(real) symplectic group* is the set

$$\mathrm{Sp}(2n,\mathbb{R}) \coloneqq \left\{ S \in \mathrm{Mat}(2n,\mathbb{R}) \;\middle|\; S^T J S = J \right\},$$

where

$$J \coloneqq J_{2n} \coloneqq \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

denotes the complex structure in $\mathbb{R}^{2n}$ and I_n is the $n \times n$ identity matrix. In the following and throughout all this thesis, J_{2n} and I_n will always be written simply as J and I, their dimensions being clear from the context. Symplectic matrices correspond to symplectic automorphism of the standard symplectic space $(\mathbb{R}^{2n}, \Omega)$, where Ω is the standard symplectic form represented by J via the standard inner product of $\mathbb{R}^{2n}$, i.e. $\Omega(u,v) \coloneqq \langle Ju, v\rangle$ for every $u, v \in \mathbb{R}^{2n}$.

By differentiating the equation $H^T JH = J$ and evaluating it at the identity matrix, we find the characterising relation of the Hamiltonian matrices: the Lie algebra of the symplectic group is defined as

$$\mathfrak{sp}(2n,\mathbb{R}) \coloneqq \left\{ H \in \mathrm{Mat}(2n,\mathbb{R}) \;\middle|\; H^T J + JH = 0 \right\},$$

and its elements are called *Hamiltonian* or *infinitesimally symplectic*.

Remark 2.1. Since $\mathrm{Sp}(2n,\mathbb{R})$ is a matrix Lie group and $\mathfrak{sp}(2n,\mathbb{R})$ is its Lie algebra, the exponential map

$$\exp : \mathfrak{sp}(2n,\mathbb{R}) \to \mathrm{Sp}(2n,\mathbb{R})$$

coincides with the usual matrix exponential, and therefore we have that H is a Hamiltonian matrix if and only if $\exp(H)$ is symplectic. It follows that $\lambda \in \sigma(H)$ if and only if $e^\lambda \in \sigma\big(\exp(H)\big)$.

The next proposition recollects the symmetries of the spectra of Hamiltonian and symplectic matrices.

Proposition 2.2. *The characteristic polynomial of a symplectic matrix is a reciprocal polynomial. Thus if λ is an eigenvalue of a real symplectic matrix, then so are λ^{-1}, $\overline{\lambda}$, $\overline{\lambda}^{-1}$.*

The characteristic polynomial of a Hamiltonian matrix is an even polynomial. Thus if λ is an eigenvalue of a Hamiltonian matrix, then so are $-\lambda$, $\overline{\lambda}$, $-\overline{\lambda}$.

Proof. See [MHO09, Proposition 3.3.1]. □

Remark 2.3. It descends directly from Proposition 2.2 that the spectrum of a Hamiltonian matrix H is, in particular, symmetric with respect to the real axis of the complex plane. Moreover, 0 has always even (possibly zero) algebraic multiplicity as a root of the characteristic polynomial of H.

We now present the definition of spectral and linear stability for Hamiltonian matrices, in view of the fact that these are the ones on which we shall focus in our analyses.

Definition 2.4. A Hamiltonian matrix $H \in \mathfrak{sp}(2n,\mathbb{R})$ is said to be *spectrally stable* if $\sigma(H) \subset i\mathbb{R}$, whereas it is *linearly stable* if $\sigma(H) \subset i\mathbb{R}$ and in addition it is diagonalisable.

This concept is easily adapted to symplectic matrices by using the exponential map, as explained in Remark 2.1, and by remembering that the imaginary axis of the complex plane is the Lie algebra of the unit circle $\mathbb{U}$ in the same plane (see Remark 2.3). Indeed, a symplectic matrix S is said to be *spectrally stable* if $\sigma(S) \subset \mathbb{U}$ and, as before, the property of *linear stability* requires in addition the diagonalisability of S.

A linear autonomous Hamiltonian system in $\mathbb{R}^{2n}$ has the form

$$\dot{\zeta}(t) = JA\zeta(t), \tag{2.1.1}$$

where A is a symmetric matrix. Being it autonomous, its fundamental solution can be written in the explicit form

$$\gamma(t) := \exp(tJA).$$

The definition of spectral and linear stability for this kind of systems is given in accord with Definition 2.4.

Definition 2.5. The linear autonomous Hamiltonian system (2.1.1) is *spectrally* (resp. *linearly*) *stable* if the symplectic matrix $\exp(JA)$ corresponding to its fundamental solution at time $t = 1$ is spectrally (resp. linearly) stable. We say that System (2.1.1) is *degenerate* if $0 \in \sigma(JA)$, or equivalently if $1 \in \sigma\big(\exp(JA)\big)$, and *non-degenerate* otherwise.

We conclude by reporting a criterion for linear stability of symplectic matrices, in order to complete our brief recollection of definitions and results on this topic. We also point out that we are not aware of any existing proof of this lemma. In the following, the symbol $\|\cdot\|_{\mathscr{L}(\mathcal{H})}$ will denote the norm of a bounded linear operator from the Hilbert space $\mathcal{H}$ to itself.

Lemma 2.6. *A matrix $S \in \mathrm{Sp}(2n, \mathbb{R})$ is linearly stable if and only if*

$$\sup_{m\in\mathbb{N}} \|S^m\|_{\mathscr{L}(\mathbb{R}^{2n})} < +\infty.$$

Proof. If S is linearly stable, then in particular it is similar to a diagonal matrix D through an invertible matrix P, so that we have

$$\begin{aligned}\sup_{m\in\mathbb{N}} \|S^m\|_{\mathscr{L}(\mathbb{R}^{2n})} &= \sup_{m\in\mathbb{N}} \left\|P^{-1}D^m P\right\|_{\mathscr{L}(\mathbb{R}^{2n})} \\ &\leqslant \left\|P^{-1}\right\|_{\mathscr{L}(\mathbb{R}^{2n})} \|P\|_{\mathscr{L}(\mathbb{R}^{2n})} \sup_{m\in\mathbb{N}} \|D^m\|_{\mathscr{L}(\mathbb{R}^{2n})} \\ &= \left\|P^{-1}\right\|_{\mathscr{L}(\mathbb{R}^{2n})} \|P\|_{\mathscr{L}(\mathbb{R}^{2n})} < +\infty,\end{aligned}$$

where the last equality holds true because all the eigenvalues of S (and hence those of D) lie on the unit circle.

Vice versa, if S is not linearly stable then it is not spectrally stable or it is not diagonalisable (or both). If it is spectrally unstable there exists, by definition, at least one eigenvalue $\lambda \notin \mathbb{U}$, and we can assume, by the properties of the spectrum of symplectic matrices, that $|\lambda| > 1$. Writing S in its Jordan form (possibly diagonal) and computing S^m yields on the diagonal a power λ^m, whose modulus diverges as $m \to +\infty$. Hence $\|S^m\|_{\mathscr{L}(\mathbb{R}^{2n})} \to +\infty$. If S is not diagonalisable, then there exists at least one Jordan block of size $k \geqslant 2$ (say) relative to the eigenvalue λ. Its m-th power has the form

$$\begin{bmatrix} \lambda^m & m\lambda^{m-1} & 0 & \cdots\cdots & & 0 \\ 0 & \lambda^m & m\lambda^{m-1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \lambda^m & m\lambda^{m-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda^m \end{bmatrix},$$

and therefore even in this case (regardless of the fact that $\lambda \in \mathbb{U}$ or not) the norm of S^m tends to $+\infty$ as m goes to $+\infty$. $\square$

2.2 DESCRIPTION OF THE PROBLEM

In this section we outline the basic definitions and properties of (singular) simple planar mechanical systems with symmetry, as well as their reduction to the quotient space.

Consider the Euclidean plane $\mathbb{R}^2$ endowed with the usual inner product $\langle\cdot,\cdot\rangle$ and let $m_1, \dots, m_n$ be $n \geqslant 3$ positive real numbers which can be thought of as masses. The configuration space of n point particles with masses m_i, with $i \in \{1, \dots, n\}$, will therefore be a suitable subset $X \subseteq \mathbb{R}^{2n}$ (equipped with its Euclidean inner product, which we denote again by $\langle\cdot,\cdot\rangle$). For any position vector $q := (q_1, \dots, q_n)^T \in \mathbb{R}^{2n}$, with $q_i \in \mathbb{R}^2$ (column vector) for every $i \in \{1, \dots, n\}$, we can define a norm in $\mathbb{R}^{2n}$ through the *moment of inertia*:[1]

$$\mathcal{I}(q) := \|q\|_M^2 := \langle Mq, q\rangle = \sum_{i=1}^{n} m_i |q_i|^2,$$

where $M := \operatorname{diag}(m_1 I_2, \dots, m_n I_2) \in \operatorname{Mat}(2n, \mathbb{R})$ is the diagonal *mass matrix* and $|q_i|$ denotes the Euclidean norm of q_i in $\mathbb{R}^2$.

A simple mechanical system of n point particles on X is described by a *Lagrangian function* $\mathscr{L} : TX \to \mathbb{R}$ of the form

$$\mathscr{L}(q, \dot{q}) := \mathcal{K}(q, \dot{q}) + \mathcal{U}(q),$$

where $\mathcal{K} : TX \to \mathbb{R}$ is the *kinetic energy* of the system and $\mathcal{U} : X \to \mathbb{R}$ is its *potential function*. This Lagrangian thus equals the difference between the kinetic energy and the potential energy ($-\mathcal{U}$); in our case we have $\mathcal{K}(q, \dot{q}) := \frac{1}{2}\mathcal{I}(\dot{q})$.

Using the mass matrix M, Newton's equations can be written as the following second-order system of ordinary differential equations on X:

$$M\ddot{q} = \nabla\mathcal{U}(q), \tag{2.2.1}$$

1 What we define here is actually the double of the moment of inertia: we drop the factor $1/2$ in order to make computations lighter in the following.

which can of course be transformed into a first-order system as follows. Let us introduce the *Hamiltonian function* $\mathscr{H} : T^*X \to \mathbb{R}$, defined by

$$\mathscr{H}(q,p) := \frac{1}{2}\langle M^{-1}p^T, p^T\rangle - \mathcal{U}(q). \tag{2.2.2}$$

In this expression $p := (p_1, \dots, p_n) \in \mathbb{R}^{2n}$, with $p_i \in \mathbb{R}^2$ (row vector) for every $i \in \{1, \dots, n\}$, is the linear momentum conjugate to q. The Hamiltonian system associated with (2.2.1) is the first-order system of ordinary differential equations on the phase space $T^*X \cong X \times \mathbb{R}^{2n}$ given by

$$\begin{cases} \dot{q} = \partial_p \mathscr{H} = M^{-1}p^T \\ \dot{p}^T = -\partial_q \mathscr{H} = \nabla\mathcal{U}(q). \end{cases} \tag{2.2.3}$$

We shall consider simple mechanical systems with an SO(2)-*symmetry*, meaning that the group SO(2) acts properly on X through isometries that leave the potential function $\mathcal{U}$ unchanged. It follows that the Lagrangian and the Hamiltonian are SO(2)-invariant under the natural lift of this action to TX and to T^*X, respectively.

2.2.1 Relative equilibria

Among all the solutions of Newton's equations (2.2.1), as already observed, the simplest are represented by a special class of periodic solutions called *relative equilibria*.

Let $e^{\omega J t} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$ be the matrix representing the rotation in the plane with angular velocity ω. In order to rewrite Hamilton's equations (2.2.3) in a frame uniformly rotating about the origin with period $2\pi/\omega$, we employ the following symplectic change of coordinates:

$$\begin{cases} x := R(t)\, q \\ y^T := R(t)\, p^T \end{cases}$$

where $R(t)$ is the $2n \times 2n$ block-diagonal matrix $\mathrm{diag}_n(e^{\omega J t}, \dots, e^{\omega J t})$. Since a symplectic change of variables preserves the Hamiltonian structure, in these new coordinates System (2.2.3) is still Hamiltonian and transforms as follows:

$$\begin{cases} \dot{x} = \partial_y \widehat{\mathscr{H}} = \omega K x + M^{-1}y^T \\ \dot{y}^T = -\partial_x \widehat{\mathscr{H}} = \nabla\mathcal{U}(x) + \omega K y^T \end{cases} \tag{2.2.4}$$

where K is the $2n \times 2n$ block-diagonal matrix $\mathrm{diag}_n(J, \dots, J)$ and $\widehat{\mathscr{H}}$ is the new Hamiltonian function given by

$$\widehat{\mathscr{H}}(x,y) := \frac{1}{2}\langle M^{-1}y^T, y^T\rangle - \mathcal{U}(x) + \omega\langle Kx, y^T\rangle. \tag{2.2.5}$$

From the physical point of view, the term involving K is due to the Coriolis force.

An equilibrium for System (2.2.4) must satisfy the conditions

$$\begin{cases} \omega K x + M^{-1} y^T = 0 \\ \nabla \mathcal{U}(x) + \omega K y^T = 0, \end{cases}$$

which, taking into account that $[K, M] = 0$ and that $K^2 = -I$, can be rewritten as

$$\begin{cases} y^T = -\omega M K x \\ M^{-1} \nabla \mathcal{U}(x) + \omega^2 x = 0. \end{cases} \tag{2.2.6}$$

Setting now $\Xi := \omega K$, it is easy to see that the Hamiltonian $\widehat{\mathscr{H}}$ defined in (2.2.5) coincides with the *augmented Hamiltonian function*

$$\mathscr{H}_\Xi(x, y) := \mathcal{K}_\Xi(x, y) - \mathcal{U}_\Xi(x),$$

where

$$\mathcal{K}_\Xi(x, y) := \frac{1}{2} \left\| M^{-1} y^T + \Xi x \right\|_M^2$$

is the *augmented kinetic energy* and

$$\mathcal{U}_\Xi(x) := \mathcal{U}(x) + \frac{1}{2} \left\| \Xi x \right\|_M^2. \tag{2.2.7}$$

is called the *augmented potential function*. In terms of these augmented quantities, System (2.2.6) becomes

$$\begin{cases} y^T = -M \Xi x \\ \nabla \mathcal{U}_\Xi(x) = 0 \end{cases}$$

and we have the following definition.

Definition 2.7. The point $(\bar{x}, \bar{y}) \in T^*X$ is a *relative equilibrium* for Newton's Equations (2.2.1) with potential $\mathcal{U}$ if both the following conditions hold:

1) $\bar{y}^T = -M \Xi \bar{x}$;

2) $\bar{x}$ is a critical point of the augmented potential function $\mathcal{U}_\Xi$.

Let us now consider the autonomous Hamiltonian system (2.2.4) in $\mathbb{R}^{4n}$: by grouping variables into $z := (x^T, y)^T$, it can be written as follows:

$$\dot{z}(t) = -J \nabla \widehat{\mathscr{H}}\big(z(t)\big). \tag{2.2.8}$$

Linearising it at the relative equilibrium $\bar{z} := (\bar{x}^T, \bar{y})^T$, we obtain the linear autonomous Hamiltonian system

$$\dot{z}(t) = -J B z(t), \tag{2.2.9}$$

where B is the constant $4n \times 4n$ symmetric matrix given by

$$B := \begin{pmatrix} -D^2 \mathcal{U}(\bar{x}) & \Xi^T \\ \Xi & M^{-1} \end{pmatrix}. \tag{2.2.10}$$

2.3 AUXILIARY RESULTS

In this section we present the lemmata and the propositions needed in the proof of the main results in Section 2.4. We first introduce some notation and definitions; for further properties we refer to Section A.1.

2.3.1 Notation and definitions

Let $\mathcal{H}$ be, throughout all this dissertation, a finite-dimensional complex Hilbert space (we shall specify its dimension when needed). We denote by $\mathscr{B}(\mathcal{H})$ the Banach algebra of all (bounded) linear operators $T : \mathcal{H} \to \mathcal{H}$ and by $\mathscr{B}^{sa}(\mathcal{H})$ the subset of all (bounded) linear self-adjoint operators on $\mathcal{H}$. For a subset $\mathscr{A} \subseteq \mathscr{B}(\mathcal{H})$, the writing $\mathscr{G}\mathscr{A}$ indicates the set of all invertible elements of $\mathscr{A}$.

Definition 2.8. For any $T \in \mathscr{B}^{sa}(\mathcal{H})$, we define its *index* $n^-(T)$, its *nullity* $\nu(T)$ and its *coindex* $n^+(T)$ as the numbers of its negative, null and positive eigenvalues, respectively. Its *extended index* and the *extended coindex* are defined as

$$n^-_{\text{ext}}(T) := n^-(T) + \nu(T), \qquad n^+_{\text{ext}}(T) := n^+(T) + \nu(T).$$

The *signature* $\operatorname{sgn}(T)$ of T is the difference between its coindex and its index:

$$\operatorname{sgn}(T) := n^+(T) - n^-(T).$$

Remark 2.9. We shall refer to the index $n^-(T)$ of a self-adjoint operator $T \in \mathscr{B}^{sa}(\mathcal{H})$ also as its *Morse index*, which will be denoted by $i_{\text{Morse}}(T)$.

Definition 2.10. Let X be a topological space, $Y \subseteq X$ a subspace and $a, b \in \mathbb{R}$, with $a < b$. We denote by $\Omega(X, Y)$ the set of all continuous paths $\gamma : [a, b] \to X$ with endpoints in Y. Instead of $\Omega(X, X)$ we simply write $\Omega(X)$. Two paths $\gamma, \delta \in \Omega(X, Y)$ are said to be *(free) homotopic* if there is a continuous map $F : [0, 1] \times [a, b] \to X$ which satisfies the following properties:

i) $F(0, \cdot) = \gamma$, $F(1, \cdot) = \delta$;

ii) $F(s, a) \in Y$, $F(s, b) \in Y$ for all $s \in [0, 1]$.

The set of homotopy classes in this sense is denoted by $\widetilde{\pi}_1(X, Y)$.

Remark 2.11. Note that the endpoints are not fixed along the homotopy; however, they are allowed to move only within Y.

Taking into account [Leso5, Corollary 3.7], we are entitled to give the following definition:

DEFINITION 2.12. Let $T \in \Omega\big(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\big)$ and $a, b \in \mathbb{R}$, with $a < b$. We define its *spectral flow* on the interval $[a, b]$ as:

$$\mathrm{sf}\left(T, [a, b]\right) \coloneqq n^{+}_{\mathrm{ext}}\big(T(b)\big) - n^{+}_{\mathrm{ext}}\big(T(a)\big).$$

REMARK 2.13. It is worth noting that

$$\mathrm{sf}\left(T, [a, b]\right) = n^{-}\left(T(a)\right) - n^{-}\left(T(b)\right).$$

We now switch to introduce the key notion of *crossing*.

DEFINITION 2.14. Let $a, b \in \mathbb{R}$, with $a < b$, and $T \in \mathscr{C}^1\left([a, b], \mathscr{B}^{sa}(\mathcal{H})\right)$. A *crossing instant* (or simply a *crossing*) for the path T is a number $t_* \in [a, b]$ for which $T(t_*)$ is not injective. We define the *crossing operator* (also called *crossing form*) $\Gamma(T, t_*) : \ker T(t_*) \to \ker T(t_*)$ of T with respect to the crossing t_* by

$$\Gamma(T, t_*) \coloneqq Q\dot{T}(t_*)Q\big|_{\ker T(t_*)}, \tag{2.3.1}$$

where $Q : \mathcal{H} \to \mathcal{H}$ denotes the orthogonal projection onto the kernel of $T(t_*)$. A crossing t_* is called *regular* if the crossing form $\Gamma(T, t_*)$ is non-degenerate. We say that the path T is *regular* if each crossing for T is regular.

REMARK 2.15. The computation of the spectral flow of a path of operators involves the signature of the crossing form. We point out here that we actually refer to the signature of the *quadratic form* associated with the linear map defined in (2.3.1), that is, we make the following implicit identification. Given an endomorphism $\Gamma : V \to V$ on a vector space V, it is associated in a natural way with a bilinear form $\mathcal{B}_\Gamma : V \times V^* \to \mathbb{R}$ defined by

$$\mathcal{B}_\Gamma(u, f) \coloneqq f(\Gamma u),$$

where $f \in V^*$ is an element of the dual space V^* of V. Since $V^* \cong V$ one can then define

$$\mathcal{B}_\Gamma(u, v) \coloneqq v^T \Gamma u.$$

The quadratic form associated with Γ is thus the quadratic form associated with $\mathcal{B}_\Gamma$. This is the justification for the abuse of language and notation that one might encounter in the following.

As last piece of information, we point out that in the rest of the dissertation we shall denote the matrix iJ by G.

2.3.2 Linear stability, spectral flow and partial signatures

Here are the properties and facts that we shall exploit later to prove our main theorem. In this subsection we identify the Hilbert space

$\mathcal{H}$ with $\mathbb{C}^{4n}$ and consider the affine path $D : [0,+\infty) \to \mathscr{B}^{sa}(\mathbb{C}^{4n})$ defined by

$$D(t) := A + tG,$$

where $A \in \mathscr{B}^{sa}(\mathbb{C}^{4n})$ is a real symmetric matrix (hence JA is Hamiltonian). Without different indication, it will be understood that $\mathcal{H}$, A and D are as defined above.

Thanks to the identification $\mathcal{H} = \mathbb{C}^{4n}$, we implicitly fix the canonical basis of $\mathbb{C}^{4n}$ and therefore every operator in $\mathscr{B}^{sa}(\mathcal{H})$ is represented by a $4n \times 4n$ complex Hermitian matrix.

We explicitly note that the spectral flow does not depend on the particular inner product chosen but only on the associated quadratic form (see [GPP04b]).

Lemma 2.16. *Assume that JA is linearly stable.*

Then if A is singular there exist $\varepsilon > 0$ and $T > \varepsilon$ such that

(i) The instant $t_* = 0$ is the only crossing for the path D on $[0,\varepsilon]$;

(ii) $\mathrm{sf}\big(D,[\varepsilon,T_1]\big) = \mathrm{sf}\big(D,[\varepsilon,T_2]\big)$ for all $T_1, T_2 \geqslant T$;

(iii) $\mathrm{sf}\big(D,[\varepsilon,T]\big)$ is an even number.

If A is non-singular there exists $T > 0$ such that

(i) $\mathrm{sf}\big(D,[0,T_1]\big) = \mathrm{sf}\big(D,[0,T_2]\big)$ for all $T_1, T_2 \geqslant T$;

(ii) $\mathrm{sf}\big(D,[0,T]\big)$ is an even number.

Proof. Since A is symmetric, the matrix JA is Hamiltonian. Therefore its spectrum is symmetric with respect to the real axis of the complex plane and $\ker A$ (which is equal to $\ker JA$ because J is an isomorphism) is even-dimensional, being JA diagonalisable. Furthermore, due to the Krein properties of G (see Section A.3), the crossing form $Q_\lambda G Q_\lambda\big|_{E_\lambda}$ is always non-degenerate on each eigenspace E_λ.

Hence the hypotheses of Proposition A.5 or of Corollary A.6 (depending whether A is invertible or not) are fulfilled and this proof reduces to the corresponding one in the appendix. □

Proposition 2.17. *Assume that $t_* > 0$ is an isolated (possibly non-regular) crossing instant for the path D. Then, for $\varepsilon > 0$ small enough,*

$$\mathrm{sf}\big(D,[t_*-\varepsilon, t_*+\varepsilon]\big) = \operatorname{sgn}\mathcal{B}_1,$$

where

$$\mathcal{B}_1 := \langle G\,\cdot,\cdot\rangle\big|_{\mathcal{H}_{t_*}}$$

and $\mathcal{H}_{t_*}$ is the generalised eigenspace given by

$$\mathcal{H}_{t_*} := \bigcup_{j=1}^{4n} \ker(GA + t_* I)^j.$$

Proof. We observe that for $t \in (0, +\infty)$

$$D(t) = t\left(\frac{1}{t}A + G\right) =: t\,\widetilde{D}\left(\frac{1}{t}\right).$$

Clearly, the spectral flow is invariant by multiplication of a path for a positive real-analytic function:

$$\mathrm{sf}\big(D, [t_* - \varepsilon, t_* + \varepsilon]\big) = \mathrm{sf}\left(\widetilde{D}, \left[\frac{1}{t_*} - \varepsilon, \frac{1}{t_*} + \varepsilon\right]\right).$$

Using now Proposition A.11, with $C := G$ and $s := \frac{1}{t}$, we obtain the thesis (observe that the difference in sign to the local contribution to the spectral flow is due to the change of variable $s := \frac{1}{t}$). ☐

We now prove the main result of this section by means of the theory of partial signatures (see Section A.2).

THEOREM 2.18. *If JA is spectrally stable, then $n^-(A)$ is even.*

Proof. If we write

$$D(t) = -J(JA - itI)$$

we see that $t_* \in [0, +\infty)$ is a crossing instant for D if and only if

$$it_* \in \sigma(JA) \cap i[0, +\infty).$$

Indeed, since $-J$ is an isomorphism,

$$\ker D(t) = \ker(JA - itI) \qquad \forall t \in [0, +\infty),$$

and thus there is a bijection between the set of crossing instants t_* of D and the set of pure imaginary eigenvalues of JA of the form it_*. Being D an affine path, it is real-analytic, and the Principle of Analytic Continuation implies that every crossing (be it regular or not) is isolated, because it can be regarded as a zero of the (real-analytic) map $\det D(t)$.

Let us examine the strictly positive crossings. By Proposition 2.17, in a suitable neighbourhood with radius $\delta > 0$ around a crossing $t_* > 0$ we see that

$$\mathrm{sf}\big(D, [t_* - \delta, t_* + \delta]\big) = \operatorname{sgn} \mathcal{B}_1,$$

where $\mathcal{B}_1$ and $\mathcal{H}_{t_*}$ are as in the aforementioned proposition. Furthermore, by the general theory of the Krein signature (see Section A.3), for any crossing $t_* \in (0, +\infty)$ the restriction $\langle G\,\cdot, \cdot\rangle|_{\mathcal{H}_{t_*}}$ of the Krein form to each generalised eigenspace $\mathcal{H}_{t_*}$ is non-degenerate. In particular, Remark A.12 yields

$$\mathrm{sf}\big(D, [t_* - \delta, t_* + \delta]\big) \equiv \dim \mathcal{H}_{t_*} \mod 2. \tag{2.3.2}$$

for every strictly positive crossing instant t_*.

When turning our attention to the instant $t = 0$, we have to distinguish two situations: one where A is singular and one where it is not. Let us start with the former and assume that A is non-invertible, so that $t_* = 0$ is a crossing for the path D. Since this is isolated, by arguing as in the proof of (T2) in Proposition A.5 we can find $\varepsilon > 0$ and $T > \varepsilon$ such that the path D has only $t_* = 0$ as crossing instant on $[0, \varepsilon]$ and $\mathrm{sf}\big(D, [\varepsilon, T_1]\big) = \mathrm{sf}\big(D, [\varepsilon, T_2]\big)$ for every $T_1, T_2 \geqslant T$. Thus, recalling Remark 2.13 and the fact that $n^-\big(D(T)\big) = n^-(G)$, we obtain

$$\begin{aligned} \mathrm{sf}\big(D, [\varepsilon, T]\big) &= n^-\big(D(\varepsilon)\big) - n^-\big(D(T)\big) \\ &= n^-\big(D(\varepsilon)\big) - 2n. \end{aligned} \tag{2.3.3}$$

We observe that the dimension of the generalised eigenspace $\mathcal{H}_0$ (which coincides with the algebraic multiplicity of the eigenvalue 0) is even, being JA Hamiltonian. Intuitively speaking, then, since the Krein form is non-degenerate on this subspace, the null eigenvalues move from 0 as t leaves 0; and since its signature at the initial instant is 0 (by Krein theory: see Section A.3, p. 99), they split evenly: half become positive and half negative. This justifies the choice of ε so small that

$$n^-\big(D(\varepsilon)\big) = n^-(A) + \frac{\dim \mathcal{H}_0}{2}. \tag{2.3.4}$$

On the other hand, we have

$$4n = 2 \sum_{it_* \in \sigma(JA) \cap i(0,+\infty)} \dim \mathcal{H}_{t_*} + \dim \mathcal{H}_0,$$

or, equally well,

$$2n - \frac{\dim \mathcal{H}_0}{2} = \sum_{it_* \in \sigma(JA) \cap i(0,+\infty)} \dim \mathcal{H}_{t_*}. \tag{2.3.5}$$

By Equation (2.3.2) and by the concatenation axiom defining the spectral flow, we get

$$\mathrm{sf}\big(D, [\varepsilon, T]\big) \equiv \sum_{it_* \in \sigma(JA) \cap i(0,+\infty)} \dim \mathcal{H}_{t_*} \mod 2, \tag{2.3.6}$$

and comparing (2.3.5) and (2.3.6) we infer

$$\mathrm{sf}\big(D, [\varepsilon, T]\big) \equiv -\frac{\dim \mathcal{H}_0}{2} \equiv \frac{\dim \mathcal{H}_0}{2} \mod 2. \tag{2.3.7}$$

Equations (2.3.3) and (2.3.4) also yield

$$\mathrm{sf}\big(D, [\varepsilon, T]\big) \equiv n^-(A) + \frac{\dim \mathcal{H}_0}{2} \mod 2, \tag{2.3.8}$$

and from the last two congruences (2.3.7) and (2.3.8), we finally conclude that

$$n^-(A) \equiv 0 \mod 2.$$

In the case where A is invertible, the initial instant $t = 0$ is not a crossing and therefore we can repeat the previous discussion in a simpler way, by considering the spectral flow directly on the interval $[0, T]$ (see Corollary A.6). □

The following corollary is a direct consequence of Theorem 2.18; however, since the case is much simpler and does not require in fact the partial signatures, we give an independent proof. In this special case in which the matrix JA is diagonalisable the result can be proved directly by arguing as in Proposition A.11 and by taking into account the local contribution to the spectral flow as discussed in Lemma A.3.

Corollary 2.19. *If A is invertible and JA is linearly stable, then $n^-(A)$ is even.*

Proof. First we observe that the second assumption implies that there is a bijection between the crossing instants t_* and the pure imaginary eigenvalues of JA of the form it_* for positive real t_*. Let us then compute the crossing form $\Gamma(D, t_*)$ in correspondence of a crossing $t_* \in (0, +\infty)$: by definition it is given by

$$\Gamma(D, t_*) := Q\dot{D}(t_*)Q\big|_{\ker D(t_*)} = QGQ\big|_{\ker D(t_*)},$$

where Q is the orthogonal projection onto the kernel of $D(t_*)$. Note that the linear map $\Gamma(D, t_*)$ coincides (in the sense of Remark 2.15) with the quadratic Krein form:

$$\Gamma(D, t_*)[u] = \langle Gu, u\rangle, \qquad \forall u \in E_{it_*}(JA),$$

since $\ker D(t_*) = E_{it_*}(JA)$ for every crossing t_*. By Krein theory and by the fact that JA is diagonalisable, for any crossing instant $t_* \in (0, +\infty)$ the Krein form $g(u, u) := \langle Gu, u\rangle$ is non-degenerate on each eigenspace $E_{it_*}(JA)$ and by Proposition A.5 there exists $T > 0$ such that $\mathrm{sf}\big(D, [0, T_1]\big) = \mathrm{sf}\big(D, [0, T_2]\big)$ for every $T_1, T_2 \geqslant T$. Thus we get

$$\begin{aligned}\mathrm{sf}\big(D, [0, T]\big) &= n^-(A) - n^-\big(D(T)\big)\\ &= n^-(A) - 2n.\end{aligned}$$

Since JA is diagonalisable we have

$$4n = 2\sum_{\lambda \in \sigma(JA) \cap i(0, +\infty)} \dim E_\lambda,$$

or, which is the same,

$$2n = \sum_{\lambda \in \sigma(JA) \cap i(0, +\infty)} \dim E_\lambda$$

Equation (A.1.5) applied to the path D yields

$$\mathrm{sf}\big(D, [0, T]\big) \equiv \sum_{\lambda \in \sigma(JA) \cap i(0, +\infty)} \dim E_\lambda \qquad \mathrm{mod}\ 2$$

and we conclude that

$$n^-(A) \equiv 0 \mod 2. \qquad \square$$

REMARK 2.20. We observe that Corollary 2.19 can be proved without using the technique of partial signatures also in the case where A is not invertible. In order to take care of the crossing instant $t = 0$ it is enough to argue as in the proof of Theorem 2.18, with the only difference that, assuming diagonalisability, $\mathcal{H}_0$ coincides with the kernel of A (and, consequently, the kernel of JA).

2.4 MAIN THEOREM

We state and prove here the main result of our research, concerning the relationship between the Morse index of a critical point and the spectral instability of an associated relative equilibrium.

Consider the matrix B defined in (2.2.10) and set

$$\begin{aligned} N &:= \begin{pmatrix} I & M\Xi \\ 0 & I \end{pmatrix} \begin{pmatrix} -D^2\mathcal{U}(\bar{x}) & \Xi^T \\ \Xi & M^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ M\Xi^T & I \end{pmatrix} \\ &= \begin{pmatrix} -(D^2\mathcal{U}(\bar{x}) + \omega^2 M) & 0 \\ 0 & M^{-1} \end{pmatrix}. \end{aligned} \tag{2.4.1}$$

Observe that $D^2\mathcal{U}(\bar{x}) + \omega^2 M$ is precisely the Hessian $D^2\mathcal{U}_\Xi(\bar{x})$ of the augmented potential $\mathcal{U}_\Xi$ evaluated at its critical point $\bar{x}$ and define then the nullity and the Morse index of $\bar{x}$ as:

$$\begin{aligned} \nu(\bar{x}) &:= \nu\big(D^2\mathcal{U}_\Xi(\bar{x})\big), \\ i_{\text{Morse}}(\bar{x}) &:= i_{\text{Morse}}\big(D^2\mathcal{U}_\Xi(\bar{x})\big). \end{aligned}$$

Thus we have the following theorem.

THEOREM 2.21. *Let $\bar{x}$ be a critical point of the augmented potential function $\mathcal{U}_\Xi$ defined in (2.2.7) and assume that $\nu(\bar{x})$ is even. If $i_{\text{Morse}}(\bar{x})$ is odd, then the relative equilibrium corresponding to $\bar{x}$ is spectrally unstable.*

Proof. Let $\mathcal{H} := \mathbb{C}^{4n}$ and define the path $D : [0, +\infty) \to \mathscr{B}^{sa}(\mathcal{H})$ as

$$D(t) := B + tG$$

with $G := iJ$, as in the previous section. We prove the contrapositive of the statement, that is, we show that if the relative equilibrium corresponding to the given critical point $\bar{x}$ is spectrally stable then its Morse index $i_{\text{Morse}}(\bar{x})$ is even. Thus, assuming spectral stability, Theorem 2.18 immediately yields

$$n^-(B) \equiv 0 \mod 2.$$

Now, by Sylvester's Law of Inertia, we observe that

$$n^-(B) = n^-(N),$$

where N is given by (2.4.1), and since $n^-(N) = 2n - i_{\text{Morse}}(\bar{x}) - \nu(\bar{x})$, it directly follows that

$$i_{\text{Morse}}(\bar{x}) \equiv 0 \mod 2. \qquad \square$$

The next corollary is an immediate consequence of the previous theorem.

Corollary 2.22. *Let $\bar{x}$ be a critical point of the augmented potential function $\mathcal{U}_\Xi$. If $i_{\text{Morse}}(\bar{x})$ or $\nu(\bar{x})$ are odd then the corresponding relative equilibrium is linearly unstable.*

Remark 2.23. Assuming linear stability we have that $\nu(\bar{x}) = \nu(JB)$, which is even due to the diagonalisability of JB.

3 AN IMPORTANT APPLICATION: GENERALISED n-BODY PROBLEMS

Contents

With reference to the notation and the setting outlined at the beginning of Section 2.2, we define two generalised n-body problems by specifying two potential functions as follows. For each pair of indices $i, j \in \{1, \dots, n\}$, $i \neq j$, we let Δ_{ij} denote the *collision set of the i-th and j-th particles*

$$\Delta_{ij} \coloneqq \{\, q \in \mathbb{R}^{2n} \mid q_i = q_j \,\};$$

we call $\Delta \coloneqq \bigcup_{i,j=1}^{n} \Delta_{ij}$ the *collision set* (by definition, then, Δ is a union of hyperplanes) and

$$X \coloneqq \mathbb{R}^{2n} \setminus \Delta = \{\, q \in \mathbb{R}^{2n} \mid q_i \neq q_j \ \forall i \neq j \,\}$$

the *(collision-free) configuration space*.

On this set (which is a cone in $\mathbb{R}^{2n}$) we define the *potential functions* $U_\alpha, U_{\log} : X \to \mathbb{R}$ (generally denoted by U) as in (1.1) by setting

$$U_\alpha(q) \coloneqq \sum_{\substack{i,j=1\\ i<j}}^{n} \frac{m_i m_j}{|q_i - q_j|^\alpha}, \qquad \alpha \in (0, 2);$$

$$U_{\log}(q) \coloneqq \sum_{\substack{i,j=1\\ i<j}}^{n} m_i m_j \log \frac{1}{|q_i - q_j|}.$$

From now on, unless otherwise specified, every reference to the contents of Section 2.2 will be intended as concerning these two potential, i.e. we consider $\mathcal{U} = U$.

Remark 3.1. Note that for $\alpha = 1$ one finds the gravitational potential of the classical n-body problem. Moreover, the logarithmic potential can be considered as a limit case of the α-homogeneous[1] one, in the following sense:

$$\frac{U_\alpha(q)-1}{\alpha} \sim U_{\log}(q), \qquad \alpha \to 0^+,$$

for every $q \in X$. Nevertheless, it displays quite a different behaviour with respect to U_α, as we shall show.

Since the centre of mass of the system moves with uniform rectilinear motion, without loss of generality we can fix it at the origin. We thus consider the *reduced (collision-free) configuration space* as follows:

$$\widehat{X} := \left\{ q \in X \;\middle|\; \sum_{i=1}^{n} m_i q_i = 0 \right\}.$$

Remark 3.2. We observe that the Hamiltonian flow of System (2.2.3) is well defined on $T^*\widehat{X}$ but it is not complete on $T^*\mathbb{R}^{2n}$, due to the existence of solutions for which the potential escapes to infinity in a finite time. This happens, for instance, for initial conditions leading to a collision between two or more particles.

3.1 CENTRAL CONFIGURATIONS AND RELATIVE EQUILIBRIA

We recall here some well-known facts about central configurations and fix our notation. For further references in the classical gravitational case, we refer to [Moe94].

Let $a, b \in \mathbb{R}$, with $a < b$. We call $\bar{q} \in \widehat{X}$ a *(planar) central configuration* if there is some smooth real-valued function $r : (a, b) \to \mathbb{R}$, with $r(t) > 0$ for all $t \in (a, b)$, such that

$$q(t) := r(t)\,\bar{q} \tag{3.1.1}$$

is a (classical) solution of Newton's Equations (2.2.1). Here $\bar{q}$ represents the constant shape of the configuration, while $r(t)$ its time-depending size. By substituting (3.1.1) into (2.2.1) we obtain:

α-homogeneous case:

$$\ddot{r} M \bar{q} = r^{-(\alpha+1)} \nabla U_\alpha(\bar{q}).$$

Taking the scalar product with $\bar{q}$ in both sides of the above equality and applying Euler's theorem on homogeneous functions, we get $\ddot{r} = -\lambda_\alpha / r^{\alpha+1}$, where

$$\lambda_\alpha := \frac{\alpha U_\alpha(\bar{q})}{\mathcal{I}(\bar{q})}. \tag{3.1.2}$$

1 The α-homogeneous potential is actually homogeneous of degree $-\alpha$; however, we call it in this way for the sake of simplicity.

Logarithmic case:

$$\ddot{r}M\bar{q} = r^{-1}\nabla U_{\log}(\bar{q}).$$

Taking again the scalar product with $\bar{q}$ as before, we get $\ddot{r} = -\lambda_{\log}/r$, where

$$\lambda_{\log} := -\frac{\langle \nabla U_{\log}(\bar{q}), \bar{q}\rangle}{\mathcal{I}(\bar{q})}.$$

A straightforward computation shows that $-\langle \nabla U_{\log}(\bar{q}), \bar{q}\rangle = \sum_{\substack{i,j=1\\ i<j}}^{n} m_i m_j =: \mathcal{M}$, so that

$$\lambda_{\log} = \frac{\mathcal{M}}{\mathcal{I}(\bar{q})}. \tag{3.1.3}$$

REMARK 3.3. It is worth noting that in the logarithmic case the Lagrange multiplier depends only on the size of the central configuration (via the moment of inertia) and not on its shape.

In both cases, a central configuration $\bar{q}$ satisfies the *central configurations equation*

$$M^{-1}\nabla U(\bar{q}) + \lambda\bar{q} = 0, \tag{3.1.4}$$

where $\lambda = \lambda_\alpha$ (resp. $\lambda = \lambda_{\log}$) when $U = U_\alpha$ (resp. $U = U_{\log}$). Thus we can also look at a central configuration as a special distribution of the bodies in which the acceleration vector of each particle lines up with its position vector, and the proportionality constant λ is the same for all particles. Equation (3.1.4) is a quite complicated system of nonlinear algebraic equations and only few solutions are known.

Let us now introduce the *ellipsoid of inertia* (also called the *standard ellipsoid*)

$$\mathcal{S} := \left\{ q \in \widehat{X} \;\middle|\; \mathcal{I}(q) = 1 \right\}.$$

If $\bar{q}$ is a central configuration, then so are $c\bar{q}$ and $R\bar{q}$, for any $c \in \mathbb{R}\setminus\{0\}$ and any $2n \times 2n$ block-diagonal matrix R with blocks given by a 2×2 fixed matrix in $SO(2)$. We observe that the rescaled configuration $c\bar{q}$ solves a system analogous to (3.1.4) obtained by replacing λ_α with $\widetilde{\lambda}_\alpha := \lambda_\alpha |c|^{-(\alpha+2)}$ and $\lambda_{\log}$ with $\widetilde{\lambda}_{\log} := \lambda_{\log}|c|^{-2}$. Because of these facts, it is standard practice to count central configurations by fixing a constant c (the "scale": this actually means to work on $\mathcal{S}$) and to identify all those which are rotationally equivalent. This amounts to take the quotient of the configuration space $\widehat{X}$ with respect to homotheties and rotations about the origin, or, which is the same, to consider the so-called *shape sphere*

$$\mathbb{S} := \mathcal{S}/SO(2).$$

Note that the second equation of System (2.2.6) (with $\mathcal{U} = U$) is precisely the Central Configurations Equation (3.1.4), with the square modulus of the angular velocity as Lagrange multiplier. Intuitively speaking, then, if we let n bodies, distributed in a planar central configuration, rotate with an angular velocity ω equal to $\sqrt{\lambda_\alpha}$ or $\sqrt{\lambda_{\log}}$ (depending on the potential they are subject to), we get a relative equilibrium, which becomes an equilibrium in a uniformly rotating coordinate system.

Motivated by the observation that one can write, for every $q \in \widehat{X}$,

$$U(q) = U\left(\sqrt{\mathcal{I}(q)}\frac{q}{\sqrt{\mathcal{I}(q)}}\right) = \begin{cases} \mathcal{I}^{-\frac{\alpha}{2}}(q)\, U_\alpha\left(\dfrac{q}{\sqrt{\mathcal{I}(q)}}\right) & \text{if } U = U_\alpha \\[2ex] U_{\log}\left(\dfrac{q}{\sqrt{\mathcal{I}(q)}}\right) - \dfrac{\mathcal{M}}{2}\log \mathcal{I}(q) & \text{if } U = U_{\log} \end{cases}$$

we define, as in [BS08], the maps $f_\alpha, f_{\log} : \widehat{X} \to \mathbb{R}$ respectively as

$$f_\alpha(q) := \mathcal{I}^{\frac{\alpha}{2}}(q) U_\alpha(q) \qquad \text{and} \qquad f_{\log}(q) := U_{\log}(q) + \frac{\mathcal{M}}{2}\log \mathcal{I}(q),$$

so that, restricting to the ellipsoid of inertia $\mathcal{S}$, we have

$$f_\alpha(q) = U_\alpha\big|_{\mathcal{S}}(q) \qquad \text{and} \qquad f_{\log}(q) = U_{\log}\big|_{\mathcal{S}}(q), \qquad \forall q \in \mathcal{S}.$$

The reason for introducing these functions lies in the fact that we want to find the critical points of the potentials $U_\alpha, U_{\log}$ *constrained to* $\mathcal{S}$: we shall now show that it is possible to compute them more easily as *free* critical points of f_α and $f_{\log}$. Since the manifold $\mathcal{S}$ is topologically a sphere, we can avoid the use of the covariant derivative for this purpose.

For every $(q, v) \in T\widehat{X}$ we calculate, in the standard basis of $\mathbb{R}^{2n}$,

$$\langle \nabla f_\alpha(q), v\rangle = \frac{\alpha}{2}\mathcal{I}^{\frac{\alpha}{2}-1}(q) U_\alpha(q)\langle \nabla \mathcal{I}(q), v\rangle + \mathcal{I}^{\frac{\alpha}{2}}(q)\langle \nabla U_\alpha(q), v\rangle, \tag{3.1.5a}$$

$$\langle \nabla f_{\log}(q), v\rangle = \langle \nabla U_{\log}(q), v\rangle + \frac{\mathcal{M}}{2\mathcal{I}(q)}\langle \nabla \mathcal{I}(q), v\rangle. \tag{3.1.5b}$$

Now, recalling that $\mathcal{I}(q) = 1$ on $\mathcal{S}$ and that $\langle \nabla \mathcal{I}(q), v\rangle = \langle 2Mq, v\rangle$, we obtain, for every $q \in \mathcal{S}$ and every $v \in T_q\widehat{X}$:

$$\langle \nabla U_\alpha\big|_{\mathcal{S}}(q), v\rangle = \langle \nabla U_\alpha(q), v\rangle + \alpha U_\alpha(q)\langle Mq, v\rangle, \tag{3.1.6a}$$
$$\langle \nabla U_{\log}\big|_{\mathcal{S}}(q), v\rangle = \langle \nabla U_{\log}(q), v\rangle + \mathcal{M}\langle Mq, v\rangle. \tag{3.1.6b}$$

It is now clear, comparing Equations (3.1.4) and (3.1.6) and using (3.1.2) and (3.1.3), that the constrained critical points of the restricted potentials $U_\alpha|_{\mathcal{S}}$ and $U_{\log}|_{\mathcal{S}}$ are precisely the central configurations.

From Equations (3.1.5) we compute the Hessians of f_α and $f_{\log}$ for every $(q,v) \in T\widehat{X}$:

$$\begin{aligned}\langle D^2 f_\alpha(q)v,v\rangle = {} & \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\mathcal{I}^{\frac{\alpha}{2}-2}(q)U_\alpha(q)\langle\nabla\mathcal{I}(q),v\rangle^2 \\ & +\alpha\mathcal{I}^{\frac{\alpha}{2}-1}(q)\langle\nabla U_\alpha(q),v\rangle\langle\nabla\mathcal{I}(q),v\rangle \\ & +\frac{\alpha}{2}\mathcal{I}^{\frac{\alpha}{2}-1}(q)U_\alpha(q)\langle D^2\mathcal{I}(q)v,v\rangle \\ & +\mathcal{I}^{\frac{\alpha}{2}}(q)\langle D^2U_\alpha(q)v,v\rangle,\end{aligned} \tag{3.1.7a}$$

$$\begin{aligned}\langle D^2 f_{\log}(q)v,v\rangle = {} & \langle D^2U_{\log}(q)v,v\rangle \\ & +\frac{\mathcal{M}}{2}\left(-\frac{\langle\nabla\mathcal{I}(q),v\rangle^2}{\mathcal{I}^2(q)}+\frac{\langle D^2\mathcal{I}(q)v,v\rangle}{\mathcal{I}(q)}\right).\end{aligned} \tag{3.1.7b}$$

Assuming that $q \in \mathcal{S}$ is a central configuration for U_α (resp. for $U_{\log}$) and recalling that $\langle D^2\mathcal{I}(q)v,v\rangle = \langle 2Mv,v\rangle$, from (3.1.4) and (3.1.7) we obtain, for every $v \in T_q\widehat{X}$:

$$\begin{aligned}\langle D^2U_\alpha\big|_{\mathcal{S}}(q)v,v\rangle &= \langle D^2U_\alpha(q)v,v\rangle+\alpha U_\alpha(q)\langle Mv,v\rangle \\ &\quad -\alpha(\alpha+2)U_\alpha(q)\langle Mq,v\rangle^2, \\ \langle D^2U_{\log}\big|_{\mathcal{S}}(q)v,v\rangle &= \langle D^2U_{\log}(q)v,v\rangle+\mathcal{M}\{\langle Mv,v\rangle-\langle Mq,v\rangle^2\}.\end{aligned}$$

Choosing $v \in T_q\mathcal{S}$, these last expressions can be simplified, since the equality $\langle Mq,v\rangle = 0$ holds:

$$\langle D^2U_\alpha\big|_{\mathcal{S}}(q)v,v\rangle = \langle D^2U_\alpha(q)v,v\rangle+\alpha U_\alpha(q)\langle Mv,v\rangle, \tag{3.1.8a}$$

$$\langle D^2U_{\log}\big|_{\mathcal{S}}(q)v,v\rangle = \langle D^2U_{\log}(q)v,v\rangle+\mathcal{M}\langle Mv,v\rangle. \tag{3.1.8b}$$

Thus, for any central configuration $q \in \mathcal{S}$, the previous equations ensure that the Hessians of the restrictions of U_α and $U_{\log}$ to $\mathcal{S}$ are restrictions to $T_q\mathcal{S}$ of quadratic forms defined on the whole $T_q\widehat{X}$.

Remark 3.4. The previous equations still hold unchanged also if we restrict the potentials to the shape sphere $\mathbb{S}$.

3.2 A SYMPLECTIC DECOMPOSITION OF THE PHASE SPACE

We continue our analysis by presenting here a symplectic splitting of the phase space which reflects the invariance of the n-body-type problems under some isometries. There are three components: the first one, denoted by E_1, represents the translational invariance, E_2 is the subspace generated by all rotations and dilations of the central configuration and the third one, E_3, is the symplectic complement of the other two. The reason behind this construction is that, due to the existence of the first integrals, there are eight eigenvalues of the linearised matrix which are always present, independently of the number of bodies n: accordingly, we isolate them and focus only on the remaining $4n-8$, the ones holding the heart of the dynamics.

When linearising around a relative equilibrium $\bar{\zeta}$, the $4n \times 4n$ Hamiltonian matrix associated with System (2.2.9) is

$$L := -JB = \begin{pmatrix} \omega K & M^{-1} \\ D^2U(\bar{x}) & \omega K \end{pmatrix}, \tag{3.2.1}$$

where, we recall, each block is a square matrix of size $2n \times 2n$. Since it will be necessary, in the following, to know the explicit expressions of the Hessians of the two potentials U_α and $U_{\log}$, we write them down here:

$$\begin{aligned} D^2U_\alpha(x) &=: \left(S_{ij}^{(\alpha)}\right), \\ D^2U_{\log}(x) &=: \left(S_{ij}^{(\log)}\right), \end{aligned}$$

with

$$\begin{cases} S_{ij}^{(\alpha)} := \alpha \dfrac{m_i m_j}{|x_i - x_j|^{\alpha+2}} \left[I_2 - (\alpha+2) u_{ij} u_{ij}^T\right] & \text{if } j \neq i \\ S_{ii}^{(\alpha)} := -\displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} S_{ij}^{(\alpha)} \end{cases} \tag{3.2.2a}$$

and

$$\begin{cases} S_{ij}^{(\log)} := \dfrac{m_i m_j}{|x_i - x_j|^{2}} \left[I_2 - 2 u_{ij} u_{ij}^T\right] & \text{if } j \neq i \\ S_{ii}^{(\log)} := -\displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{n} S_{ij}^{(\log)} \end{cases} \tag{3.2.2b}$$

where $u_{ij} := \dfrac{x_i - x_j}{|x_i - x_j|}$ and the indices i and j vary in $\{1, \dots, n\}$.

Going back to the linearisation, we note that the first integrals of motion and the symmetries of the system generate two linear symplectic subspaces of the phase space $T^*X \cong X \times \mathbb{R}^{2n}$ which are invariant under L. Indeed, a basis for the position and momentum of the centre of mass is given by the four vectors in $\mathbb{R}^{4n}$

$$v_1 := \begin{pmatrix} v \\ 0 \end{pmatrix}, \quad v_2 := \begin{pmatrix} Kv \\ 0 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 0 \\ Mv \end{pmatrix}, \quad v_4 := \begin{pmatrix} 0 \\ KMv \end{pmatrix}$$

with $v := (1,0,1,0,\dots,1,0)^T \in \mathbb{R}^{2n}$. If we let E_1 denote the space spanned by these vectors, with the following computations we see that it is L-invariant:

$$\begin{aligned}
Lv_1 &= \begin{pmatrix} \omega K v \\ D^2U(\bar{x})v \end{pmatrix} = \omega v_2 + \begin{pmatrix} 0 \\ D^2U(\bar{x})v \end{pmatrix} = \omega v_2, \\
Lv_2 &= \begin{pmatrix} \omega K^2 v \\ D^2U(\bar{x})Kv \end{pmatrix} = -\omega v_1 + \begin{pmatrix} 0 \\ D^2U(\bar{x})Kv \end{pmatrix} = -\omega v_1, \\
Lv_3 &= \begin{pmatrix} M^{-1}Mv \\ \omega KMv \end{pmatrix} = v_1 + \begin{pmatrix} 0 \\ \omega KMv \end{pmatrix} = v_1 + \omega v_4, \\
Lv_4 &= \begin{pmatrix} M^{-1}KMv \\ \omega K^2 Mv \end{pmatrix} = v_2 + \begin{pmatrix} 0 \\ \omega K^2 Mv \end{pmatrix} = v_2 - \omega v_3,
\end{aligned}$$

since K and M commute and $D^2U(\bar{x})v = D^2U(\bar{x})Kv = 0$ for both U_α and $U_{\log}$, due to their matrix structure (3.2.2). The invariant space E_1 is also symplectic, because the standard symplectic form[2] $\Omega_1 := \Omega|_{E_1 \times E_1}$ of $(\mathbb{R}^{4n}, \Omega)$ restricted to E_1 is non-degenerate: we have indeed that $\Omega_1(v_1, v_3) = \langle Jv_1, v_3 \rangle = v^T M v \neq 0$ and $\Omega_1(v_2, v_4) = \langle Jv_2, v_4 \rangle = v^T M v \neq 0$. We denote by L_1 the restriction $L|_{E_1}$ of L to E_1; from the calculations performed above to show the invariance of E_1, it follows that it is given, in the basis (v_1, v_2, w_1, w_2), by the 4×4 matrix

$$L_1 := \begin{pmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{pmatrix}.$$

Its eigenvalues are $\pm i\omega$, each with algebraic multiplicity 2; however, the dimension of the associated eigenspaces is 1, and therefore L_1 is not diagonalisable. Note that the symplectic complement $E_1^{\perp\Omega}$ of E_1 is the space where the centre of mass of the system is fixed at the origin and the total linear momentum is zero.

The scaling symmetry and the conservation of the angular momentum generate another linear symplectic L-invariant subspace E_2, a basis of which is given by the four vectors in $\mathbb{R}^{4n}$

$$w_1 := \begin{pmatrix} \bar{x} \\ 0 \end{pmatrix}, \quad w_2 := \begin{pmatrix} K\bar{x} \\ 0 \end{pmatrix}, \quad w_3 := \begin{pmatrix} 0 \\ M\bar{x} \end{pmatrix}, \quad w_4 := \begin{pmatrix} 0 \\ KM\bar{x} \end{pmatrix}$$

To show that this is L-invariant, we compute:

$$\begin{aligned}
Lw_1 &= \begin{pmatrix} \omega K\bar{x} \\ D^2U(\bar{x})\bar{x} \end{pmatrix} = \omega w_2 + \begin{pmatrix} 0 \\ D^2U(\bar{x})\bar{x} \end{pmatrix} \\
&= \begin{cases} \omega w_2 + (\alpha+1)\omega^2 w_3 & \text{if } U = U_\alpha \\ \omega w_2 + \omega^2 w_3 & \text{if } U = U_{\log} \end{cases},
\end{aligned}$$

2 We take as standard symplectic form Ω on $\mathbb{R}^{4n}$ that one induced by the complex structure J_{4n} (denoted again by J).

$$Lw_2 = \begin{pmatrix} \omega K^2\bar{x} \\ D^2U(\bar{x})K\bar{x} \end{pmatrix} = -\omega w_1 + \begin{pmatrix} 0 \\ D^2U(\bar{x})K\bar{x} \end{pmatrix} = -\omega w_1 - \omega^2 w_4,$$

$$Lw_3 = \begin{pmatrix} M^{-1}M\bar{x} \\ \omega KM\bar{x} \end{pmatrix} = w_1 + \begin{pmatrix} 0 \\ \omega KM\bar{x} \end{pmatrix} = w_1 + \omega w_4,$$

$$Lw_4 = \begin{pmatrix} M^{-1}KM\bar{x} \\ \omega K^2M\bar{x} \end{pmatrix} = w_2 + \begin{pmatrix} 0 \\ \omega K^2M\bar{x} \end{pmatrix} = w_2 - \omega w_3.$$

The first relation is obtained from Euler's theorem on homogeneous functions applied to $\nabla U(x)$:

$$D^2U(x)x = D\big(\nabla U(x)\big)x = \begin{cases} -(\alpha+1)\nabla U_\alpha(x) & \text{if } U = U_\alpha \\ -\nabla U_{\log}(x) & \text{if } U = U_{\log} \end{cases}$$

and using the central configurations equation for relative equilibria. The second one comes from the invariance of the potentials under rotations: following [Moe94], we have indeed that $U\big(R(t)x\big) = U(x)$ for every $R(t) := e^{\omega K t}$, and differentiating this relation with respect to x we obtain[3] $DU\big(R(t)x\big)R(t) = DU(x)$. If we differentiate again with respect to t at $t = 0$ and divide by ω, we get

$$(Kx)^T D^2U(x) + DU(x)K = 0.$$

When $x = \bar{x}$ is a central configuration associated with a relative equilibrium, as it is in this case, $DU(\bar{x}) = -\omega^2(M\bar{x})^T$ and the equation above becomes, dividing both sides by ω:

$$(K\bar{x})^T D^2U(\bar{x}) - \omega^2(M\bar{x})^T K = 0,$$

or, equivalently,

$$(K\bar{x})^T D^2U(\bar{x}) + \omega^2(KM\bar{x})^T = 0.$$

Because of the symmetry of the Hessian it is now sufficient to take the transpose of both sides to conclude.

The space E_2 is again symplectic: the non-degeneracy of the form $\Omega_2 := \Omega|_{E_2\times E_2}$ can be verified in a way that is completely analogous to the one that we performed above for E_1. The matrices of $L|_{E_2}$ with respect to the basis (w_1, w_2, w_3, w_4) are

$$L_2^{(\alpha)} := \begin{pmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ (\alpha+1)\omega^2 & 0 & 0 & -\omega \\ 0 & -\omega^2 & \omega & 0 \end{pmatrix} \qquad \text{if } U = U_\alpha$$

and

$$L_2^{(\log)} := \begin{pmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ \omega^2 & 0 & 0 & -\omega \\ 0 & -\omega^2 & \omega & 0 \end{pmatrix} \qquad \text{if } U = U_{\log}.$$

3 We denote by $DU(x)$ the transpose of $\nabla U(x)$.

Their eigenvalues are 0 (with algebraic multiplicity 2) and $\pm i\omega\sqrt{2-\alpha}$ in the homogeneous case (U_α), and 0 (with algebraic multiplicity 2) and $\pm i\omega\sqrt{2}$ in the logarithmic case ($U_{\log}$). Again, these matrices are not diagonalisable because the eigenspace associated with 0 is only one-dimensional. Table 3.1 on the following page summarises the information obtained thus far about these first eight eigenvalues.

Thus, a relative equilibrium is always degenerate and not linearly stable in the classical sense. For this reason, we shall consider the restriction $L_3 := L|_{E_3}$ of L to the skew-orthogonal complement

$$E_3 := (E_1 \oplus E_2)^{\perp_\Omega}, \tag{3.2.3}$$

which is a linear symplectic subspace of dimension $4n-8$ of $\mathbb{R}^{4n}$. Following [Moe94], we adopt the following terminology.

Definition 3.5. A relative equilibrium is *non-degenerate* if the remaining $4n-8$ eigenvalues (relative to L_3) are different from 0; we say that it is *spectrally stable* if these eigenvalues are pure imaginary and *linearly stable* if, in addition to this condition of spectral stability, L_3 is diagonalisable.

In order to understand the structure of L_3, let us now consider the following change of variables:

$$\begin{cases} x \mapsto C\xi, \\ y^T \mapsto (C^{-1})^T \eta^T, \end{cases}$$

where C is a $2n \times 2n$ invertible matrix such that $[C, K] = 0$ and $C^T M C = I$. Then we have, for every $(x, y) \in T^*X$:

$$\begin{aligned} L\begin{pmatrix} x \\ y^T \end{pmatrix} &= \begin{pmatrix} \omega K & M^{-1} \\ D^2U(\bar{x}) & \omega K \end{pmatrix} \begin{pmatrix} C\xi \\ (C^{-1})^T \eta^T \end{pmatrix} \\ &= \begin{pmatrix} \omega K C\xi + M^{-1}(C^{-1})^T \eta^T \\ D^2U(\bar{x}) C\xi + \omega K (C^{-1})^T \eta^T \end{pmatrix}. \end{aligned}$$

From the first condition on C we find that also $(C^{-1})^T$ commutes with K, while from the second one we get that $(C^{-1})^T = MC$, so that we can write

$$\begin{aligned} L\begin{pmatrix} x \\ y^T \end{pmatrix} &= \begin{pmatrix} C(\omega K\xi + \eta^T) \\ (C^{-1})^T (C^T D^2U(\bar{x}) C\xi + \omega K \eta^T) \end{pmatrix} \\ &= \begin{pmatrix} C & 0 \\ 0 & (C^{-1})^T \end{pmatrix} \begin{pmatrix} \omega K & I \\ C^T D^2U(\bar{x}) C & \omega K \end{pmatrix} \begin{pmatrix} \xi \\ \eta^T \end{pmatrix}. \end{aligned}$$

The matrix C can be thought of as made up of 2×2 blocks of the form (b, Jb), for any vector $b \in \mathbb{R}^2$; furthermore, it can be shown [see MS05] that, using a Gram-Schmidt-type algorithm, the first four

Table 3.1. Eigenvalues of L_1 and L_2 for both potentials.

Potential		Eigenvalue	Multiplicity
U_α	$L_1^{(\alpha)}$	$i\omega$	2
		$-i\omega$	2
	$L_2^{(\alpha)}$	0	2
		$i\omega\sqrt{2-\alpha}$	1
		$-i\omega\sqrt{2-\alpha}$	1
$U_{\log}$	$L_1^{(\log)}$	$i\omega$	2
		$-i\omega$	2
	$L_2^{(\log)}$	0	2
		$i\omega\sqrt{2}$	1
		$-i\omega\sqrt{2}$	1

columns of C can be chosen as $(\nu, K\nu, \bar{x}, K\bar{x})$, where ν is, as before, the vector $(1, 0, \dots, 1, 0)^T \in \mathbb{R}^{2n}$. Looking now at the structures of (the first columns of) C and K, one can recover the restrictions L_1 and L_2 from the equation above and derive the expression for the $(4n-8) \times (4n-8)$ matrix representing L_3:

$$L_3 \coloneqq \begin{pmatrix} \omega K & I \\ \mathcal{D} & \omega K \end{pmatrix},$$

where every block has dimension $(2n-4) \times (2n-4)$ and $\mathcal{D}$ is the Hessian $C^T D^2 U(\bar{x}) C$ restricted to E_3, acting on the last $2n-4$ components of ξ. The study of the linear stability of the relative equilibrium $\bar{z}$ amounts then to determine whether or not this matrix is spectrally stable and/or diagonalisable.

3.3 AN EXAMPLE: THE EQUILATERAL TRIANGLE

It is easy to see that the Lagrangian triangle with equal masses is a central configuration both for the α-homogeneous potential and the logarithmic one. Indeed, both of them give rise to a central force field and the symmetry of a regular polygon is a sufficient condition for the bodies to satisfy Equation (3.1.4). We analyse here the behaviour of this relative equilibrium with respect to linear stability for both potentials.

For simplicity of computation we set

$$m_1 \coloneqq m_2 \coloneqq m_3 \coloneqq 1$$

in both situations. The centre of mass is fixed at the origin and the setting is as described previously, specially in Section 2.2.1.

In the α-homogeneous case we have that $\omega = \sqrt{3\alpha}$, hence the matrix $L^{(\alpha)}$ of the linearised problem is

$$L^{(\alpha)} = \begin{pmatrix}
0 & -\sqrt{3\alpha} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{3\alpha} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \sqrt{3\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3\alpha} & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \sqrt{3\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\frac{1}{2}a & 0 & -\frac{1}{4}a & -\frac{\sqrt{3}}{4}b & -\frac{1}{4}a & \frac{\sqrt{3}}{4}b & 0 & -\sqrt{3\alpha} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2}c & -\frac{\sqrt{3}}{4}b & -\frac{1}{4}c & \frac{\sqrt{3}}{4}b & -\frac{1}{4}c & \sqrt{3\alpha} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4}a & -\frac{\sqrt{3}}{4}b & \frac{1}{2}\alpha+\frac{5}{4}\alpha^2 & \frac{\sqrt{3}}{4}b & -\alpha(\alpha+1) & 0 & 0 & 0 & 0 & -\sqrt{3\alpha} & 0 & 0 \\
-\frac{\sqrt{3}}{4}b & -\frac{1}{4}c & \frac{\sqrt{3}}{4}b & -\frac{1}{2}\alpha+\frac{3}{4}\alpha^2 & 0 & \alpha & 0 & 0 & \sqrt{3\alpha} & 0 & 0 & 0 \\
-\frac{1}{4}a & \frac{\sqrt{3}}{4}b & -\alpha(\alpha+1) & 0 & \frac{1}{2}\alpha+\frac{5}{4}\alpha^2 & -\frac{\sqrt{3}}{4}b & 0 & 0 & 0 & 0 & 0 & -\sqrt{3\alpha} \\
\frac{\sqrt{3}}{4}b & -\frac{1}{4}c & 0 & \alpha & -\frac{\sqrt{3}}{4}b & -\frac{1}{2}\alpha+\frac{3}{4}\alpha^2 & 0 & 0 & 0 & 0 & \sqrt{3\alpha} & 0
\end{pmatrix},$$

where $a := \alpha(\alpha-2)$, $b := \alpha(\alpha+2)$ and $c := \alpha(3\alpha+2)$. Its eigenvalues are

$$\begin{aligned}
\lambda_1 &:= i\sqrt{3\alpha}, & \lambda_5 &:= 0, \\
\lambda_2 &:= -i\sqrt{3\alpha}, & \lambda_6 &:= 0, \\
\lambda_3 &:= i\sqrt{3\alpha}, & \lambda_7 &:= i\sqrt{3\alpha(2-\alpha)}, \\
\lambda_4 &:= -i\sqrt{3\alpha}, & \lambda_8 &:= -i\sqrt{3\alpha(2-\alpha)},
\end{aligned}$$

$$\begin{aligned}
\lambda_9 &:= \frac{1}{2}\sqrt{6\alpha^2 + 12\alpha(i\sqrt{2\alpha}-1)}, \\
\lambda_{10} &:= -\frac{1}{2}\sqrt{6\alpha^2 + 12\alpha(i\sqrt{2\alpha}-1)}, \\
\lambda_{11} &:= \frac{1}{2}\sqrt{6\alpha^2 - 12\alpha(i\sqrt{2\alpha}+1)}, \\
\lambda_{12} &:= -\frac{1}{2}\sqrt{6\alpha^2 - 12\alpha(i\sqrt{2\alpha}+1)}.
\end{aligned}$$

The first four are those relative to the subspace E_1, the second four are related to the subspace E_2 and the last four are linked to the essential part of the dynamics, the subspace E_3. It is immediate to see that, for any value of $\alpha \in (0,2)$, none of these last four eigenvalues is pure imaginary: their square is indeed a complex number with non-zero imaginary part, and not a negative real number as it should be. Therefore we conclude that the equilateral triangle is spectrally (hence linearly) unstable for every $\alpha \in (0,2)$. This accords with the fact that every regular polygon is linearly unstable in the gravitational case, as showed by Moeckel in [Moe95]. We also verified (only for $\alpha = 1/2$ and $\alpha = 1$) that the matrix $L_3^{(\alpha)}$ is diagonalisable; unfortunately, due to lack of computational power, we could not check if this property is maintained for every other value of the homogeneity parameter in the range of investigation.

As for the logarithmic potential, the angular velocity of the bodies is $\omega = \sqrt{3}$ and the matrix of the linearisation is the following:

$$L^{(\log)} = \begin{pmatrix}
0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} & -1 & 0 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\
-\frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 1 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{\sqrt{3}}{2} & -1 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & \sqrt{3} & 0
\end{pmatrix}.$$

Its eigenvalues are

$$\begin{aligned}
&\lambda_1 := i\sqrt{3}, && \lambda_5 := 0, && \lambda_9 := i\sqrt{3}, \\
&\lambda_2 := -i\sqrt{3}, && \lambda_6 := 0, && \lambda_{10} := -i\sqrt{3}, \\
&\lambda_3 := i\sqrt{3}, && \lambda_7 := i\sqrt{6}, && \lambda_{11} := i\sqrt{3}, \\
&\lambda_4 := -i\sqrt{3}, && \lambda_8 := -i\sqrt{6}, && \lambda_{12} := -i\sqrt{3}
\end{aligned}$$

and as before the last four are connected to the essential subspace E_3. Here it is clear that the relative equilibrium is spectrally stable, since every eigenvalue is pure imaginary. Nevertheless, it is not linearly stable, because the matrix $L_3^{(\log)}$ is not diagonalisable.

This simple example shows the deep contrast between the α-homogeneous potential and the logarithmic one, as well as their similarities: in both cases, indeed, there is linear instability, but for opposite reasons.

3.4 LINEAR INSTABILITY RESULTS

We now present a theorem on spectral (hence linear) instability of relative equilibria, valid both in the α-homogenous and in the logarithmic case. This constitutes an improvement, even in the gravitational case ($\alpha = 1$), of the result found by X. Hu and S. Sun in [HS09b]. Since their proof was only sketched, we provide here a complete demonstration and, at the same time, we show that it holds for more general singular potentials. In what follows U can be indifferently substituted by U_α or $U_{\log}$.

Let $B_3 \in \mathrm{Mat}(4n-8, \mathbb{R})$ be the restriction of the $4n \times 4n$ matrix B of System (2.2.9) to the invariant symplectic subspace E_3 of the phase space defined by (3.2.3). It can be written as

$$B_3 := \begin{pmatrix} -\mathcal{D} & \omega K^T \\ \omega K & I \end{pmatrix},$$

where each block is of dimension $(2n-4)\times(2n-4)$ and $\mathcal{D}$ is the restriction of $C^T D^2 U(\bar{x}) C$ to E_3. Following the authors in [HS09b], we have:

$$\begin{pmatrix} I & \omega K \\ 0 & I \end{pmatrix}\begin{pmatrix} -\mathcal{D} & \omega K^T \\ \omega K & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -\omega K & I \end{pmatrix} = \begin{pmatrix} -\left(\mathcal{D}+\omega^2 I\right) & 0 \\ 0 & I \end{pmatrix} =: N_3. \tag{3.4.1}$$

Note that

$$\mathcal{D}+\omega^2 I := C^T D^2 U(\bar{x}) C\big|_{E_3} + \omega^2 I = C^T\left(D^2 U(\bar{x}) + \omega^2 M\right) C\big|_{E_3}$$

is precisely the Hessian of $U|_S$ evaluated at the central configuration $\bar{x}$ (cf. Equations (3.1.8), keeping in mind that $\omega^2 = \lambda_\alpha$ if $U = U_\alpha$ and $\omega^2 = \lambda_{\log}$ if $U = U_{\log}$) and restricted to E_3.

Define then the nullity and the Morse index of $\bar{x}$ as

$$\nu(\bar{x}) := \nu\left(\mathcal{D}+\omega^2 I\right)$$

and

$$i_{\mathrm{Morse}}(\bar{x}) := i_{\mathrm{Morse}}\left(\mathcal{D}+\omega^2 I\right),$$

respectively.

Theorem 3.6. *Let $\bar{x} \in S$ be a central configuration for U_α or $U_{\log}$ such that its nullity $\nu(\bar{x})$ is even. If $i_{\mathrm{Morse}}(\bar{x})$ is odd, then the corresponding relative equilibrium is spectrally unstable.*

Proof. Let $\mathcal{H} := \mathbb{C}^{4n-8}$ and define the path $D : [0,+\infty) \to \mathscr{B}^{sa}(\mathcal{H})$ as

$$D(t) := B_3 + tG$$

with $G := iJ$, as above. The proof is then completely analogous to that of Theorem 2.21, taking into account (3.4.1) rather than (2.4.1). □

Remark 3.7. A case occurring quite frequently is $\nu(\bar{x}) = 0$: this happens, for instance, in regular n-gons, at least for small values of n.

An immediate corollary of this theorem is the following, which, in the gravitational case $\alpha = 1$, is the main result of [HS09b].

Corollary 3.8. *Let $\bar{x} \in S$ be a central configuration for U_α or $U_{\log}$. If $i_{\mathrm{Morse}}(\bar{x})$ or $\nu(\bar{x})$ are odd then the corresponding relative equilibrium is linearly unstable.*

We shall now derive a useful condition to detect spectral instability of a relative equilibrium utilising only the associated central configuration. Consider again the matrix L of the linearised problem given by Equation (3.2.1). In the wake of [Rob99], we study the eigenvalue problem $Lu = \lambda u$, with $\lambda \in \mathbb{C}$ and $u := \left(\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right)$ belonging to $\mathbb{C}^{2n} \times \mathbb{C}^{2n}$ (both u_1 and u_2 are column vectors):

$$Lu := \begin{pmatrix} \omega K & M^{-1} \\ D^2U(\bar{x}) & \omega K \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix},$$

which corresponds to the system

$$\begin{cases} u_2 = M(\lambda I - \omega K)u_1 \\ Pu_1 = 0, \end{cases}$$

where

$$P := M^{-1}D^2U(\bar{x}) + (\omega^2 - \lambda^2)I + 2\lambda\omega K.$$

Thus, in order to compute the eigenvalues of L it is enough to find those of P.

Note that the diagonal 2×2 blocks of P are of the form

$$\begin{pmatrix} d_{ii} + \omega^2 - \lambda^2 & d_{i,i+1} - 2\lambda\omega \\ d_{i+1,i} + 2\lambda\omega & d_{i+1,i+1} + \omega^2 - \lambda^2 \end{pmatrix},$$

where the d_{ij}'s are the entries of the symmetric matrix $M^{-1}D^2U(\bar{x})$ — hence i is odd. The determinant of each diagonal block is (setting $\mu := \lambda^2$)

$$\mu^2 + (2\omega^2 - d_{ii} - d_{i+1,i+1})\mu + (d_{ii} + \omega^2)(d_{i+1,i+1} + \omega^2),$$

so that we have

$$\det P = \mu^{2n} + \left(2n\omega^2 - \operatorname{tr}\left[M^{-1}D^2U(\bar{x})\right]\right)\mu^{2n-1} + \cdots, \tag{3.4.2}$$

because the only contribution to the coefficient of μ^{2n-1} comes from the diagonal blocks. Now, since the characteristic polynomial of L is even (being L Hamiltonian), from Equation (3.4.2) we can derive an expression for the sum of the squares of its roots, i.e. the eigenvalues λ_i of L:

$$\sum_{i=1}^{2n} \mu_i = \sum_{i=1}^{2n} (\lambda^2)_i = \frac{1}{2}\sum_{i=1}^{4n} (\lambda_i)^2 = \operatorname{tr}\left[M^{-1}D^2U(\bar{x})\right] - 2n\omega^2.$$

Recalling the structure of the Hessians of the potentials (3.2.2), we obtain

$$\operatorname{tr}\left[M^{-1}D^2U(\bar{x})\right] = \begin{cases} \displaystyle\sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{\alpha^2(m_i + m_j)}{|\bar{x}_i - \bar{x}_j|^{\alpha+2}} & \text{if } U = U_\alpha \\[2em] 0 & \text{if } U = U_{\log}. \end{cases}$$

The computation is easily done, noting that $\mathrm{tr}(u_{ij}u_{ij}^T) = 1$:

$$\begin{aligned}\mathrm{tr}\left[M^{-1}D^2U_\alpha(\bar{x})\right] &= \sum_{i=1}^{n}\left\{-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\alpha m_j}{|\bar{x}_i-\bar{x}_j|^{\alpha+2}}\left[2-(\alpha+2)\right]\right\}\\ &= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\alpha^2 m_j}{|\bar{x}_i-\bar{x}_j|^{\alpha+2}} = \sum_{\substack{i,j=1\\ i<j}}^{n}\frac{\alpha^2(m_i+m_j)}{|\bar{x}_i-\bar{x}_j|^{\alpha+2}},\end{aligned}$$

$$\mathrm{tr}\left[M^{-1}D^2U_{\log}(\bar{x})\right] = \sum_{i=1}^{n}\left\{-\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{m_j}{|\bar{x}_i-\bar{x}_j|^{2}}\left[2-2\right]\right\} = 0. \qquad (3.4.3)$$

This discussion proves the following claim.

THEOREM 3.9. *Let $\bar{z} := (\bar{x}^T, \bar{y})^T$, with $\bar{x} \in \mathcal{S}$ a central configuration, be a relative equilibrium for System (2.2.8) related to U_α (resp. $U_{\log}$), with angular velocity $\omega = \sqrt{\alpha U_\alpha(\bar{x})}$ (resp. $\omega = \sqrt{\mathcal{M}}$), and let L be the matrix (3.2.1) of the associated linearised System (2.2.9), with eigenvalues λ_i ($i = 1, \ldots, 4n$). Then we have*

i) α-homogeneous case:

$$\sum_{i=1}^{4n}(\lambda_i)^2 = 2\alpha^2\sum_{\substack{i,j=1\\ i<j}}^{n}\frac{m_i+m_j}{|\bar{x}_i-\bar{x}_j|^{\alpha+2}} - 4n\alpha U_\alpha(\bar{x}); \qquad (3.4.4)$$

ii) Logarithmic case:

$$\sum_{i=1}^{4n}(\lambda_i)^2 = -4n\mathcal{M}.$$

For a relative equilibrium to be spectrally stable, its eigenvalues must be pure imaginary and therefore their squares must be non-positive. We know the first eight of them, listed in Table 3.1 on page 38: the sum of their squares in the α-homogeneous case is

$$\sum_{i=1}^{8}(\lambda_i)^2 = (2\alpha-8)\omega^2 = 2\alpha(\alpha-4)U_\alpha(\bar{x}). \qquad (3.4.5)$$

We are now in the position to formulate the following sufficient condition for spectral (hence linear) instability.

COROLLARY 3.10. *With the hypotheses of Theorem 3.9 (for $U = U_\alpha$), if the following inequality holds:*

$$\sum_{\substack{i,j=1\\ i<j}}^{n}\frac{m_i+m_j}{|\bar{x}_i-\bar{x}_j|^{\alpha+2}} > \frac{2n+\alpha-4}{\alpha}U_\alpha(\bar{x}) \qquad (3.4.6)$$

then the relative equilibrium $\bar{z}$ is spectrally unstable.

REMARK 3.11. Observe that the relative equilibrium may be degenerate, i.e. the matrix L_3 may have some zero eigenvalues. We rule out, however, the possibility of complete degeneracy ($L_3 = 0$): this would correspond indeed to a spectrally stable scenario.

Proof of Corollary 3.10. We prove the contrapositive statement: suppose that the relative equilibrium $\bar{z}$ is spectrally stable. This assumption implies that the sum of the squares of the remaining $4n - 8$ eigenvalues must be non-positive:

$$\sum_{i=9}^{4n} (\lambda_i)^2 \leqslant 0,$$

where equality corresponds to the completely degenerate case where all the eigenvalues of L_3 are equal to zero. Adding to both sides the first eight eigenvalues we obtain

$$\sum_{i=1}^{4n} (\lambda_i)^2 \leqslant \sum_{i=1}^{8} (\lambda_i)^2.$$

Therefore, by Equations (3.4.4) and (3.4.5), we get

$$2\alpha^2 \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{m_i + m_j}{\left|\bar{x}_i - \bar{x}_j\right|^{\alpha+2}} - 4n\alpha U_\alpha(\bar{x}) \leqslant 2\alpha(\alpha-4)U_\alpha(\bar{x}).$$

Solving for the summation yields the result. □

REMARK 3.12. Note that Corollary 3.10 provides a tool to detect spectral instability only for the α-homogeneous potential U_α. In the logarithmic case, indeed, it is not possible to derive a similar useful condition because of Equation (3.4.3). As a justification of this fact, if we let $\alpha \to 0^+$ in (3.4.6) we see that the left-hand side remains finite, as well as $U_\alpha(\bar{x})$, whereas the coefficient on the right-hand side tends to $+\infty$, thus shrinking the solution set of the inequality to $\varnothing$. This is not surprising, and is actually in accord with Remark 3.3.

As an example of application of Corollary 3.10, we examine regular n-gons (with $n \geqslant 3$, as before), employing Roberts' estimates in [Rob99]. For the sake of simplicity, set $m_j := 1$ for every $j \in \{1, \dots, n\}$ and let all the bodies lie at distance 1 from the origin of the reference frame, positioned at the vertices of a regular n-gon. In this way $\bar{x}_j = \left(\cos\frac{2j\pi}{n}, \sin\frac{2j\pi}{n}\right)^T$ denotes the position of the j-th body. Because of the symmetry of this configuration, we have that $\left|\bar{x}_i - \bar{x}_{i+j}\right| = \left|\bar{x}_n - \bar{x}_j\right|$ for all $i, j \in \{1, \dots, n\}$, where the indices are understood modulo n. Through elementary trigonometry we find

$$\left|\bar{x}_n - \bar{x}_j\right| = 2\sin\frac{j\pi}{n}, \qquad \forall\, j \in \{1, \dots, n-1\},$$

and after a few simplifications Inequality (3.4.6) becomes

$$\sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha+2}\left(\frac{j\pi}{n}\right)} - \frac{4n+2\alpha-8}{n\alpha} \sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha}\left(\frac{j\pi}{n}\right)} > 0,$$

where we have taken into account the moment of inertia, $\mathcal{I}(\bar{x}) = n$, so that $\bar{x} \in \mathcal{S}$. We make use of the following estimates:

$$\sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha+2}\left(\frac{j\pi}{n}\right)} \geqslant \frac{2}{\sin^{\alpha+2}\left(\frac{\pi}{n}\right)}, \qquad \sum_{j=1}^{n-1} \frac{1}{\sin^{\alpha}\left(\frac{j\pi}{n}\right)} \leqslant \frac{n-1}{\sin^{\alpha}\left(\frac{\pi}{n}\right)}$$

and impose the stronger condition

$$\frac{2}{\sin^{\alpha+2}\left(\frac{\pi}{n}\right)} - \frac{4n+2\alpha-8}{n\alpha} \left[\frac{n-1}{\sin^{\alpha}\left(\frac{\pi}{n}\right)}\right] > 0.$$

Collecting the common factor $1/\sin^{\alpha}\left(\frac{\pi}{n}\right)$, which is positive for every $n \geqslant 3$, this is equivalent to asking

$$\frac{2}{\sin^{2}\left(\frac{\pi}{n}\right)} - (n-1)\frac{4n+2\alpha-8}{n\alpha} > 0.$$

Exploiting the fact that $\frac{1}{\sin^2 x} > \frac{1}{x^2}$ for every $x \in \mathbb{R} \setminus \pi\mathbb{Z}$, we obtain the solution

$$\bar{\alpha}(n) := \frac{2\pi^2(n^2-3n+2)}{n^3-\pi^2 n+\pi^2} < \alpha < 2,$$

which is meaningful only for $n \geqslant 8$. Therefore, for every $n \geqslant 8$ we see that there exists a real number $\bar{\alpha}(n) \in (0,2)$ such that for any $\alpha \in \big(\bar{\alpha}(n), 2\big)$ the regular n-gon is spectrally unstable. Moreover, we observe that $\bar{\alpha}(n)$ monotonically tends to 0 as $n \to +\infty$.

4 MASLOV-TYPE INDEX THEORIES

Contents

The aim of this chapter is to briefly describe some Maslov-type index theories for paths of symplectic matrices as well as for paths of Lagrangian subspaces. In Section 4.1 we recall a geometric definition of the Maslov index for symplectic paths exploiting the intersection number of a curve and a singular cycle (an algebraic variety of codimension 1 in the symplectic group). Then, in Section 4.2, we recollect the basic definitions of the ω-index theory, essentially developed by Long and his school, and exhibit the relation with the geometric Maslov-type index. Our main sources for these two subsections are [CZ84; LZ90; LZ00] and references therein. Section 4.3 is devoted to a brief presentation of other Maslov-type index theories defined through a suitable intersection theory in the Lagrangian Grassmannian manifold by means of the crossing forms. We also show the relationship with the Maslov-type index theories previously introduced in the symplectic context. Our basic references for all this are [RS93; CLM94; Por08; GPP04a; Lon02; HS10; HS11; Arn67; PPT04; MPP05; Por10].

4.1 MASLOV-TYPE INDEX THEORY FOR SYMPLECTIC PATHS

Following Long and Zhu in [LZ00], we define for all $n \in \mathbb{N} \setminus \{0\}$ the *complex and real symplectic groups*

$$\mathrm{Sp}(2n,\mathbb{C}) := \left\{ M \in \mathrm{GL}(2n,\mathbb{C}) \,\middle|\, M^\dagger J M = J \right\}$$
$$\mathrm{Sp}(2n) := \mathrm{Sp}(2n,\mathbb{R}) := \left\{ M \in \mathrm{GL}(2n,\mathbb{R}) \,\middle|\, M^T J M = J \right\}$$

and for $0 \leqslant k \leqslant 2n$ we set

$$\mathrm{Sp}_k(2n,\mathbb{C}) := \{ M \in \mathrm{Sp}(2n,\mathbb{C}) \mid \dim_{\mathbb{C}} \ker_{\mathbb{C}}(M - I) = k \}$$
$$\mathrm{Sp}_k(2n) := \{ M \in \mathrm{Sp}(2n,\mathbb{R}) \mid \dim \ker(M - I) = k \}.$$

It is clear that one has the following stratifications:

$$\mathrm{Sp}(2n,\mathbb{C}) = \bigcup_{k=0}^{2n} \mathrm{Sp}_k(2n,\mathbb{C}), \qquad \mathrm{Sp}(2n) = \bigcup_{k=0}^{2n} \mathrm{Sp}_k(2n).$$

We recall the following well-known result, which gives the properties of the stratification.

Proposition 4.1. *The subsets* $\mathrm{Sp}_k(2n,\mathbb{C})$ *and* $\mathrm{Sp}_k(2n)$ *are, respectively speaking, smooth submanifolds of* $\mathrm{Sp}(2n,\mathbb{C})$ *and* $\mathrm{Sp}(2n)$, *with codimension* k^2 *and* $\frac{1}{2}k(k+1)$. *Moreover,* $\mathrm{Sp}_1(2n,\mathbb{C})$ *and* $\mathrm{Sp}_1(2n)$ *are co-oriented, the transverse orientation being given by the vector field* $\frac{d}{dt}(Me^{Jt})\big|_{t=0}$. *We have in addition that*

$$\overline{\mathrm{Sp}_k(2n,\mathbb{C})} = \bigcup_{l\geqslant k} \mathrm{Sp}_l(2n,\mathbb{C}) \qquad \text{and} \qquad \overline{\mathrm{Sp}_k(2n)} = \bigcup_{l\geqslant k} \mathrm{Sp}_l(2n).$$

By Proposition 4.1 the intersection points of the curve

$$\gamma(t) := Me^{Jt}, \qquad M \in \mathrm{Sp}_1(2n,\mathbb{C})$$

with the cycle $\overline{\mathrm{Sp}_1(2n,\mathbb{C})}$ form a discrete subset of $\gamma(\mathbb{R})$. We recall that a matrix in $\mathrm{Sp}(2n,\mathbb{C})$ is called *non-degenerate* if it does not admit 1 as an eigenvalue. A straightforward computation allows to see that for a continuous path $\gamma : [a,b] \to \mathrm{Sp}(2n,\mathbb{C})$ there exists $\delta > 0$ such that for any $\varepsilon \in (-\delta,\delta)\setminus\{0\}$ the (perturbed) path $t \mapsto \gamma(t)e^{-\varepsilon J}$ is non-degenerate, meaning that it has non-degenerate endpoints.

Definition 4.2. Let $\gamma : [a,b] \to \mathrm{Sp}(2n,\mathbb{C})$. We define its *geometric Maslov-type index* to be the intersection number of $t \mapsto \gamma(t)e^{-\varepsilon J}$ with $\overline{\mathrm{Sp}_1(2n,\mathbb{C})}$ for all $\varepsilon \in (0,\delta)$ (where δ is such that the perturbed path is non-degenerate).

$$i_{\mathrm{geo}}(\gamma) := \left[\gamma e^{-\varepsilon J} : \overline{\mathrm{Sp}_1(2n,\mathbb{C})}\right],$$

where the right-hand side is the usual homotopy intersection number.

For any $\omega \in \mathbb{U} := \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$ and $T > 0$ it is convenient to define the set

$$\mathscr{P}_T(2n) := \left\{\, \gamma \in \mathscr{C}^0\big([0,T];\mathrm{Sp}(2n,\mathbb{C})\big) \;\middle|\; \gamma(0) = I_{2n} \,\right\}$$

and its subset

$$\mathscr{P}^*_{T,\omega}(2n) := \left\{\, \gamma \in \mathscr{P}_T(2n) \;\middle|\; \gamma(T) \in \omega\mathrm{Sp}_0(2n,\mathbb{C}) \,\right\}.$$

Consider now two square matrices M_1 and M_2 of sizes $2m_1 \times 2m_1$ and $2m_2 \times 2m_2$ respectively (with $m_1, m_2 \in \mathbb{N}\setminus\{0\}$) such that they can both be written in the form

$$M_k := \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}, \qquad k = 1,2,$$

each block being of size $m_k \times m_k$. The *diamond product* of M_1 and M_2 is defined [see Lon02, p. 17] as the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix:

$$M_1 \diamond M_2 := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}. \tag{4.1.1}$$

The k-fold diamond product of M with itself is denoted by $M^{\diamond k}$. The *symplectic sum* of two paths $\gamma_j \in \mathscr{P}_T(2n_j)$, with $j = 1, 2$ and $n_1, n_2 \in \mathbb{N} \setminus \{0\}$, is defined in a natural way:

$$(\gamma_1 \diamond \gamma_2)(t) := \gamma_1(t) \diamond \gamma_2(t), \quad \forall t \in [0, T].$$

Here is a list pf the basic properties of the geometric Maslov-type index that we shall need later.

i) *Path additivity:* Let $\gamma : [a, b] \to \mathrm{Sp}(2n, \mathbb{C})$ and $c \in [a, b]$. Then

$$i_{geo}(\gamma) = i_{geo}(\gamma|_{[a,c]}) + i_{geo}(\gamma|_{[c,b]}).$$

ii) *$\diamond$-additivity:* Let $\gamma_1 : [a, b] \to \mathrm{Sp}(2k, \mathbb{C})$ and $\gamma_2 : [a, b] \to \mathrm{Sp}(2l, \mathbb{C})$ be two symplectic paths. Then we have

$$i_{geo}(\gamma_1 \diamond \gamma_2) = i_{geo}(\gamma_1) + i_{geo}(\gamma_2).$$

iii) *Homotopy invariance:* For any two paths γ_1 and γ_2, if γ_1 is homotopic to γ_2 (written $\gamma_1 \sim \gamma_2$) in $\mathrm{Sp}(2n, \mathbb{C})$ with either fixed or always non-degenerate endpoints, there holds

$$i_{geo}(\gamma_1) = i_{geo}(\gamma_2).$$

iv) *Normalisation:* If $n = 1$ then

$$i_{geo}(e^{it} I, t \in [0, a]) = \begin{cases} 1 & \text{if } a \in (0, 2\pi) \\ 0 & \text{if } a = 2\pi. \end{cases}$$

v) *Affine scale invariance:* For all $k > 0$ and $\gamma \in \mathscr{P}_{kT}(2n)$ we have

$$i_{geo}\big(\gamma(kt), t \in [0, \tau]\big) = i_{geo}\big(\gamma(t), t \in [0, k\tau]\big).$$

4.2 THE ω-INDEX THEORY AND THE ITERATION FORMULA

For any two continuous paths $\gamma, \delta : [0, T] \to \mathrm{Sp}(2n, \mathbb{C})$ such that $\gamma(T) = \delta(0)$, we define their *concatenation* $\gamma * \delta : [0, T] \to \mathrm{Sp}(2n, \mathbb{C})$ as

$$(\gamma * \delta)(t) := \begin{cases} \gamma(2t) & \text{if } 0 \leqslant t \leqslant \frac{T}{2} \\ \delta(2t - T) & \text{if } \frac{T}{2} \leqslant t \leqslant T. \end{cases}$$

For any $n \in \mathbb{N} \setminus \{0\}$ we also define a special continuous symplectic path $\xi_n : [0, T] \to \mathrm{Sp}(2n)$ as follows:

$$\xi_n(t) := \begin{pmatrix} 2 - \dfrac{t}{T} & 0 \\ 0 & \left(2 - \dfrac{t}{T}\right)^{-1} \end{pmatrix}^{\diamond n}, \qquad \forall\, t \in [0, T].$$

DEFINITION 4.3 ([Lon99; HS09a]). Let $\omega \in \mathbb{U}$. If $\gamma \in \mathscr{P}_T(2n)$, we define

$$\nu_\omega(\gamma) := \dim_{\mathbb{C}} \ker_{\mathbb{C}} \big(\gamma(T) - \omega I_{2n}\big).$$

If $\gamma \in \mathscr{P}^*_{T,\omega}(2n)$ the ω-*index* is defined as

$$i_\omega(\gamma) := \big[\overline{\omega}\gamma * \xi_n : \overline{\mathrm{Sp}_1(2n, \mathbb{C})}\big]. \tag{4.2.1}$$

If $\gamma \in \mathscr{P}_T(2n) \setminus \mathscr{P}^*_{T,\omega}(2n)$, we let $\mathscr{F}(\gamma)$ be the set of all open neighbourhoods U of γ in $\mathscr{P}_T(2n)$, and define

$$i_\omega(\gamma) := \sup_{U \in \mathscr{F}(\gamma)} \inf \left\{ i_\omega(\delta) \,\middle|\, \delta \in U \cap \mathscr{P}^*_{T,\omega}(2n) \right\}.$$

Finally the ω-*geometric Maslov index* is defined as

$$i_{\mathrm{geo},\omega}(\gamma) := \big[\overline{\omega}\gamma e^{-\varepsilon J} : \overline{\mathrm{Sp}_1(2n, \mathbb{C})}\big]. \tag{4.2.2}$$

The right-hand side of (4.2.1) and (4.2.2) is the usual homotopy intersection number, the orientation of $\overline{\omega}\gamma * \xi_n$ is its positive time direction under homotopies with fixed end-points and ε is a positive real number sufficiently small.

We list the basic properties of the ω-index that we need in the sequel.

i) *Lower semicontinuity:* For all $\gamma : [a, b] \to \mathscr{P}_T(2n)$ and $c \in [a, b]$ we have

$$i_\omega(\gamma) = \inf \left\{ i_\omega(\beta) \,\middle|\, \beta \in \mathscr{P}^*_T(2n) \text{ is sufficiently } \mathscr{C}^0\text{-close to } \gamma \right\}.$$

ii) $\diamond$-*additivity:* Let $\gamma_1 : [a, b] \to \mathrm{Sp}(2k, \mathbb{C})$ and $\gamma_2 : [a, b] \to \mathrm{Sp}(2l, \mathbb{C})$ be two symplectic paths. Then we have

$$i_\omega(\gamma_1 \diamond \gamma_2) = i_\omega(\gamma_1) + i_\omega(\gamma_2).$$

iii) *Homotopy invariance:* For any two paths γ_1 and γ_2, if $\gamma_1 \sim \gamma_2$ in $\mathrm{Sp}(2n, \mathbb{C})$ with either fixed or always non-degenerate endpoints, there holds

$$i_\omega(\gamma_1) = i_\omega(\gamma_2).$$

iv) *Affine scale invariance:* For all $k > 0$ and $\gamma \in \mathscr{P}_{kT}(2n)$, we have

$$i_\omega\big(\gamma(kt), t \in [0, T]\big) = i_\omega\big(\gamma(t), t \in [0, kT]\big).$$

The proofs of these properties are consequences of [LZ00, Lemma 2.2 (3), Corollary 2.1, Theorem 2.1] and of the index theory contained in [Lon99].

Let $\gamma \in \mathscr{P}_T(2n)$ and $m \in \mathbb{N} \setminus \{0\}$. The m-*th iteration of* γ is $\gamma^m : [0, mT] \to \mathrm{Sp}(2n)$ defined as

$$\gamma^m(t) \coloneqq \gamma(t - jT)\big(\gamma(T)\big)^j, \quad \text{for } jT \leqslant t \leqslant (j+1)T, \quad j = 0, \dots, m-1.$$

The next Bott-type iteration formula is crucial in order to study the geometric multiplicity of periodic orbits and plays a big role in the question of linear stability.

LEMMA 4.4 (Bott-Long iteration formula, [Lon02, Theorem 9.2.1]). *For any* $z \in \mathbb{U}$, $\gamma \in \mathscr{P}_T(2n)$ *and* $m \in \mathbb{N} \setminus \{0\}$ *the following formula holds:*

$$i_z(\gamma^m) = \sum_{\omega^m = z} i_\omega(\gamma).$$

In particular one has $i_1(\gamma^2) = i_1(\gamma) + i_{-1}(\gamma)$.

4.3 MORSE INDEX AND RELATIONSHIP WITH OTHER MASLOV-TYPE INDICES

Let $(\mathbb{C}^{2n}, \{\cdot,\cdot\})$ be the complex symplectic space whose complex symplectic structure can be represented through the Hermitian product $(\cdot,\cdot)$ as

$$\{v, w\} \coloneqq (Jv, w), \qquad \forall\, v, w \in \mathbb{C}^{2n}.$$

We denote by $\mathrm{Lag}(\mathbb{C}^{2n})$ the space of all Lagrangian subspaces in $\mathbb{C}^{2n}$.

Let $l : [a, b] \to \mathrm{Lag}(\mathbb{C}^{2n})$ be a $\mathscr{C}^1$-curve of Lagrangian subspaces and let $L_0 \in \mathrm{Lag}(\mathbb{C}^{2n})$. Fix $t \in [a, b]$ and let W be a fixed Lagrangian complement of $l(t)$. If s belongs to a suitable small neighbourhood of t for every $v \in l(t)$ we can find a unique vector $w(s) \in W$ in such a way that $v + w(s) \in l(s)$.

DEFINITION 4.5. The *crossing form* $\Gamma(l, L_0, t_*)$ at t_* is the quadratic form $\Gamma(l, L_0, t_*) : l(t^*) \cap L_0 \to \mathbb{R}$ defined by

$$\Gamma(l, L_0, t_*)[v] \coloneqq \frac{d}{ds}\{v, w(s)\}\Big|_{s=t_*}. \tag{4.3.1}$$

The number t_* is said to be a *crossing instant for* l *with respect to* L_0 if $l(t_*) \cap L_0 \neq \{0\}$ and it is called *regular* if the crossing form is non-degenerate.

Let us remark that regular crossings are isolated and hence on a compact interval they are finitely many. Following [LZ00, Definition 3.1, Theorem 3.1] we give the next definition.

DEFINITION 4.6. If l has only regular crossings with respect to L_0, the *Maslov index of* l *with respect to* L_0 is defined as

$$\begin{aligned} i_{\mathrm{CLM}}(L_0, l, [a,b]) := m^+\big(\Gamma(l, L_0, a)\big) \\ + \sum_{t_* \in (a,b)} \operatorname{sgn} \Gamma(l, L_0, t_*) \\ - m^-\big(\Gamma(l, L_0, b)\big), \end{aligned} \tag{4.3.2}$$

where the summation runs over all crossings $t_* \in (a,b)$, the symbols m^+, m^- denote the dimension of the positive and negative spectral subspaces respectively and $\operatorname{sgn} := m^+ - m^-$ is the signature.

Let $V := \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$, and $(\cdot,\cdot)$ be the standard Hermitian product of V. We define

$$\{v, w\}_{\mathcal{J}} := (\mathcal{J}v, w), \qquad \forall v, w \in V$$

where

$$\mathcal{J} := \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$

By a direct calculation it follows that if $M \in \mathrm{Sp}(2n, \mathbb{C})$ then the complex subspace

$$\mathrm{Gr}(M) := \left\{ \begin{pmatrix} x \\ Mx \end{pmatrix} \middle| \, x \in \mathbb{C}^{2n} \right\}$$

is a Lagrangian subspace of the (complex) symplectic space $(V, \{\cdot,\cdot\}_{\mathcal{J}})$.

Given a path of symplectic matrices $\gamma : [a,b] \to \mathrm{Sp}(2n, \mathbb{C})$ then the *graph of the path* γ, $\mathrm{Gr}(\gamma)$, is defined as the path of graphs: $\mathrm{Gr}(\gamma)(t) := \mathrm{Gr}(\gamma(t))$, $t \in [a,b]$, and it is indeed a path of Lagrangian subspaces of $(V, \{\cdot,\cdot\}_{\mathcal{J}})$. The next result gives the relationship between the geometric index of a path of symplectic matrices and the Maslov index of the corresponding path of Lagrangian subspaces with respect to the diagonal $\Delta := \mathrm{Gr}(I_{2n})$.

PROPOSITION 4.7. *For all paths* $\gamma : [a,b] \to \mathrm{Sp}(2n, \mathbb{C})$ *we have*

$$i_{\mathrm{geo}}(\gamma) = i_{\mathrm{CLM}}(\Delta, \mathrm{Gr}(\gamma), [a,b]),$$

where the crossing forms involved in the right-hand side are calculated using the symplectic structure $\{\cdot,\cdot\}_{\mathcal{J}}$ *in* $\mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$.

Proof. The proof of this proposition follows from [LZ00, Formula (3.4) in Proposition 3.1, Theorem 3.1 (ii), Definition 3.1] and from [CLM94, Proposition 4.1]. □

By [HS09a, Lemma 4.6, Formulæ (2.15)–(2.16)] we immediately obtain

LEMMA 4.8. *For any path* $\gamma \in \mathscr{P}_T(2n)$ *we have the following equalities:*

1) $i_1(\gamma) + n = i_{\mathrm{CLM}}\big(\Delta, \mathrm{Gr}(\gamma), [0,T]\big)$;

2) $i_\omega(\gamma) = i_{\mathrm{CLM}}\big(\mathrm{Gr}(\omega I_{2n}), \mathrm{Gr}(\gamma), [0,T]\big)$ *for all* $\omega \in \mathbb{U} \setminus \{1\}$.

Remark 4.9. We observe that the integer i_1 is sometimes denoted by i_{CZ} and it is called the *Conley-Zehnder index*. For further details we refer to [LZ00] and references therein.

We now show some examples of computation of $i_1(\gamma)$ — passing through i_{CLM} — of some paths of matrices in $\mathrm{Sp}(2) \subset \mathrm{Sp}(2,\mathbb{C})$. Let $\gamma : [a,b] \to \mathrm{Sp}(2)$ be the path

$$\gamma(t) := \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix},$$

with $a, b, c, d \in \mathscr{C}^1([a,b], \mathbb{R})$, let l be the induced path of Lagrangian subspaces in $\mathbb{R}^4$ defined by $l(t) := \mathrm{Gr}\big(\gamma(t)\big)$. Let us assume that t_* is a crossing instant for l such that $l(t_*) = \Delta$. In order to compute the crossing form (4.3.1) we consider the Lagrangian subspace complementary to Δ:

$$W := \{0\} \times \mathbb{R} \times \mathbb{R} \times \{0\}.$$

Thus the Lagrangian splitting $\mathbb{R}^4 = \Delta \oplus W$ holds and for any $v := (x_0, y_0, x_0, y_0) \in \Delta$ let us choose $w(t) := (0, \eta(t), \xi(t), 0) \in W$ in order that $v + w(t) \in l(t)$. This means that $\eta(t)$ and $\xi(t)$ solve the equations

$$\begin{aligned} x_0 + \xi(t) &= a(t)x_0 + b(t)\big(y_0 + \eta(t)\big), \\ y_0 &= c(t)x_0 + d(t)\big(y_0 + \eta(t)\big). \end{aligned}$$

Since in a crossing instant t_* we have $\xi(t_*) = \eta(t_*) = 0$, differentiating the above identities gives

$$\xi'(t_*) = a'(t_*)x_0 + b'(t_*)y_0 - \frac{b(t_*)}{d(t_*)}\big[c'(t_*)x_0 + d'(t_*)y_0\big], \tag{4.3.3a}$$

$$\eta'(t_*) = -\frac{1}{d(t_*)}\big[c'(t_*)x_0 + d'(t_*)y_0\big]. \tag{4.3.3b}$$

By a direct computation we obtain

$$\begin{aligned} \{v, w(t)\}_{\mathcal{J}} &= \big(\mathcal{J}v, w(t)\big) \\ &= -\left\langle J\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} 0 \\ \eta(t) \end{pmatrix}\right\rangle + \left\langle J\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} \xi(t) \\ 0 \end{pmatrix}\right\rangle \\ &= -x_0\eta(t) - y_0\xi(t). \end{aligned}$$

Hence the crossing form at the crossing instant $t = t_*$ is given by

$$\Gamma(l, \Delta, t_*)(v) = \frac{d}{dt}\{v, w(t)\}_{\mathcal{J}}\Big|_{t=t_*} = -x_0\eta'(t_*) - y_0\xi'(t_*).$$

EXAMPLE 4.10. Let us consider the path $R_\alpha : [0, 2\pi] \to Sp(2)$ with

$$R_\alpha(t) = \begin{pmatrix} \cos(\sqrt{2-\alpha}\,t) & -\sqrt{2-\alpha}\,\sin(\sqrt{2-\alpha}\,t) \\ \frac{1}{\sqrt{2-\alpha}}\sin(\sqrt{2-\alpha}\,t) & \cos(\sqrt{2-\alpha}\,t) \end{pmatrix}, \quad \alpha \in (0,2),$$

that means $a = 0$, $b = 2\pi$, and

$$a(t) = d(t) = \cos(\sqrt{2-\alpha}\,t)$$
$$b(t) = -\sqrt{2-\alpha}\sin(\sqrt{2-\alpha}\,t), \qquad c(t) = \frac{1}{\sqrt{2-\alpha}}\sin(\sqrt{2-\alpha}\,t)$$

For any value of the parameter α, $t_* = 0$ is a crossing instant and $a(0) = 1$, $a'(0) = 0$, $b(0) = 0$, $b'(0) = -(2-\alpha)$, $c(0) = 0$, $c'(0) = 1$. Using Equations (4.3.3) we get

$$\Gamma(l_\alpha, \Delta, 0)[v] = -x_0\eta'_\alpha(0) - y_0\xi'_\alpha(0) = x_0^2 + (2-\alpha)y_0^2$$

where l_α is the path of Lagrangian subspaces associated with R_α. Since $\Gamma(l_\alpha, \Delta, 0)$ is a positive definite quadratic form, its signature is 2. Thus, according to Formula (4.3.2), the contribution to i_{CLM} at the starting point of the path is 2.

In order to find out all the crossing instants, we observe that they are in one-to-one correspondence with the zeros of the function

$$\det\big(R_\alpha(t) - I_2\big),$$

and hence with the solutions in $[0, 2\pi]$ of the equation

$$\cos(\sqrt{2-\alpha}\,t) = 1, \tag{4.3.4}$$

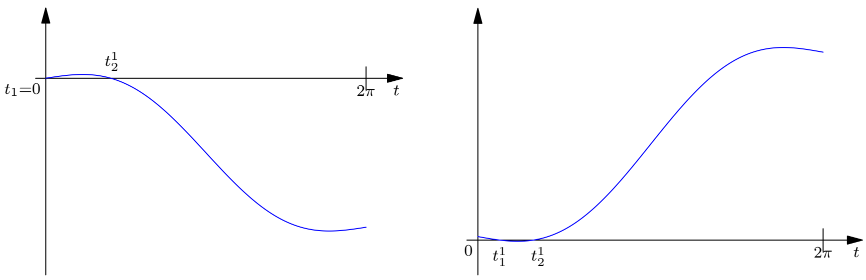
that we write as $t_k^\alpha := 2k\pi/\sqrt{2-\alpha}$, with $k \in \mathbb{Z}$. It is readily seen that

- if $\alpha \in (1,2)$ then the only solution of (4.3.4) is $t_0^\alpha = 0$, hence there are no other contributions to the computation of i_{CLM}.
- if $\alpha = 1$ then we have two solutions: $t_0^1 = 0$ and $t_1^1 = 2\pi$. We need to add $m^-\big(\Gamma(l_\alpha, \Delta, 2\pi)\big)$ to the contribution of the initial instant, but this quantity is actually 0.
- if $\alpha \in (0,1)$ then (4.3.4) admits also the non-zero solution[1] $t_1^\alpha = \dfrac{2\pi}{\sqrt{2-\alpha}}$. The contribution of this crossing is $\operatorname{sgn}\Gamma(l_\alpha, \Delta, t_1^\alpha) = 2$.

Summing up all these computations we obtain

$$i_{CLM}(R_\alpha) = \begin{cases} 2 & \text{if } \alpha \in [1,2) \\ 4 & \text{if } \alpha \in [0,1). \end{cases}$$

1 We observe that this value coincides with the apsidal angle for the α-homogeneous potential.



(a) The function $f_1(t)$ in the interval $[0, 2\pi]$.

(b) The function $2 - 2\cos\varepsilon - f_1(t)\sin\varepsilon$ in the interval $[0, 2\pi]$.

Figure 4.1. The function f_α (a) and its deformation (b) for $\alpha = 1$.

EXAMPLE 4.11. We now consider the path $N_\alpha : [0, 2\pi] \to \mathrm{Sp}(2)$ with

$$N_\alpha(t) = \begin{pmatrix} 1 & 0 \\ f_\alpha(t) & 1 \end{pmatrix}$$

where the function

$$f_\alpha(t) := \frac{1}{36\pi^2}\left(\frac{4\sin(\sqrt{2-\alpha}\,t)}{(2-\alpha)^{3/2}} - \frac{2+\alpha}{2-\alpha}t\right)$$

is drawn in Figure 4.1a for $\alpha = 1$ (the other cases for different α's are all similar).

We first observe that we are in a very degenerate situation, in the sense that $N_\alpha(t) \subset \overline{\mathrm{Sp}_1(2)}$. Furthermore, the function f_α admits two zeros in the interval $[0, 2\pi]$, $t_1 = 0$ and $t_2^\alpha \in (0, 2\pi)$. Thus the path is not contained in a fixed stratum of the Maslov cycle.

However, by taking into account the very definition of the Maslov index in the degenerate case given in Definition 4.3, we need to compute the contributions of the crossing of the graph of the perturbed matrix

$$N_{\varepsilon,\alpha}(t) := N_\alpha(t)\, e^{-\varepsilon J} \qquad \forall\, t \in [0, 2\pi]$$

and for $\varepsilon > 0$ sufficiently small. By direct computation we get:

$$N_{\varepsilon,\alpha} := \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon + f_\alpha(t)\cos\varepsilon & \cos\varepsilon + f_\alpha(t)\sin\varepsilon \end{pmatrix}$$

The crossing instants are the zeros of the equation

$$2 - 2\cos\varepsilon - f_\alpha(t)\sin\varepsilon = 0.$$

The function whose zeros we are searching is depicted in Figure 4.1b.

It is easy to see that for ε sufficiently small and for any $\alpha \in (0, 2)$ this equation admits two distinct solutions t_1^α and t_2^α in $(0, 2\pi)$.

Denoting by t^α a generic solution (crossing) we easily compute

$$\eta'_\alpha(t^\alpha) = -f'_\alpha(t^\alpha)x_0$$
$$\xi'_\alpha(t^\alpha) = 0,$$

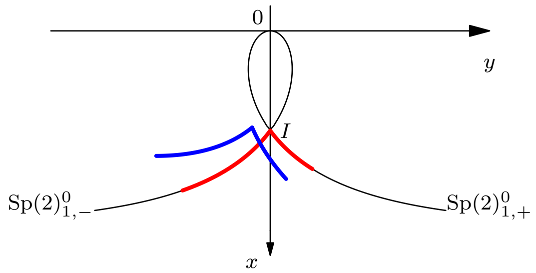


Figure 4.2. The path N_α (in red) and its deformation $N_{\varepsilon,\alpha}$ (in blue). The first path starts at the identity, then goes downwards right, then comes back to the identity and finally bends downwards left. The second follows the same trajectory, just rotated clockwise by an angle ε. See Section A.4 for more details about the coordinates and the underlying curves.

whence

$$\Gamma(N_{\varepsilon,\alpha}, \Delta, t^\alpha) = f'_\alpha(t^\alpha)x_0^2.$$

Summing up the two contributions, from the monotonicity of f_α we immediately obtain that $i_{CLM}(N_\alpha, \Delta, [0, 2\pi]) = 0$. The path N_α and its deformation $N_{\varepsilon,\alpha}$ are represented in Figure 4.2.

4.4 COMPUTATION OF THE MASLOV INDEX

In the case of autonomous Hamiltonian systems and under the assumption of non-degeneracy it is possible, at least theoretically, to compute the Maslov index (see for instance [Abb01] and references therein). Let $M \in \mathrm{Sp}(2n, \mathbb{R})$ act on $\mathbb{C}^{2n}$ in the usual way:

$$M(\xi + i\eta) := M\xi + iM\eta, \qquad \forall\, \xi, \eta \in \mathbb{R}^{2n}$$

and consider the Hermitian form g on $\mathbb{C}^{2n}$ defined as

$$g(v, w) := (iJv, w).$$

Definition 4.12. Let $\lambda \in \mathbb{U}$ be an eigenvalue of a complex symplectic matrix. The *Krein signature* of λ is the signature of the restriction of the Hermitian form g to the generalised eigenspace associated with λ. If g is positive definite on this subspace then λ is said to be *Krein-positive*.

The next result will be useful in the following.

Proposition 4.13 ([Abb01, Theorem 1.5.1]). *Let* B *be a real symmetric matrix. Let* $i\theta_1, \dots, i\theta_k$ *be the Krein-positive purely imaginary eigenvalues* *of* JB*, counted with their algebraic multiplicity. Then the linear autonomous Hamiltonian system*

$$\zeta'(t) = JB\zeta(t)$$

is non-degenerate at time T *if and only if* $\theta_j T \notin 2\pi\mathbb{Z}$*, for any* $j = 1, \dots, k$*. If* ψ *denotes the fundamental solution, we get:*

$$i_1(\psi) = -\sum_{j=1}^{k} \left[\left[\frac{T\theta_j}{\pi}\right]\right]$$

provided that it is non-degenerate at time T*. The function* $[[\cdot]]$ *is defined as follows:*

$$[[\theta]] := \begin{cases} \theta & \text{if } \theta \in \mathbb{Z} \\ \text{the closest odd integer} & \text{if } \theta \in \mathbb{R} \setminus \mathbb{Z}. \end{cases}$$

Now, following [Lon02], we recall the definition of the so-called *splitting numbers* as well as their basic properties, which will be crucial later. For this we refer to [Lon02, Chapter 6, pp. 190–199].

DEFINITION 4.14. For any $M \in \mathrm{Sp}(2n)$ and every $\omega \in \mathbb{U}$, the *splitting numbers* $S^{\pm}_M(\omega)$ of M at ω are defined by

$$S^{\pm}_M(\omega) := \lim_{\varepsilon \to 0^+} i_{\omega \exp(\pm i\varepsilon)}(\gamma) - i_\omega(\gamma), \tag{4.4.1}$$

where $\gamma \in \mathscr{P}_T(2n)$ is such that $\gamma(T) = M$.

In the next proposition we recall the basic properties of the splitting numbers. For their computation we introduce the *normal forms*

$$R(\theta) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in (0, 2\pi) \setminus \{0\},$$
$$N_1(\lambda, a) := \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}^*, \ a \in \mathbb{R}.$$

PROPOSITION 4.15 ([Lon02, Chapter 6]). *For* $M, M_0, M_1 \in \mathrm{Sp}(2n)$ *and all* $\omega \in \mathbb{U}$, $\theta \in (0, \pi)$*, the following properties hold:*

1. *The splitting numbers* $S^{\pm}_M(\omega)$ *are well defined, i.e. they are independent of the choice of the path* $\gamma \in \mathscr{P}_\tau(2n)$ *satisfying* $\gamma(\tau) = M$ *in Definition (4.4.1).*
2. *The splitting numbers* $S^{\pm}_M(\omega)$ *are constant in the set* $\Omega^0(M)$*, that is the path-connected component containing* M *of the set*

$$\Omega(M) := \big\{ N \in \mathrm{Sp}(2n) \;\big|\; \sigma(N) \cap \mathbb{U} = \sigma(M) \cap \mathbb{U} \text{ and } \nu_\lambda(N) = \nu_\lambda(M)\ \forall \lambda \in \sigma(M) \cap \mathbb{U} \big\}.$$

3. $S^{\pm}_M(\omega) = 0$ *if* $\omega \notin \sigma(M)$.

4. $S^{\pm}_M(\overline{\omega}) = S^{\mp}_M(\omega)$.

5. $0 \leqslant S^{\pm}(\omega) \leqslant \dim\ker(M - \omega I)$.

6. $S^+_M(\omega) + S^-_M(\omega) \leqslant \dim\ker(M - \omega I)^{2n}$ *if* $\omega \in \sigma(M)$.

7. $S^{\pm}_{M_0 \diamond M_1}(\omega) = S^{\pm}_{M_0}(\omega) + S^{\pm}_{M_1}(\omega)$.

8. $i_\omega(\gamma) - i_1(\gamma) = S^+_M(1) + \sum_{\omega_0}\big(S^+_M(\omega_0) - S^-_M(\omega_0)\big) - S^-_M(\omega)$, *where* $\mathfrak{I}(\omega) \geqslant 0$ *and* $\omega_0 \in \sigma(M)$ *lies in the interior of the arc of the upper unit semicircle connecting* 1 *and* ω.

9. $\left(S^+_{N_1(1,a)}(1),\, S^-_{N_1(1,a)}(1)\right) = \begin{cases} (1,1) & \text{if } a \in \{0,1\} \\ (0,0) & \text{if } a = -1. \end{cases}$

10. $\left(S^+_{N_1(-1,a)}(-1),\, S^-_{N_1(-1,a)}(-1)\right) = \begin{cases} (1,1) & \text{if } a \in \{-1,0\} \\ (0,0) & \text{if } a = 1. \end{cases}$

11. $\left(S^+_{R(\theta)}(e^{i\theta}),\, S^-_{R(\theta)}(e^{i\theta})\right) = (0,1)$.

12. $\left(S^+_{R(2\pi-\theta)}(e^{i\theta}),\, S^-_{R(2\pi-\theta)}(e^{i\theta})\right) = (1,0)$.

4.5 VARIATIONAL SETTING: AN INDEX THEOREM

We recall here some basic facts about the Lagrangian and Hamiltonian dynamics (for further details see for instance [Fat08; AF07; APS08]). The elements of the tangent bundle $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ are denoted by (q, v) where $q \in U$ and $v \in T_q U$. Let $\mathscr{L} \in \mathscr{C}^\infty(T\mathbb{R}^n; \mathbb{R})$ be a *regular Lagrangian*, meaning that $\mathscr{L}$ is assumed to satisfy

(L1) $\partial_{vv}\mathscr{L}(q, v) > 0$ for all $(q, v) \in T\mathbb{R}^n$;

(L2) There is a constant $l_1 > 0$ such that

$$\|\partial_{vv}\mathscr{L}(q,v)\| \leqslant l_1, \qquad \|\partial_{qv}\mathscr{L}(q,v)\| \leqslant l_1(1+|v|),$$
$$\|\partial_{qq}\mathscr{L}(q,v)\| \leqslant l_1(1+|v|^2).$$

As a direct consequence of the Inverse Function Theorem, under Condition (L1) the Legendre transformation

$$\mathcal{L}_{\mathscr{L}} : T\mathbb{R}^n \to T^*\mathbb{R}^n, \qquad (q,v) \mapsto \big(D_v\mathscr{L}(q,v), q\big),$$

is a smooth local diffeomorphism. The Fenchel transformation of $\mathscr{L}$ is the autonomous Hamiltonian on $T^*\mathbb{R}^n$

$$\mathcal{H}(p,q) := \max_{v \in T_q\mathbb{R}^n} \big(p[v] - \mathscr{L}(q,v)\big) = p\big[v(p,q)\big] - \mathscr{L}\big(q, v(p,q)\big),$$

where $\big(q, v(p,q)\big) = \mathcal{L}_{\mathscr{L}}^{-1}(p,q)$. Under the above assumptions on $\mathscr{L}$, the function $\mathcal{H}$ is smooth on $T^*\mathbb{R}^n$. The associated autonomous Hamiltonian vector field $X_{\mathcal{H}}$ on $T^*\mathbb{R}^n$, defined by

$$\langle JX_{\mathcal{H}}(p,q), \xi\rangle = -D\mathcal{H}(p,q)[\xi], \quad \forall\,(p,q) \in T^*\mathbb{R}^n,\ \forall\,\xi \in T_{(p,q)}T^*\mathbb{R}^n,$$

is then smooth, so it defines an autonomous smooth local flow on $T^*\mathbb{R}^n$. The corresponding flow on $T\mathbb{R}^n$ obtained by conjugating the Hamiltonian flow $\varphi^{\mathcal{H}}$ by the Legendre transform $\mathcal{L}_{\mathscr{L}}$ is denoted by

$$\varphi^{\mathscr{L}} : T\mathbb{R}^n \to T\mathbb{R}^n$$

and its orbits have the form $t \mapsto \big(\gamma(t), \gamma'(t)\big)$, where γ solves the Euler-Lagrange equation

$$\frac{d}{dt}\partial_v\mathscr{L}\big(\gamma(t),\gamma'(t)\big) = \partial_q\mathscr{L}\big(\gamma(t),\gamma'(t)\big). \tag{4.5.1}$$

Let us consider the *Lagrangian action functional*

$$\mathbb{A} : W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}) \to \mathbb{R}$$

defined by

$$\mathbb{A}(\gamma) := \int_0^{2\pi} \mathscr{L}\big(\gamma(t),\gamma'(t)\big)\,dt.$$

We recall that if $\mathscr{L}$ satisfies (L2) then $\mathbb{A}$ is of class $\mathscr{C}^2$ [cf. AF07, Proposition 4.1]. Moreover if the first variation of $\mathbb{A}$ vanishes at $\gamma \in W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X})$ for every $\xi \in W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X})$, then γ is a (classical) solution of class $\mathscr{C}^2$ of the Euler-Lagrange equation (4.5.1) such that $\gamma(2\pi) = \gamma(0)$. Given a classical solution γ of (4.5.1) the second variation of $\mathbb{A}$ is given by

$$d^2\mathbb{A}(\gamma)[\xi,\eta] = \int_0^{2\pi} \Big[\big(P(t)\xi' + Q(t)\xi\big)\eta' + Q^T(t)\xi'\eta + R(t)\xi\eta\Big]\,dt, \tag{4.5.2}$$

where

$$P(t) := D_{vv}\mathscr{L}\big(\gamma(t),\gamma'(t)\big), \qquad Q(t) := D_{qv}\mathscr{L}\big(\gamma(t),\gamma'(t)\big),$$
$$R(t) := D_{qq}\mathscr{L}\big(\gamma(t),\gamma'(t)\big).$$

Linearising the Euler-Lagrange equations (4.5.1) around a critical point γ we obtain the Sturm system

$$-\big(P(t)\gamma'(t) + Q(t)\gamma(t)\big)' + Q^T(t)\gamma'(t) + R(t)\gamma(t) = 0 \tag{4.5.3}$$

Let now $\zeta(t) := \big(D_v\mathscr{L}(\gamma(t),\gamma'(t),\gamma(t)\big)$ be the solution of the Hamiltonian system associated with (4.5.3), whose fundamental solution ϕ satisfies

$$\begin{cases} \phi'(t) = JB(t)\phi(t) \\ \phi(0) = I_{2n}, \end{cases} \tag{4.5.4}$$

with

$$B(t) := \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q^T(t)P^{-1}(t) & Q^T(t)P^{-1}(t)Q(t) - R(t) \end{pmatrix}.$$

For any $\omega \in \mathbb{U}$ let $h(\gamma)$ be the quadratic form on

$$D(\omega) := \left\{ \xi \in W^{1,2}\big([0,2\pi],\mathbb{C}^n\big) \;\middle|\; \xi(2\pi) = \omega\xi(0) \right\}$$

induced by $d^2\mathbb{A}(\gamma)$. Then it is possible to show, arguing as in [MPP05, Proposition 3.1], that $h(\gamma)$ is an essentially positive Fredholm quadratic form in the sense specified in Section A.5.

Definition 4.16. Let $\gamma \in D(\omega)$ be a critical point of $\mathbb{A}$. We define the ω-*Morse index* of γ, denoted by $i^{\omega}_{\text{Morse}}(\gamma)$, as the dimension of the largest subspace of $D(\omega)$ such that the quadratic form $h(\gamma)$ is negative definite.

We observe that the ω-Morse index is the number of negative eigendirections — counted according to their multiplicities — on which $h(\gamma)$ is negative definite. We also define

$$n_\omega(\gamma) := \dim \ker h(\gamma).$$

The following Morse-type index theorem relates the Morse index of a solution with the ω-index introduced in Section 4.2.

Lemma 4.17 (Morse Index Theorem, [Lon02, p. 172]). *Let γ be a critical point of the Lagrangian action functional (hence a classical solution of the Euler-Lagrange equation (4.5.1)) and let ϕ be the fundamental solution of the linearised system around γ (that is, ϕ satisfies (4.5.4)). Then*

$$i^{\omega}_{\text{Morse}}(\gamma) = i_\omega(\phi), \qquad n_\omega(\gamma) = \nu_\omega(\gamma), \qquad \forall\, \omega \in \mathbb{U}.$$

We close this section by recalling two important results about the minimising properties of the circular periodic solutions of the α-homogeneous Kepler problem and the circular Lagrangian solution of the three-body problem under α-homogeneous potential. The first one is due to Gordon [cf. Gor77] for the case $\alpha = 1$ and was generalised to different homogeneity degrees by Venturelli in [Ven02, Proposition 2.2.3].

Lemma 4.18. *In the α-homogeneous Kepler problem with $\alpha \in [1, 2)$, circular solutions are local minimisers of the Lagrangian action functional in the space of loops with winding number ± 1 around the origin.*

As regards the circular Lagrangian solution for the α-homogeneous 3-body problem (without any restriction on the choice of the masses), from [Ven02, Theorem 3.1.17] we infer the following result.

LEMMA 4.19. *In the α-homogeneous 3-body problem the circular Lagrange relative equilibrium is a local minimum of the Lagrangian action functional when $\alpha \in [1, 2)$ (it is actually a strict minimiser if $\alpha \neq 1$). It is a non-degenerate saddle point when $\alpha \in (0, 1)$.*

5 LINEAR STABILITY OF THE LAGRANGE CIRCULAR SOLUTION IN THE GENERALISED 3-BODY PROBLEM

Contents

Consider three bodies with positive masses m_1, m_2, m_3 moving in the Euclidean plane $\mathbb{R}^2$ and denote by $q := \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \in \mathbb{R}^6$ the column vector of all positions, where each q_i is a column vector in $\mathbb{R}^2$.

We are interested in finding periodic solutions of the Newtonian system (2.2.1) that we rewrite here:

$$M\ddot{q} = \nabla U(q),$$

where $U : X \subset \mathbb{R}^6 \to \mathbb{R}$ is one of the two potential functions (1.1):

$$U_\alpha(q) := \sum_{\substack{i,j=1 \\ i<j}}^{3} \frac{m_i m_j}{|q_i - q_j|^\alpha}, \qquad \alpha \in (0,2),$$

$$U_{\log}(q) := \sum_{\substack{i,j=1 \\ i<j}}^{3} m_i m_j \log \frac{1}{|q_i - q_j|},$$

($\alpha = 1$ corresponds to the gravitational case) defined on the *collision-free configuration space*

$$X := \mathbb{R}^6 \setminus \{\, q \in \mathbb{R}^6 \mid q_i = q_j \text{ for some } i \neq j \,\}.$$

Proceeding as in Section 2.2, in order to rewrite the second-order system (2.2.1) as a first-order Hamiltonian system we define the *Hamiltonian function* $\mathscr{H} : T^*X \to \mathbb{R}$ to be[1]

$$\mathscr{H}(p, q) := \frac{1}{2}\langle M^{-1}p^T, p^T\rangle - U(q),$$

where $p := (p_1, p_2, p_3) \in \mathbb{R}^6$ is the row vector of the linear momenta conjugate to q. Hence System (2.2.1) becomes

$$\begin{cases} \dot{p}^T = -\partial_q \mathscr{H} = \nabla U(q) \\ \dot{q} = \partial_p \mathscr{H} = M^{-1}p^T. \end{cases}$$

Let us remark that by summing up the equations of (2.2.1) we obtain that the centre of mass of the system moves uniformly along a straight line; therefore, without loss of generality, we can fix it at the origin and study the dynamics on the *reduced (collision-free) configuration space*

$$\widehat{X} := \left\{ q \in X \;\middle|\; \sum_{i=1}^{3} m_i q_i = 0 \right\}.$$

The reduced phase space $T^*\widehat{X}$ is therefore 8-dimensional.

Again we consider relative equilibria precisely as in Subsection 2.2.1 and Section 3.1, moving to a uniformly rotating reference frame (x, y) through a rigid rotation of angular velocity ω given by

$$\omega^2 = \begin{cases} \lambda_\alpha := \dfrac{\alpha U_\alpha(\bar{x})}{\mathcal{I}(\bar{x})} & \text{if } U = U_\alpha \\ \lambda_{\log} := \dfrac{1}{\mathcal{I}(\bar{x})} \displaystyle\sum_{\substack{i,j=1 \\ i<j}}^{n} m_i m_j & \text{if } U = U_{\log}, \end{cases}$$

ending up with the new Hamiltonian function

$$\widehat{\mathscr{H}}(y, x) := \frac{1}{2}\langle M^{-1}y^T, y^T\rangle - U(x) - \omega\langle K y^T, x\rangle. \tag{5.1}$$

It is well known from Lagrange's and Euler's works that on the shape sphere $\mathcal{S}$ (for any choice of the masses) there are exactly five central configurations: three of them are collinear (the three bodies lie on the same line), while in the other two the bodies are arranged at the vertices of an equilateral triangle. We focus on this last one.

1 Note that we swapped the positions of the variables q and p: we wrote (q, p) in Section 2.2, whilst we write (p, q) here. The reason is that we intend to allow a clearer and quicker comparison of our results with the other papers in the literature which we refer to. This choice is also responsible of the change of sign in the Hamiltonian (5.1), expressed in rotating coordinates.

5.1 A SYMPLECTIC DECOMPOSITION OF THE PHASE SPACE FOR THE LINEARISED SYSTEM

Consider the Hamiltonian System (2.2.3) in $\mathbb{R}^{12}$

$$\dot{\zeta}(t) = J\nabla\mathscr{H}\big(\zeta(t)\big), \tag{5.1.1}$$

where $\zeta := (p, q^T)^T$ and $\mathscr{H}$ is the Hamiltonian of the 3-body problem defined in (2.2.2) (taking $n = 3$ there). We linearise it around a relative equilibrium $\bar{\zeta}$ and write

$$\dot{\zeta}(t) = JD^2\mathscr{H}(\bar{\zeta})\,\zeta(t). \tag{5.1.2}$$

Arguing as in Section 3.2, we see that the presence of the first integrals of motion and the invariance of the problem under some isometries gives rise to three symplectic invariant subspaces of the phase space: E_1, carrying the information about the translational invariance, E_2, generated by the conservation of the angular momentum and by the invariance by dilations, and E_3, defined as the symplectic orthogonal complement of the first two.

Indeed, a basis for the position and momentum of the centre of mass is given by the four vectors in $\mathbb{R}^{12}$

$$G_1 := \begin{pmatrix} Mv \\ 0 \end{pmatrix}, \quad G_2 := \begin{pmatrix} KMv \\ 0 \end{pmatrix}, \quad g_1 := \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad g_2 := \begin{pmatrix} 0 \\ Kv \end{pmatrix}$$

with $v := (1, 0, 1, 0, 1, 0)^T \in \mathbb{R}^6$. If we let E_1 be the space spanned by these vectors, it turns out that it is invariant and also symplectic. Note that the symplectic complement of E_1 is the space where the barycentre of the system is fixed at the origin and the total linear momentum is zero. The scaling and rotational symmetries generate another linear symplectic invariant subspace E_2, a basis of which is given by the four vectors in $\mathbb{R}^{12}$

$$Z_1 := \begin{pmatrix} M\bar{q} \\ 0 \end{pmatrix}, \quad Z_2 := \begin{pmatrix} KM\bar{q} \\ 0 \end{pmatrix}, \quad z_1 := \begin{pmatrix} 0 \\ \bar{q} \end{pmatrix}, \quad z_2 := \begin{pmatrix} 0 \\ K\bar{q} \end{pmatrix}.$$

The coordinates on third subspace E_3 will be denoted by $(W, w^T)^T$; note that this also is 4-dimensional.

We now derive a useful expression of the matrix of the linearised system by adapting Meyer and Schmidt's proof in [MS05, Lemma 3.1, pp. 271–273] to the case of the α-homogeneous potential, but restricting ourselves to the circular case, i.e. with zero eccentricity. In order to simplify the computations we set, without loss of generality,

$$m_1 + m_2 + m_3 = 1;$$

furthermore we introduce the key parameter

$$\begin{aligned}\beta &:= \frac{27(m_1m_2 + m_1m_3 + m_2m_3)}{(m_1 + m_2 + m_3)^2} \\ &= 27(m_1m_2 + m_1m_3 + m_2m_3) \in (0, 9].\end{aligned}$$

PROPOSITION 5.1 (α-homogeneous case). *There exists a system of symplectic coordinates* $\xi := (\bar{Z}, \bar{W}, \bar{z}^T, \bar{w}^T)^T \in \mathbb{R}^8$ *and a rescaled time* τ *such that the linearised System (5.1.2) restricted to* $E_2 \oplus E_3 = T^*\widehat{X}$ *has the form*

$$\frac{d\xi}{d\tau} = \Lambda\xi, \tag{5.1.3}$$

where

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & 0 & \alpha+1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{2}\left(\alpha + \frac{\alpha+2}{3}\sqrt{9-\beta}\right) & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1}{2}\left(\alpha - \frac{\alpha+2}{3}\sqrt{9-\beta}\right) \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5.1.4}$$

Proof. The Hamiltonian of the system in the fixed reference frame is

$$\mathscr{H}(p, q) := \frac{1}{2}\langle M^{-1}p^T, p^T\rangle - U_\alpha(q),$$

We make the following symplectic change of coordinates:

$$p^T = C^{-T}\begin{pmatrix} G^T \\ Z^T \\ W^T \end{pmatrix}, \qquad q = C\begin{pmatrix} g \\ z \\ w \end{pmatrix}, \tag{5.1.5}$$

where C is given by [cf. MS05, pp. 268–269]

$$C := \begin{pmatrix} 1 & 0 & \frac{9(m_2+m_3)}{2\sqrt{\beta}} & \frac{3\sqrt{3}(m_2-m_3)}{2\sqrt{\beta}} & 0 & -\frac{3\sqrt{3}\sqrt{m_2m_3}}{\sqrt{\beta}\sqrt{m_1}} \\ 0 & 1 & -\frac{3\sqrt{3}(m_2-m_3)}{2\sqrt{\beta}} & \frac{9(m_2+m_3)}{2\sqrt{\beta}} & \frac{3\sqrt{3}\sqrt{m_2m_3}}{\sqrt{\beta}\sqrt{m_1}} & 0 \\ 1 & 0 & -\frac{9m_1}{2\sqrt{\beta}} & -\frac{3\sqrt{3}(m_1+2m_3)}{2\sqrt{\beta}} & \frac{9\sqrt{m_1m_3}}{2\sqrt{\beta}\sqrt{m_2}} & \frac{3\sqrt{3}\sqrt{m_1m_3}}{2\sqrt{\beta}\sqrt{m_2}} \\ 0 & 1 & \frac{3\sqrt{3}(m_1+2m_3)}{2\sqrt{\beta}} & -\frac{9m_1}{2\sqrt{\beta}} & -\frac{3\sqrt{3}\sqrt{m_1m_3}}{2\sqrt{\beta}\sqrt{m_2}} & \frac{9\sqrt{m_1m_3}}{2\sqrt{\beta}\sqrt{m_2}} \\ 1 & 0 & -\frac{9m_1}{2\sqrt{\beta}} & \frac{3\sqrt{3}(m_1+2m_2)}{2\sqrt{\beta}} & -\frac{9\sqrt{m_1m_2}}{2\sqrt{\beta}\sqrt{m_3}} & \frac{3\sqrt{3}\sqrt{m_1m_2}}{2\sqrt{\beta}\sqrt{m_3}} \\ 0 & 1 & -\frac{3\sqrt{3}(m_1+2m_2)}{2\sqrt{\beta}} & -\frac{9m_1}{2\sqrt{\beta}} & -\frac{3\sqrt{3}\sqrt{m_1m_2}}{2\sqrt{\beta}\sqrt{m_3}} & -\frac{9\sqrt{m_1m_2}}{2\sqrt{\beta}\sqrt{m_3}} \end{pmatrix}.$$

It is a straightforward computation to verify that C is invertible and it satisfies the relations

$$C^TMC = I, \qquad C^{-1}JC = J.$$

After fixing the centre of mass at the origin (i.e. setting $g = G^T = 0$, thus restricting the system to $E_2 \oplus E_3$), the Hamiltonian of the system becomes

$$\mathscr{H}(Z, W, z, w) = \frac{1}{2}(Z_1^2 + Z_2^2 + W_1^2 + W_2^2) - U_\alpha(z, w).$$

Consider now the rotation in the plane

$$R(t) := \begin{pmatrix} \cos(\lambda_\alpha t) & -\sin(\lambda_\alpha t) \\ \sin(\lambda_\alpha t) & \cos(\lambda_\alpha t) \end{pmatrix},$$

where λ_α is the Lagrange multiplier (3.1.2) of the central configuration, corresponding to the square of the angular velocity of each body. Accordingly, we move to a uniformly rotating reference frame in the following way:

$$\begin{cases} Z^T = R(t)\widetilde{Z}^T \\ W^T = R(t)\widetilde{W}^T \\ z = R(t)\widetilde{z} \\ w = R(t)\widetilde{w}. \end{cases} \tag{5.1.6}$$

Since we are moving to a new set of canonical coordinates (see for instance [GPS80, Chapter 9]) via the time-depending generating function

$$F\big(Z, W, \widetilde{z}, \widetilde{w}, t\big) := -ZR(t)\widetilde{z} - WR(t)\widetilde{w},$$

the new Hamiltonian function (still denoted by $\mathscr{H}$) must contain the extra term $\dfrac{dF}{dt}$:

$$\begin{aligned} \mathscr{H}\big(\widetilde{Z}, \widetilde{W}, \widetilde{z}, \widetilde{w}\big) &= \frac{1}{2}\big(\widetilde{Z}_1^2 + \widetilde{Z}_2^2 + \widetilde{W}_1^2 + \widetilde{W}_2^2\big) - U_\alpha(\widetilde{z}, \widetilde{w}) \\ &\quad + \lambda_\alpha\big(\widetilde{Z}_1\widetilde{z}_2 - \widetilde{Z}_2\widetilde{z}_1 + \widetilde{W}_1\widetilde{w}_2 - \widetilde{W}_2\widetilde{w}_1\big). \end{aligned}$$

Then we operate the following symplectic scaling with multiplier $\lambda_\alpha^{-\frac{\alpha}{\alpha+2}}$:

$$\begin{cases} \widetilde{Z} = \lambda_\alpha^{\frac{\alpha+1}{\alpha+2}}\widehat{Z} \\ \widetilde{W} = \lambda_\alpha^{\frac{\alpha+1}{\alpha+2}}\widehat{W} \\ \widetilde{z} = \lambda_\alpha^{-\frac{1}{\alpha+2}}\widehat{z} \\ \widetilde{w} = \lambda_\alpha^{-\frac{1}{\alpha+2}}\widehat{w} \end{cases} \tag{5.1.7}$$

obtaining thus

$$\begin{aligned} \mathscr{H}\big(\widehat{Z}, \widehat{W}, \widehat{z}, \widehat{w}\big) &= \frac{\lambda_\alpha}{2}\big(\widehat{Z}_1^2 + \widehat{Z}_2^2 + \widehat{W}_1^2 + \widehat{W}_2^2\big) - U_\alpha(\widehat{z}, \widehat{w}) \\ &\quad + \lambda_\alpha\big(\widehat{Z}_1\widehat{z}_2 - \widehat{Z}_2\widehat{z}_1 + \widehat{W}_1\widehat{w}_2 - \widehat{W}_2\widehat{w}_1\big). \end{aligned}$$

The next step consists in a time scaling: define $\tau := \lambda_\alpha t$ and rewrite System (5.1.1) as

$$\frac{d\zeta\big(\tau(t)\big)}{d\tau}\frac{d\tau(t)}{dt} = J\nabla\mathscr{H}\big(\zeta(\tau(t))\big),$$

or equivalently as

$$\zeta'(\tau)\lambda_\alpha = J\nabla\mathscr{H}\big(\zeta(\tau)\big), \tag{5.1.8}$$

where the prime $'$ denotes the derivative with respect to τ. Hence a division of both sides of (5.1.8) by λ_α yields the equivalent system

$$\zeta'(\tau) = J\nabla\widehat{\mathscr{H}}\big(\zeta(\tau)\big),$$

where

$$\begin{aligned}\widehat{\mathscr{H}}\big(\widehat{Z},\widehat{W},\widehat{z},\widehat{w}\big) &= \frac{1}{2}\big(\widehat{Z}_1^2+\widehat{Z}_2^2+\widehat{W}_1^2+\widehat{W}_2^2\big)-\frac{1}{\lambda_\alpha}U_\alpha(\widehat{z},\widehat{w})\\ &\quad+\widehat{Z}_1\widehat{z}_2-\widehat{Z}_2\widehat{z}_1+\widehat{W}_1\widehat{w}_2-\widehat{W}_2\widehat{w}_1\\ &=\frac{1}{\lambda_\alpha}\mathscr{H}\big(\widehat{Z},\widehat{W},\widehat{z},\widehat{w}\big).\end{aligned}$$

Finally, in order to shift the equilibrium point into the origin, we operate a translation and set

$$\begin{cases}\bar{Z}_1 := \widehat{Z}_1\\ \bar{Z}_2 := \widehat{Z}_2-1\\ \bar{W}_1 := \widehat{W}_1\\ \bar{W}_2 := \widehat{W}_2\\ \bar{z}_1 := \widehat{z}_1-1\\ \bar{z}_2 := \widehat{z}_2\\ \bar{w}_1 := \widehat{w}_1\\ \bar{w}_2 := \widehat{w}_2\end{cases}, \tag{5.1.9}$$

whence

$$\begin{aligned}\widehat{\mathscr{H}}\big(\bar{Z},\bar{W},\bar{z},\bar{w}\big) &= \frac{1}{2}\big[\bar{Z}_1^2+(\bar{Z}_2+1)^2+\bar{W}_1^2+\bar{W}_2^2\big]-\frac{1}{\lambda_\alpha}U_\alpha(\bar{z},\bar{w})\\ &\quad+\bar{Z}_1\bar{z}_2-(\bar{Z}_2+1)(\bar{z}_1+1)+\bar{W}_1\bar{w}_2-\bar{W}_2\bar{w}_1.\end{aligned}$$

The matrix of the linearised system is (J times) the Hessian of this Hamiltonian, evaluated at the origin. In order to write it down we need the Hessian of the potential U_α expressed in the coordinates $(\bar{z},\bar{w})$, but since the computations are quite long and tedious we shall omit them and indicate only the way in which we obtained the result. We have that

$$U_\alpha(\bar{z},\bar{w}) = \frac{m_1m_2}{d_{12}^\alpha}+\frac{m_1m_3}{d_{13}^\alpha}+\frac{m_2m_3}{d_{23}^\alpha},$$

where

$$\begin{aligned}d_{12} := \frac{3\sqrt{3}}{\sqrt{\beta}}\Bigg[&(\bar{z}_1+1)^2+\bar{z}_2^2+\frac{m_3(m_1^2+m_1m_2+m_2^2)}{m_1m_2}(\bar{w}_1^2+\bar{w}_2^2)\\ &+\sqrt{\frac{3m_2m_3}{m_1}}\big(\bar{z}_2\bar{w}_1-(\bar{z}_1+1)\bar{w}_2\big)\\ &-(2m_1+m_2)\sqrt{\frac{m_3}{m_1m_2}}\big((\bar{z}_1+1)\bar{w}_1+\bar{z}_2\bar{w}_2\big)\Bigg]^{\frac{1}{2}},\end{aligned}$$

$$d_{13} := \frac{3\sqrt{3}}{\sqrt{\beta}}\Bigg[(\bar{z}_1+1)^2+\bar{z}_2^2+\frac{m_2(m_1^2+m_1m_3+m_3^2)}{m_1m_3}(\bar{w}_1^2+\bar{w}_2^2)$$
$$+\sqrt{\frac{3m_2m_3}{m_1}}(\bar{z}_2\bar{w}_1-(\bar{z}_1+1)\bar{w}_2)$$
$$+(2m_1+m_3)\sqrt{\frac{m_2}{m_1m_3}}\big((\bar{z}_1+1)\bar{w}_1+\bar{z}_2\bar{w}_2\big)\Bigg]^{\frac{1}{2}},$$

$$d_{23} := \frac{3\sqrt{3}}{\sqrt{\beta}}\Bigg[(\bar{z}_1+1)^2+\bar{z}_2^2+\frac{m_1(m_2^2+m_2m_3+m_3^2)}{m_2m_3}(\bar{w}_1^2+\bar{w}_2^2)$$
$$-(m_2+m_3)\sqrt{\frac{3m_1}{m_2m_3}}(\bar{z}_2\bar{w}_1-(\bar{z}_1+1)\bar{w}_2)$$
$$+(m_2-m_3)\sqrt{\frac{m_1}{m_2m_3}}\big((\bar{z}_1+1)\bar{w}_1+\bar{z}_2\bar{w}_2\big)\Bigg]^{\frac{1}{2}}.$$

Now, calculating the Hessian of $\frac{1}{\lambda_\alpha}U_\alpha$ and evaluating it at the origin yields

$$\frac{1}{\lambda_\alpha}D^2U_\alpha(0,0) = \begin{pmatrix} \alpha+1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix},$$

with $a := \frac{1}{4}\big[4(\alpha+1)m_1+(\alpha-2)(m_2+m_3)\big]$, $b := \frac{1}{4}\big[\sqrt{3}(\alpha+2)(m_2-m_3)\big]$ and $c := \frac{1}{4}\big[-4m_1+(3\alpha+2)(m_2+m_3)\big]$. The matrix of the linearised system is thus

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \alpha+1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & a & b \\ 0 & 0 & -1 & 0 & 0 & 0 & b & c \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5.1.10}$$

Extracting from it the submatrix representing the dynamics on E_3 (i.e. the one acting on the W's and w's only):

$$\begin{pmatrix} 0 & 1 & \frac{1}{4}[4(\alpha+1)m_1+(\alpha-2)(m_2+m_3)] & \frac{1}{4}[\sqrt{3}(\alpha+2)(m_2-m_3)] \\ -1 & 0 & \frac{1}{4}[\sqrt{3}(\alpha+2)(m_2-m_3)] & \frac{1}{4}[-4m_1+(3\alpha+2)(m_2+m_3)] \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix},$$

we apply a rotation to both positions w and momenta W and obtain

$$\begin{pmatrix} 0 & 1 & \frac{1}{2}\Big(\alpha+\frac{\alpha+2}{3}\sqrt{9-\beta}\Big) & 0 \\ -1 & 0 & 0 & \frac{1}{2}\Big(\alpha-\frac{\alpha+2}{3}\sqrt{9-\beta}\Big) \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{5.1.11}$$

so that the final matrix depends only on α and β. Now substitute (5.1.11) back into (5.1.10) to get (5.1.4). □

In the logarithmic case there is a completely similar result.

PROPOSITION 5.2 (Logarithmic case). *There exist a system of symplectic coordinates* $\xi := (\bar{Z}, \bar{W}, \bar{z}^T, \bar{w}^T)^T \in \mathbb{R}^8$ *and a rescaled variable* τ *such that the linearised System* (5.1.2) *restricted to* $E_2 \oplus E_3 = T^*\widehat{X}$ *has the form*

$$\frac{d\xi}{d\tau} = \Lambda \xi,$$

where

$$\Lambda := \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{3}\sqrt{9-\beta} & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -\frac{1}{3}\sqrt{9-\beta} \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{5.1.12}$$

Proof. We proceed exactly as in Proposition 5.1 with some slight modifications. After the symplectic change of coordinates (5.1.5), we have of course to replace λ_α with $\lambda_{\log}$. Hence we apply the rotation in the plane

$$R(t) := \begin{pmatrix} \cos \lambda_{\log} t & -\sin \lambda_{\log} t \\ \sin \lambda_{\log} t & \cos \lambda_{\log} t \end{pmatrix},$$

in the same way as in (5.1.6), getting

$$\begin{aligned} \mathscr{H}(\widetilde{Z}, \widetilde{W}, \widetilde{z}, \widetilde{w}) = {} & \frac{1}{2}(\widetilde{Z}_1^2 + \widetilde{Z}_2^2 + \widetilde{W}_1^2 + \widetilde{W}_2^2) - U_{\log}(\widetilde{z}, \widetilde{w}) \\ & + \lambda_{\log}(\widetilde{Z}_1 \widetilde{z}_2 - \widetilde{Z}_2 \widetilde{z}_1 + \widetilde{W}_1 \widetilde{w}_2 - \widetilde{W}_2 \widetilde{w}_1). \end{aligned}$$

Transformation (5.1.7) is now the following:

$$\begin{cases} \widetilde{Z} = \lambda_{\log}^{1/2} \widehat{Z} \\ \widetilde{W} = \lambda_{\log}^{1/2} \widehat{W} \\ \widetilde{z} = \lambda_{\log}^{-1/2} \widehat{z} \\ \widetilde{w} = \lambda_{\log}^{-1/2} \widehat{w} \end{cases}$$

and gives

$$\begin{aligned} \mathscr{H}(\widehat{Z}, \widehat{W}, \widehat{z}, \widehat{w}) = {} & \frac{\lambda_{\log}}{2}(\widehat{Z}_1^2 + \widehat{Z}_2^2 + \widehat{W}_1^2 + \widehat{W}_2^2) - U_{\log}(\lambda_{\log}^{-1/2} \widehat{z}, \lambda_{\log}^{-1/2} \widehat{w}) \\ & + \lambda_{\log}(\widehat{Z}_1 \widehat{z}_2 - \widehat{Z}_2 \widehat{z}_1 + \widehat{W}_1 \widehat{w}_2 - \widehat{W}_2 \widehat{w}_1). \end{aligned}$$

Then we rescale time by setting $\tau := \lambda_{\log} t$ and obtain

$$\begin{aligned}\widehat{\mathscr{H}}(\widehat{Z},\widehat{W},\widehat{z},\widehat{w}) &= \frac{1}{2}(\widehat{Z}_1^2+\widehat{Z}_2^2+\widehat{W}_1^2+\widehat{W}_2^2) - \frac{1}{\lambda_{\log}}U_{\log}(\lambda_{\log}^{-1/2}\widehat{z},\lambda_{\log}^{-1/2}\widehat{w})\\ &\quad + \widehat{Z}_1\widehat{z}_2 - \widehat{Z}_2\widehat{z}_1 + \widehat{W}_1\widehat{w}_2 - \widehat{W}_2\widehat{w}_1\\ &= \frac{1}{\lambda_{\log}}\mathscr{H}(\widehat{Z},\widehat{W},\widehat{z},\widehat{w}).\end{aligned}$$

Translation (5.1.9) sets the equilibrium point at the origin and we have[2]

$$\begin{aligned}\widehat{\mathscr{H}}(\bar{Z},\bar{W},\bar{z},\bar{w}) &= \frac{1}{2}[\bar{Z}_1^2+(\bar{Z}_2+1)^2+\bar{W}_1^2+\bar{W}_2^2]\\ &\quad - \frac{1}{\lambda_{\log}}U_{\log}(\lambda_{\log}^{-1/2}\bar{z},\lambda_{\log}^{-1/2}\bar{w})\\ &\quad + \bar{Z}_1\bar{z}_2 - (\bar{Z}_2+1)(\bar{z}_1+1) + \bar{W}_1\bar{w}_2 - \bar{W}_2\bar{w}_1.\end{aligned}$$

The Hessian of $\frac{1}{\lambda_{\log}}U_{\log}$ evaluated at the origin is

$$\frac{1}{\lambda_{\log}}D^2U_{\log}(0,0) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & \frac{1}{2}(2m_1-m_2-m_3) & \frac{\sqrt{3}}{2}(m_2-m_3)\\ 0 & 0 & \frac{\sqrt{3}}{2}(m_2-m_3) & -\frac{1}{2}(2m_1-m_2-m_3)\end{pmatrix};$$

an orthogonal transformation applied on the subspace E_3 to both positions $\bar{w}$ and momenta $\bar{W}$ diagonalises the lower right corner of $\lambda_{\log}^{-1}D^2U_{\log}(0,0)$, making it dependent only on β:

$$\frac{1}{\lambda_{\log}}D^2U_{\log}(0,0) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & \frac{1}{3}\sqrt{9-\beta} & 0\\ 0 & 0 & 0 & -\frac{1}{3}\sqrt{9-\beta}\end{pmatrix}.$$

By computing the Hessian of $\mathscr{H}$ and multiplying on the left by J, we find the matrix Λ of the statement. $\square$

Remark 5.3. We observe that (5.1.12) can be obtained from (5.1.4) simply by setting $\alpha = 0$. Therefore in the analysis that will follow we shall consider the logarithmic case as a subcase of the α-homogeneous one. Note that this is a remark *a posteriori*, since we could not deduce it directly from the relation

$$\frac{U_\alpha(q)-1}{\alpha} \sim U_{\log}(q) \qquad \text{as } \alpha \to 0^+,$$

which is only asymptotic.

2 Here and in the following, with a slight abuse of notation, we denote by $\lambda_{\log}^{-1/2}\bar{z}$ the vector $\left(\lambda_{\log}^{-1/2}(\bar{z}_1+1),\lambda_{\log}^{-1/2}\bar{z}_2\right)^T$.

5.2 LINEAR AND SPECTRAL STABILITY OF THE LAGRANGIAN SOLUTION

Recall that in Section 5.1 we established that there exists a system of symplectic coordinates such that the linearised system restricted to $E_2 \oplus E_3$ is represented in the standard basis of $\mathbb{R}^8$ by the matrix Λ defined in (5.1.4). Note now that Λ can be expressed as the diamond product $\Lambda_2 \diamond \Lambda_3$ of two matrices Λ_2 and Λ_3 defined by

$$\Lambda_2 \coloneqq \begin{pmatrix} 0 & 1 & \alpha+1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{5.2.1a}$$

$$\Lambda_3 \coloneqq \begin{pmatrix} 0 & 1 & \frac{1}{2}\left(\alpha + \frac{\alpha+2}{3}\sqrt{9-\beta}\right) & 0 \\ -1 & 0 & 0 & \frac{1}{2}\left(\alpha - \frac{\alpha+2}{3}\sqrt{9-\beta}\right) \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{5.2.1b}$$

the range of α now being $[0, 2)$ (cf. Remark 5.3). The former encodes the dynamics on the symplectic invariant subspace E_2, whereas the latter governs the motion on E_3.

System (5.1.3) thus decouples into two linear autonomous Hamiltonian subsystems on E_2 and E_3 respectively, and it follows that its fundamental solution $\Phi \in \mathscr{P}_{2\pi}(8)$ can be written as the diamond product of the fundamental solutions $\phi_2 \in \mathscr{P}_{2\pi}(4)$ and $\phi_3 \in \mathscr{P}_{2\pi}(4)$ of these subsystems.

REMARK 5.4. By virtue of the discussion given in Section 2.2 (cf. also [MS05, p. 271] for the gravitational case), the Hamiltonian system on the invariant subspace E_2 is equivalent to the generalised α-homogeneous and logarithmic Kepler problem. It is worth noting that the matrix Λ_3 coincides with Λ_2 when[3] $\beta = 0$: in this case then the essential part of the fundamental solution of the Lagrangian circular orbit coincides with the fundamental solution of the Kepler orbit.

The linear autonomous Hamiltonian system (5.1.3) is *spectrally stable* if the spectrum $\sigma(\Lambda)$ of Λ is contained in the imaginary axis $i\mathbb{R}$; we call it *linearly stable* if in addition the matrix Λ is diagonalisable. We say that System (5.1.3) is *degenerate* if $\ker \Lambda \neq \{0\}$.

3 Technically speaking we ruled out the possibility that the parameter β could be equal to 0 for two reasons. The first is that at some point of the derivation of the matrix of the linearised system we divided by β (cf. Section 2.2); the second is due to the fact that if $\beta = 0$ then two masses would vanish and therefore there would be no dynamics at all. However we consider the limit $\beta \to 0$ and the extension by continuity.

Note that the spectrum $\sigma(\Lambda)$ of Λ is the union $\sigma(\Lambda_2) \cup \sigma(\Lambda_3)$ of the spectra of Λ_2 and Λ_3 respectively. The eigenvalues of Λ_2 are

$$0,\ 0,\ \pm i\sqrt{2-\alpha};$$

hence the system is always degenerate for every $n \geqslant 3$. It is then natural, following Moeckel in [Moe94], to adopt the following terminology.

REMARK 5.5. We observe that when $\alpha = 2$ the spectrum of Λ_2 reduces to $\{0\}$, while for $\alpha > 2$ such matrix admits also two non-zero real eigenvalues. More precisely, when $\alpha = 2$ the two non-zero purely imaginary eigenvalues of Λ_2 collapse into the origin (this corresponds to a Krein collision in 1 for the eigenvalues of the monodromy matrix) and split into a pair of non-zero real eigenvalues when $\alpha > 2$. The value $\alpha = 2$ is then the threshold of linear stability on E_2.

As before, a relative equilibrium is *non-degenerate* if the remaining 4 eigenvalues (relative to Λ_3) are different from 0; we say that it is *spectrally stable* if these eigenvalues are purely imaginary and *linearly stable* if, in addition to this condition of spectral stability, Λ_3 is diagonalisable.

The eigenvalues of the Hamiltonian matrix Λ_3 are

$$\lambda_1^{\pm} := \pm\frac{1}{6}i\sqrt{36 - 18\alpha + 6\sqrt{9(\alpha-2)^2 - \beta(\alpha+2)^2}}$$
$$\lambda_2^{\pm} := \pm\frac{1}{6}i\sqrt{36 - 18\alpha - 6\sqrt{9(\alpha-2)^2 - \beta(\alpha+2)^2}}$$

and their direct study leads to a picture of the zones of stability and instability in the parameter space (see Figure 1.1 on page 7).

PROPOSITION 5.6. *The rectangle $(0, 9] \times [0, 2)$ is divided into three regions, depending on the stability of the relative equilibrium determined by the parameters α and β:*

1. *Region of* linear stability
$$LS := \left\{ (\beta, \alpha) \in (0, 9] \times [0, 2) \;\middle|\; \beta < 9\left(\frac{\alpha-2}{\alpha+2}\right)^2 \right\};$$
2. *Curve of* spectral (but not linear) stability
$$SS := \left\{ (\beta, \alpha) \in (0, 9] \times [0, 2) \;\middle|\; \beta = 9\left(\frac{\alpha-2}{\alpha+2}\right)^2 \right\};$$
3. *Region of* spectral instability
$$SI := \left\{ (\beta, \alpha) \in (0, 9] \times [0, 2) \;\middle|\; \beta > 9\left(\frac{\alpha-2}{\alpha+2}\right)^2 \right\}.$$

Proof. A direct computation shows that the eigenvalues of Λ_3 are purely imaginary in $LS \cup SS$; however on the stability curve SS they collide and form two pairs of purely imaginary eigenvalues which give rise to two Jordan blocks, so that diagonalisability is lost. In the region SI their real part is different from 0. □

Remark 5.7. Let us observe that as β is arbitrarily small (which corresponds to the presence of a dominant mass) and α is bounded away from 2 we lie in the region of linear stability. Such a result agrees with Moeckel's conjecture on the dominant mass, according to which relative equilibria with a dominant mass are linearly stable.

5.3 MASLOV INDEX OF THE GENERALISED KEPLER PROBLEM

The aim of this section is to compute the ω-index of the restriction of the Hamiltonian system (5.1.3) to the invariant subspace E_2 of the phase space. As already observed, the Hamiltonian function on this subspace coincides with the Hamiltonian of the generalised (i.e. α-homogeneous and logarithmic) Kepler problem.

5.3.1 Computation of the Maslov index

Consider the linear autonomous Hamiltonian initial value problem

$$\begin{cases} \dot{\phi}_2(\tau) = \Lambda_2 \phi_2(\tau) \\ \phi_2(0) = I_4. \end{cases} \tag{5.3.1}$$

Here ϕ_2 is the restriction to E_2 of the fundamental solution Φ of the Lagrangian circular orbit.

Proposition 5.8. *The Maslov index of the fundamental solution ϕ_2 of System (5.3.1) is*

$$i_1(\phi_2) = \begin{cases} 0 & \text{if } \alpha \in [1, 2) \\ 2 & \text{if } \alpha \in [0, 1). \end{cases}$$

Proof. Here is the fundamental solution $\phi_2(\tau) := \exp(\tau\Lambda_2)$, with $\tau \in [0, 2\pi]$:

$$\phi_2(\tau) = \begin{pmatrix} \frac{2-\alpha\cos(\sqrt{2-\alpha}\,\tau)}{2-\alpha} & \frac{2+\alpha}{2-\alpha}\tau - \frac{2\alpha\sin(\sqrt{2-\alpha}\,\tau)}{(2-\alpha)^{3/2}} & \frac{2+\alpha}{2-\alpha}\tau - \frac{\alpha^2\sin(\sqrt{2-\alpha}\,\tau)}{(2-\alpha)^{3/2}} & \frac{\alpha[1-\cos(\sqrt{2-\alpha}\,\tau)]}{2-\alpha} \\ -\frac{\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} & \frac{2\cos(\sqrt{2-\alpha}\,\tau)-\alpha}{2-\alpha} & \frac{\alpha[\cos(\sqrt{2-\alpha}\,\tau)-1]}{2-\alpha} & -\frac{\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} \\ \frac{\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} & \frac{2-2\cos(\sqrt{2-\alpha}\,\tau)}{2-\alpha} & \frac{2-\alpha\cos(\sqrt{2-\alpha}\,\tau)}{2-\alpha} & \frac{\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} \\ \frac{2\cos(\sqrt{2-\alpha}\,\tau)-2}{2-\alpha} & \frac{4\sin(\sqrt{2-\alpha}\,\tau)}{(2-\alpha)^{3/2}} - \frac{2+\alpha}{2-\alpha}\tau & \frac{2\alpha\sin(\sqrt{2-\alpha}\,\tau)}{(2-\alpha)^{3/2}} - \frac{2+\alpha}{2-\alpha}\tau & \frac{2\cos(\sqrt{2-\alpha}\,\tau)-\alpha}{2-\alpha} \end{pmatrix}.$$

Following [HLS14], if we consider the symplectic matrix

$$P := \begin{pmatrix} 1 & 0 & 0 & 6\pi \\ 0 & -\frac{1}{6\pi} & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6\pi \end{pmatrix}$$

we see that $\phi_2(\tau)$ is symplectically equivalent to $\widetilde{\phi}_2(\tau) := P^{-1}\phi_2(\tau)P$, which is given by

$$\widetilde{\phi}_2(\tau) := \begin{pmatrix} \cos(\sqrt{2-\alpha}\,\tau) & -\frac{2\sin(\sqrt{2-\alpha}\,\tau)}{6\pi\sqrt{2-\alpha}} & -\sqrt{2-\alpha}\sin(\sqrt{2-\alpha}\,\tau) & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} & \frac{2\cos(\sqrt{2-\alpha}\,\tau)-2}{6\pi(2-\alpha)} & \cos(\sqrt{2-\alpha}\,\tau) & 0 \\ \frac{2-2\cos(\sqrt{2-\alpha}\,\tau)}{6\pi(2-\alpha)} & \frac{1}{36\pi^2}\left(\frac{4\sin(\sqrt{2-\alpha}\,\tau)}{(2-\alpha)^{3/2}} - \frac{2+\alpha}{2-\alpha}\tau\right) & \frac{2\sin(\sqrt{2-\alpha}\,\tau)}{6\pi\sqrt{2-\alpha}} & 1 \end{pmatrix};$$

it follows, by the naturality property, that $i_1(\phi_2) = i_1(\widetilde{\phi}_2)$. Take now the homotopy $F : [0,1] \times [0,2\pi] \to \mathrm{Sp}(4)$ defined by

$$F(s,\tau) := \begin{pmatrix} \cos(\sqrt{2-\alpha}\,\tau) & -s\frac{2\sin(\sqrt{2-\alpha}\,\tau)}{6\pi\sqrt{2-\alpha}} & -\sqrt{2-\alpha}\sin(\sqrt{2-\alpha}\,\tau) & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} & s\frac{2\cos(\sqrt{2-\alpha}\,\tau)-2}{6\pi(2-\alpha)} & \cos(\sqrt{2-\alpha}\,\tau) & 0 \\ s\frac{2-2\cos(\sqrt{2-\alpha}\,\tau)}{6\pi(2-\alpha)} & \frac{1}{36\pi^2}\left(\frac{4\sin(\sqrt{2-\alpha}\,\tau)}{(2-\alpha)^{3/2}} - \frac{2+\alpha}{2-\alpha}\tau\right) & s\frac{2\sin(\sqrt{2-\alpha}\,\tau)}{6\pi\sqrt{2-\alpha}} & 1 \end{pmatrix}.$$

It is admissible because we have that $F(1,\tau) = \widetilde{\phi}_2(s,\tau) \in \mathrm{Sp}(4)$ and $F(s,0) = I_4$ for all $s \in [0,1]$ and all $\tau \in [0,2\pi]$. Moreover, $F(1,\tau) = \widetilde{\phi}_2(\tau)$ and

$$\begin{aligned}\widetilde{\phi}_2(0,\tau) &= \begin{pmatrix} \cos(\sqrt{2-\alpha}\,\tau) & 0 & -\sqrt{2-\alpha}\sin(\sqrt{2-\alpha}\,\tau) & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} & 0 & \cos(\sqrt{2-\alpha}\,\tau) & 0 \\ 0 & \frac{1}{36\pi^2}\left(\frac{4\sin(\sqrt{2-\alpha}\,\tau)}{(2-\alpha)^{3/2}} - \frac{2+\alpha}{2-\alpha}\tau\right) & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\sqrt{2-\alpha}\,\tau) & -\sqrt{2-\alpha}\sin(\sqrt{2-\alpha}\,\tau) \\ \frac{\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} & \cos(\sqrt{2-\alpha}\,\tau) \end{pmatrix} \diamond \begin{pmatrix} 1 & 0 \\ \frac{1}{36\pi^2}\left(\frac{4\sin(\sqrt{2-\alpha}\,\tau)}{(2-\alpha)^{3/2}} - \frac{2+\alpha}{2-\alpha}\tau\right) & 1 \end{pmatrix} \\ &=: R_\alpha(\tau) \diamond N_\alpha(\tau).\end{aligned}$$

Therefore, being the Maslov index a homotopic invariant, we have

$$i_1(\phi_2) = i_1(R_\alpha) + i_1(N_\alpha). \tag{5.3.2}$$

From Example 4.10, Example 4.11 and Lemma 4.8 we find

$$i_1(R_\alpha) = \begin{cases} 1 & \text{if } \alpha \in (1,2) \\ 3 & \text{if } \alpha \in [0,1), \end{cases} \qquad i_1(N_\alpha) = -1 \qquad \forall\, \alpha \in [0,2),$$

and the thesis follows. □

5.3.2 Computation of the ω-index on E_2

Next we compute the ω-index $i_\omega(\phi_2)$ for all $\omega \in \mathbb{U}\setminus\{1\}$. To this end we have to compute first the splitting numbers of the monodromy matrix

$$M_2 := \phi_2(2\pi) \sim R_\alpha(2\pi) \diamond N_1(1,1) \qquad \text{for every } \alpha \in [0,2).$$

We note that $R_\alpha(\tau)$ is not a normal form for every $\tau \in [0,2\pi]$; however, it is homotopic to the rotation $R(\sqrt{2-\alpha}\,\tau)$ via the map $G : [0,1] \times [0,2\pi] \to \Omega^0(R_\alpha)$ defined by

$$G(s,\tau) := \begin{pmatrix} \cos(\sqrt{2-\alpha}\,\tau) & -\frac{\sqrt{2-\alpha}\sin(\sqrt{2-\alpha}\,\tau)}{1-s+s\sqrt{2-\alpha}} \\ \frac{(1-s+s\sqrt{2-\alpha})\sin(\sqrt{2-\alpha}\,\tau)}{\sqrt{2-\alpha}} & \cos(\sqrt{2-\alpha}\,\tau) \end{pmatrix}.$$

Accordingly, for all $\alpha \in [0,2)$

$$M_2 \sim R(\theta_\alpha) \diamond N_1(1,1), \tag{5.3.3}$$

where, modulo 2π,

$$\theta_\alpha := 2\pi\sqrt{2-\alpha} \in \begin{cases} \{0\} & \text{if } \alpha = 1 \\ (0,\pi) & \text{if } \alpha \in [0,1) \cup \left(\frac{7}{4},2\right) \\ \{\pi\} & \text{if } \alpha = \frac{7}{4} \\ (\pi,2\pi) & \text{if } \alpha \in \left(1,\frac{7}{4}\right) \end{cases} \tag{5.3.4}$$

Proposition 5.9. *The ω-index $i_\omega(\phi_2)$ of the fundamental solution ϕ_2 is given by:*

(i) $\alpha \in \left[\frac{7}{4},2\right)$*:*

$$i_\omega(\phi_2) = \begin{cases} 1 & \textit{if } 0 < \theta < \theta_\alpha \\ 0 & \textit{if } \theta_\alpha \leqslant \theta \leqslant \pi \end{cases}$$

(ii) $\alpha \in \left(1,\frac{7}{4}\right)$*:*

$$i_\omega(\phi_2) = \begin{cases} 1 & \textit{if } 0 < \theta \leqslant -\theta_\alpha \\ 2 & \textit{if } -\theta_\alpha < \theta \leqslant \pi \end{cases}$$

(iii) $\alpha = 1$*:*

$$i_\omega(\phi_2) = 2 \quad \textit{for all } \theta \in (0,\pi]$$

(iv) $\alpha \in [0,1)$*:*

$$i_\omega(\phi_2) = \begin{cases} 3 & \textit{if } 0 < \theta < \theta_\alpha \\ 2 & \textit{if } -\theta_\alpha \leqslant \theta \leqslant \pi \end{cases}$$

where $\omega = e^{i\theta} \neq 1$.

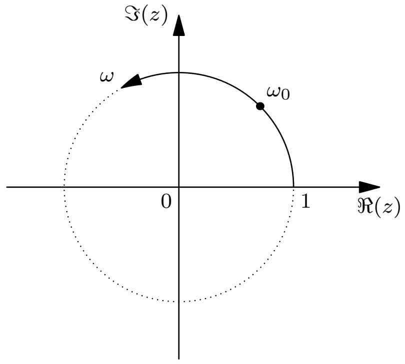


Figure 5.1. Position of ω and ω_0.

Proof. Item 8 of Proposition 4.15 gives

$$i_\omega(\phi_2) = i_1(\phi_2) + S^+_{M_2}(1) + \sum_{\omega_0}\big(S^+_{M_2}(\omega_0) - S^-_{M_2}(\omega_0)\big) - S^-_{M_2}(\omega), \tag{5.3.5}$$

where $\omega \in \mathbb{U}\setminus\{1\}$ is such that $\mathfrak{I}(\omega) \geqslant 0$ and $\omega_0 \in \sigma(M_2)$ lies in the interior of the arc of the upper unit semicircle connecting 1 and ω (see Figure 5.1). Note that the assumption $\mathfrak{I}(\omega) \geqslant 0$ does not imply any loss of generality: by virtue of Item 4 of Proposition 4.15 we have indeed that

$$i_{\overline{\omega}}(\phi_2) = i_\omega(\phi_2).$$

From (5.3.3) we find that for every $\omega \in \mathbb{U}$ with $\mathfrak{I}(\omega) \geqslant 0$

$$S^\pm_{M_2}(\omega) = \begin{cases} S^\pm_{R(\theta_\alpha)}(\omega) + S^\pm_{N_1(1,1)}(\omega) & \text{if } \alpha \in [0,2)\setminus\{1,\frac{7}{4}\}, \\ S^\pm_{-I_2}(\omega) + S^\pm_{N_1(1,1)}(\omega) & \text{if } \alpha = \frac{7}{4}, \\ S^\pm_{I_2}(\omega) + S^\pm_{N_1(1,1)}(\omega) & \text{if } \alpha = 1. \end{cases}$$

Thanks to the results collected in Proposition 4.15 we know that if $\omega \notin \sigma(M_2) = \{1, 1, e^{i\theta_\alpha}, e^{-i\theta_\alpha}\}$ then $S^\pm_{M_2}(\omega) = 0$; moreover the splitting numbers involved are the following:

$$\begin{aligned} \big(S^+_{N_1(1,1)}(1), S^-_{N_1(1,1)}(1)\big) &= (1,1), \\ \big(S^+_{R(\theta_\alpha)}(e^{i\theta_\alpha}), S^-_{R(\theta_\alpha)}(e^{i\theta_\alpha})\big) &= (0,1), \qquad \forall\, \alpha \in [0,2)\setminus\{1,\tfrac{7}{4}\} \\ \big(S^+_{I_2}(1), S^-_{I_2}(1)\big) &= (1,1), \\ \big(S^+_{-I_2}(-1), S^-_{-I_2}(-1)\big) &= (1,1). \end{aligned}$$

Writing $\omega := e^{i\theta}$, we are now able to compute the ω-index depending on α and on the position of ω with respect to the eigenvalues $e^{\pm i\theta_\alpha}$ (modulo 2π). Using Formula (5.3.5), we distinguish the following cases:

(i) $\alpha \in \left(\frac{7}{4}, 2\right)$:

$$i_\omega(\phi_2) = \begin{cases} i_1(\phi_2) + S^+_{M_2}(1) & \text{if } \theta \in (0, \theta_\alpha) \\ i_1(\phi_2) + S^+_{M_2}(1) - S^-_{M_2}(e^{i\theta_\alpha}) & \text{if } \theta = \theta_\alpha \\ i_1(\phi_2) + S^+_{M_2}(1) + S^+_{M_2}(e^{i\theta_\alpha}) - S^-_{M_2}(e^{i\theta_\alpha}) & \text{if } \theta \in (\theta_\alpha, \pi], \end{cases}$$

leading to

$$i_\omega(\phi_2) = \begin{cases} 1 & \text{if } \theta \in (0, \theta_\alpha) \\ 0 & \text{if } \theta \in [\theta_\alpha, \pi]. \end{cases}$$

(ii) $\alpha = \frac{7}{4}$:

$$i_\omega(\phi_2) = \begin{cases} i_1(\phi_2) + S^+_{M_2}(1) & \text{if } \theta \in (0, \pi) \\ i_1(\phi_2) + S^+_{M_2}(1) - S^-_{M_2}(-1) & \text{if } \theta = \pi, \end{cases}$$

giving

$$i_\omega(\phi_2) = \begin{cases} 1 & \text{if } \theta \in (0, \pi) \\ 0 & \text{if } \theta = \pi. \end{cases}$$

(iii) $\alpha \in \left(1, \frac{7}{4}\right)$:

$$i_\omega(\phi_2) = \begin{cases} i_1(\phi_2) + S^+_{M_2}(1) & \text{if } \theta \in (0, -\theta_\alpha) \\ i_1(\phi_2) + S^+_{M_2}(1) - S^-_{M_2}(e^{-i\theta_\alpha}) & \text{if } \theta = -\theta_\alpha \\ i_1(\phi_2) + S^+_{M_2}(1) + S^+_{M_2}(e^{-i\theta_\alpha}) - S^-_{M_2}(e^{-i\theta_\alpha}) & \text{if } \theta \in (-\theta_\alpha, \pi], \end{cases}$$

yielding

$$i_\omega(\phi_2) = \begin{cases} 1 & \text{if } \theta \in (0, -\theta_\alpha] \\ 2 & \text{if } \theta \in (-\theta_\alpha, \pi]. \end{cases}$$

(iv) $\alpha = 1$:

$$i_\omega(\phi_2) = i_1(\phi_2) + S^+_{M_2}(1) = 2 \qquad \text{for all } \theta \in (0, \pi].$$

(v) $\alpha \in [0, 1)$:

$$i_\omega(\phi_2) = \begin{cases} i_1(\phi_2) + S^+_{M_2}(1) & \text{if } \theta \in (0, \theta_\alpha) \\ i_1(\phi_2) + S^+_{M_2}(1) - S^-_{M_2}(e^{i\theta_\alpha}) & \text{if } \theta = \theta_\alpha \\ i_1(\phi_2) + S^+_{M_2}(1) + S^+_{M_2}(e^{i\theta_\alpha}) - S^-_{M_2}(e^{i\theta_\alpha}) & \text{if } \theta \in (\theta_\alpha, \pi], \end{cases}$$

obtaining

$$i_\omega(\phi_2) = \begin{cases} 3 & \text{if } \theta \in (0, \theta_\alpha) \\ 2 & \text{if } \theta \in [\theta_\alpha, \pi]. \end{cases}$$ □

The following result is a direct consequence of Lemma 4.4 and generalises [HS10, Proposition 3.6] to the α-homogeneous case.

Proposition 5.10. *Let ϕ_2 be the fundamental solution of System (5.3.1) and $k \in \mathbb{N} \setminus \{0\}$. Then the Maslov index of the k-th iteration ϕ_2^k of ϕ_2 is given by $i_1(\phi_2^k) = \sum_{\omega^k=1} i_\omega(\phi_2)$ and is equal to:*

(i) $\alpha \in (\frac{7}{4}, 2)$:

$$i_1(\phi_2^k) = 2(n^-_{k,\alpha} - 1),$$

where $n^-_{k,\alpha}$ is the number of k-th roots of unity in the arc $[1, e^{i\theta_\alpha})$;

(ii) $\alpha \in \left(1, \frac{7}{4}\right)$:

$$i_1(\phi_2^k) = \begin{cases} 2(n^-_{k,\alpha} - 1) + 4(n^+_{k,\alpha} - 1) + 2 & \text{if } k \text{ is even} \\ 2(n^-_{k,\alpha} - 1) + 4n^+_{k,\alpha} & \text{if } k \text{ is odd} \end{cases}$$

where $n^-_{k,\alpha}$ is the number of k-th roots of unity in the arc $[1, e^{-i\theta_\alpha}]$ and $n^+_{k,\alpha}$ is the number of k-th roots of unity in the arc $(e^{-i\theta_\alpha}, -1]$;

(iii) $\alpha = 1$:

$$i_1(\phi_2^k) = 2(k-1)$$

(iv) $\alpha \in [0, 1)$:

$$i_1(\phi_2^k) = \begin{cases} 6(n^-_{k,\alpha} - 1) + 4(n^+_{k,\alpha} - 1) + 4 & \text{if } k \text{ is even} \\ 6(n^-_{k,\alpha} - 1) + 4n^+_{k,\alpha} + 2 & \text{if } k \text{ is odd,} \end{cases}$$

where $n^-_{k,\alpha}$ is the number of k-th roots of unity in the arc $[1, e^{\pm i\theta_\alpha})$ and $n^+_{k,\alpha}$ is the number of k-th roots of unity in the arc $[e^{\pm i\theta_\alpha}, -1]$.

We observe that, for fixed k, the index $i_1(\phi_2^k)$ is constant on horizontal bands of the rectangle $(0, 9] \times [0, 2)$, since it is independent of β (see Figure 1.3 on page 10). From the previous proposition it is evident that the index is monotonically non-increasing as α increases for every $k \in \mathbb{N} \setminus \{0\}$.

Since the computation of the Maslov index of the iterate is based on the Bott-Long formula, it is clear that the only contributions to this value are given by those ω-indices for which ω is a root of unity. This means that one has a jump in the index of the k-th iterate only when the angle θ_α (defined in (5.3.4)) is a rational multiple of 2π, i.e. $\theta_\alpha = \frac{2l\pi}{k}$ for some $l \in \mathbb{N} \setminus \{0\}$. Now, since $\theta_\alpha \in [0, 2\sqrt{2}\pi]$ it follows that l actually ranges in the set $\{1, \dots, [\sqrt{2}k]\}$.

In particular the Maslov index vanishes when $0 < \theta_\alpha < \frac{2\pi}{k}$, that is when $\alpha > 2 - \frac{1}{k^2}$. As k increases, the horizontal lines corresponding to the jumps of $i_1(\phi_2^k)$, which are characterised by the double sequence $(\alpha_{k,l})$ with $\alpha_{k,l} \coloneqq 2 - \frac{l^2}{k^2}$, accumulate at the stability threshold $\alpha = 2$ as $k \to +\infty$ (see Remark 5.5).

Let us now fix $\alpha \in [0, 2)$. The number of k-th roots of unity in the arc $[1, e^{\pm i\theta_\alpha})$ increases with k and diverges to $+\infty$ as $k \to +\infty$, hence $i_1(\phi_2^k) \to +\infty$ as $k \to +\infty$.

5.4 THE ω-INDEX ASSOCIATED WITH THE RESTRICTION TO E_3

In this section we perform the computation of the ω-index of the restriction ϕ_3 to E_3 of the fundamental solution Φ of the Lagrangian circular orbit. This will be achieved, as before, by means of the splitting numbers.

5.4.1 Computation of the Maslov index

The restriction ϕ_3 to E_3 of the fundamental solution Φ of the Lagrangian circular orbit satisfies the linear autonomous Hamiltonian initial value problem

$$\begin{cases} \dot{\phi}_3(\tau) = \Lambda_3 \phi_3(\tau) \\ \phi_3(0) = I_4. \end{cases} \tag{5.4.1}$$

By taking into account Proposition 5.6, we immediately get the following result.

Proposition 5.11. *The Maslov index $i_1(\phi_3)$ is zero for all $(\beta, \alpha) \in SI$.*

Proof. The eigenvalues that contribute to the Maslov index are only the ones contained in $\mathbb{U}$. If $9(\alpha-2)^2 - \beta(\alpha+2)^2 < 0$ (i.e. in the region *SI*) the spectrum is contained in $\mathbb{C} \setminus (\mathbb{U} \cup \mathbb{R})$ and the result follows. □

The monodromy matrix $M_3 := \phi_3(2\pi) := \exp(2\pi\Lambda_3)$ is non-degenerate in the whole region *LS* of linear stability, except on the curve of equation

$$\beta = \frac{36(1-\alpha)}{(\alpha+2)^2}, \tag{5.4.2}$$

where two of the four eigenvalues are equal to 1. On the stability curve *SS* of equation

$$\beta = 9\left(\frac{\alpha-2}{\alpha+2}\right)^2,$$

instead, M_3 is non-degenerate but not diagonalisable. We can compute its Maslov index in the non-degenerate subzone of *LS* by using again the formula of Proposition 4.13: the Krein-positive eigenvalues of Λ_3 are

$$\lambda_1^- = -\frac{1}{6} i \sqrt{36 - 18\alpha + 6\sqrt{9(\alpha-2)^2 - \beta(\alpha+2)^2}}$$

and

$$\lambda_2^+ = \frac{1}{6} i \sqrt{36 - 18\alpha - 6\sqrt{9(\alpha-2)^2 - \beta(\alpha+2)^2}}$$

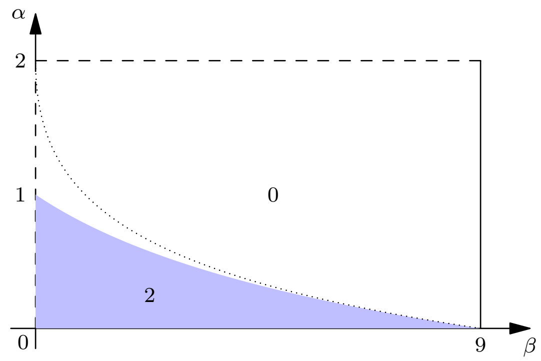


Figure 5.2. Values of $i_1(\phi_3)$. The dotted curve is the stability curve.

for all $(\beta, \alpha) \in LS$, so that

$$i_1(\phi_3) = \begin{cases} 0 & \text{if } \dfrac{36(1-\alpha)}{(\alpha+2)^2} < \beta < 9\dfrac{(\alpha-2)^2}{(\alpha+2)^2} \\ 2 & \text{if } 0 < \beta < \dfrac{36(1-\alpha)}{(\alpha+2)^2}. \end{cases}$$

However, since the Maslov index is a lower semicontinuous function, we conclude that $i_1(\phi_3) = 0$ also on the curve (5.4.2) and on the stability curve:

$$i_1(\phi_3) = \begin{cases} 0 & \text{if } \beta \geqslant \dfrac{36(1-\alpha)}{(\alpha+2)^2} \\ 2 & \text{if } 0 < \beta < \dfrac{36(1-\alpha)}{(\alpha+2)^2}. \end{cases}$$

The result is depicted in Figure 5.2.

5.4.2 Computation of the ω-index on E_3

The monodromy matrix $M_3 := \exp(2\pi\Lambda_3)$ is similar to the diagonal matrix

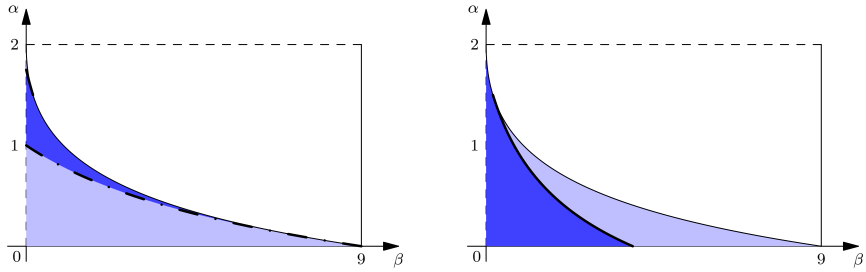
$$\operatorname{diag}(e^{2\pi\lambda_1^-}, e^{2\pi\lambda_2^+}, e^{2\pi\lambda_1^+}, e^{2\pi\lambda_2^-})$$

and can consequently be expressed as

$$M_3 = R(\theta^{(1)}_{\alpha,\beta}) \diamond R(\theta^{(2)}_{\alpha,\beta}),$$

with $\theta^{(1)}_{\alpha,\beta} := \Im(2\pi\lambda_1^+)$ and $\theta^{(2)}_{\alpha,\beta} := \Im(2\pi\lambda_2^-)$.

Remark 5.12. Note that these two angles correspond to the Krein-negative eigenvalues; the reason is the following. When $\beta \to 0$ the dynamics of the problem reduces to that of a generalised Kepler problem, i.e. to the restriction to E_2 previously analysed. The values of the ω-index must then agree with the ones found in the previous study when approaching the segment $\{0\} \times [0, 2)$ as β tends to 0, and this forces the choice of the two eigenvalues.



(a) *Dashed-dotted line:* $\theta^{(1)}_{\alpha,\beta} = 0$; *Solid line:* $\theta^{(1)}_{\alpha,\beta} = \pi$; *Light shade:* $\theta^{(1)}_{\alpha,\beta} \in (0,\pi)$, *Dark shade:* $\theta^{(1)}_{\alpha,\beta} \in (\pi, 2\pi)$.

(b) *Solid line:* $\theta^{(2)}_{\alpha,\beta} = \pi$; *Light shade:* $\theta^{(2)}_{\alpha,\beta} \in (0,\pi)$; *Dark shade:* $\theta^{(2)}_{\alpha,\beta} \in (\pi, 2\pi)$.

Figure 5.3. Values of $\theta^{(1)}_{\alpha,\beta}$ (a) and $\theta^{(2)}_{\alpha,\beta}$ (b) modulo 2π.

Observe that in the region *LS* these angles take the following values (modulo 2π):

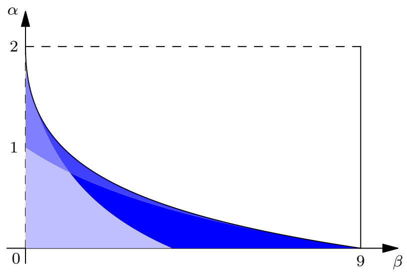
$$\theta^{(1)}_{\alpha,\beta} \in \begin{cases} \{0\} & \text{if } \beta = \dfrac{36(1-\alpha)}{(\alpha+2)^2} \\ (0,\pi) & \text{if } \beta < \dfrac{36(1-\alpha)}{(\alpha+2)^2} \text{ or } \left(\beta > \dfrac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha > \dfrac{3}{2}\right) \\ \{\pi\} & \text{if } \beta = \dfrac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha > \dfrac{3}{2} \\ (\pi, 2\pi) & \text{if } \beta > \dfrac{36(1-\alpha)}{(\alpha+2)^2} \text{ and } \left(\beta < \dfrac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ or } \alpha < \dfrac{3}{2}\right) \end{cases} \tag{5.4.3}$$

$$\theta^{(2)}_{\alpha,\beta} \in \begin{cases} (0,\pi) & \text{if } \beta > \dfrac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha < \dfrac{3}{2} \\ \{\pi\} & \text{if } \beta = \dfrac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha < \dfrac{3}{2} \\ (\pi, 2\pi) & \text{if } \beta < \dfrac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ or } \alpha > \dfrac{3}{2}. \end{cases} \tag{5.4.4}$$

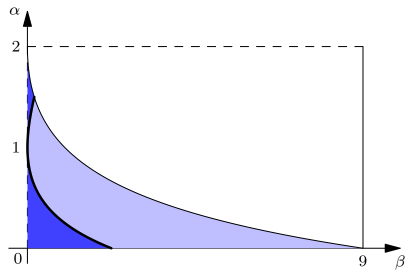
Figure 5.3a and Figure 5.3b show the involved regions, and they are superposed in Figure 5.4a. In order to compute the splitting numbers and eventually find the ω-index we have to determine not only the absolute position of $\theta^{(1)}_{\alpha,\beta}$ and $\theta^{(2)}_{\alpha,\beta}$ on $\mathbb{U}$ (which is the one given above), but also how their *relative* position changes as the parameters α and β vary. This is represented in Figure 5.4b.

Now, for every $\omega \in \mathbb{U}$ we have that

$$S^{\pm}_{M_3}(\omega) = S^{\pm}_{R(\theta^{(1)}_{\alpha,\beta})}(\omega) + S^{\pm}_{R(\theta^{(2)}_{\alpha,\beta})}(\omega)$$



(a) *Light shade:* $\theta^{(1)}_{\alpha,\beta} \in (0,\pi)$ and $\theta^{(2)}_{\alpha,\beta} \in (\pi, 2\pi)$; *Medium shade:* $\theta^{(1)}_{\alpha,\beta}, \theta^{(2)}_{\alpha,\beta} \in (\pi, 2\pi)$; *Heavy shade:* $\theta^{(1)}_{\alpha,\beta} \in (\pi, 2\pi)$ and $\theta^{(2)}_{\alpha,\beta} \in (0,\pi)$; *Dark shade:* $\theta^{(1)}_{\alpha,\beta}, \theta^{(2)}_{\alpha,\beta} \in (0,\pi)$.



(b) If $\widetilde{\theta}^{(1)}_{\alpha,\beta}$ and $\widetilde{\theta}^{(2)}_{\alpha,\beta}$ are the representatives of $\pm\theta^{(1)}_{\alpha,\beta}$ and $\pm\theta^{(2)}_{\alpha,\beta}$ in the upper unit semicircle, then the colours have to be interpreted in the following way. *Light shade:* $\widetilde{\theta}^{(1)}_{\alpha,\beta} < \widetilde{\theta}^{(2)}_{\alpha,\beta}$; *Solid line:* $\widetilde{\theta}^{(1)}_{\alpha,\beta} = \widetilde{\theta}^{(2)}_{\alpha,\beta}$; *Dark shade:* $\widetilde{\theta}^{(1)}_{\alpha,\beta} > \widetilde{\theta}^{(2)}_{\alpha,\beta}$.

Figure 5.4. Values of $\theta^{(1)}_{\alpha,\beta}$ and $\theta^{(2)}_{\alpha,\beta}$ (a) and their relative position (b) modulo 2π.

and $S^{\pm}_{M_3}(\omega) = 0$ if $\omega \notin \sigma(M_3) = \{e^{\pm i\theta^{(1)}_{\alpha,\beta}}, e^{\pm i\theta^{(2)}_{\alpha,\beta}}\}$. In order to compute the ω-index we use the formula

$$i_\omega(\phi_3) - i_1(\phi_3) = S^{+}_{M_3}(1) + \sum_{\omega_0} \big(S^{+}_{M_3}(\omega_0) - S^{-}_{M_3}(\omega_0)\big) - S^{-}_{M_3}(\omega),$$

where $\omega \in \mathbb{U} \setminus \{1\}$ is such that $\Im(\omega) \geqslant 0$ and $\omega_0 \in \sigma(M_3)$ lies in the interior of the arc of the upper unit semicircle connecting 1 and ω (see Figure 5.1). The splitting numbers involved are the following:

$$\big(S^{+}_{M_3}(1), S^{-}_{M_3}(1)\big) = \begin{cases} (1,1) & \text{if } \beta = \dfrac{36(1-\alpha)}{(\alpha+2)^2} \\ (0,0) & \text{otherwise} \end{cases}$$

$$\big(S^{+}_{M_3}(-1), S^{-}_{M_3}(-1)\big) = \begin{cases} (1,1) & \text{if } \beta = \dfrac{9(7-4\alpha)}{4(\alpha+2)^2} \text{ and } \alpha \neq \dfrac{3}{2} \\ (0,0) & \text{otherwise} \end{cases}$$

$$\big(S^{+}_{M_3}(e^{i\theta^{(1)}_{\alpha,\beta}}), S^{-}_{M_3}(e^{i\theta^{(1)}_{\alpha,\beta}})\big) = \begin{cases} (0,1) & \text{for all } \theta^{(1)}_{\alpha,\beta} \notin \{0, \pi, \pm\theta^{(2)}_{\alpha,\beta}\} \\ (0,2) & \text{if } \theta^{(1)}_{\alpha,\beta} = \theta^{(2)}_{\alpha,\beta} \\ (1,1) & \text{if } \theta^{(1)}_{\alpha,\beta} = -\theta^{(2)}_{\alpha,\beta} \end{cases}$$

$$\big(S^{+}_{M_3}(e^{i\theta^{(2)}_{\alpha,\beta}}), S^{-}_{M_3}(e^{i\theta^{(2)}_{\alpha,\beta}})\big) = \begin{cases} (0,1) & \text{for all } \theta^{(2)}_{\alpha,\beta} \notin \left\{ 0, \pi, \pm\theta^{(1)}_{\alpha,\beta} \right\} \\ (0,2) & \text{if } \theta^{(2)}_{\alpha,\beta} = \theta^{(1)}_{\alpha,\beta} \\ (1,1) & \text{if } \theta^{(2)}_{\alpha,\beta} = -\theta^{(1)}_{\alpha,\beta} \end{cases}$$

The ω-index depends therefore on the values of α and β. Writing $\omega := e^{i\theta}$, we have

i) $\beta > \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$ and $\alpha > \dfrac{3}{2}$:

$$i_\omega(\phi_3) = \begin{cases} 0 & \text{if } 0 < \theta \leqslant -\theta^{(2)}_{\alpha,\beta} \\ 1 & \text{if } -\theta^{(2)}_{\alpha,\beta} < \theta < \theta^{(1)}_{\alpha,\beta} \\ 0 & \text{if } \theta^{(1)}_{\alpha,\beta} \leqslant \theta \leqslant \pi \end{cases}$$

ii) $\beta = \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$ and $\alpha > \dfrac{3}{2}$:

$$i_\omega(\phi_3) = \begin{cases} 0 & \text{if } 0 < \theta \leqslant -\theta^{(2)}_{\alpha,\beta} \\ 1 & \text{if } -\theta^{(2)}_{\alpha,\beta} < \theta < \pi \\ 0 & \text{if } \theta = \pi \end{cases}$$

iii) $\beta < \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$ and $\beta < 9\dfrac{(\alpha-1)^2}{(\alpha+2)^2}$ and $\alpha > 1$:

$$i_\omega(\phi_3) = \begin{cases} 0 & \text{if } 0 < \theta \leqslant -\theta^{(2)}_{\alpha,\beta} \\ 1 & \text{if } -\theta^{(2)}_{\alpha,\beta} < \theta \leqslant -\theta^{(1)}_{\alpha,\beta} \\ 2 & \text{if } -\theta^{(1)}_{\alpha,\beta} < \theta \leqslant \pi \end{cases}$$

iv) $\beta = 9\dfrac{(\alpha-1)^2}{(\alpha+2)^2}$ and $\alpha > 1$:

$$i_\omega(\phi_3) = \begin{cases} 0 & \text{if } 0 < \theta \leqslant \theta^{(2)}_{\alpha,\beta} = \theta^{(1)}_{\alpha,\beta} \\ 2 & \text{if } \theta^{(2)}_{\alpha,\beta} = \theta^{(1)}_{\alpha,\beta} < \theta \leqslant \pi \end{cases}$$

v) $\beta < \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$ and $\beta > 9\dfrac{(\alpha-1)^2}{(\alpha+2)^2}$ and $\beta > \dfrac{36(1-\alpha)}{4(\alpha+2)^2}$:

$$i_\omega(\phi_3) = \begin{cases} 0 & \text{if } 0 < \theta < -\theta^{(1)}_{\alpha,\beta} \\ 1 & \text{if } -\theta^{(1)}_{\alpha,\beta} < \theta \leqslant -\theta^{(2)}_{\alpha,\beta} \\ 2 & \text{if } -\theta^{(2)}_{\alpha,\beta} < \theta \leqslant \pi \end{cases}$$

vi) $\beta = \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$ and $\beta > \dfrac{36(1-\alpha)}{(\alpha+2)^2}$ and $\alpha < \dfrac{3}{2}$:

$$i_\omega(\phi_3) = \begin{cases} 0 & \text{if } 0 < \theta \leqslant -\theta^{(1)}_{\alpha,\beta} \\ 1 & \text{if } -\theta^{(1)}_{\alpha,\beta} < \theta < \pi \\ 0 & \text{if } \theta = \pi \end{cases}$$

vii) $\beta > \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$ and $\beta > \dfrac{36(1-\alpha)}{(\alpha+2)^2}$:

$$i_\omega(\phi_3) = \begin{cases} 0 & \text{if } 0 < \theta \leqslant -\theta^{(1)}_{\alpha,\beta} \\ 1 & \text{if } \theta^{(1)}_{\alpha,\beta} < \theta < \theta^{(2)}_{\alpha,\beta} \\ 0 & \text{if } \theta^{(2)}_{\alpha,\beta} \leqslant \theta \leqslant \pi \end{cases}$$

viii) $\beta = \dfrac{36(1-\alpha)}{(\alpha+2)^2}$ and $\beta < \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$:

$$i_\omega(\phi_3) = \begin{cases} 1 & \text{if } 0 < \theta \leqslant -\theta^{(2)}_{\alpha,\beta} \\ 2 & \text{if } -\theta^{(2)}_{\alpha,\beta} < \theta \leqslant \pi \end{cases}$$

ix) $\beta = \dfrac{36(1-\alpha)}{(\alpha+2)^2}$ and $\beta = \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$:

$$i_\omega(\phi_3) = \begin{cases} 1 & \text{if } 0 < \theta < \pi \\ 0 & \text{if } \theta = \pi \end{cases}$$

x) $\beta = \dfrac{36(1-\alpha)}{(\alpha+2)^2}$ and $\beta > \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$:

$$i_\omega(\phi_3) = \begin{cases} 1 & \text{if } 0 < \theta < \theta^{(2)}_{\alpha,\beta} \\ 0 & \text{if } \theta^{(2)}_{\alpha,\beta} \leqslant \theta \leqslant \pi \end{cases}$$

xi) $\beta < 9\dfrac{(\alpha-1)^2}{(\alpha+2)^2}$ and $\alpha < 1$:

$$i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta \leqslant -\theta^{(2)}_{\alpha,\beta} \\ 3 & \text{if } -\theta^{(2)}_{\alpha,\beta} < \theta < \theta^{(1)}_{\alpha,\beta} \\ 2 & \text{if } -\theta^{(1)}_{\alpha,\beta} \leqslant \theta \leqslant \pi \end{cases}$$

xii) $\beta = 9\dfrac{(\alpha-1)^2}{(\alpha+2)^2}$ and $\alpha < 1$:

$$i_\omega(\phi_3) = \begin{cases} 2 & \text{if } \theta \neq \theta^{(1)}_{\alpha,\beta} = -\theta^{(2)}_{\alpha,\beta} \\ 1 & \text{if } \theta = \theta^{(1)}_{\alpha,\beta} = -\theta^{(2)}_{\alpha,\beta} \end{cases}$$

xiii) $\beta > 9\dfrac{(\alpha-1)^2}{(\alpha+2)^2}$ and $\beta < \dfrac{36(1-\alpha)}{(\alpha+2)^2}$ and $\beta < \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$:

$$i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta < \theta^{(1)}_{\alpha,\beta} \\ 1 & \text{if } \theta^{(1)}_{\alpha,\beta} \leqslant \theta \leqslant -\theta^{(2)}_{\alpha,\beta} \\ 2 & \text{if } -\theta^{(2)}_{\alpha,\beta} < \theta \leqslant \pi \end{cases}$$

xiv) $\beta < \dfrac{36(1-\alpha)}{(\alpha+2)^2}$ and $\beta = \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$:

$$i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta < \theta^{(1)}_{\alpha,\beta} \\ 1 & \text{if } \theta^{(1)}_{\alpha,\beta} \leqslant \theta < \pi \\ 0 & \text{if } \theta = \pi \end{cases}$$

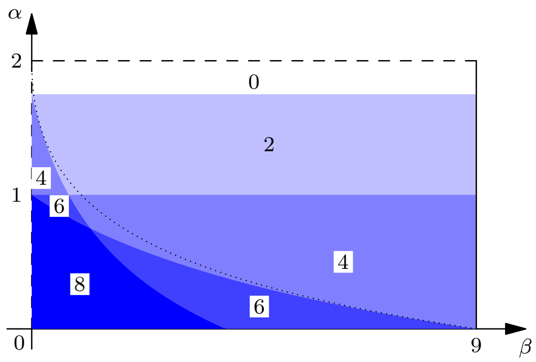


Figure 5.5. Values of the Maslov index $i_1(\phi_3^2)$ of the second iteration of ϕ_3. The dotted curve is the stability curve.

xv) $\beta < \dfrac{36(1-\alpha)}{(\alpha+2)^2}$ and $\beta > \dfrac{9(7-4\alpha)}{4(\alpha+2)^2}$:

$$i_\omega(\phi_3) = \begin{cases} 2 & \text{if } 0 < \theta < \theta^{(1)}_{\alpha,\beta} \\ 1 & \text{if } \theta^{(1)}_{\alpha,\beta} \leqslant \theta < \theta^{(2)}_{\alpha,\beta} \\ 0 & \text{if } \theta^{(2)}_{\alpha,\beta} \leqslant \theta \leqslant \pi \end{cases}$$

As we did analogously for E_2, we now turn our attention to the computation of the Maslov index $i_1(\phi_3^k)$ of the iterates of ϕ_3. Once again we have that the Maslov index jumps in correspondence of those ω that are roots of unity, due to the structure of Bott-Long formula. Hence, in the region *LS*, there are jumps of the index of the k-th iterate if and only if

$$\theta^{(i)}_{\alpha,\beta} = \frac{2l\pi}{k}, \tag{5.4.5}$$

for some $i = 1, 2$ and $l \in \mathbb{N} \setminus \{0\}$ (here $\theta^{(i)}_{\alpha,\beta}$ are the angles defined in (5.4.3) and (5.4.4)). In actual fact $\theta^{(2)}_{\alpha,\beta}$ ranges in $(0, 2\pi)$, whereas $\theta^{(1)}_{\alpha,\beta}$ varies in $(0, 2\sqrt{2}\pi)$: this implies that l takes values in the finite set $\{1, \dots, [\sqrt{2}k]\}$.

Condition (5.4.5) defines a family of curves $\{f_{k,l}\}$ in the plane (β, α), parameterised by k and l, that are defined by the equations

$$\beta = -\frac{36}{(\alpha+2)^2}\frac{l^2}{k^2}\left(\frac{l^2}{k^2} + \alpha - 2\right).$$

Each of these curves is convex and for $l \in \{1, \dots, k\}$ they are tangent at exactly one point to *SS*, namely

$$\left(\frac{9l^4}{(2k^2 - l^2)^2}, 2\left(1 - \frac{l^2}{k^2}\right)\right), \tag{5.4.6}$$

and it turns out that the stability curve is actually the envelope of the one-parameter family $\{f_t\}_{t\in(0,1]}$ consisting of curves of equations

$$\beta = -\frac{36}{(\alpha+2)^2}t^2(t^2 + \alpha - 2),$$

into which the collection $\{f_{k,l}\}$ is contained. We observe that at every point in *LS* the Maslov index $i_1(\phi_3^k)$ increases with k and that, for each fixed $k \in \mathbb{N} \setminus \{0\}$, it decreases along half-lines from the origin. The index is also monotonically increasing when one crosses any of the curves $f_{k,l}$ (going towards the origin). Note that the intersections of these curves with the line $\beta = 0$ yield exactly the values of the sequence $(\alpha_{k,l})$ introduced in E_2 that tends to $\alpha = 2$ as $k \to +\infty$.

In Figure 5.5 we present, as an example, a complete computation of $i_1(\phi_3^2)$, whereas Figure 1.4 on page 11 shows some of the curves $f_{k,l}$ for some values of k.

5.5 THE ω-MORSE INDEX OF THE LAGRANGIAN CIRCULAR ORBIT

Let $\mathscr{L} \in \mathscr{C}^\infty(T\widehat{X}, \mathbb{R})$ and $\mathbb{A} : W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}) \to \mathbb{R}$ be the Lagrangian function and the Lagrangian action functional respectively, as given in (1.3) and (1.4). Since the Euler-Lagrange equation for $\mathbb{A}$, which is smooth on collisionless loops, coincides with the Newton's equations given in (1.2), for each pair $(\beta, \alpha) \in (0, 9] \times [0, 2)$ the Lagrangian circular solution $\gamma_{\alpha,\beta}$ of Newton's equation can be found (up to a standard bootstrap argument) as a critical point of $\mathbb{A}$.

From Equation (4.5.2) we see that the second variation at the critical point $\gamma_{\alpha,\beta}$ is

$$d^2\mathbb{A}(\gamma_{\alpha,\beta})[\xi, \eta] = \int_0^{2\pi} \langle M\xi', \eta' \rangle + \langle D^2U\big(\gamma_{\alpha,\beta}(t)\big)\xi, \eta \rangle \, dt. \tag{5.5.1}$$

Using the Sobolev Embedding Theorem it follows that the second variation is a (bounded) essentially positive Fredholm quadratic form, being a weakly compact perturbation of an invertible quadratic form (see for instance [MPP05, Section 2, Proposition 3.1] and references therein). This in particular ensures that the ω-Morse index i^ω_{Morse} is finite.

By taking into account the Morse index theorem (Lemma 4.17), in order to compute the $i^\omega_{\text{Morse}}(\gamma_{\beta,\alpha})$ it is enough to compute the ω-index $i_\omega(\psi)$, where $\psi : [0, 2\pi] \to \mathrm{Sp}(8)$ is the fundamental solution of the first-order Hamiltonian system obtained from the associated Sturm system through the Legendre transformation, i.e. ψ satisfies

$$\begin{cases} \psi'(t) = JB_{\alpha,\beta}(t)\psi(t) \\ \psi(0) = I_{2n} \end{cases} \tag{5.5.2}$$

where

$$B_{\alpha,\beta}(t) := \begin{pmatrix} M & 0 \\ 0 & -D^2U\big(\gamma_{\alpha,\beta}(t)\big) \end{pmatrix}.$$

Taking into account [MS05, Theorem 2.1] there exists a linear symplectomorphism between $T^*\widehat{X}$ and $E_2 \oplus E_3$. By the symplectic invariance of i_{CLM} [see CLM94, Property V, p. 128] and hence of i_ω (as a direct consequence of Lemma 4.8), it follows that

$$i_\omega(\psi) = i_\omega(\Phi),$$

where Φ was defined in Section 5.2. Since $\Phi = \phi_2 \diamond \phi_3$, by using the symplectic additivity property of i_ω and considering the previous discussion it follows that

$$i^{\omega}_{\text{Morse}}(\gamma_{\alpha,\beta}) = i_\omega(\phi_2) + i_\omega(\phi_3).$$

Remark 5.13. We assume that H is a Hilbert space and there exist $H_1, \dots, H_n$ such that $H = \bigoplus_{k=1}^{n} H_k$. Let A be a self-adjoint essentially positive bounded Fredholm operator such that $A(H_k) \subseteq H_k$ for $i = 1, \dots, n$. Setting $A_k := A|_{H_k}$ we have

$$i^{\omega}_{\text{Morse}}(A) = \sum_{k=1}^{n} i^{\omega}_{\text{Morse}}(A_k).$$

It is worth noting that in correspondence of the 4-dimensional subspaces E_2 and E_3 there exist two 2-dimensional subspaces $\widehat{X}_2$ and $\widehat{X}_3$ of $\widehat{X}$ such that $E_2 = T^*\widehat{X}_2$ and $E_3 = T^*\widehat{X}_3$. Hence

$$W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}) = W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}_2) \times W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}_3).$$

In the next two subsections we shall compute the Lagrangian functions on the aforementioned subspaces $\widehat{X}_2$ and $\widehat{X}_3$ as well as the differential operators on such subspaces.

5.5.1 The ω-Morse index of the generalised Kepler problem

Define the Lagrangian function on $W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}_2)$ as

$$\mathscr{L}_2(x, \dot{x}) := \frac{1}{2}\|\dot{x}\|^2 + \langle Jx, \dot{x}\rangle + \frac{1}{2}\langle S_2 x, x\rangle, \tag{5.5.3}$$

where $S_2 := \begin{pmatrix} \alpha+2 & 0 \\ 0 & 0 \end{pmatrix}$. By a straightforward calculation it follows that the origin in the configuration space is a solution of the corresponding Euler-Lagrange equation

$$-\ddot{x} - 2J\dot{x} + S_2 x = 0 \tag{5.5.4}$$

associated with $\mathscr{L}_2$. Let

$$\mathcal{B}_2 : W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}_2) \times W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}_2) \to \mathbb{R}$$

be defined as follows:

$$\mathcal{B}_2(x, y) := \int_0^{2\pi} \left[\langle \dot{x}, \dot{y}\rangle + \langle Jy, \dot{x}\rangle + \langle J\dot{x}, y\rangle + \langle S_2 x, y\rangle\right] dt.$$

Once again it follows from the Sobolev Embedding Theorem that $\mathcal{B}_2$ is a (bounded) essentially positive Fredholm quadratic form, being a weakly compact perturbation of an invertible quadratic form. This in particular ensures that the Morse index $i^{\omega}_{\text{Morse}}$ is finite.

By taking into account the Legendre transformation, the corresponding autonomous Hamiltonian function is

$$\mathscr{H}_2(v) := \frac{1}{2}\langle B_2 v, v\rangle, \qquad \forall\, v \in \mathbb{R}^4,$$

where

$$B_2 := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -(\alpha+1) & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly the origin in the phase space is the corresponding solution of the linear autonomous Hamiltonian initial value problem

$$\begin{cases} \phi_2'(\tau) = \Lambda_2 \phi_2(\tau) \\ \phi_2(0) = I_4 \end{cases} \tag{5.5.5}$$

where $\Lambda_2 = JB_2$ agrees with the one given in formula (5.2.1).

Theorem 5.14. *For all $\omega \in \mathbb{U}$, the ω-Morse index of the circular solution $\gamma_{\alpha,0}$ of the generalised Kepler problem coincides with $i_\omega(\phi_2)$, which has been computed in Propositions 5.8 and 5.9.*

Proof. First of all we observe that as a direct consequence of the results proved in Section 2.2 the subspace E_2 is invariant under the phase flow of the Hamiltonian (2.2.2). Moreover on this subspace the aforementioned Hamiltonian reduces to the Hamiltonian of the generalised Kepler problem. Now, by the above construction System (5.5.5) is the Legendre transformation of the Euler-Lagrange system (5.5.4). The thesis is then a direct consequence of Lemma 4.17. □

Remark 5.15. It is worth noting that this result perfectly agrees with [HS10, Proposition 3.6] and [Ven02, Proposition 2.2.3]. Moreover we point out that in the last quoted reference the author only states that for $\alpha \in (0, 1)$ the circular solutions are not local minimisers, without any further information on the Morse index. The logarithmic case has not been treated thus far from this point of view.

5.5.2 The ω-Morse index of the Lagrange circular orbit

We proceed exactly as in the previous subsection, by introducing the Lagrangian

$$\mathscr{L}_3(x, \dot{x}) := \frac{1}{2}\|\dot{x}\|^2 + \langle Jx, \dot{x}\rangle + \frac{1}{2}\langle S_3 x, x\rangle$$

on the Sobolev space $W^{1,2}(\mathbb{R}/2\pi\mathbb{Z}, \widehat{X}_3)$, with

$$S_3 := \begin{pmatrix} \frac{1}{6}\left[6+3\alpha+(\alpha+2)\sqrt{9-\beta}\right] & 0 \\ 0 & \frac{1}{6}\left[6+3\alpha-(\alpha+2)\sqrt{9-\beta}\right] \end{pmatrix}.$$

Defining a symmetric bilinear form $\mathcal{B}_3$ in a completely analogous way as above, we obtain the Hamiltonian system

$$\begin{cases} \phi_3'(\tau) = \Lambda_3 \phi_3(\tau) \\ \phi_3(0) = I_4, \end{cases}$$

where $\Lambda_3 = JB_3$, being

$$B_3 := \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -\frac{1}{2}\left(\alpha+\frac{\alpha+2}{3}\sqrt{9-\beta}\right) & 0 \\ 1 & 0 & 0 & -\frac{1}{2}\left(\alpha-\frac{\alpha+2}{3}\sqrt{9-\beta}\right) \end{pmatrix}.$$

THEOREM 5.16. *For all $\omega \in \mathbb{U}$ the ω-Morse index of the Lagrangian circular solution $\gamma_{\alpha,\beta}$ is given by $i_\omega(\Phi) = i_\omega(\phi_2) + i_\omega(\phi_3)$. In particular for $\omega = 1$ we have*

$$i_{\text{Morse}}(\gamma_{\alpha,\beta}) = \begin{cases} 0 & \text{if } \alpha \in [1,2) \\ 2 & \text{if } \beta \geqslant \dfrac{36(1-\alpha)}{(\alpha+2)^2} \text{ and } \alpha \in [0,1) \\ 4 & \text{if } 0 < \beta < \dfrac{36(1-\alpha)}{(\alpha+2)^2}. \end{cases}$$

Proof. Arguing as in the proof of Theorem 5.14, it is enough to apply Lemma 4.17, use the calculations performed in Subsections 5.4.1 and 5.3.1 and the additivity of the Maslov index i_1. □

5.5.3 Relation between linear stability and Morse index

We have shown how both in E_2 and in E_3 there is a sequence of curves (possibly straight lines) that "converge", in a suitable sense, to the boundary of the region of linear stability. By virtue of the Index Theorem also the Morse index of the iterates jumps when crossing each of those curves.

Since the angles $\theta^{(1)}_{\alpha,\beta}$ and $\theta^{(2)}_{\alpha,\beta}$ introduced in Subsection 5.4.2 cover the whole of $\mathbb{U}$ as α and β vary, it may happen that for some values of these parameters one of them is a rational multiple of 2π (so that its exponential is a root of unity). When this occurs then the corresponding curve in the plane (β, α) is tangent to the stability curve at the point whose coordinates are given by (5.4.6). Instead, in the case when the aforementioned angles do not give rise to roots of unity, one obtains tangency to the stability curve at some point only after taking the limit as $k \to +\infty$. The reason of this fact is simply due to the density of roots of unity in $\mathbb{U}$.

A ANALYTIC AND SYMPLECTIC FRAMEWORK

Contents

This appendix is conceived with the aim of examining more in depth the analytic and symplectic setting which is used throughout the dissertation, reporting some properties and results supporting and completing the previous propositions.

A.1 ON THE SPECTRAL FLOW

We present here some important properties of the spectral flow. Our basic reference is [Les05].

Theorem A.1 ([Les05]). *Let*

$$\mu : \Omega\big(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\big) \to \mathbb{Z},$$

be a map which satisfies the following properties:

i) Concatenation: *If $\gamma, \delta \in \Omega\big(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\big)$, with $\gamma(b) = \delta(a)$, then*

$$\mu(\gamma * \delta) = \mu(\gamma) + \mu(\delta).$$

ii) Homotopy invariance: *The map μ descends to a map*

$$\widetilde{\mu} : \widetilde{\pi}_1\left(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\right) \to \mathbb{Z},$$

that is, the following diagram is commutative (p denotes the quotient map):

$$\begin{tikzcd}
\Omega\big(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\big) \arrow[r, "\mu"] \arrow[d, "p"'] & \mathbb{Z} \\
\widetilde{\pi}_1\left(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\right). \arrow[ur, "\widetilde{\mu}"'] &
\end{tikzcd}$$

iii) Normalisation: *There exists an orthogonal projector $P \in \mathscr{B}^{sa}(\mathcal{H})$ of rank 1 such that* *a) the restriction* $(I-P)A(I-P)|_{\ker P}$ *of the operator* $(I-P)A(I-P) \in \mathscr{B}^{sa}(\mathcal{H})$ *to the kernel of* P *is invertible for every* $A \in \mathscr{B}^{sa}(\mathcal{H})$;

b) the path $\zeta \in \Omega\big(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\big)$ *defined by*

$$\zeta(t) := \left(t - \frac{1}{2}\right)P + (I-P)A(I-P) \qquad \text{for all } t \in [0,1]$$

verifies

$$\mu(\zeta) = 1.$$

Then

$$\mu(\gamma) = \operatorname{sf}\big(\gamma, [a,b]\big)$$

for all $\gamma \in \Omega\big(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\big)$.

Remark A.2. If we fix a basis $(e_1, \dots, e_n)$ in $\mathcal{H}$, then the axiom of normalisation in the previous theorem can be stated as follows. Let $P \in \mathscr{B}^{sa}(\mathcal{H})$ be an orthogonal projector whose image is generated by e_1 and for a fixed $k \in \{2, \dots n-1\}$ define two other orthogonal projectors P_k^+ and P_k^- by $\operatorname{im} P_k^+ := \operatorname{span}\{e_2, \dots, e_k\}$ and $\operatorname{im} P_k^- := \operatorname{span}\{e_{k+1}, \dots, e_n\}$. Choose $A := P_k^+ - P_k^-$. Then the path

$$\zeta \in \Omega\big(\mathscr{B}^{sa}(\mathcal{H}), \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})\big)$$

given by

$$\zeta(t) := \left(t - \frac{1}{2}\right)P + A \qquad \text{for all } t \in [0,1]$$

satisfies $\mu(\zeta) = 1$.

This is actually a particular case of what we wrote in Theorem A.1, but we observe that it can be used as well to declare which paths have spectral flow equal to 1.

We note that our formulation of this axiom corrects the statement of [Les05, Theorem 5.7], in which there is clearly just an oversight: the condition of invertibility of $(I-P)A(I-P)$ is indeed missing there.

Lemma A.3. *Let* $t_* \in \mathbb{R}$ *and consider a path* $T \in \mathscr{C}^1\big([t_*-\varepsilon, t_*+\varepsilon], \mathscr{B}^{sa}(\mathcal{H})\big)$, *for some* $\varepsilon > 0$. *Suppose that* T *has a unique regular crossing at* $t = t_*$. *Then*

$$\operatorname{sf}\big(T, [t_*-\varepsilon, t_*+\varepsilon]\big) = \operatorname{sgn}\Gamma(T, t_*).$$

Proof. Let $Q : \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto the kernel of $T(t_*)$. Since t_* is a regular crossing instant for T, the operator $Q\dot{T}(t_*)Q|_{\mathcal{H}_{t_*}}$ is invertible on $\mathcal{H}_{t_*} := \ker T(t_*)$. Therefore there exists a number $\beta > 0$ such that $Q\big(\dot{T}(t_*) + B\big)Q|_{\mathcal{H}_{t_*}}$ is also invertible on $\mathcal{H}_{t_*}$ for every $B \in \mathscr{B}^{sa}(\mathcal{H})$ such that $\|B\| < \beta$. On the other hand, being $T(t_*)|_{\mathcal{H}_{t_*}} = 0$, we may choose a number $\varepsilon > 0$ such that

$$\left\| \left(\frac{T(t) - T(t_*)}{t - t_*} - \dot{T}(t_*) \right)\Big|_{\mathcal{H}_{t_*}} \right\| = \left\| \left(\frac{T(t)}{t - t_*} - \dot{T}(t_*) \right)\Big|_{\mathcal{H}_{t_*}} \right\| < \beta$$

for every $t \in [t_* + \varepsilon, t_* + \varepsilon] \setminus \{t_*\}$.

Define then a homotopy $F : [0, 1] \times [t_* - \varepsilon, t_* + \varepsilon] \to \mathscr{B}^{sa}(\mathcal{H}_{t_*})$ by

$$\begin{aligned} F(s, t) &:= sT(t)\big|_{\mathcal{H}_{t_*}} + (1 - s)(t - t_*)\dot{T}(t_*)\big|_{\mathcal{H}_{t_*}} \\ &= (t - t_*)\left[s\left(\frac{T(t)}{t - t_*} - \dot{T}(t_*) \right)\Bigg|_{\mathcal{H}_{t_*}} + \dot{T}(t_*)\big|_{\mathcal{H}_{t_*}} \right]. \end{aligned}$$

The previous choice of ε is thus sufficient to guarantee that $F(s, t)$ is invertible for every $s \in [0, 1]$ and every $t \in [t_* - \varepsilon, t_* + \varepsilon]$. Hence, by the homotopy invariance of the spectral flow,

$$\operatorname{sf}\big(T, [t_* - \varepsilon, t_* + \varepsilon]\big) = \operatorname{sf}\big((t - t_*)\dot{T}(t_*)\big|_{\mathcal{H}_{t_*}}, [t_* - \varepsilon, t_* + \varepsilon]\big). \quad \text{(A.1.1)}$$

Here we actually use the fact that, for all $t \in [t_* - \varepsilon, t_* + \varepsilon]$, the operator $T(t)$ splits into $T|_{\mathcal{H}_{t_*}}(t) + T|_{\mathcal{H}_{t_*}^{\perp}}(t)$ on $\mathcal{H} = \mathcal{H}_{t_*} \oplus \mathcal{H}_{t_*}^{\perp}$ and that the spectral flow is compatible with this splitting, i.e.

$$\begin{aligned} \operatorname{sf}\big(T\big|_{\mathcal{H}_{t_*}} + T\big|_{\mathcal{H}_{t_*}^{\perp}}, [t_* - \varepsilon, t_* + \varepsilon]\big) &= \operatorname{sf}\big(T\big|_{\mathcal{H}_{t_*}}, [t_* - \varepsilon, t_* + \varepsilon]\big) \\ &\quad + \operatorname{sf}\big(T\big|_{\mathcal{H}_{t_*}^{\perp}}, [t_* - \varepsilon, t_* + \varepsilon]\big). \end{aligned}$$

The last addendum is of course zero and this justifies equality (A.1.1).

Finally, by Remark 2.13,

$$\begin{aligned} \operatorname{sf}\big((t - t_*)\dot{T}(t_*)\big|_{\mathcal{H}_{t_*}}, [t_* - \varepsilon, t_* + \varepsilon]\big) &= n^{-}\big(-\varepsilon\dot{T}(t_*)\big|_{\mathcal{H}_{t_*}}\big) - n^{-}\big(\varepsilon\dot{T}(t_*)\big|_{\mathcal{H}_{t_*}}\big) \\ &= n^{+}\big(\dot{T}(t_*)\big|_{\mathcal{H}_{t_*}}\big) - n^{-}\big(\dot{T}(t_*)\big|_{\mathcal{H}_{t_*}}\big) \\ &= \operatorname{sgn} \dot{T}(t_*)\big|_{\mathcal{H}_{t_*}} \\ &= \operatorname{sgn} Q\dot{T}(t_*)Q|_{\mathcal{H}_{t_*}}. \quad \square \end{aligned}$$

From this Lemma immediately follows the next Proposition.

Proposition A.4. *Let $T \in \mathscr{C}^1\big([0, 1], \mathscr{B}^{sa}(\mathcal{H})\big)$ be a regular curve with invertible endpoints. Then the spectral flow is computed as:*

$$\operatorname{sf}\big(T, [0, 1]\big) = \sum_{\substack{t_* \in [0,1] \\ t_* \text{ crossing}}} \operatorname{sgn} \Gamma(T, t_*). \quad \text{(A.1.2)}$$

Proof. Since every crossing is regular by assumption, the corresponding crossing forms are all non-degenerate and we can use the Inverse Function Theorem to deduce that the crossings are isolated. Then we can apply Lemma A.3 to each isolated crossing and sum up every contribution by means of the concatenation property of the spectral flow. The compactness of the interval $[0, 1]$ ensures that there are only finitely many crossing and that the sum on the right-hand side of (A.1.2) is well defined. $\square$

In the following proposition we investigate the parity of the spectral flow of an affine path of Hermitian matrices.

Proposition A.5. *Let $\mathcal{H}$ be a complex Hilbert space of dimension $4n$, let $A \in \mathscr{B}^{sa}(\mathcal{H}) \setminus \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})$ be a real symmetric non-invertible matrix and take $C \in \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})$ of the form $C := iB$, where B is a real skew-symmetric invertible matrix. Consider the affine path $D : [0, +\infty) \to \mathscr{B}^{sa}(\mathcal{H})$ defined by*

$$D(t) := A + tC.$$

Let $E_\lambda := \ker(B^{-1}A + \lambda I)$ be the eigenspace of $-B^{-1}A$ relative to the eigenvalue λ and let $Q_\lambda : \mathcal{H} \to \mathcal{H}$ be the eigenprojection onto E_λ. We assume that

(H1) The quadratic form $Q_\lambda C Q_\lambda\big|_{E_\lambda}$ is non-degenerate for every $\lambda \in \sigma(-B^{-1}A)$;

(H2) $-B^{-1}A$ is diagonalisable;

(H3) $\sigma(-B^{-1}A) \subset i\mathbb{R}$ and it is symmetric with respect to the real axis;

(H4) $\nu(A)$ is even.

Then there exist $\varepsilon > 0$ and $T > \varepsilon$ such that

(T1) The instant $t = 0$ is the only crossing for D on $[0, \varepsilon]$;

(T2) $\mathrm{sf}\big(D, [\varepsilon, T_1]\big) = \mathrm{sf}\big(D, [\varepsilon, T_2]\big)$ for all $T_1, T_2 \geqslant T$;

(T3) $\mathrm{sf}\big(D, [\varepsilon, T]\big)$ is even.

Proof. Statement (T1) follows by assumption (H1). Indeed, $t = 0$ is a crossing instant because A is singular and it is regular because the crossing form $Q_0 C Q_0|_{E_0}$ is non-degenerate. Thus, by the Inverse Function Theorem, it is isolated and the number $\varepsilon > 0$ claimed in the first thesis exists.

In order to prove (T2), we observe that there exist $T > \varepsilon$ such that

$$\operatorname{sgn} D(t) = \operatorname{sgn} C, \qquad \forall\, t \geqslant T.$$

To prove this claim, we analyse the following two cases (note that $\sigma(C) \subset \mathbb{R} \setminus \{0\}$, being C hermitian and invertible):

- $\lambda_* \in \sigma(C) \cap \mathbb{R}^-$. If $u_* \in \ker(C - \lambda_* I)$ is an eigenvector related to λ_*, we have

  $$\langle D(t)u_*, u_*\rangle = \langle Au_*, u_*\rangle + t\lambda_* \|u_*\|^2 .$$

  Thus

  $$\sup_{\substack{\|u\|=1 \\ u \in \ker(C - \lambda_* I)}} \langle D(t)u, u\rangle \leqslant \lambda_{max} + t\lambda_*,$$

  where λ_{max} is the maximum of the quadratic form $\langle Au, u\rangle$ on the unit sphere of the eigenspace of C relative to λ_* (which is attained by Weierstraß theorem). If we choose $T_{max} := \dfrac{1+\lambda_{max}}{|\lambda_*|}$, we obtain that

$$\sup_{\substack{\|u\|=1\\ u\in\ker(C-\lambda_* I)}} \langle D(t)u,u\rangle \leqslant -1, \qquad \forall t \geqslant T_{max},$$

so that λ_* eventually defines a negative eigendirection for $D(t)$.

- $\lambda_* \in \sigma(C)\cap\mathbb{R}^+$. If $u_* \in \ker(C-\lambda_* I)$ is an eigenvector related to λ_*, we have

  $$\langle D(t)u_*,u_*\rangle = \langle Au_*,u_*\rangle + t\lambda_* \|u_*\|^2 .$$

  Thus

  $$\sup_{\substack{\|u\|=1\\ u\in\ker(C-\lambda_* I)}} \langle D(t)u,u\rangle \geqslant \lambda_{min} + t\lambda_*,$$

  where λ_{min} is the minimum of the quadratic form $\langle Au,u\rangle$ on the unit sphere of the eigenspace of C relative to λ_* (which is attained by Weierstraß theorem). If we choose $T_{min} := \dfrac{1-\lambda_{min}}{|\lambda_*|}$, we obtain that

  $$\sup_{\substack{\|u\|=1\\ u\in\ker(C-\lambda_* I)}} \langle D(t)u,u\rangle \geqslant 1, \qquad \forall t \geqslant T_{min},$$

  so that λ_* eventually defines a positive eigendirection for $D(t)$.

Define then $T := \max\{T_{min}, T_{max}\}$. Without loss of generality we may assume that $T_1 < T_2$. By means of the concatenation property of the spectral flow we get

$$\mathrm{sf}\big(D,[\varepsilon,T_2]\big) = \mathrm{sf}\big(D,[\varepsilon,T_1]\big) + \mathrm{sf}\big(D,[T_1,T_2]\big) = \mathrm{sf}\big(D,[\varepsilon,T_1]\big),$$

where the last equality comes from the fact that $D(t)$ is an isomorphism for every $t \geqslant T$. As we showed, indeed, for any $t \geqslant T$ each eigenvalue of C determines an eigendirection (and hence an eigenvalue) of $D(t)$ of the same sign. Being C invertible, the claim follows.

We now prove (T3). Let us first make the link between t_* and λ explicit: writing

$$D(t) = -B(-B^{-1}A - itI) \qquad \forall t \in [0,+\infty)$$

it is clear that t_* is a crossing for D if and only if $\lambda = it_*$ is an eigenvalue of $-B^{-1}A$. Now, by construction both ε and T are not crossing instants for D, hence we can apply Proposition A.4 to $\mathrm{sf}\big(D,[\varepsilon,T]\big)$ and write

$$\mathrm{sf}\big(D,[\varepsilon,T]\big) = \sum_{\substack{t_*\in[\varepsilon,T]\\ t_*\ \text{crossing}}} \mathrm{sgn}\Big(Q_\lambda C Q_\lambda\big|_{E_\lambda}\Big). \tag{A.1.3}$$

(Note that this summation is meaningful because of our brief discussion a few lines above.) Since the crossing forms are non-degenerate by (H1) and since $-B^{-1}A$ is diagonalisable by (H2), we have that

$$\begin{aligned}\operatorname{sgn}\left(Q_\lambda C Q_\lambda|_{E_\lambda}\right) &\coloneqq n^+\left(Q_\lambda C Q_\lambda|_{E_\lambda}\right) - n^-\left(Q_\lambda C Q_\lambda|_{E_\lambda}\right)\\ &\equiv n^+\left(Q_\lambda C Q_\lambda|_{E_\lambda}\right) + n^-\left(Q_\lambda C Q_\lambda|_{E_\lambda}\right) \qquad \mod 2\\ &= \dim E_\lambda\end{aligned} \tag{A.1.4}$$

for all $\lambda \in \sigma(-B^{-1}A)$. As a consequence of (A.1.3) and (A.1.4) we infer that

$$\operatorname{sf}\big(D, [\varepsilon, T]\big) \equiv \sum_{\substack{t_* \in [\varepsilon,T]\\ t_* \text{ crossing}}} \dim E_\lambda = \sum_{\lambda \in \sigma(-B^{-1}A)\cap i[\varepsilon,+\infty)} \dim E_\lambda \qquad \mod 2. \tag{A.1.5}$$

Finally, taking into account (H2), (H3) and (H4), we deduce

$$\sum_{\lambda \in \sigma(-B^{-1}A)\cap i[\varepsilon,+\infty)} \dim E_\lambda = 2n - \nu(A) \equiv 0 \qquad \mod 2. \qquad \square$$

The next corollary is directly derived from the previous proposition and it deals with the case when the matrix A is invertible.

Corollary A.6. *In the same setting of Proposition A.5, assume that* $A \in \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})$ *and that (H1), (H2) and (H3) hold. Then there exists* $T > 0$ *such that*

(T2') $\operatorname{sf}\big(D, [0, T_1]\big) = \operatorname{sf}\big(D, [0, T_2]\big)$ *for all* $T_1, T_2 \geqslant T$;

(T3') $\operatorname{sf}\big(D, [0, T]\big)$ *is even.*

Proof. Since A is invertible, the instant $t = 0$ is not a crossing for L and $\nu(A) = 0$. Consequently, we can compute the spectral flow of L directly on the interval $[0, T]$ and apply Proposition A.5 with obvious modifications. $\square$

A.2 ROOT FUNCTIONS, PARTIAL SIGNATURES AND SPECTRAL FLOW

The aim of this section is to derive a formula for computing the spectral flow of an affine path at a possibly degenerate (i.e. non-regular) crossing instant. The main references are [GPP04a; GPP04b] and references therein.

Let $t_* \in \mathbb{R}$, $\varepsilon > 0$ and $T : [t_* - \varepsilon, t_* + \varepsilon] \to \mathscr{B}^{sa}(\mathcal{H})$ be a real-analytic path such that $t = t_*$ is an isolated crossing for T. We are interested in computing the “jumps” of the functions $n^+\big(T(t)\big)$ and

$n^-\big(T(t)\big)$ as t passes through t_* also in the degenerate case, in order to generalise Lemma A.3. Recall that, given $k \in \mathbb{N} \setminus \{0\}$, a smooth map $f : [t_* - \varepsilon, t_* + \varepsilon] \to \mathcal{H}$ is said to have a *zero of order* k at $t = t_*$ if $f(t_*) = f'(t_*) = \ldots = f^{(k-1)}(t_*) = 0$ and $f^{(k)}(t_*) \neq 0$. We recall that in this case both the eigenvalues and the eigenvectors of $T(t)$ are real-analytic functions defined on the domain of T [see Kat95, Chapter 2]; we denote them by $\lambda_i(t)$ and $v_i(t)$ respectively, for $i \in \{1, \ldots, \dim \mathcal{H}\}$. Moreover, if $\lambda_i(t)$ vanishes at $t = t_*$ for some index i then it has a zero of finite order, and for each i the v_i's are pairwise orthogonal unit eigenvectors relative to the $\lambda_i(t)$'s.

Definition A.7. A *root function* for $T(t)$ at $t = t_*$ is a smooth map $u : [t_* - \varepsilon, t_* + \varepsilon] \to \mathcal{H}$ such that $u(t_*) \in \ker T(t_*)$. The *order* $\operatorname{ord}(u)$ of the root function u is the (possibly infinite) order of the zero at $t = t_*$ of the map $t \mapsto T(t)u(t)$.

In correspondence of the (possibly non-regular) crossing instant t_* for T we define, for every $k \in \mathbb{N} \setminus \{0\}$, a descending filtration $(\mathcal{W}_k)$ of vector spaces $\mathcal{W}_k \subset \mathcal{H}$ and a sequence $(\mathcal{B}_k)$ of sesquilinear forms $\mathcal{B}_k : \mathcal{W}_k \times \mathcal{W}_k \to \mathbb{C}$ as follows:

$$\mathcal{W}_k \coloneqq \{\, u_* \in \mathcal{H} \mid \exists \text{ a root function } u \text{ with } \operatorname{ord}(u) \geqslant k \text{ and } u(t_*) = u_* \,\},$$

$$\mathcal{B}_k(u_*, v_*) \coloneqq \frac{1}{k!}\left\langle \frac{d^k}{dt^k}\big[T(t)u(t)\big]\Big|_{t=t_*}, v_* \right\rangle \qquad \forall\, u_*, v_* \in \mathcal{W}_k, \tag{A.2.1}$$

where u in (A.2.1) is any root function with $\operatorname{ord}(u) \geqslant k$ and $u(t_*) = u_*$. The right-hand side of the equality in (A.2.1) is well defined and indeed it turns out to be independent of the choice of the root function u (see [GPP04b, Proposition 2.4]).

Definition A.8. For all $k \in \mathbb{N} \setminus \{0\}$, the integer number

$$\operatorname{sgn}_k(T, t_*) \coloneqq \operatorname{sgn} \mathcal{B}_k$$

is called the k*-th partial signature* of $T(t)$ at $t = t_*$.

Proposition A.9. *Let* $t_* \in \mathbb{R}$, $\varepsilon > 0$ *and* $T : [t_* - \varepsilon, t_* + \varepsilon] \to \mathscr{B}^{sa}(\mathcal{H})$ *be a real-analytic path having a unique (possibly non-regular) crossing at* $t = t_*$. *Then*

(i) $\mathcal{W}_k = \operatorname{span}\left\{ v_i(t_*) \in \mathcal{H} \;\middle|\; \lambda_i^{(j)}(t_*) = 0 \text{ for all } j < k \text{ and } \lambda_i^{(k)}(t_*) \neq 0 \right\}$;

(ii) If $v \in \mathcal{W}_k$ *is an eigenvector of* $\lambda(t_*)$ *then* $\mathcal{B}_k(v, w) = \frac{1}{k!}\lambda^{(k)}(t_*)\langle v, w\rangle$, *for all* $w \in \mathcal{W}_k$;

(iii) $\operatorname{sf}\big(T, [t_* - \varepsilon, t_* + \varepsilon]\big) = \displaystyle\sum_{k=1}^{+\infty} \operatorname{sgn}_{2k-1}(T, t_*)$, *where the sum has only finitely many non-zero terms.*

Proof. It follows verbatim from [GPP04b, Proposition 2.9, Corollary 2.14]: the results there contained hold also if the underlying Hilbert space is complex. ☐

Remark A.10. Part (iii) of Proposition A.9 is the generalisation of Lemma A.3 to the degenerate case that we were seeking.

We close this subsection with the following central result, which computes the spectral flow for a path of Hermitian matrices in terms of partial signatures.

Proposition A.11. *Let* $A \in \mathscr{B}^{sa}(\mathcal{H})$ *and* $C \in \mathscr{G}\mathscr{B}^{sa}(\mathcal{H})$. *Consider the affine path* $\widetilde{D} : (0, +\infty) \to \mathscr{B}^{sa}(\mathcal{H})$ *defined by*

$$\widetilde{D}(s) \coloneqq sA + C$$

and assume that $s_* \in (0, +\infty)$ *is an isolated (possibly non-regular) crossing instant for* $\widetilde{D}$, *so that* $1/s_*$ *is an eigenvalue of* $-C^{-1}A$. *Then for* $\varepsilon > 0$ *small enough*

$$\mathrm{sf}\big(\widetilde{D}, [s_* - \varepsilon, s_* + \varepsilon]\big) = -\operatorname{sgn} \mathcal{B}_1,$$

where

$$\mathcal{B}_1 \coloneqq \langle C\,\cdot, \cdot\rangle\big|_{\mathcal{H}_{s_*}}$$

and $\mathcal{H}_{s_*}$ *is the generalised eigenspace*

$$\mathcal{H}_{s_*} \coloneqq \bigcup_{j=1}^{\dim \mathcal{H}} \ker\left(C^{-1}A + \frac{1}{s_*} I\right)^j.$$

Proof. See [GPP04b, Corollary 3.30]. ☐

A.3 KREIN SIGNATURE OF A COMPLEX SYMPLECTIC MATRIX

We now briefly recall some basic facts about the Krein signature of a symplectic matrix. Our main references are the books [Abb01, Chapter 1] and [Lon02].

Let $S \in \mathrm{Sp}(2n, \mathbb{R})$ be a real symplectic matrix. In order to define the *Krein signature* of the eigenvalues of S, we consider the usual action of S on $\mathbb{C}^{2n}$

$$S(\xi + i\eta) \coloneqq S\xi + iS\eta, \qquad \forall\, \xi, \eta \in \mathbb{R}^{2n},$$

and the Hermitian form $g : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{R}$ given by

$$g(v, w) \coloneqq \langle Gv, w\rangle \qquad \forall\, v, w \in \mathbb{C}^{2n},$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{C}^{2n}$. The *complex symplectic group* $\mathrm{Sp}(2n, \mathbb{C})$ is the set of all complex linear automorphisms of $\mathbb{C}^{2n}$ which preserve g or, equivalently, the set of all complex matrices S satisfying the condition $S^\dagger J S = J$. A matrix is an element of $\mathrm{Sp}(2n,\mathbb{R})$ if and only if it belongs to $\mathrm{Sp}(2n,\mathbb{C})$ and it is real. Following the discussion in [Abb01, pp. 12–13] and [Lon02, Chapter 1], it is possible to show that the spectral decomposition of $\mathbb{C}^{2n}$

$$\mathbb{C}^{2n} = \bigoplus_{\substack{\lambda\in\sigma(S)\\ |\lambda|\geqslant 1}} F_\lambda,$$

where

$$F_\lambda := \begin{cases} E_\lambda & \text{if } |\lambda| = 1 \\ E_\lambda \oplus E_{\overline{\lambda}^{-1}} & \text{if } |\lambda| > 1 \end{cases}$$

and

$$E_\lambda := \bigcup_{j=1}^{2n} \ker(S-\lambda I)^j,$$

is g-orthogonal. Therefore each restriction $g|_{F_\lambda}$ is non-degenerate for all $\lambda \in \sigma(S)$.

Remark A.12. Because of the non-degeneracy of g on each space F_λ we obtain that

$$\begin{aligned} \operatorname{sgn} g\big|_{F_\lambda} &:= n^+(g\big|_{F_\lambda}) - n^-(g\big|_{F_\lambda}) \\ &\equiv n^+(g\big|_{F_\lambda}) + n^-(g\big|_{F_\lambda}) \qquad \mod 2 \\ &= \dim F_\lambda. \end{aligned}$$

If $\lambda \in \sigma(S)\setminus\mathbb{U}$ has algebraic multiplicity d, then g restricted to the $2d$-dimensional subspace $E_\lambda \oplus E_{\overline{\lambda}^{-1}}$ has a d-dimensional isotropic subspace. Thus g has zero signature on $E_\lambda \oplus E_{\overline{\lambda}^{-1}}$. On the contrary, an eigenvalue $\lambda \in \sigma(S)\cap\mathbb{U}$ may have any signature on E_λ, and therefore we are entitled to give the following definition.

Definition A.13. Let $S \in \mathrm{Sp}(2n,\mathbb{C})$ be a complex symplectic matrix and let $\lambda \in \sigma(S)\cap\mathbb{U}$ be a unitary eigenvalue of S. The *Krein signature* of λ is the signature of the restriction $g|_{E_\lambda}$ of the Hermitian form g to the generalised eigenspace E_λ.

Assume that $S \in \mathrm{Sp}(2n,\mathbb{R})$. If an eigenvalue $\lambda \in \sigma(S)\cap\mathbb{U}$ has Krein signature p, then its complex conjugate $\overline{\lambda}$ (which is again an eigenvalue of S because of the properties of the spectrum of symplectic matrices, see Proposition 2.2) has Krein signature $-p$. This implies, in particular, that 1 and -1 always have Krein signature 0.

A.4 THE GEOMETRIC STRUCTURE OF $\mathrm{Sp}(2)$

The symplectic group $\mathrm{Sp}(2)$ captured the attention of I. M. Gel'fand and V. B. Lidskii first, who in 1958 described a toric representation of it [GL55; GL58]. The $\mathbb{R}^3$-cylindrical coordinate representation of Sp(2) was instead introduced by Y. Long in 1991 [Lon91], and what follows, including Figure A.1 and Figure A.2 (although we re-drew them ourselves), already appeared in [Lon02, Section 2.1].

Every real invertible matrix A can be decomposed in *polar form*

$$A = PO,$$

where $P := (AA^T)^{1/2}$ is symmetric and positive definite and $O := P^{-1}A$ is orthogonal. If $A \in \mathrm{Sp}(2)$ then $\det P = 1$ and therefore $P \in \mathrm{Sp}(2)$ as well. This entails that $O \in \mathrm{Sp}(2)$; in fact, being orthogonal, it belongs to $\mathrm{SO}(2) \cong \mathbb{U}$, i.e. it is a proper rotation:

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Let $u : \mathrm{Sp}(2) \to \mathbb{U}$ be the map which associates every 2×2 real symplectic matrix with the angle of rotation of its orthogonal part:

$$u(A) = u(PO) := e^{i\theta}.$$

Now, the eigenvalues of P are all real, positive and reciprocal of each other. Therefore we have that $\operatorname{tr} P \geqslant 2$ and we may introduce a coordinate ξ ranging in $[0, +\infty)$ by setting $\operatorname{tr} P = 2\cosh\xi$. Hence we can write

$$P = \begin{pmatrix} \cosh\xi + a & b \\ b & \cosh\xi - a \end{pmatrix}$$

for some $a, b \in \mathbb{R}$ such that $\cosh^2\xi - a^2 - b^2 = 1$. Thus $b^2 = \sinh^2\xi - a^2$, which is meaningful if and only if $|a| \leqslant |\sinh\xi|$. Hence we are allowed to set $a := \sinh\xi\cos\eta$ for some $\eta \in \mathbb{R}$, so that $b = \sinh\xi\sin\eta$ and P becomes

$$P = \begin{pmatrix} \cosh\xi + \sinh\xi\cos\eta & \sinh\xi\sin\eta \\ \sinh\xi\sin\eta & \cosh\xi - \sinh\xi\cos\eta \end{pmatrix}.$$

Setting now $r := \cosh\xi + \sinh\xi\cos\eta$ and $z := \sinh\xi\sin\eta$ yields

$$P = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix}$$

and then every symplectic matrix M of size 2 can be written as the product

$$M = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{A.4.1}$$

where $(r, \theta, z) \in (0, +\infty) \times [0, 2\pi) \times \mathbb{R}$. Viewing (r, θ, z) as cylindrical coordinates in $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ we obtain a representation of Sp(2) in $\mathbb{R}^3$; more precisely, we obtain a smooth global diffeomorphism $\psi : \mathrm{Sp}(2) \to \mathbb{R}^3 \setminus \{z\text{-axis}\}$. We shall henceforth identify elements in Sp(2) with their image under ψ.

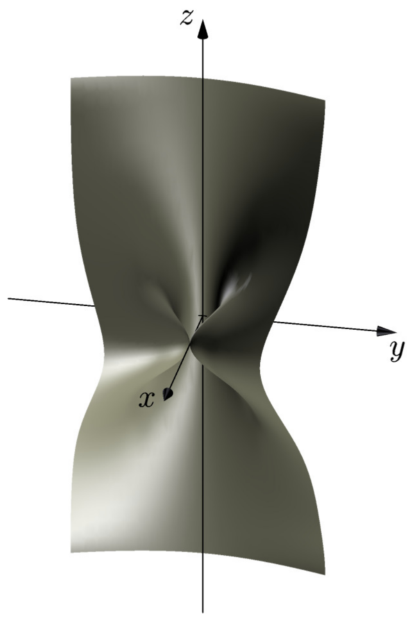


Figure A.1. The singular surface $\mathrm{Sp}(2)^0_1$. The representation is in Cartesian coordinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$.

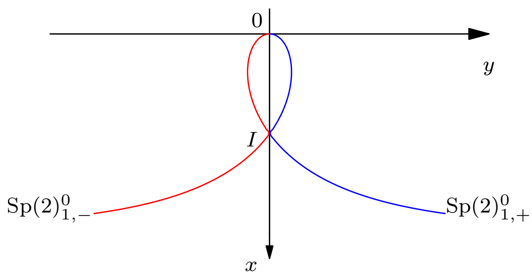


Figure A.2. Intersection of $\mathrm{Sp}(2)^0_1$ with the plane $z = 0$. The representation is in Cartesian coordinates $(x, y) = (r\cos\theta, r\sin\theta)$.

The eigenvalues of a symplectic matrix M written as in (A.4.1) are

$$\lambda_\pm := \frac{1}{2r}\Big[(1+r^2+z^2)\cos\theta \pm \sqrt{(1+r^2+z^2)^2\cos^2\theta - 4r^2}\Big].$$

For $\omega := e^{i\varphi} \in \mathbb{U}$ we get

$$\begin{aligned} D_\omega(M) &:= (-1)^{n-1}\omega^{-n}\det(M-\omega I)\big|_{n=1} \\ &= e^{-i\varphi}\det(M - e^{i\varphi} I) \\ &= 2\cos\varphi - \left(r + \frac{1+z^2}{r}\right)\cos\theta \end{aligned}$$

and define

$$\begin{aligned} \mathrm{Sp}(2)^\pm_\omega &:= \left\{ (r,\theta,z) \in (0,+\infty)\times[0,2\pi)\times\mathbb{R} \;\middle|\; \pm(1+r^2+z^2)\cos\theta > 2r\cos\varphi \right\}, \\ \mathrm{Sp}(2)^0_\omega &:= \left\{ (r,\theta,z) \in (0,+\infty)\times[0,2\pi)\times\mathbb{R} \;\middle|\; \pm(1+r^2+z^2)\cos\theta = 2r\cos\varphi \right\}. \end{aligned}$$

The set $\mathrm{Sp}(2)^+_\omega \cup \mathrm{Sp}(2)^-_\omega$ is named the ω-*regular part* of $\mathrm{Sp}(2)$, while $\mathrm{Sp}(2)^0_\omega$ is its ω-*singular part*; the former corresponds to the subset of 2×2 symplectic matrices which do not have ω as an eigenvalue, whereas those matrices admitting ω in their spectrum belong to the latter.

We are particularly interested in $\mathrm{Sp}(2)^0_1$, the singular part of $\mathrm{Sp}(2)$ associated with the eigenvalue 1, a representation of which is depicted in Figure A.1. The "pinched" point is the identity matrix, and it is the only element satisfying $\dim\ker(M - I) = 2$. If we denote by

$$\mathrm{Sp}(2)^0_{\omega,\pm} := \left\{ (r, \theta, z) \in \mathrm{Sp}(2)^0_\omega \;\middle|\; \pm \sin\theta > 0 \right\},$$

we see that $\mathrm{Sp}(2)^0_1 \setminus \{I\} = \mathrm{Sp}(2)^0_{1,+} \cup \mathrm{Sp}(2)^0_{1,-}$, and each subset is a path-connected component diffeomorphic to $\mathbb{R}^2 \setminus \{0\}$.

The *stratum homotopy property* of the Maslov index states that the Maslov index of a path does not change if to that path is applied a homotopy that maintains each endpoint in its original stratum. Thanks to this property we can simplify the visualisation of paths involving $\mathrm{Sp}(2)^0_1$ by considering only their deformation (in the sense just described) onto the intersection of the surface with the plane $z = 0$ (which is the curve represented in Figure A.2).

A.5 MORSE INDEX OF FREDHOLM QUADRATIC FORMS

In this section we recall the definition of Morse index of Fredholm quadratic forms acting on a (real) separable Hilbert space (for further details see [PW14]). Let $\big(\mathcal{H}, \langle\cdot,\cdot\rangle\big)$ be a real separable Hilbert space. As usual we denote by $\mathscr{L}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$ and by $\mathscr{F}(\mathcal{H}) \subset \mathscr{L}(\mathcal{H})$ the subspace consisting of all (bounded) Fredholm operators. An operator in $\mathscr{L}(\mathcal{H})$ defined on all of $\mathcal{H}$ is self-adjoint if and only if it is symmetric. We denote by $\mathscr{F}^s(\mathcal{H})$ the subspace of all (bounded) self-adjoint Fredholm operators. For $T \in \mathscr{F}^s(\mathcal{H})$, if 0 belongs to the spectrum $\sigma(T)$, then (being T Fredholm) 0 is an isolated point of $\sigma(T)$ and therefore it follows from the Spectral Decomposition Theorem that there is an orthogonal decomposition of H,

$$H = E_-(T) \oplus \ker T \oplus E_+(T),$$

that reduces the operator T and has the property that

$$\sigma(T) \cap (-\infty, 0) = \sigma(T|_{E_-(T)})$$

and

$$\sigma(T) \cap (0, +\infty) = \sigma(T|_{E_+(T)}).$$

If $\dim E_-(T) < +\infty$, then T is called *essentially positive* and if it is also an isomorphism its Morse index $i_{\text{Morse}}(T)$ is defined as

$$i_{\text{Morse}}(T) := \dim E_-(T).$$

Let us consider a bounded quadratic form $q : \mathcal{H} \to \mathbb{R}$ and we let $b = b_q : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be the bounded symmetric bilinear form such that

$$q(u) = b(u, u), \qquad \forall\, u \in \mathcal{H}.$$

By the Riesz Representation Theorem there exists a bounded self-adjoint operator $A_q : \mathcal{H} \to \mathcal{H}$ such that $b_q(u, v) = \langle A_q u, v\rangle$, $u, v \in \mathcal{H}$.

Definition A.14. We call $q : \mathcal{H} \to \mathbb{R}$ a *Fredholm quadratic form* if A_q is Fredholm; i.e. $\ker A_q$ is finite-dimensional and $\operatorname{Ran} A_q$ is closed.

Recall that the space $Q(\mathcal{H})$ of bounded quadratic forms is a Banach space with respect to the norm

$$\|q\| := \sup_{\|u\|=1} |q(u)|.$$

The subset $Q_F(\mathcal{H})$ of all Fredholm quadratic forms is an open subset of $Q(\mathcal{H})$ which is stable under perturbations by weakly continuous quadratic forms. A quadratic form $q \in Q_F(\mathcal{H})$ is called *non-degenerate* if the corresponding Riesz representation A_q is invertible.

Remark A.15. It is worth noting that if the representation of a quadratic form on $\mathcal{H}$ is either invertible, Fredholm or compact then so is its representation with respect to any other Hilbert product on the (real) vector space $\mathcal{H}$.

Proposition A.16. *A quadratic form on the Hilbert space $\mathcal{H}$ is weakly continuous if and only if one (and hence any by Remark A.15) of its representations is a compact (self-adjoint) operator in $\mathscr{L}(\mathcal{H})$.*

Proof. Recall that K is compact if and only if it maps weakly convergent sequences to strongly convergent sequences. We prove $(\Leftarrow)$. Suppose that K is compact and let (u_n) be a sequence in $\mathcal{H}$ such that $u_n \xrightarrow{w} u_0$. Then (Ku_n) strongly converges to Ku_0. Thus we getting

$$\lim_{n\to+\infty} q(u_n) = \lim_{n\to+\infty} \langle Ku_n, u_n\rangle = \langle Ku_0, u_0\rangle = q(u_0),$$

so the quadratic form is weakly sequentially continuous (and hence weakly continuous because $\mathcal{H}$ is first-countable). Now suppose that q is weakly sequentially continuous. By the polarisation identity applied to the bilinear form $(u, v) \mapsto \langle Ku, v\rangle$ with $v = KU$ we get

$$\langle Ku, Ku\rangle = \frac{1}{4}\Big[\big\langle K(u + Ku), u + Ku\big\rangle - \big\langle K(u - Ku), u - Ku\big\rangle\Big] \tag{A.5.1}$$

for all $u \in \mathcal{H}$. Let us assume that $(u_n) \subset \mathcal{H}$ weakly converges to u_0. Since $K \in \mathscr{L}(\mathcal{H})$ then $Ku_n \xrightarrow{w} Ku_0$. Thus $(u_n \pm Ku_n)$ weakly converges to $u_0 \pm Ku_0$. Therefore by the weak sequential continuity of q and by the identity (A.5.1) applied to $u = u_n$ and $u = u_0$ we get

$$\lim_{n \to +\infty} \|Ku_n\|^2 = \|Ku_0\|^2 .$$

Since (Ku_n) converges to Ku_0 weakly and in norm, it follows that it converges pointwise to Ku_0 (strongly) in $\mathcal{H}$. Thus K is compact and this conclude the proof. □

From this proposition we immediately get that Fredholm quadratic forms remain Fredholm under perturbations by weakly continuous quadratic forms (since by definition a Fredholm operator is the pre-image of the invertibles of the Calkin algebra under the projection on the quotient) and that any Fredholm quadratic form is weakly continuous perturbation of a non-degenerate Fredholm quadratic form.

Definition A.17. A Fredholm quadratic form $q : \mathcal{H} \to \mathbb{R}$ is said *essentially positive* if it is the perturbation of a positive definite Fredholm quadratic form by a weakly continuous quadratic form.

This discussion entails the following proposition.

Proposition A.18. *A Fredholm quadratic form q is essentially positive if and only if it is represented by an essentially positive self-adjoint Fredholm operator A_q.*

Proof. By the Riesz representation theorem there exists a bounded self-adjoint Fredholm operator $A_q : \mathcal{H} \to \mathcal{H}$ such that $b_q(u, v) = \langle A_q u, v\rangle$ for all $u, v \in \mathcal{H}$. Now since a bounded self-ajoint Fredholm operator is essentially positive if and only if it is a self-adjoint compact perturbation of a self-adjoint positive definite (and hence Fredholm, being invertible) operator, the conclusion follows by applying Proposition A.16. □

Definition A.19. The *Morse index of an essentially positive Fredholm quadratic form* $q : \mathcal{H} \to \mathbb{R}$ is the Morse index of the (self-adjoint) bounded Fredholm operator $A_q : \mathcal{H} \to \mathcal{H}$ uniquely determined by the Riesz Representation Theorem, i.e.

$$b_q(u, v) = \langle A_q u, v\rangle \text{ for all } u, v \in \mathcal{H}$$

where b_q is the bounded symmetric form induced by q through the polarisation identity.

Remark A.20. It is worth noting that it is possible to show that the *Morse index* of an essentially positive Fredholm quadratic form depends only on the quadratic form and not on the Hilbert structure on $\mathcal{H}$.

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