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TESI DI DOTTORATO

**SPECTRAL ANALYSIS AND PROBLEMS WITH
OSCILLATING CONSTRAINTS IN THE THEORY OF
HOMOGENIZATION**

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Introduction

The theory of homogenization has a quite recent history, which dates back between the end of 1960s and the beginning of 1970s. The term *homogenization* appeared for the first time in 1976 in works by Babuška [8], [7], [9] concerning nuclear engineering, but many important mathematical tools used in homogenization had been already considered since the end of 1960s.

The aim of this theory can be briefly explained with a quotation that I took from [6], which was one of my first encounters with this theory: “homogenization theory aims to find an effective description of materials whose heterogeneities scale is much smaller than the size of the body”. This means that it allows to treat heterogeneous materials, with complicated structure, also from the numerical point of view, approximating them by a homogenized one. The original point of this theory, as Tartar underlined many times [52], is that it can fit to problems with a certain degree of randomness (the structure of a heterogeneous material) avoiding the use of probabilities.

In this thesis we focus our attention on two homogenization problems, both involving the geometry of the domain: an elliptic spectral problem in a perforated domain and problems with oscillating constraints.

The structure of perforated domains will be described thoroughly in section 2.1; in order to fix the ideas, we can consider the simplest case, defined as

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i \in I_\varepsilon} B_\varepsilon^i,$$

where Ω is an open bounded set of \mathbb{R}^d with Lipschitz boundary and B_ε^i , for $i \in I_\varepsilon \subset \mathbb{Z}^d$ are the holes, obtained from a given closed and \mathcal{C}^2 set $B \subset Q = (0, 1)^d$ by means of translations and homothety, as follows

$$B_\varepsilon^i = \varepsilon(B + i), \quad I_\varepsilon = \left\{ i \in \mathbb{Z}^d : \varepsilon(B + i) \subset \Omega \right\}.$$

Hence, by construction, the boundary of perforated domain will be

$$\partial\Omega_\varepsilon = \partial\Omega \cup \left(\bigcup_{i \in I_\varepsilon} \partial B_\varepsilon^i \right) = \partial\Omega \cup \Sigma_\varepsilon.$$

In such a domain we consider elliptic PDEs and related spectral problems: in our case

$$-\operatorname{div}(a_\varepsilon(x)\nabla u(x)) = \lambda_\varepsilon u_\varepsilon,$$

where $a_\varepsilon(x) = a(x/\varepsilon)$ and $a \in \mathcal{M}^{d \times d}$ is a Q -periodic and symmetric matrix satisfying a standard ellipticity condition, so that the problem is actually equivalent to

$$-\Delta u_\varepsilon(x) = \lambda_\varepsilon u_\varepsilon. \quad (1)$$

This type of problems has been treated since the 1970s: the reader can find many examples in books as [4], [28], [31], [49]. In particular for our analysis, the crucial work on spectral problems by Vanninathan [54] collects several results about the asymptotics of eigenpairs with Dirichlet, Neumann and Steklov boundary conditions. There, the author points out that the behavior of the eigenpairs $(\lambda_\varepsilon, u_\varepsilon)$ of problems in a perforated domain Ω_ε , as $\varepsilon \rightarrow 0$, strongly depends on the boundary conditions on $\partial\Omega_\varepsilon$.

Starting from this paper, many authors have worked on similar problems, changing boundary conditions or hypotheses on the geometry of the perforated domain, adding weight functions or analyzing localization effects. The boundary-value problem or spectral problem with Fourier boundary condition was treated by several authors ([55], [29], [32], [30], [12], [24]).

In our work we consider Fourier type boundary conditions with variable coefficients:

$$\nabla u_\varepsilon(x) \cdot n_\varepsilon = -q(x)u_\varepsilon(x), \quad x \in \Sigma_\varepsilon, \quad u_\varepsilon(x) = 0, \quad x \in \partial\Omega. \quad (2)$$

whose behavior depends on the assumptions on the weight function $q(x)$. The problem was suggested by a work by Chiadò Piat, Pankratova, Piatniski [27], where the authors consider problem (1), (2) with $q \in C^2(\overline{\Omega})$ strictly positive and realizing its global minimum at a unique point $x_0 \in \Omega$. Moreover, they assume the Hessian matrix in x_0 to be positive definite. Here, on the contrary, we suppose that

$$q(x) = \begin{cases} 0 & x \in K, \\ 1 & x \in \Omega \setminus K, \end{cases}$$

where $K \Subset \Omega$ is a compact set with non empty interior part A and Lipschitz boundary.

A physical interpretation of the weight q in our work is as the insulating power of the holes located inside K . Namely, the homogeneous Neumann boundary condition at the boundary of the holes Σ_ε means that they represent completely insulating inclusions; the presence of the weight q , which is zero in K and positive outside it, allows these inclusions to conduct only in region $\Omega \setminus K$ and to be insulating in K . Homogenization can describe the behavior of such a material when the number of these holes tends to infinity and their size tends to zero.

We consider the functional $F_\varepsilon : \mathbb{L}^2(\Omega) \rightarrow [0, +\infty]$ defined as

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega_\varepsilon} |\nabla u|^2 + \int_{\Sigma_\varepsilon \setminus K} |u|^2 & u \in H_0^1(\Omega_\varepsilon; \partial\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $H_\varepsilon = H_0^1(\Omega_\varepsilon; \partial\Omega) = \{u \in H^1(\Omega_\varepsilon) : u = 0 \text{ in } \partial\Omega\}$ is a Hilbert space, equipped with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_\varepsilon} \nabla u \nabla v dx.$$

It is convenient to represent the first eigenvalue through its variational characterization

$$\lambda_\varepsilon^1 = \min \left\{ F_\varepsilon(u) : u \in H_\varepsilon, \int_{\Omega_\varepsilon} u^2 dx = 1 \right\}.$$

More generally, we can describe any eigenvalue λ_ε^j in a variational way. by introducing a basis of eigenfunctions u_ε^i of our problem (1), (2), and by taking the minimum over the spaces

$$H_\varepsilon^j = \{u \in H_\varepsilon : (u, u_\varepsilon^i)_{H_\varepsilon} = 0, i = 1, \dots, j-1\}.$$

We first prove equiboundedness of the first eigenvalue λ_ε^1 . Then we compute the Γ -limit of F_ε :

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u) = F(u) = \begin{cases} \int_{\Omega} f^{\text{hom}}(\nabla u) dx, & u \in H_0^1(A) \\ +\infty & \text{otherwise.} \end{cases}$$

Here $f^{\text{hom}} : \mathbb{R}^d \rightarrow [0, +\infty]$ is defined by

$$f^{\text{hom}}(\xi) = \inf \left\{ \int_Y |\xi + \nabla u|^2 dx : u \in H_{\text{per}}^1(\mathbb{R}^d) \right\}.$$

and gives the homogenized spectral problem, with Dirichlet conditions,

$$\begin{cases} -\text{div}(a^{\text{hom}} \nabla u) = |Y| \lambda u & u \in A \\ u = 0 & u \in \partial A, \end{cases} \quad (3)$$

where $a^{\text{hom}} \xi \xi = f^{\text{hom}}(\xi)$, A is the interior part of K and $Y = Q \setminus B$ is the perforated periodicity cell.

This Γ -convergence result, together with the equicoerciveness of F_ε and the variational formulation of eigenvalues, implies the convergence

$$\lambda_\varepsilon^j \xrightarrow{\varepsilon \rightarrow 0} \lambda^j,$$

with λ^j eigenvalues of the homogenized problem (3). We also study the asymptotics of the eigenfunctions u_ε^j with respect to the homogenized ones u^j , proving the convergence of eigenspaces, in the sense of Mosco; moreover, we investigate the rate of convergence of eigenfunctions in the L^2 -norm, and we show that is of the order $\sqrt{\varepsilon}$. This last result is obtained following a classical procedure similar to the one in [26], that exploits Višik lemma [41].

The second and third parts of this thesis are devoted to the homogenization of problems with oscillating constraint, that can be included in the general theory of periodic or almost-periodic homogenization of integral functions. There is a wide literature concerning homogenization problems for singular structures and for functions defined on networks and periodic manifolds ([3, 15, 23, 35, 42, 46, 47, 56, 45, 44]). In most of those problems the geometric complexity is in the domain of definition, and the functions are considered as defined on the whole space or as limits of functions defined on full-dimensional sets as those sets tend to a lower dimensional (possibly multidimensional) structure. In our case we consider similar geometries, but the geometrical complexity is in the codomain, as we consider instead functions with values in a periodic manifold, and we analyze the behavior of the corresponding energies as the geometry of the target manifold gets increasingly oscillating. Homogenization problems with a fixed target manifold have been considered in [6, 5].

Our results concern the behavior of energies defined on functions constrained to take their values on manifolds V_ε with a finely oscillating geometry as these manifolds converge to a smoother manifold V as $\varepsilon \rightarrow 0$.

Since we are interested in highlighting the effects of the constraint, we will focus on the simplest energy functional, i.e., the Dirichlet integral. Namely, for $u : \Omega \subset \mathbb{R}^n \rightarrow V_\varepsilon$ we will consider

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega} |\nabla u|^2 dx & u \in H^1(\Omega; V_\varepsilon), \\ +\infty & \text{otherwise.} \end{cases}$$

We suppose that the limit V is a smooth m -dimensional manifold and the oscillating V_ε are manifolds of the same dimension lying in a tubular neighborhood of V with vanishing radius as $\varepsilon \rightarrow 0$. A localization and blow-up argument leads to a problem where V is an m -dimensional linear subspace of an Euclidean space $\mathbb{R}^{m+m'}$. We will treat the *cartesian case*; i.e., when the manifolds V_ε can be seen as graphs of a function defined on V (identified with \mathbb{R}^m); i.e., there exist functions $\varphi_\varepsilon : \mathbb{R}^m \rightarrow \mathbb{R}^{m'}$:

$$V_\varepsilon = \{(x, \varphi_\varepsilon(x)) : x \in \mathbb{R}^m\} \subseteq \mathbb{R}^{m+m'}.$$

Hence the assumption that V_ε converges to V as $\varepsilon \rightarrow 0$ is translated into

$$\lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon = 0, \quad V_\varepsilon = \{(x, \varphi_\varepsilon(x)) : x \in \mathbb{R}^m\} \rightarrow \{(x, 0) : x \in \mathbb{R}^m\} = V.$$

The general case, where V is not necessarily a hyperplane, can be explained in the same way, imposing that V_ε converges to the tangent space to V at X , so that the model case is a “local” description of the general one.

Our modeling assumption is that the description of the oscillations of φ_ε is obtained through a single periodic function $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^{m'}$ satisfying

1. $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}^{m'}$ is $(0, 1)^m$ -periodic;
2. $\varphi_\varepsilon(x) = \delta_\varepsilon \varphi\left(\frac{x}{\varepsilon}\right)$, with $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

In such a setting a function $u \in H^1(\Omega; V_\varepsilon)$ can be rewritten as

$$u(x) = (u_1(x), u_2(x)),$$

with $u_1 : \Omega \rightarrow \mathbb{R}^m$ and

$$u_2(x) = \varphi_\varepsilon(u_1(x)).$$

Hence we can write F_ε in an unconstrained form:

$$\begin{aligned} F_\varepsilon(u) &= \int_\Omega |\nabla u|^2 dx = \int_\Omega |\nabla u_1|^2 dx + \int_\Omega |\nabla \varphi_\varepsilon(u_1)|^2 dx = \\ &= \int_\Omega |\nabla u_1|^2 dx + \left(\frac{\delta}{\varepsilon}\right)^2 \int_\Omega |\nabla \varphi\left(\frac{u_1}{\varepsilon}\right) \nabla u_1|^2 dx. \end{aligned}$$

The coefficient in front of the second term of F_ε , suggests three different behaviors depending on the scale of the coefficient δ :

1. $\delta/\varepsilon \rightarrow 0$. In this case the homogenization becomes trivial, the second term can be neglected and the Γ -limit is just the Dirichlet integral of the function u_1 , which in particular is independent of the constraint φ ;
2. $\delta/\varepsilon \rightarrow c \in (0, +\infty)$. In this case we can consider the energy density

$$f(v, \xi) = |\xi|^2 dx + c^2 |\nabla \varphi(v) \xi|^2$$

so that

$$F_\varepsilon(u_1) = \int_\Omega f\left(\frac{u_1}{\varepsilon}, \nabla u_1\right) dx.$$

Since f is periodic and satisfies a standard growth condition the homogenization of these energies can be then performed by using general almost periodic homogenization theorems [19];

3. $\delta/\varepsilon \rightarrow +\infty$. In this case the energy density f_ε of F_ε does not satisfy standard growth conditions and we cannot use known results. The fact that the coefficient of the second term blows up as $\varepsilon \rightarrow 0$, suggests that the behavior of the homogenized functional is related to conditions that make the second integral negligible as $\varepsilon \rightarrow 0$. Upon scaling the

variable, this leads to the condition that u_1 makes $D_y\varphi(u_1)$ almost zero; i.e. u_1 is very close to lying on a level set of φ . Therefore, the Γ -limit will strongly depend on the geometry of the constraint φ ; in particular on its level sets.

The result in the thesis deals with the description of asymptotic *metric properties* of V_ε , for which we deal with *curves* in \mathbb{R}^m (i.e., $n = 1$) and hypersurfaces (i.e., $m' = 1$). The general vectorial case seems to include additional effect than in the case of curves, which require the use of notions as quasiconvexity, polyconvexity or rank-1-convexity, and is beyond the scope of this thesis. Generalization to surfaces of higher codimension $m' \geq 1$ on the contrary seems to be analytically more standard.

In order to describe the Γ -limit in the case $\delta \gg \varepsilon$ we have to introduce several types of homogenization. With a slight abuse of notation from now on we will directly use the variable u in place of u_1 .

First, with fixed z we consider the strict constraint

$$\varphi\left(\frac{u}{\varepsilon}\right) = z,$$

which is meaningful only if z is in the image of φ . For functions satisfying this constraint the functional $F_\varepsilon(u)$ reduces to the Dirichlet integral. Moreover, the limit of functions satisfying this constraint is not constant only if the set $\{x \in \mathbb{R}^m : \varphi(x) = z\}$ contains an infinite connected curve with locally finite length. We will make the stronger assumption:

- **(uniform connectedness)** there is a single infinite connected component of this set and all pairs of points x, x' in this component are connected with a curve of length proportional to the distance between x and x' .

This property is easily verified if $\{x \in \mathbb{R}^m : \varphi(x) = z\}$ is composed of unions of periodic C^1 hypersurfaces.

Under this assumption the homogenization of the Dirichlet integral with the strict constraint is an integral functional

$$\int_{\Omega} \psi_{\text{hom}}^z(u') dt,$$

with $\psi_{\text{hom}}^z : \mathbb{R}^m \rightarrow [0, +\infty)$ a two-homogeneous convex function. Moreover, ψ_{hom}^z satisfies the *asymptotic homogenization formula*

$$\psi_{\text{hom}}^z(w) = \lim_{T \rightarrow +\infty} \psi_T^z(w),$$

where

$$\psi_T^z(w) = \frac{1}{T} \min \left\{ \int_0^T |u'|^2 dt : |u(0)| \leq \sqrt{m}, |u(T) - Tw| \leq \sqrt{m}, \varphi(u) = z \right\}. \quad (4)$$

This is a variation on corresponding formulas for the homogenization of functionals with energy densities $f(u/\varepsilon, u')$. In this case such energy densities are infinite outside the constraint, so that some extra care must be taken. Note in particular that we cannot easily impose strict boundary conditions (that in the usual case would read $u(0) = 0$ and $u(T) = Tw$). We prefer to substitute those conditions with an inequality, which in particular is satisfied when the $u(0)$ lies in the periodicity cube $(0, 1)^m$.

In order to derive the homogenization theorem for our energies from those strictly constrained energies we make the following assumptions:

- for all z in the image of φ either $\{x \in \mathbb{R}^m : \varphi(x) = z\}$ is uniformly connected (in the sense defined above) or it has no infinite connected component;

- curves satisfying the weaker constraint $u(t) \in \{x \in \mathbb{R}^m : |\varphi(x) - z| \leq c\}$ are close to curves satisfying a strict constraint for some z' for c small enough.

Before stating more precisely the latter condition, we consider the model example of $n = 2$ and

$$\varphi(x, y) = \sin(2\pi x) \sin(2\pi y).$$

In this case, the only connected level set is with $z = 0$. A set $\{(x, y) : |\varphi(x, y) - z| \leq c\}$ is either composed of disconnected components (when $c < |z|$), or contains a tubular neighborhood of $\{(x, y) : \varphi(x, y) = 0\}$. In any case, given a curve u taking values in that set, we can find a curve u_0 satisfying the strict condition $\varphi(u_0(t)) = 0$ close to the original curve and with energy not greater than the energy of u times $1 + o(1)$ as $c \rightarrow 0$.

With this example in mind we can state the condition above as follows. For $w \in \mathbb{R}^m$, $z \in \text{Im}(\varphi) \subset \mathbb{R}$, $c, T > 0$, we consider the minimum problems

$$\psi_T^{z,c}(w) = \frac{1}{T} \min \left\{ \int_0^T |u'|^2 : |u(0)| \leq \sqrt{m}, |u(T) - Tw| \leq \sqrt{m}, |\varphi(u) - z| \leq c \right\}. \quad (5)$$

Then we require that there exists z' such that $|z - z'| \leq c$ and for all w there exists $w' = w + o_T(1)$ such that

$$\psi_T^{z,c}(w) \geq (1 + o_c(1)) \psi_T^{z'}(w') + o_T(1),$$

where $o_c(1) \rightarrow 0$ as $c \rightarrow 0$ and $o_T(1) \rightarrow 0$ as $T \rightarrow +\infty$.

This is the (rather complex) variational formulation of a geometric stability property of level sets, which is easily proved for ordinary constraints. By using this property it is possible to prove the homogenization result by reducing to strict constraints. Summarizing the main arguments in the proof, from energy bounds we deduce that functions u_ε with $F_\varepsilon(u_\varepsilon)$ equibounded locally must lie in some set $\{x : |\varphi(u_\varepsilon/\varepsilon) - z| \leq c\}$ with c small. By a scaling argument then the energy is estimated using $\psi_T^{z,c}(w)$, where w is the local averaged slope of u_ε , and eventually with $\psi_T^z(w')$. A particular care

has to be taken in these computations in order to reduce to “local” estimate which nevertheless, after scaling, can be estimated by problems with T large enough.

Finally, we get our homogenized energy density by optimizing over z and on mesoscopic oscillations between level sets, producing a convex envelope of the minima between homogenization formulas on strict constraint:

$$\psi_{\text{hom}} = \left(\min_{z \in \text{Im}(\varphi)} \psi_{\text{hom}}^z \right)^{**}.$$

Note that under our assumptions on level sets, in dimension two there is only one infinite connected level set for some $z = z_0$, so that this formula simplifies to $\psi_{\text{hom}}^{z_0}$. In dimension three or higher it is instead possible to give examples where there are more than one infinite connected level set, and the formula above must indeed be applied. Eventually, the Γ -limit is then given by

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = \int_0^1 \psi_{\text{hom}}(u') dt.$$

Working on oscillating constraint problems, we found out that the functionals F_ε in the scalar case, actually their Γ -limit, can be geometrically interpreted as a norm over \mathbb{R}^m : the existence of at least one level set of φ containing an unbounded connected component, which is necessary for the boundedness of the homogenized function, creates a sort of periodic unbounded and connected ε -network over \mathbb{R}^m , that represents the “allowed” zones for curves u_ε in F_ε . Indeed, if u_ε lies on this lattice, the gradient of $\varphi(u_\varepsilon/\varepsilon)$ will be zero and the Γ -limit can be finite on their limit u . In this metric standpoint, we can interpret ψ_{hom} as measuring the distance between the origin and the point w , not with the euclidean norm, but with the length of a curve that microscopically lies in the lattice defined by the constraint.

Our last result regards the characterization of all metrics on \mathbb{R}^m that can be obtained as a Γ -limit following the procedure above for some φ . This problems takes inspiration by a work by Braides, Buttazzo and Fragalà [17], where the authors consider the density of Finsler metrics in the Riemannian ones, with respect to Γ -convergence. By a Finsler metrics in \mathbb{R}^m we mean any convex and 2-homogeneous function ψ on \mathbb{R}^m , controlled from below by the Euclidean norm: $\psi(w) \geq c|w|^2$.

Note as a preliminary trivial observation that all limit metrics ψ_{hom} will satisfy $\psi(z) \geq |w|^2$, and that the equality is achieved with $\varphi = 0$. Moreover we already proved that ψ_{hom} is 2-homogeneous and convex, so that it is a Finsler metric. Following this observations, we conjectured that we could approximate, via Γ -convergence, any symmetric Finsler metric larger than the Euclidean metric by a ψ_{hom} defined as the homogenized of an oscillating constraint problem. We prove that the answer is positive if $m = 2$. The case

$m > 2$, is more complicated because of the geometry of the level sets, but can be, together with the vectorial case $n > 1$, a starting point for future works.

We consider more in detail the approximation of two dimensional non-degenerate Finsler metrics; the degenerate case, i.e., if there exists $\nu \in S^1$, $s \in \mathbb{R}$ and $C > 1$ such that

$$F(u) = \begin{cases} C \int_{\Omega} |u'|^2 dt & \text{if } \langle u(t), \nu \rangle = s \text{ for all } t \\ +\infty & \text{otherwise,} \end{cases}$$

being dealt with by hand. As remarked above, in this case the desired φ will only have one non-empty connected level set. The construction of φ is equivalued to the construction of this level set, which we may suppose to be $\{x : \varphi(x) = 0\}$.

By approximation we first reduce to a crystalline case; i.e., when the set $\{w : \psi(w) \leq 1\}$ is a polygon. If $\nu_1, \dots, \nu_N \in S^1$ denote the directions of the extremal points of this polygon, again by approximation we can suppose that ν_j also are *rational*; i.e., there exist $t_j > 0$ such that $t_j \nu_j \in \mathbb{Z}^2$: look to Figure 1 for example.

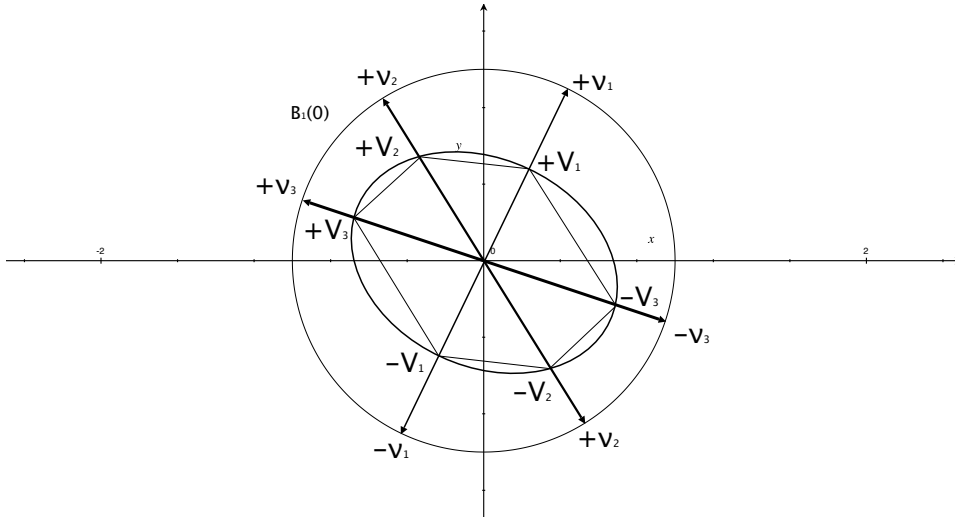


Figure 1: The polygon $\{w : \psi(w) \leq 1\}$, with extremal points V_j in the directions ν_j .

This allows to construct a periodic network composed of lines in the directions ν_j , as Figure 2 shows.

By perturbing this network we obtain a periodic network of curves \mathcal{L} , as in Figure 3, such that the homogenization of the Dirichlet integral on the curves satisfying the constraint $u_\varepsilon \in \varepsilon \mathcal{L}$ gives ψ . The desired φ is then given for example by $\varphi(x) = \text{dist}^2(x, \mathcal{L})$.

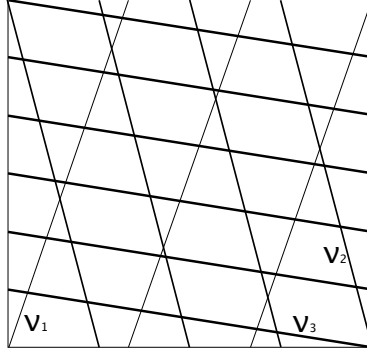


Figure 2: The network with lines in the directions ν_j .

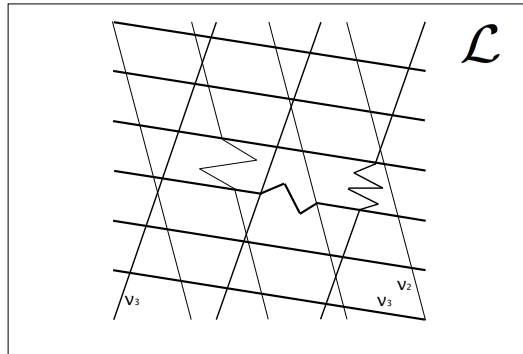


Figure 3: The network \mathcal{L} with perturbed lines in the directions ν_j .

The thesis is divided in four parts according to different problems treated. The first part is dedicated to preliminary statements: in chapter 1 we introduce some results of the general theory of homogenization of integral functions, of Γ -convergence and of convex analysis; in chapter 2 we describe the structure of perforated domains with three examples of problems, that are Dirichlet, Neumann homogeneous, section 2.2, and Fourier boundary conditions, section 2.3.

The second part concerns the asymptotics of eigenpairs for a Fourier boundary condition problem, with the weight function q described above: we define this problem in section 3.1, then we consider an upper bound of first eigenvalue λ_ε^1 in section 3.2; hence, by meanings of Γ -convergence theory, we consider the Rayleigh quotient and its Γ -limit in section 3.3; finally we state two important results about convergence of eigenvalues and eigenfunctions in

section 3.4, the first one exploits the properties of Γ -convergence and Mosco convergence, while the second uses a general result of spectral theory, known as Višik lemma, that allows to study the rate of this convergence.

Homogenization of oscillating constraint problems is treated in chapter 4: we start with the general cartesian case, in the hypothesis $\delta/\varepsilon \rightarrow 0$ and $\delta/\varepsilon \rightarrow c$, respectively in section 4.2 and 4.3; then, in chapter 5 we consider the case $\delta/\varepsilon \rightarrow +\infty$, for the scalar case, $n = 1 = m'$, treating the existence of the limit in the homogenization formula and the Γ -limit's proof.

Chapter 6 is dedicated to the density of \mathbb{R}^2 oscillating constraint problems for curves in Finsler metrics: we prove the density result in the most interesting case $\sup_{\|w\|=1} \psi(w) < +\infty$, and we consider some examples.

Chapter 1

Homogenization of integral functionals

We present some preliminary tools and fix the notation that we will use in the next chapters. In section 1.1 we recall the definition of Γ -convergence with its main properties, and we point out some property of convex functions and sets; in section 1.2 we will state some classic results about periodic and almost periodic homogenization of integral functions.

1.1 Preliminary tools

The notion of Γ -convergence, due to De Giorgi and Franzoni, [34], dates in 1975; it is a suitable type of convergence in the variational way, because it satisfies the important property of preserving minima and minimizers, as we will see in theorem 1.1.1. We will consider for its definition a metric space (X, d) . We refer to books as [19] or [33] for details.

Definition 1.1.1. *Let $f_j : X \rightarrow \overline{\mathbb{R}}$, $j \in \mathbb{N}$, be a sequence of functions. We say that f_j Γ -converges to $f : X \rightarrow \overline{\mathbb{R}}$, in the topology generated by the metric d , writing*

$$\Gamma(d)\text{-}\lim_{j \rightarrow +\infty} f_j = f,$$

if, for any $x \in X$, we have

i) (Γ -lim inf) for every sequence x_j converging to x with respect the d -topology

$$f(x) \leq \liminf_{j \rightarrow \infty} f_j(x_j); \tag{1.1}$$

ii) (Γ -lim sup) there exists a sequence $\{x_j\}_j$, often called recovery sequence, converging to x , always in the d -topology, such that

$$f(x) \geq \limsup_{j \rightarrow \infty} f_j(x_j), \tag{1.2}$$

or, equivalently

$$\lim_{j \rightarrow \infty} f_j(x_j) = f(x).$$

By the definition of Γ -convergence we can obtain the following properties:

Proposition 1.1.1. *Let $f_j : X \rightarrow \overline{\mathbb{R}}$ Γ -converge to f , then we have*

i) *f is d -lower semicontinuous (l.s.c. for short). This is a consequence of (1.1).*

ii) *For every increasing sequence of indices j_k , $f = \Gamma(d)\text{-}\lim_k f_{j_k}$.*

iii) *If g is a continuous function then*

$$\Gamma(d)\text{-}\lim_{j \rightarrow \infty} f_j + g = f + g.$$

iv) *If $f_j = F$ for any $j \in \mathbb{N}$, then f_j Γ -converges to \overline{F} , that is the lower semicontinuous envelope of F , i.e.*

$$\overline{F}(x) = \sup \{g(x) : g \text{ l.s.c.}, g \leq f\}.$$

We can state two further properties of the Γ -limit: the first about compactness in separable metric spaces

Property 1.1.1. *Let (X, d) be a separable metric space and $f_j : X \rightarrow \overline{\mathbb{R}}$, $j \in \mathbb{N}$ a sequence of function. Then there is an increasing sequence of indices $\{j_k\}$, such that $\Gamma(d)\text{-}\lim_k f_{j_k}$ exists.*

The second is about the convergence for subsequence

Property 1.1.2. *One has $\Gamma(d)\text{-}\lim_j f_j = f$ if and only if for every subsequence $\{f_{j_k}\}$ $\Gamma(d)\text{-}\lim_k f_{j_k} = f$*

This last property allow us to treat the convergence of a sequence depending on a continuous parameter: let $\{f_\varepsilon\}_{\varepsilon > 0}$ be a sequence of functions depending on $\varepsilon \in \mathbb{R}^+$; then we say that

$$\Gamma(d)\text{-}\lim_{\varepsilon \rightarrow 0} f_\varepsilon = f$$

if, for any decreasing sequence $\{\varepsilon_j\}_{j \in \mathbb{N}}$ such that $\lim_{j \rightarrow \infty} \varepsilon_j = 0$, we have

$$\Gamma(d)\text{-}\lim_{j \rightarrow \infty} f_{\varepsilon_j} = f.$$

Finally we show the main property of the Γ -limit: the convergence of minima and minimizers.

Theorem 1.1.1. *Let (X, d) be a metric space and f_j, f functionals from X to $\overline{\mathbb{R}}$, and let x_j be a minimizer for f_j , i.e.*

$$f_j(x_j) = \min_{x \in X} f_j(x).$$

If f_j $\Gamma(d)$ -converges to f and $x_j \xrightarrow{d} x_0 \in X$, then

$$f(x_0) = \lim_{j \rightarrow \infty} f_j(x_j) = \min_{x \in X} f(x).$$

In this last part of the section we want to recall some results about convexity. Let us first consider the definition of a convex scalar function.

Definition 1.1.2. *Consider the function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$; we say that f is convex if*

$$f(tu + (1-t)v) \leq tf(u) + (1-t)f(v),$$

for all $u, v \in \mathbb{R}^n$ and $t \in (0, 1)$ for which the right-hand side is defined.

Remark 1.1.1. *Definition 1.1.2 can be extended to a convex linear combination: let $\lambda_i \in \mathbb{R}$, for $i = 1, \dots, m$ be such that $\sum_{i=1}^m \lambda_i = 1$ and let $x_i \in \mathbb{R}^n$ be such that $f(x_i)$ is defined for $i = 1, \dots, m$; we say that $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ is convex if*

$$f\left(\sum_{i=1}^m \lambda_i x_i\right) \leq \sum_{i=1}^m \lambda_i f(x_i).$$

There are equivalent definitions of convexity: let us recall the epigraph of a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$, that is the set

$$\text{epi}(f) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t \geq f(x)\}.$$

Then the following theorem holds true, see [53], theorem 5.10 for the proof:

Theorem 1.1.2. *Consider the function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$. The following conditions are equivalent*

- i) f is convex;*
- ii) $\text{epi}(f)$ is convex.*

Now we recall two important properties of convex function:

Proposition 1.1.2. *Let $f : \mathbb{R}^n \rightarrow (-\infty, +\infty]$ be a convex function, then Jensen's inequality holds:*

$$f\left(\frac{1}{\mu(\Omega)} \int_{\Omega} u d\mu\right) \leq \frac{1}{\mu(\Omega)} \int_{\Omega} f(u) d\mu,$$

for any finite positive measure μ on $\Omega \subseteq \mathbb{R}^n$ and $u \in L^1_{\mu}(\Omega)$.

Proposition 1.1.3. *Let consider $p \geq 1$ and $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ a convex function such that, for all $u \in \mathbb{R}^n$*

$$|f(u)| \leq (1 + |u|^p).$$

Then f is locally Lipschitz continuous, i.e. there exists a constant $c > 0$ such that for any $x_1, x_2 \in \mathbb{R}^n$

$$|f(x_1) - f(x_2)| < c(1 + |x_1|^{p-1} + |x_2|^{p-1})|x_1 - x_2|.$$

We will also use the convex hull of a function, also known as convex envelope, that is the largest convex minorant:

$$(f)^{**}(x) = \sup \{g(x) : g \text{ convex and l.s.c.}, g \leq f\},$$

and the convex envelope of a set $A \in \mathbb{R}^x$, that is the smallest convex set containing A ; it can be proved that it is the set of all finite convex combination of elements in A :

$$co(A) = \left\{ \sum_{i=1}^n \lambda_i x_i, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, x_i \in A, i = 1, \dots, n, n \in \mathbb{N} \right\}$$

Finally we recall the Caratheodory theorem for convex sets:

Theorem 1.1.3. *Let consider the set $A \subseteq \mathbb{R}^n$. Then for any $x \in co(A)$ there exist $n + 1$ vectors in A , x_1, \dots, x_{n+1} , such that x is a convex combination of these vectors, i.e.*

$$x = \sum_{i=1}^{n+1} \lambda_i x_i, \quad \sum_{i=1}^{n+1} \lambda_i = 1, \quad x_i \in A.$$

1.2 Periodic and almost periodic homogenization

We want to consider homogenization of integral functions, i.e. functions of the type

$$F_\varepsilon(u) = \int_{\Omega} f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx,$$

with f satisfying a periodicity (almost periodicity) and a standard growth condition.

Let $f : \Omega \subseteq \mathbb{R}^n \times M^{m \times n} \rightarrow [0, +\infty)$ be a Borel function; we say that f satisfies a standard p -growth condition if there exist $1 \leq p < +\infty$, $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha|\xi|^p \leq f(x, \xi) \leq \beta(1 + |\xi|^p). \quad (1.3)$$

Hence we can consider the class of integral functions $\mathcal{F}(\alpha, \beta, p)$ defined as

Definition 1.2.1. *Let consider $\alpha, \beta > 0$ and $1 \leq p < +\infty$. We say that $F : W^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega) \rightarrow [0, +\infty)$ belongs to $\mathcal{F}(\alpha, \beta, p)$ if there exists a Borel function $f : \mathbb{R}^n \times M^{m \times n} \rightarrow [0, +\infty)$ satisfying the p -growth condition (1.3), such that*

$$F(u, A) = \int_A f(x, \nabla u(x)) dx,$$

for $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and $A \in \mathcal{A}(\Omega)$ that is the family of open subsets of Ω .

We will also consider the periodicity condition: $f(\cdot, \xi)$ is 1-periodic for all $\xi \in M^{m \times n}$:

$$f(x + e_i, \xi) = f(x, \xi), \quad (1.4)$$

for all $x \in \mathbb{R}^n$, $\xi \in M^{m \times n}$ and $i = 1, \dots, n$.

In order to consider Γ -convergence of F_ε , there are two important issues to be treated:

- i) is the Γ -limit an integral type function? More precisely, we ask if can we find a function $\varphi : \Omega \times M^{m \times n} \rightarrow [0, +\infty)$ such that

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = F(u) = \int_\Omega \varphi(x, \nabla u(x)) dx;$$

- ii) if such a φ exists, is it possible to have an explicit formula independent of ε to compute it?

Our assumptions on f ensure a positive answer to both the questions:

Theorem 1.2.1. *Let $f : \mathbb{R}^n \times M^{m \times n} \rightarrow [0, +\infty)$ be a Borel function satisfying the periodicity condition (1.4) and the p -growth condition (1.3), for $1 \leq p < +\infty$. If $\Omega \subset \mathbb{R}^n$ is a bounded open set and we consider the integral function*

$$F_\varepsilon(u) = \int_\Omega f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx,$$

for $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, then

$$\Gamma(L^p)\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \int_\Omega f_{hom}(\nabla u(x)) dx,$$

for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$, where $f_{hom} : M^{m \times n} \rightarrow [0, +\infty)$ satisfies the asymptotic homogenization formula:

$$f_{hom}(\xi) = \lim_{T \rightarrow +\infty} \frac{1}{T^n} \inf \left\{ \int_{(0,T)^n} f(x, \xi + \nabla u(x)) dx : u \in W_0^{1,p}((0,T)^n; \mathbb{R}^m) \right\}, \quad (1.5)$$

for any $\xi \in M^{m \times n}$.

The proof of this important result, that can be found in [19] chapter 14, for example, goes through four fundamental steps:

- Step 1. using the p -growth condition, one shows that the Γ -limit of a integral function F_ε admits an integral representation;
- Step 2. one proves that the density of the Γ -limit, i.e. the function φ of question *i*) does not depend on the variable $x \in \mathbb{R}^n$;
- Step 3. using the periodicity of problem's structure, the existence of the limit in (1.5) is considered;
- Step 4. finally the two inequalities (1.1) and (1.2) are proved.

Theorem 1.2.1 can be generalized to function f non periodic. Let us consider the following

Definition 1.2.2. *Let $(X, \|\cdot\|)$ be a complex Banach space. We say that a measurable function $u : \mathbb{R}^n \rightarrow X$ is uniformly almost periodic (u.a.p. for short), if it is the uniform limit of a sequence of trigonometric polynomials on X :*

$$\lim_k \|P_k(\cdot) - u(\cdot)\|_\infty = 0,$$

for suitable

$$P_k(y) = \sum_{j=1}^{r_k} x_j^k e^{i(\lambda_j^k, y)},$$

with $x_j^k \in X$, $\lambda_j^k \in \mathbb{R}^n$ and $r^k \in \mathbb{N}$.

For these type of density functions the following homogenization theorem holds true, see [19], for example, for the proof:

Theorem 1.2.2. *Let $p > 1$ and $f : \mathbb{R}^n \times \mathbb{R}^m \times M^{m \times n} \rightarrow \mathbb{R}$ satisfy the following conditions*

i)

$$\alpha|\xi|^p \leq f(x, s, \xi) \leq \beta(1 + |\xi|^p), \quad (1.6)$$

for any $(x, s, \xi) \in \mathbb{R}^n \times \mathbb{R}^m \times M^{m \times n}$ and for suitable $\alpha, \beta > 0$;

ii) f is uniformly almost periodic in the first two variables $(x, s) \in \mathbb{R}^n \times \mathbb{R}^m$.

Then there exists a function $f_{hom} : M^{m \times n} \rightarrow \mathbb{R}$ such that for every bounded open subset Ω of \mathbb{R}^n and every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ the following limit exists

$$\Gamma(L^p)\text{-}\lim_{\varepsilon \rightarrow 0} \int_{\Omega} f\left(\frac{x}{\varepsilon}, \frac{u(x)}{\varepsilon}, \nabla u(x)\right) dx = \int_{\Omega} f_{hom}(\nabla u(x)) dx,$$

and the function f_{hom} is defined by the asymptotic homogenization formula

$$f_{hom}(\xi) = \lim_{T \rightarrow +\infty} \inf \left\{ \frac{1}{T^n} \int_{(0,T)^N} f(x, u(x) + \xi x, \nabla u(x) + \xi) dx : \right. \quad (1.7)$$
$$\left. u \in W^{1,p}((0,T)^n; \mathbb{R}^m) \right\},$$

for any $\xi \in M^{m \times n}$.

Chapter 2

Homogenization in periodically perforated domains

In this preliminary chapter we will give the reader a brief overview on the theory of homogenization in perforated domains, presenting the standard methods in section 2.1, two classical examples in section 2.2 and a particular problem in section 2.3, that will be our starting point for the one treated in chapter 3.

The literature on this topic is very large and for this presentation we refer to works as [4], [28], [31], [49].

2.1 Introduction and problem setup

Homogenization theory is a very important mathematical method that allows to treat a lot of problems, where the structures considered are too complicated from the numerical point of view, but they can be well approximated by a homogenized one. It was born in the late 70s, thanks to the contribution of works by Tartar, Murat, Spagnolo, Sanchez Palencia and others. The reader can refer to [52] for a good and interesting introduction, also from the historical point of view, on Homogenization theory.

In this work and in particular in this chapter we want to consider homogenization of perforated domains, that is strictly related to the study of P.D.Es. defined on particular domains with a large number of small holes: this means that boundary conditions on such a domain, i.e. on the boundary of these holes, make the problem completely unable to be treated with numerical methods.

From the physical point of view homogenization theory plays a fundamental role. To understand better we can consider as an example a composite material, which occupies an open bounded set $\Omega \subseteq \mathbb{R}^3$, made of two different

materials A and B with two conductivity coefficients λ_A , and λ_B :

$$\lambda = \begin{cases} \lambda_A & x \in A \\ \lambda_B & x \in B. \end{cases}$$

We can consider the temperature as a function $u(x)$ at a point $x \in \Omega$. Suppose that the portion of type B is a union of small insulating inclusions, compared to the dimension of the sample, periodically distributed inside Ω and its conductivity is zero. This means that u will satisfy an equation such as

$$\begin{cases} -\operatorname{div}(\lambda \nabla u) = G & x \in A \\ u = f & x \in \partial\Omega \\ u \cdot n = 0 & x \in \partial B \end{cases}$$

where f is the fixed temperature in the boundary of the material, n is the unite external norm in the boundary of the inclusions B , so that the third condition means that the holes are completely isolated.

Here it is clear that, the more the number of holes growths, the more it will be difficult to treat the equation numerically.

The aim of homogenization is to replace such a microscopically complicated structure with a macroscopic homogeneous one, which approximate the behavior of the original material. This means, from the mathematical point of view, that one has to introduce a small parameter ε , which is, for example, the size of holes, and formulate the problem in a perforated domain Ω_ε , then, with homogenization theory, one has to studied the problem as ε tends to zero.

Let us start with the definition of a periodically perforated set. We consider the sets $Q = [0, 1)^d$ and $E \subset \mathbb{R}^d$, which is Q -periodic, open and connected, with Lipschitz boundary $\Sigma = \partial E$. We define the complement of E , $B = \mathbb{R}^d \setminus E$, that represents the holes. At this point we have to distinguish two different cases: the first one is obtained assuming $Q \cap E$ connected, $Q \cap B \Subset Q$, so that B consists of disjoint components; the second, that is more general, avoids this hypothesis.

The first case is classical and it has been treated since 1977, by Tartar, Murat, Sanchez Palencia and other authors. The hypothesis on the connectedness of the perforated cell, allows us to use classic extension results for Sobolev spaces on regular open sets, see theorem 2.1.1, but it avoids to treat many interesting situations, as the one in figure 2.1, or in \mathbb{R}^3 the one in figure 2.2.

The second and more general case can be considered in a different way: in the work by Acerbi et al. [1] (1992), the authors show the existence of an extension operator, without the assumption on the connectedness of the perforated cell, so that this hypothesis can be avoided.

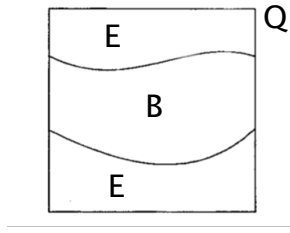


Figure 2.1: Non connected periodicity cell.

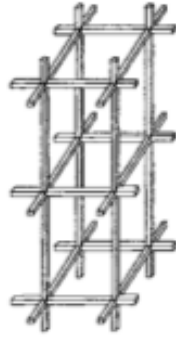


Figure 2.2: Non connected periodicity cell in \mathbb{R}^3 .

To simplify our presentation, in the sequel we will assume that $Q \cap E$ is connected and $Q \cap B \Subset Q$, remembering that this hypothesis is not necessary. So we denote by $Y = Q \cap E$ the periodicity perforated cell and $\Sigma^0 = Q \cap \partial B = Q \cap \Sigma$, the boundary of the hole.

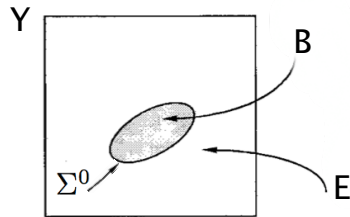


Figure 2.3: Admissible perforated cell.

For every $i \in \mathbb{Z}^d$ and for fixed $\varepsilon > 0$, we denote $Y_\varepsilon^i = \varepsilon(i + Y)$, $\Sigma_\varepsilon^i = \varepsilon \Sigma \cap Y_\varepsilon^i$, and $B_\varepsilon^i = \varepsilon B \cap Y_\varepsilon^i$, that are respectively the periodicity cells, the boundary of the holes, and the holes themselves. Given a bounded open set $\Omega \subset \mathbb{R}^d$, with Lipschitz boundary $\partial\Omega$, our perforated domain is

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{i \in I_\varepsilon} B_\varepsilon^i, \quad I_\varepsilon = \left\{ i \in \mathbb{Z}^d : Y_\varepsilon^i \subset \Omega \right\}. \quad (2.1)$$

By the hypothesis on the connectedness of $Q \cap E$, we can assume that Ω_ε is still connected; by definition of I_ε we get that the holes don't intersect $\partial\Omega$; we have

$$\partial\Omega_\varepsilon = \partial\Omega \cup \Sigma_\varepsilon, \quad \Sigma_\varepsilon = \bigcup_{i \in I_\varepsilon} \Sigma_\varepsilon^i.$$

This last assumption can be avoided, using a different extension operator, as we will see in the sequel.

In this setting we want to consider elliptic P.D.Es, with Dirichlet boundary condition on the external boundary of Ω , so that it is natural to defined the following space

$$H_\varepsilon = H_0^1(\Omega_\varepsilon; \partial\Omega) = \{u_\varepsilon \in H^1(\Omega_\varepsilon) : u_\varepsilon = 0 \text{ on } \partial\Omega\}, \quad (2.2)$$

that is a Hilbert space equipped with the norm

$$\|u_\varepsilon\|_{H_\varepsilon}^2 = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx.$$

The main difficulty about solving problems on perforated domains is due to the fact that this particular norm strongly depends on ε , in particular there is no any inclusion relation between H_ε and $H_{\varepsilon'}$, so that a priori estimates independent of ε , that we need to apply Lax-Milgram theorem, for example, can't be obtained. A natural and classical way to solve this issue is to defined a suitable continuous extension operator, that allows to consider u_ε defined in $H^1(\Omega)$ and to get, by continuity, a priori estimate in the whole Ω , in order to pass to the limit as $\varepsilon \rightarrow 0$. In other words, as we will see in our problem in section 3.3, we will define the problem in a variational way, i.e. as a minimum of a functional, and we will study its gamma limit.

Let us consider the first method and define the extension operator. The idea, due to Tartar (1977, 1978), is to define an extension on the periodicity cell Y and repeat it in the whole Ω .

Theorem 2.1.1. *Let $\Omega_\varepsilon \subseteq \mathbb{R}^d$ be an admissible periodically perforated domain as in our setting, i.e.*

- i) the set of the holes B is a smooth open set with C^2 boundary;*
- ii) the perforated cell Y is locally on one side of the hole of B , so that sets like the one in figure 2.1 are not admissible;*
- iii) the holes does not intersect the boundary of Y ;*
- iv) the holes does not intersect the boundary of Ω ;*

Then for every $\varepsilon > 0$, there exists a linear and continuous extension operator $T_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow H^1(\Omega)$ such that, for any $u \in H^1(\Omega_\varepsilon)$

i) $T_\varepsilon u_\varepsilon = u_\varepsilon$ in Ω_ε ,

ii) $\|T_\varepsilon u_\varepsilon\|_{H^1(\Omega)} \leq c \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}$,

where the constant $c > 0$ depends on Y , but is independent of ε .

Proof. Consider $u \in H^1(Y)$, in the perforated cell: by hypothesis i) and iii), Y has a regular boundary, then, there exists an extension operator

$$P : H^1(Y) \rightarrow H^1(Y \cup B),$$

such that

$$\|Pu\|_{L^2(Y \cup B)} \leq C \|u\|_{L^2(Y)},$$

$$\|Pu\|_{H^1(Y \cup B)} \leq C \|u\|_{H^1(Y)}.$$

This is a classical result on extension in Sobolev spaces, see for example [21], theorem IX.7. Consider now a function $u \in H^1(Y)$; we have

$$u = \mathcal{M}_Y(u) + \psi,$$

with

$$\mathcal{M}_Y(u) = \frac{1}{|Y|} \int_Y u(x) dx.$$

Being ψ with zero mean value, we can apply Poincaré-Wirtinger inequality:

$$\|\psi\|_{H^1(Y)} \leq C' \|\nabla \psi\|_{L^2(Y)},$$

and, being $\nabla u = \nabla \psi$, one has

$$\|P\psi\|_{H^1(Y \cup B)} \leq C \|\psi\|_{H^1(Y)} \leq C' \|\nabla \psi\|_{L^2(Y)} = C' \|\nabla u\|_{L^2(Y)}.$$

Now define $Tu = \mathcal{M}_Y(u) + P\psi$, hence

$$\|Tu\|_{H^1(Y \cup B)} \leq \|P\psi\|_{H^1(Y \cup B)} \leq C' \|\nabla u\|_{L^2(Y)} \leq C \|u\|_{H^1(Y)}$$

Consider now a function $u_\varepsilon \in H^1(\Omega_\varepsilon)$, by the periodicity of the domain we have

$$\int_{\Omega_\varepsilon} |u_\varepsilon(x)|^2 dx \approx \sum_{i \in I_\varepsilon} \int_{Y_\varepsilon^i} |u_\varepsilon(x)|^2 dx$$

and the same holds true for ∇u_ε . Note that the number of the cells in Ω is

$$N_\varepsilon = \frac{|\Omega|}{|Y_\varepsilon|} \approx \varepsilon^{-d} \frac{|\Omega|}{|Y|}.$$

Therefore we can take $u_\varepsilon \in H^1(\Omega_\varepsilon)$; for any $x \in \Omega_\varepsilon$ there exists $i \in I_\varepsilon$ such that $x = \varepsilon(i + y)$, with $y \in Y_\varepsilon$. Hence define $u_{\varepsilon,k}(x) = u_\varepsilon(\varepsilon(k + y))$ that

belongs to $H^1(Y)$, for any $x \in \Omega_\varepsilon$ such that $x = \varepsilon(k + y)$. For any $i \in I_\varepsilon^i$ we can extend $u_{\varepsilon,i}$ as in the previous part, and, defining

$$T_\varepsilon u_\varepsilon = T u_{\varepsilon,i} \left(\frac{x - \varepsilon i}{\varepsilon} \right), \quad x = \varepsilon(i + Y) \in \Omega, i \in \mathbb{Z}^d,$$

we get

$$\begin{aligned} \|T_\varepsilon u_\varepsilon\|_{H^1(\Omega)} &= \sum_{i \in I_\varepsilon} \int_{Y_\varepsilon^i} |\nabla T u_{\varepsilon,i}|^2 \leq C' \varepsilon^{-d} \frac{|\Omega|}{|Y|} \int_Y |\varepsilon^{-1} \nabla_y u_{\varepsilon,k}(y)|^2 \varepsilon^d dy = \\ &= C' \varepsilon^{-d} \frac{|\Omega|}{|Y|} \int_{\varepsilon Y} |\nabla u_\varepsilon(x)|^2 dx = C \int_{\Omega_\varepsilon} |\nabla u_\varepsilon(y)|^2 dy \leq \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \end{aligned}$$

□

Remark 2.1.1. *It is possible to relax hypothesis iv) and get the existence of an extension operator not in the whole Ω , but in a retracted domain. Let us define the set*

$$\Omega(k) = \{x \in \Omega : \text{dist}(x, \partial\Omega) > k\}.$$

The following result holds true

Theorem 2.1.2. *Let $\Omega_\varepsilon \subseteq \mathbb{R}^d$ be a periodic perforated domain satisfying hypothesis i), ii) and iii) of theorem 2.1.1. There exists an extension operator $T_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow H_{loc}^1(\Omega)$ and two constants k, C , such that*

$$\int_{\Omega(k\varepsilon)} |\nabla T_\varepsilon u_\varepsilon|^2 \leq C \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)},$$

for every $u_\varepsilon \in H^1(\Omega_\varepsilon)$, with constants k and C independent of ε .

Hence, if holes intersect the boundary of Ω , we can't construct an extension operator in the whole Ω , but, being k independent of ε , in many situation it is enough to get similar result than the case of Ω satisfying iv). A proof of theorem 2.1.2 can be found in [1].

2.2 Dirichlet and Neumann spectral problems in perforated domains

In this section we will show some example of homogenization in perforated domains. Proves of next results and more details can be found in [54].

By elliptic spectral problem in a perforated domain we mean to find eigenpairs $(\lambda_\varepsilon, u_\varepsilon)$ satisfying the following equation

$$- \text{div}(a_\varepsilon(x) \nabla u_\varepsilon) = \lambda_\varepsilon u_\varepsilon \quad x \in \Omega_\varepsilon. \quad (2.3)$$

Here we assume classical hypothesis on the matrix $a^\varepsilon(x) = a(x/\varepsilon)$: $a(y)$ is a $d \times d$ positive and symmetric matrix satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^d a_{i,j}(y)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad (2.4)$$

for some $\alpha > 0$. Moreover coefficients $a_{i,j}(y)$ are in $L^\infty(\mathbb{R}^d)$ and Q -periodic.

In order to get a well posed problem, we need to defined boundary conditions; in this section we will consider two examples of spectral problems: the Dirichlet problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon(x)) = \lambda_\varepsilon^D u_\varepsilon(x), & x \in \Omega_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \quad (2.5)$$

and the homogeneous Neumann one

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon(x)) = \lambda_\varepsilon^N u_\varepsilon(x), & x \in \Omega_\varepsilon, \\ a_\varepsilon(x)\nabla u_\varepsilon(x) \cdot n_\varepsilon = 0, & x \in \Sigma_\varepsilon, \\ u_\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases} \quad (2.6)$$

Remark 2.2.1. *Observe that, by ellipticity condition (2.4), we can study simpler problems where the matrix $a^{ij}(x) = \delta_{ij}$: estimates obtained in this particular case, can be generalized to problems (2.5) and (2.6), using (2.4).*

Hence in the sequel we will consider for simplicity the following problems:

$$\begin{cases} -\Delta u_\varepsilon(x) = \lambda_\varepsilon^D u_\varepsilon(x), & x \in \Omega_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega_\varepsilon, \end{cases} \quad (2.7)$$

$$\begin{cases} -\Delta u_\varepsilon(x) = \lambda_\varepsilon^N u_\varepsilon(x), & x \in \Omega_\varepsilon, \\ a_\varepsilon(x)\nabla u_\varepsilon(x) \cdot n_\varepsilon = 0, & x \in \Sigma_\varepsilon, \\ u_\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases} \quad (2.8)$$

Remark 2.2.2. *We have to choose two different perforated domains, because of the boundary conditions that we assume: in the Dirichlet problem (2.7) we set $\Omega_\varepsilon = \Omega \setminus \bigcup_{i \in \mathbb{Z}^d} B_\varepsilon^i$, while in the Neumann problem (2.8) we take $\Omega_\varepsilon = \Omega \setminus \bigcup_{i \in I_\varepsilon} B_\varepsilon^i$, as in equation (2.1). The difference is that in the Dirichlet case we allow the holes to intersect the boundary of Ω ; in the Neumann one we “fill” the holes that intersect $\partial\Omega$, so that we have $\partial\Omega \cap \Sigma_\varepsilon = \emptyset$.*

We can state the respective weak formulation, that is to find $\lambda_\varepsilon^D \in \mathbb{C}$ and $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi dx = \lambda_\varepsilon^D \int_{\Omega_\varepsilon} u_\varepsilon \varphi dx, \quad \varphi \in H_0^1(\Omega_\varepsilon) \quad (2.9)$$

for the Dirichlet problem, and $\lambda_\varepsilon^N \in \mathbb{C}$, $u_\varepsilon \in H_\varepsilon$ such that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla \varphi dx = \lambda_\varepsilon^N \int_{\Omega_\varepsilon} u_\varepsilon \varphi dx, \quad \varphi \in H_\varepsilon \quad (2.10)$$

for the homogeneous Neumann one. Here the space H_ε is the one defined in (2.2) and $H_0^1(\Omega_\varepsilon) = \{u \in H^1(\Omega_\varepsilon), u(x) = 0, x \in \partial\Omega_\varepsilon\}$. By classical spectral theory, we have that, for a fixed $\varepsilon > 0$, the spectrum of these two problem is a sequence of real eigenvalues tending to infinity:

$$\begin{aligned} 0 < \lambda_\varepsilon^{1,D} \leq \lambda_\varepsilon^{2,D} \leq \dots \lambda_\varepsilon^{D,j} \dots \rightarrow +\infty, \\ 0 < \lambda_\varepsilon^{1,N} \leq \lambda_\varepsilon^{2,N} \leq \dots \lambda_\varepsilon^{N,j} \dots \rightarrow +\infty, \end{aligned}$$

moreover, for such eigenvalues, there exist associated eigenfuntions u_ε^D and u_ε^N respectively.

The aim of homogenization is to study the behavior of these solutions $(\lambda_\varepsilon^D, u_\varepsilon)$ and $(\lambda_\varepsilon^N, u_\varepsilon)$ as $\varepsilon \rightarrow 0$. Let us first consider the Dirichlet boundary conditions and defined the following cell problem:

$$\begin{cases} -\Delta \chi = \lambda \chi & x \in Y \\ \chi = 0 & \partial B \\ \chi \in H_{\text{per}}^1(Y) \end{cases} \quad (2.11)$$

and its weak formulation

$$\int_Y \nabla \chi \nabla \varphi = \lambda \int_Y \chi \varphi, \quad \varphi \in H_{\text{per}}^1(Y).$$

It is known by spectral theory that the first eigenvalue of problem (2.11) is simple and the corresponding eigenfunction has a constant sign in Y . Then we can choose the one with positive sign, for example, and extend it by zero in the interior of the hole B , naming again χ . If we consider χ_ε defined from χ periodically, we have

$$\begin{cases} -\Delta \chi_\varepsilon = \varepsilon^{-2} \lambda \chi_\varepsilon & x \in \Omega_\varepsilon \\ \chi_\varepsilon = 0 & x \in \partial\Omega_\varepsilon. \end{cases} \quad (2.12)$$

Before stating the homogenization result we have to define another auxiliary problem: let us consider the weighted Sobolev space

$$H_\varepsilon^\chi = \{u \in \mathcal{D}'(\Omega_\varepsilon) : u \chi_\varepsilon \in L^2(\Omega_\varepsilon), \nabla u \chi_\varepsilon \in L^2(\Omega_\varepsilon), u = 0 \text{ on } \partial\Omega\};$$

with χ_ε defined by problem (2.12). Hence the following problem is well define

$$\int_{\Omega_\varepsilon} (\chi_\varepsilon)^2 \nabla u_\varepsilon \nabla \varphi = \mu_\varepsilon \int_{\Omega_\varepsilon} (\chi_\varepsilon)^2 u_\varepsilon \varphi, \quad u_\varepsilon, \varphi \in H_\varepsilon^\chi, \quad (2.13)$$

and it admits a sequence of positive eingevalues μ_ε^j tending to infinity. Hence the following homogenization result holds true

Theorem 2.2.1. Let $\lambda_\varepsilon^{1,D}$ be the first eigenvalue of problem (2.7) and $\lambda^{1,D}$ the first eigenvalue of the cell problem (2.11); then one has

$$\lambda_\varepsilon^{1,D} = \frac{1}{\varepsilon^2} \lambda^{1,D} + \mu_\varepsilon + o(\varepsilon),$$

where μ_ε is the first eigenvalue of problem (2.13).

Remark 2.2.3. Theorem 2.2.1 is formulated only on the first eigenvalue of Dirichlet problem, but it can be generalized to any λ_ε^j of problem (2.7), and the result is still true:

$$\lambda_\varepsilon^{j,D} = \frac{1}{\varepsilon^2} \lambda^{j,D} + \mu_\varepsilon^j + o(\varepsilon).$$

Here it is clear that, for Dirichlet boundary conditions, eigenvalues are not bounded as $\varepsilon \rightarrow 0$.

For the Neumann problem the result is much different and it involves eigenfunctions too. We consider the homogenized problem

$$\begin{cases} -\Delta u = |Y| \lambda u & x \in \Omega \\ u \neq 0, \end{cases} \quad (2.14)$$

which comes from the cell problem

$$\begin{cases} -\Delta \chi = |Y| \lambda \chi & x \in Y \\ \chi \in H_{\text{per}}^1(Y), \end{cases} \quad (2.15)$$

Then we have

Theorem 2.2.2. Let $(\lambda_\varepsilon^{N,j}, u_\varepsilon^{N,j})$ be the sequence of eigenpairs of problem (2.8). Hence

i)

$$\lambda_\varepsilon^{N,j} \xrightarrow{\varepsilon \rightarrow 0} \lambda^{N,j},$$

with $\lambda^{N,j}$ eigenvalues of the homogenized problem (2.14);

ii) Up to subsequence we have

$$T_\varepsilon u_\varepsilon^{N,j} \xrightarrow{\varepsilon \rightarrow 0} u^{N,j}$$

weakly in $H^1(\Omega)$, where $u^{N,j}$ are eigenfunctions associated to $\lambda^{N,j}$.

iii) if $\lambda^{N,j}$ is simple, for any $u^{N,j}$ eigenfunction associated to $\lambda^{N,j}$, there exists a sequence of eigenfunctions $u_\varepsilon^{N,j}$ corresponding to $\lambda_\varepsilon^{N,j}$ of problem (2.14), such that

$$T_\varepsilon u_\varepsilon^{N,j} \rightharpoonup u^{N,j}$$

weakly in $H^1(\Omega)$.

Here T_ε is the extension operator described in theorem 2.1.1.

2.3 A Fourier boundary condition problem

In this section we present a result due to Chiadò-Piat, Pankratova and Piatniski [27] on a particular Fourier problem. The setting of the perforated domain Ω_ε is the one described in section 2.1, and we consider the following problem

$$\begin{cases} -\operatorname{div}(a^\varepsilon(x)\nabla u^\varepsilon(x)) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ a^\varepsilon(x)\nabla u^\varepsilon(x) \cdot n_\varepsilon = -q(x)u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega, \end{cases} \quad (2.16)$$

with $a^\varepsilon(x) = a(x/\varepsilon)$, where $a(y)$ is a $d \times d$ matrix; n_ε is the outward unit normal at the boundary Σ_ε , and \cdot denotes the usual scalar product in \mathbb{R}^d .

We will assume the following hypothesis:

(H1) $a(y)$ is a real symmetric matrix satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^d a_{i,j}(y)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

for some $\alpha > 0$.

(H2) The coefficients $a_{i,j}(y)$ are in $L^\infty(\mathbb{R}^d)$ and Q -periodic.

(H3) The function $q \in C^2(\overline{\Omega})$ is strictly positive and it realizes its only global minimum in $x_0 \in \Omega$:

$$q(x) \geq q(x_0) > 0.$$

(H4) The Hessian matrix $\partial^2 q / \partial x^2$ evaluated in the minimum point is positive definite.

Under assumption (H3) it occurs a localization phenomenon of the eigenfunctions: for any $j \in \mathbb{N}$, the j -th eigenfunction of problem (2.16) is asymptotically localized, as $\varepsilon \rightarrow 0$, in a neighborhood of x_0 ; in other words the properly normalized principal eigenfunction converges to a δ -function supported at x_0 , as ε tends to 0.

In order to study homogenization, let us first consider the weak formulation of problem (2.16), that is find $\lambda_\varepsilon \in \mathbb{C}$ and $u_\varepsilon \in H_\varepsilon$, $u_\varepsilon \neq 0$, such that

$$\int_{\Omega_\varepsilon} a_\varepsilon \nabla u_\varepsilon \nabla v dx + \int_{\Sigma_\varepsilon} q u_\varepsilon v d\sigma = \lambda_\varepsilon \int_{\Omega_\varepsilon} u_\varepsilon v dx, \quad v \in H_\varepsilon \quad (2.17)$$

Hence we can define any eigenvalues of problem (2.16) by Rayleigh quotient:

$$\lambda_\varepsilon^j = \min \left\{ \int_{\Omega_\varepsilon} a_\varepsilon |\nabla u_\varepsilon|^2 dx + \int_{\Sigma_\varepsilon} q u_\varepsilon^2 d\sigma, \quad u_\varepsilon \in H_\varepsilon^j, \|u_\varepsilon\|_{H_\varepsilon} = 1 \right\}, \quad (2.18)$$

with $H_\varepsilon^j = \{v_\varepsilon \in H_\varepsilon : (u_\varepsilon^i, v_\varepsilon) = 0, i = 1, \dots, j-1\}$, for u_ε^i eigenfunctions associated to λ_ε^i .

Remark 2.3.1. *Using variational formulation of eigenvalues, we can compare them with eigenvalues of other problems, such as Dirichlet or Neumann. Consider the min-max principle for problem (2.5) and (2.6):*

$$\lambda_\varepsilon^{j,D} = \max_{\dim H_0^{1,j} = j} \left\{ \min \left\{ \int_{\Omega_\varepsilon} a_\varepsilon |\nabla u_\varepsilon|^2 dx, u_\varepsilon \in H_0^{1,j}(\Omega_\varepsilon), \|u_\varepsilon\|_{H_0^1(\Omega_\varepsilon)} = 1 \right\} \right\},$$

where $H_0^{1,j}(\Omega_\varepsilon)$ is a subspace of $H_0^1(\Omega_\varepsilon)$;

$$\lambda_\varepsilon^{j,N} = \max_{\dim H_\varepsilon^{j,N} = j} \left\{ \min \left\{ \int_{\Omega_\varepsilon} a_\varepsilon |\nabla u_\varepsilon|^2 dx, u_\varepsilon \in H_\varepsilon^{j,N}, \|u_\varepsilon\|_{H_\varepsilon} = 1 \right\} \right\},$$

with $H_\varepsilon^{j,N}$ subspace of H_ε .

Now, being $H_0^1(\Omega_\varepsilon) \subseteq H_\varepsilon$, we simply get

$$\lambda_\varepsilon^j \leq \lambda_\varepsilon^{j,D},$$

and, on the other hand, being $q > 0$, one has

$$\lambda_\varepsilon^j \geq \lambda_\varepsilon^{j,N}.$$

Therefore, using theorems 2.2.1 and 2.2.2 we have

$$\lambda_\varepsilon^j \geq \lambda_\varepsilon^{j,N} \rightarrow \lambda^{j,N}$$

and

$$\lambda_\varepsilon^j \leq \frac{1}{\varepsilon^2} \lambda^{j,D} + \mu^j + o(\varepsilon),$$

so that we get a lower bound, but we don't have a suitable estimate from above.

Consider now

$$\rho(x) = \frac{|\Sigma^0|}{|Y|} q(x), \quad Q(x) = \frac{|\Sigma^0|}{|Y|} H(q),$$

with $H(q)$ the Hessian matrix of q in $x = x_0$. We can assume, being the problem invariant under translation, that $x_0 = 0$. We define the rescaled problem in the domain

$$\tilde{\Omega}_\varepsilon = \varepsilon^{-1/4} \Omega_\varepsilon, \quad \tilde{\Sigma}_\varepsilon = \varepsilon^{-1/4} \Sigma_\varepsilon,$$

that is

$$\begin{cases} -\operatorname{div}(a_\varepsilon(z) \nabla v_\varepsilon(z)) - \frac{\rho(0)}{\sqrt{\varepsilon}} v_\varepsilon(z) = \mu_\varepsilon v_\varepsilon(z) & z \in \tilde{\Omega}_\varepsilon, \\ a_\varepsilon(z) \nabla v_\varepsilon(z) \cdot n_\varepsilon = -\varepsilon^{1/4} q(\varepsilon^{1/4} z) v_\varepsilon(z) & z \in \tilde{\Sigma}_\varepsilon, \\ v_\varepsilon(z) = 0 & z \in \varepsilon^{-1/4} \partial\Omega, \end{cases} \quad (2.19)$$

with

$$v_\varepsilon(z) = u_\varepsilon(\varepsilon^{1/4}z), \quad a_\varepsilon(z) = a_\varepsilon\left(\frac{z}{\varepsilon^{3/4}}\right), \quad \mu_\varepsilon = \sqrt{\varepsilon}\left(\lambda_\varepsilon - \frac{\rho(0)}{\varepsilon}\right).$$

For the rescaled problem we have the following homogenized form:

$$- \operatorname{div}(a_{\text{hom}}^N \nabla v) + (z^T Q z)v = \mu v, \quad v \in L^2(\mathbb{R}^d), \quad (2.20)$$

where a_{hom}^N is the homogenized matrix of the Neumann problem:

$$a_{\text{hom}}^N \xi \xi = \frac{1}{|Y|} \int_Y a(y) |\xi + \nabla N|^2 dy,$$

and N is the vector of solutions of the cell problem, as in (2.11).

Hence we can state the following homogenization result, see [27],

Theorem 2.3.1. *Consider problem (2.16) and let conditions (H1) – H(4) be satisfied. Naming $(\lambda_\varepsilon^j, u_\varepsilon^j)$ the j -th eigenpair of problem 2.16, we have*

$$\lambda_\varepsilon^j = \frac{1}{\varepsilon} \frac{|\Sigma^0|}{|Y|} q(0) + \frac{\mu_j^\varepsilon}{\sqrt{\varepsilon}}, \quad u_\varepsilon^j(x) = v_\varepsilon^j\left(\frac{x}{\varepsilon^{1/4}}\right).$$

Moreover, for $(\mu_\varepsilon^j, v_\varepsilon^j)$, the eigenpairs of the rescaled problem (2.19), one has

i)

$$\mu_\varepsilon^j \xrightarrow{\varepsilon \rightarrow 0} \mu^j,$$

with μ^j j -th eigenvalue of the homogenized problem (2.20).

ii) if μ^j is simple, then, there exists ε_0 such that, for $\varepsilon < \varepsilon_0$, μ_ε^j is simple too, and we get the convergence of the corresponding normalized eigenfunction, extended in the whole Ω :

$$T_\varepsilon v_\varepsilon^j \xrightarrow{\varepsilon \rightarrow 0} v^j,$$

strongly in $L^2(\Omega)$.

Finally we present the localization effect for the first eigenfunction: let us consider the following definition

Definition 2.3.1. *The family $\{w_\varepsilon(x)\}_{\varepsilon > 0}$, with $c_1 \leq \|w_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c_2$, is concentrated at x_0 , as $\varepsilon \rightarrow 0$, if for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that*

$$\int_{\Omega_\varepsilon \setminus B_\delta(x_0)} |w_\varepsilon|^2 < \delta, \quad \forall \varepsilon < \varepsilon_0.$$

Therefore

Lemma 2.3.1. *The first eigenfunction u_ε^1 of problem (2.16) is concentrated in x_0 , the minimum point of $q(x)$.*

In next chapter, starting from this problem, we will modify the function q , allowing it to be zero and expanding the minimum zone, from a point x_0 to a compact set $K \subset \Omega$.

Chapter 3

Asymptotics of eigenpairs for an elliptic spectral problem in a perforated domain

In this chapter we will consider an elliptic spectral problem in a perforated domain, with Fourier boundary conditions imposed on the boundary of perforation. The presence of a different function q in the boundary operator, will change result obtained in [27], see section 2.3: namely the localization effect will be still satisfied, in the sense that our solutions of the homogenized problem will be concentrated in the compact set K , where q is zero. Moreover we will not need to use the rescaled problem: we will directly get the convergence of eigenpairs.

Here we use a slightly different method from the one explained in chapter 2: we will define eigenvalues as minima of Rayleigh quotient and, in section 3.3, we will study the Γ -convergence of the respective defined functional. Finally, in section 3.4, we present the convergence of eigenspaces in the sense of Mosco and the rate of convergence of eigenpairs, using Viřik lemma.

3.1 Problem statement

Let Ω_ε be a perforated domain as defined in section 2.1; we consider the following spectral problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x)\nabla u_\varepsilon(x)) = \lambda_\varepsilon u_\varepsilon(x), & x \in \Omega_\varepsilon, \\ a_\varepsilon(x)\nabla u_\varepsilon(x) \cdot n_\varepsilon = -q(x)u_\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u_\varepsilon(x) = 0, & x \in \partial\Omega. \end{cases} \quad (3.1)$$

Here $a_\varepsilon(x) = a(x/\varepsilon)$, where $a(y)$ is a $d \times d$ matrix; n_ε is the outward unit normal at the boundary Σ_ε , and \cdot denotes the usual scalar product in \mathbb{R}^d .

We will state the following hypothesis:

(H1) $a(y)$ is a real symmetric matrix satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^d a_{ij}(y)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

for some $\alpha > 0$.

(H2) The coefficients $a_{ij}(y)$ are in $L^\infty(\mathbb{R}^d)$ and Q -periodic.

(H3) The function $q(x)$ is defined as

$$q(x) = \begin{cases} 0, & x \in K, \\ 1, & x \in \Omega \setminus K, \end{cases}$$

where $K \Subset \Omega$ is a compact subset of Ω , with non empty interior $A = \overset{\circ}{K}$ and Lipschitz boundary.

We can consider the weak formulation of problem (3.1), that is to find $\lambda_\varepsilon \in \mathbb{C}$ (eigenvalues) and $u_\varepsilon \in H_\varepsilon$, $u_\varepsilon \neq 0$ (eigenfunctions), such that

$$\int_{\Omega_\varepsilon} a_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q(x) u_\varepsilon v \, d\sigma = \lambda_\varepsilon \int_{\Omega_\varepsilon} u_\varepsilon v \, dx, \quad v \in H_0^1(\Omega) \quad (3.2)$$

where

$$H_\varepsilon = H_0^1(\Omega_\varepsilon, \partial\Omega) = \{u \in H^1(\Omega_\varepsilon) : u = 0 \text{ on } \partial\Omega\}$$

is a Hilbert space, equipped with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_\varepsilon} \nabla u \nabla v \, dx.$$

For this spectral problem we have the following classical result:

Theorem 3.1.1. *For any $\varepsilon > 0$, the spectrum of problem (3.2) is real and consists of a countable set of values*

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \dots \leq \lambda_j^\varepsilon \leq \dots + \infty.$$

Every eigenvalue has a finite multiplicity. The corresponding eigenfunctions normalized by

$$\int_{\Omega_\varepsilon} u_\varepsilon^i u_\varepsilon^j \, dx = \delta_{ij},$$

form an orthonormal basis in $\mathbb{L}^2(\Omega_\varepsilon)$. Furthermore λ_ε^1 is simple.

The proof of this proposition will be given in Section 3.4

Under hypothesis (H1), (H2), (H3), we want to study the asymptotic behavior of eigenpairs $(\lambda_\varepsilon, u_\varepsilon)$, as $\varepsilon \rightarrow 0$.

3.2 The upper bound

In this section we prove that the first eigenvalue λ_1^ε of problem (3.1) is uniformly bounded with respect to $\varepsilon > 0$.

Let us start with a simpler case, where we consider the matrix $a_{i,j}(y) = \delta_{ij}$, so that we deal with the spectrum of the Laplacian operator. We may easily generalize the results to our problem (3.1), using the ellipticity condition and boundedness of $a(y)$. So we can consider the first eigenpair $(\lambda_\varepsilon^1, u_\varepsilon^1)$ of problem

$$\begin{cases} -\Delta u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Omega_\varepsilon, \\ \nabla u^\varepsilon(x) \cdot n_\varepsilon = -q(x)u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega, \end{cases} \quad (3.3)$$

that is, in the weak formulation, find $\lambda_\varepsilon \in \mathbb{C}$ (eigenvalues) and $u_\varepsilon \in H_\varepsilon$, $u_\varepsilon \neq 0$, such that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx + \int_{\Sigma_\varepsilon} q(x)u_\varepsilon v \, d\sigma = \lambda_\varepsilon \int_{\Omega_\varepsilon} u_\varepsilon v \, dx, \quad v \in H_\varepsilon. \quad (3.4)$$

Lemma 3.2.1. *For the first eigenvalue of the problem (3.2), as $\varepsilon \rightarrow 0$, we have the following inequality*

$$\limsup_{\varepsilon \rightarrow 0} \lambda_\varepsilon^1 \leq \lambda^1, \quad (3.5)$$

where λ^1 is the first eigenvalue of the Laplace operator on the set $A = \dot{K}$, with homogenous Dirichlet condition on ∂A :

$$\lambda^1 = \inf_{u \in H_0^1(A)} \frac{\int_A |\nabla u|^2}{\int_A |u|^2}, \quad (3.6)$$

Proof. By the Rayleigh equation we have

$$\lambda_\varepsilon^1 = \inf_{u \in H_\varepsilon} \frac{\int_{\Omega_\varepsilon} |\nabla u(x)|^2 dx + \int_{\Sigma_\varepsilon} q(x)|u(x)|^2 d\sigma}{\int_{\Omega_\varepsilon} |u(x)|^2}. \quad (3.7)$$

Let u be a normalized solution of the minimum problem (3.6) on the set A :

$$\frac{\int_A |\nabla u|^2}{\int_A |u|^2} = \lambda^1, \quad \int_A |u|^2 = 1, \quad u \in H_0^1(A), \quad u = 0 \text{ on } \partial A.$$

We can extend u in $H_0^1(\Omega)$ by

$$\begin{cases} u(x) & x \in A \\ 0 & x \in \Omega \setminus A. \end{cases}$$

Now define the function

$$u_\varepsilon = \frac{u}{\|u\|_{\mathbb{L}^2(\Omega_\varepsilon)}}, \quad u_\varepsilon \in H_\varepsilon.$$

Since, as $\varepsilon \rightarrow 0$, one has $\chi_{\Omega_\varepsilon} \rightharpoonup^* |Y|\chi_\Omega$, in L^∞ -weak*, where $|Y|$ is the d -dimensional measure of the perforated cell Y , we have that

$$\|u\|_{\mathbb{L}^2(\Omega_\varepsilon)}^2 = \int_{\Omega_\varepsilon} |u|^2 = \int_{\Omega} |u|^2 \chi_{\Omega_\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} |Y| \int_{\Omega} |u|^2 = |Y| \int_A |u|^2 = |Y|.$$

We can use u_ε as a test function in the functional (3.7), remembering that u_ε is equal 0 out of the set A , getting

$$\begin{aligned} \lambda_\varepsilon^1 &\leq \frac{\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx + \int_{\Sigma_\varepsilon} q(x) |u_\varepsilon(x)|^2 dx}{\int_{\Omega_\varepsilon} |u_\varepsilon(x)|^2} = \\ &= \frac{\int_{\Omega_\varepsilon} |\nabla u_\varepsilon(x)|^2 dx + \int_{\Sigma_\varepsilon \setminus K} |u_\varepsilon(x)|^2 dx}{\int_{\Omega_\varepsilon} |u_\varepsilon(x)|^2} = \frac{\frac{1}{\|u\|_{\mathbb{L}^2(\Omega_\varepsilon)}^2} \int_{\Omega_\varepsilon} |\nabla u(x)|^2 dx}{\frac{1}{\|u\|_{\mathbb{L}^2(\Omega_\varepsilon)}^2} \|u\|_{\mathbb{L}^2(\Omega_\varepsilon)}^2}. \end{aligned}$$

Since

$$\frac{\int_{\Omega} |\nabla u(x)|^2 \chi_{\Omega_\varepsilon} dx}{\|u\|_{\mathbb{L}^2(\Omega_\varepsilon)}^2} \xrightarrow{\varepsilon \rightarrow 0} \frac{|Y| \int_{\Omega} |\nabla u(x)|^2 dx}{|Y|} = \lambda^1,$$

then we get the thesis. \square

It is easy to show a different upper bound, that is greater than λ_1 and it depends on ε , which requires to introduce the spectral problem for the Laplace operator in the perforated set $A_\varepsilon = A \cap \Omega_\varepsilon$. Let us set

$$\mu_\varepsilon^1(A) = \inf \left\{ \int_{A_\varepsilon} |\nabla u|^2 : u \in H_0^1(A_\varepsilon), \int_{A_\varepsilon} |u|^2 = 1 \right\}. \quad (3.8)$$

Lemma 3.2.2. *For any $\varepsilon > 0$, the following inequality holds*

$$\lambda_\varepsilon^1 \leq \mu_\varepsilon^1(A).$$

Proof. By the definition (3.8) of $\mu_\varepsilon^1(A)$, for any $\eta > 0$ and $\varepsilon > 0$, there exists a function $u_{\varepsilon,\eta} \in H_0^1(A_\varepsilon)$, with $\int_{A_\varepsilon} |u_{\varepsilon,\eta}|^2 = 1$, such that

$$\int_{A_\varepsilon} |\nabla u_{\varepsilon,\eta}|^2 < \mu_\varepsilon^1(A) + \eta.$$

Let us set

$$v_{\varepsilon,\eta} = \begin{cases} u_{\varepsilon,\eta}(x) & x \in A \\ 0 & x \in \Omega \setminus A. \end{cases}$$

Hence $v_{\varepsilon,\eta} \in H_\varepsilon$, and then, by equation (3.7),

$$\lambda_\varepsilon^1 \leq \int_{\Omega_\varepsilon} |\nabla v_{\varepsilon,\eta}|^2 = \int_{A \cap \Omega_\varepsilon} |\nabla u_{\varepsilon,\eta}|^2 < \mu_\varepsilon^1(A) + \eta.$$

By arbitrariness of η it follows that

$$\lambda_\varepsilon^1 \leq \mu_\varepsilon^1.$$

Finally note that, by definition, the following inclusion holds true

$$\left\{ u \in H_0^1(A_\varepsilon), \int_{A \cap \Omega_\varepsilon} |u|^2 = 1 \right\} \subseteq \left\{ u \in H_0^1(A), \int_A |u|^2 = 1 \right\}$$

therefore

$$\mu_\varepsilon^1(A) \geq \lambda^1$$

□

3.3 An approach by Γ -convergence

In this section we will consider the first eigenvalue of problem (3.4) as the minimum of a functional, depending on the small parameter ε , and we will discuss the Γ -convergence of this functional as $\varepsilon \rightarrow 0$. Let us define $F_\varepsilon : \mathbb{L}^2(\Omega) \rightarrow [0, +\infty]$, with

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega_\varepsilon} |\nabla u|^2 + \int_{\Sigma_\varepsilon \setminus K} |u|^2 & u \in H_0^1(\Omega_\varepsilon; \partial\Omega), \\ +\infty & \text{otherwise.} \end{cases} \quad (3.9)$$

We have that

$$\lambda_\varepsilon^1 = \inf_{u \in H_\varepsilon} \frac{\int_{\Omega_\varepsilon} |\nabla u(x)|^2 dx + \int_{\Sigma_\varepsilon} q(x) |u(x)|^2 dx}{\int_{\Omega_\varepsilon} |u(x)|^2} =$$

$$\inf_{\substack{u \in H_\varepsilon, \\ \int_{\Omega_\varepsilon} |u|^2 = 1}} \int_{\Omega_\varepsilon} |\nabla u(x)|^2 dx + \int_{\Sigma_\varepsilon} q(x) |u(x)|^2 dx = \inf_{\substack{u \in L^2(\Omega), \\ \int_{\Omega_\varepsilon} |u|^2 = 1}} F_\varepsilon(u).$$

Now consider the set $X_\varepsilon = \{u \in L^2(\Omega) : \int_{\Omega_\varepsilon} |u|^2 = \|u\|_{\mathbb{L}^2(\Omega_\varepsilon)}^2 = 1\}$, and the function

$$I_{X_\varepsilon}(u) = \begin{cases} 0 & u \in X_\varepsilon \cap L^2(\Omega), \\ +\infty & u \in L^2(\Omega) \setminus X_\varepsilon. \end{cases}$$

Hence

$$\lambda_\varepsilon^1 = \inf_{\substack{u \in L^2(\Omega), \\ \int_{\Omega_\varepsilon} |u|^2 = 1}} F_\varepsilon(u) = \inf_{u \in L^2(\Omega)} [F_\varepsilon(u) + I_{X_\varepsilon}(u)].$$

Remark 3.3.1. Observe that a natural limit of the constraint X_ε , as $\varepsilon \rightarrow 0$, is the set $X = \{u \in L^2(\Omega) : \int_\Omega |u|^2 = 1/|Y|\}$. Infact, if $X_\varepsilon \ni u_\varepsilon \rightarrow u$ in $L^2(\Omega)$, then

$$1 = \int_{\Omega_\varepsilon} |u_\varepsilon|^2 = \int_\Omega |u_\varepsilon|^2 \chi_{\Omega_\varepsilon} \xrightarrow{\varepsilon \rightarrow 0} |Y| \int_\Omega |u|^2$$

so that we get the condition $\int_\Omega |u|^2 = 1/|Y|$.

Now we can prove the following preliminary result:

Lemma 3.3.1. If the functional F_ε , defined in (3.9), Γ -converges in the strong $L^2(\Omega)$ topology to a functional F , then we have

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} (F_\varepsilon + I_{X_\varepsilon}) = F + I_X,$$

in the same topology, with I_X defined as I_{X_ε} :

$$I_X(u) = \begin{cases} 0 & u \in X, \\ +\infty & u \in L^2(\Omega) \setminus X. \end{cases}$$

Proof. Let us define the functionals $\mathcal{F}_\varepsilon = F_\varepsilon + I_{X_\varepsilon}$ and $\mathcal{F} = F + I_X$. We will consider the two conditions (1.1) and (1.2) of Γ -convergence separately.

- i) We have to show that for every sequence u_ε converging to u , in the strong topology of $L^2(\Omega)$, one has

$$\mathcal{F}(u) \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon). \quad (3.10)$$

First of all we can suppose, possibly passing to a subsequence, that exists the $\lim \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$, otherwise our inequality (3.10) is trivial. By the definition and finiteness of \mathcal{F}_ε , we get $u_\varepsilon \in X_\varepsilon$, that is $\int_{\Omega_\varepsilon} |u_\varepsilon|^2 = 1$ and $\mathcal{F}_\varepsilon(u_\varepsilon) = F_\varepsilon(u_\varepsilon)$. By hypothesis we know that F_ε Γ -converges to F , then $F(u) \leq \lim F_\varepsilon(u_\varepsilon) < +\infty$.

We have

$$\int_\Omega |u_\varepsilon|^2 = \int_{\Omega_\varepsilon} |u_\varepsilon|^2 + \int_{\Omega \setminus \Omega_\varepsilon} |u_\varepsilon|^2 = 1 + \int_\Omega |u_\varepsilon|^2 \chi_{\Omega \setminus \Omega_\varepsilon}.$$

Now, by the weak* convergence $\chi_{\Omega \setminus \Omega_\varepsilon} \rightharpoonup^* 1 - |Y|$ and the strong $L^2(\Omega)$ convergence $u_\varepsilon \rightarrow u$, one has

$$\int_\Omega |u|^2 \xleftarrow{\varepsilon \rightarrow 0} \int_\Omega |u_\varepsilon|^2 \xrightarrow{\varepsilon \rightarrow 0} 1 + (1 - |Y|) \int_\Omega |u|^2,$$

therefore

$$\frac{1}{|Y|} = \int_\Omega |u|^2 \quad \Rightarrow \quad u \in X,$$

so that

$$\mathcal{F}(u) = F(u) \leq \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

- ii) We will prove that, for every $u \in H_\varepsilon$, there exists a sequence u_ε , converging to u in $\mathbb{L}^2(\Omega)$, such that

$$\mathcal{F}(u) \geq \limsup_{\varepsilon \rightarrow 0} \mathcal{F}_\varepsilon(u_\varepsilon).$$

As before we can suppose that $\mathcal{F}(u) < +\infty$, i.e. $u \in X$, that is $\int_\Omega |u|^2 = 1/|Y|$, and $\mathcal{F}(u) = F(u) < +\infty$. By hypothesis one has $\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon = F$, hence there exists a sequence v_ε , converging in $\mathbb{L}^2(\Omega)$ to u , such that

$$F(u) = \lim_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon).$$

Now define the sequence $u_\varepsilon = v_\varepsilon / \|v_\varepsilon\|_{\mathbb{L}^2(\Omega_\varepsilon)}$. We have

$$u_\varepsilon = \frac{v_\varepsilon}{\int_\Omega |v_\varepsilon|^2 \chi_{\Omega_\varepsilon}} \xrightarrow{\mathbb{L}^2(\Omega)} \frac{u}{|Y| \int_\Omega |u|^2} = u.$$

Observe that, by construction, $\|u_\varepsilon\|_{\mathbb{L}^2(\Omega_\varepsilon)}^2 = 1$, that is $u_\varepsilon \in X_\varepsilon$, and, by Γ -convergence, one has

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\|v_\varepsilon\|^2} F_\varepsilon(u_\varepsilon) = \frac{1}{|Y| \int_\Omega |u|^2} F(u).$$

Therefore, being $u \in X$,

$$\mathcal{F}_\varepsilon(u_\varepsilon) = F_\varepsilon(u_\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} \frac{F(u)}{|Y| \int_\Omega |u|^2} = \mathcal{F}(u).$$

□

Thanks to Lemma 3.3.1 we can consider the Γ -convergence of F_ε only, ignoring the oscillating constraint I_{X_ε} . In order to do this, we will follow the procedure used in [1]. In our case it will be simpler, because, by our hypothesis on the perforated domain, the holes don't intersect the boundary of Ω : $\partial\Omega \cap \Sigma_\varepsilon = \emptyset$.

Remark 3.3.2. *It is well known, see section 2.1 theorem 2.1.1, that, under the present assumptions on Ω_ε , for every $\varepsilon > 0$, there exists a linear and continuous extension operator $T_\varepsilon : H^1(\Omega_\varepsilon) \rightarrow H^1(\Omega)$ such that, for any $u \in H^1(\Omega_\varepsilon)$*

- i) $T_\varepsilon u_\varepsilon = u_\varepsilon$ in Ω_ε ,

$$ii) \|T_\varepsilon u_\varepsilon\|_{H^1(\Omega)} \leq c \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)},$$

where the constant $c > 0$ depends on Y , but is independent of ε .

We will use the following lemma.

Lemma 3.3.2. *Let $\Omega_\varepsilon^K = (\Omega \setminus K) \setminus \{\cup_i B_\varepsilon^i : Y_\varepsilon^i \subseteq (\Omega \setminus K)\}$, where $K = \bar{A}$ and A is, by our hypothesis, a non empty open set $A \Subset \Omega$ with Lipschitz boundary. Let $\Sigma_\varepsilon^K = \{\cup \partial B_\varepsilon^i : Y_\varepsilon^i \subseteq (\Omega \setminus K)\}$, so that $\partial\Omega_\varepsilon^K = \partial\Omega \cup \partial K \cup \Sigma_\varepsilon^K$. Hence there exists two constants $c = c(K)$, independent of ε , and $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $w \in H_0^1(\Omega_\varepsilon; \partial\Omega)$*

$$\left| \frac{C^*}{\varepsilon} \int_{\Omega_\varepsilon^K} |w|^2 dx - \int_{\Sigma_\varepsilon^K} |w|^2 d\sigma \right| \leq c(K) \int_{\Omega_\varepsilon^K} |\nabla w|^2, \quad (3.11)$$

where $C^* = \frac{|\partial Y|}{|Y|}$.

Proof. We proceed as in proof of lemma (4.1) in [27]. Let $\chi \in H_{\text{per}}^1(Y)$ the solution of

$$\begin{cases} -\text{div}_y \chi(y) = C^* & y \in Y \\ \chi(y) \cdot n = -1 & y \in \Sigma_0 \\ \chi \in H_{\text{per}}^1(Y). \end{cases}$$

Then consider its periodic extension over the whole Ω_ε and the rescaled function $\varepsilon \chi(x/\varepsilon)$: one has

$$-\varepsilon \text{div}_x \chi(x/\varepsilon) = C^*.$$

Multiplying by w^2 , for any $w \in H_\varepsilon$, and integrating over Ω_ε^K , we get

$$-\varepsilon \int_{\Omega_\varepsilon^K} \text{div} \chi(x/\varepsilon) w^2(x) dx = C^* \int_{\Omega_\varepsilon^K} w^2(x);$$

integrating by part we have

$$\begin{aligned} \varepsilon \int_{\Omega_\varepsilon^K} \chi(x/\varepsilon) \nabla(w^2) dx - \varepsilon \int_{\Sigma_\varepsilon^K} \chi(x/\varepsilon) \cdot n w^2 d\sigma &= C^* \int_{\Omega_\varepsilon^K} w^2 dx + \\ &+ \varepsilon \int_{\partial K} \chi(x/\varepsilon) \cdot n w^2 d\sigma. \end{aligned}$$

Now, being $\chi \in L^\infty(\Omega_\varepsilon)$, and $\chi(x/\varepsilon) \cdot n = -1$ in Σ_ε , one has

$$\left| \frac{C^*}{\varepsilon} \int_{\Omega_\varepsilon^K} w^2 dx - \int_{\Sigma_\varepsilon^K} w^2 d\sigma \right| \leq \|\chi\|_{L^\infty} \left(\int_{\Omega_\varepsilon^K} |\nabla(w^2)| dx + \int_{\partial K} |w^2| d\sigma \right)$$

Finally note that, by Cauchy-Schwarz and Poincaré inequalities,

$$\int_{\Omega_\varepsilon^K} |\nabla(w^2)| dx \leq 2 \int_{\Omega_\varepsilon^K} |w \nabla w| dx \leq c'(K) \int_{\Omega_\varepsilon^K} |\nabla w|^2,$$

and, by trace inequality,

$$\int_{\partial K} w^2 \leq c''(K) \int_{\Omega_\varepsilon^K} |\nabla w|^2.$$

Therefore

$$\left| \frac{C^*}{\varepsilon} \int_{\Omega_\varepsilon^K} w^2 dx - \int_{\Sigma_\varepsilon^K} w^2 d\sigma \right| \leq c(K) \int_{\Omega_\varepsilon^K} |\nabla w|^2.$$

□

Before considering the Γ -limit of $F_\varepsilon(u)$, we can prove the following useful compactness property:

Lemma 3.3.3. *Let $u_\varepsilon \in L^2(\Omega_\varepsilon)$ be a sequence such that $F_\varepsilon(u_\varepsilon) \leq c \forall \varepsilon > 0$, then, up to subsequence,*

- a) $T_\varepsilon u_\varepsilon \rightarrow u \in H_0^1(\Omega)$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$;
- b) $u = 0$ in $\Omega \setminus K$;
- c) If $\int_{\Omega_\varepsilon} |u_\varepsilon - u_0|^2 \rightarrow 0$, as $\varepsilon \rightarrow 0$, with $u_0 \in L^2(\Omega)$, then $u = u_0$ $L^2(\Omega)$ -almost everywhere and the convergence a) holds for the whole sequence $T_\varepsilon u_\varepsilon$.

Proof. By the equiboundedness of the functional $F_\varepsilon(u_\varepsilon) \leq c$, it follows that $u_\varepsilon \in H_\varepsilon$ and $\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \leq c$. By the extension property of Ω_ε in remark 3.3.2, we have that $T_\varepsilon u_\varepsilon \in H_0^1(\Omega)$ and

$$\int_{\Omega} |\nabla T_\varepsilon u_\varepsilon|^2 \leq c_1 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \leq c_2.$$

Hence, up to subsequence, there exists a function $u \in H_0^1(\Omega)$, such that a) holds.

To prove b) remember that also $\int_{\Sigma_\varepsilon^K} |u_\varepsilon|^2 \leq c$, then, by (3.11) in Lemma 3.3.2, it follows that, for any $\varepsilon > 0$,

$$\int_{\Omega_\varepsilon^K} |u_\varepsilon|^2 \leq \varepsilon c.$$

Now note that

$$\varepsilon c \geq \int_{\Omega_\varepsilon^K} |u_\varepsilon|^2 = \int_{\Omega \setminus K} |T_\varepsilon u_\varepsilon|^2 \chi_{\Omega_\varepsilon^K} \xrightarrow{\varepsilon \rightarrow 0} |Y| \int_{\Omega \setminus K} |u|^2,$$

so that, taking the limit as $\varepsilon \rightarrow 0$, we get

$$\int_{\Omega \setminus K} |u|^2 = 0.$$

Finally, from hypothesis $\int_{\Omega_\varepsilon} |u_\varepsilon - u_0|^2 \rightarrow 0$ in *c*), it follows that

$$|Y| \int_{\Omega} |u - u_0|^2 \xleftarrow{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |u - u_0|^2 \leq 2 \int_{\Omega_\varepsilon} |u - T_\varepsilon u_\varepsilon|^2 + 2 \int_{\Omega_\varepsilon} |T_\varepsilon u_\varepsilon - u_0|^2.$$

Now, from *a*) we have that the first addend tends to 0 as $\varepsilon \rightarrow 0$; the second one, being $T_\varepsilon u_\varepsilon = u_\varepsilon$ in Ω_ε , tends to 0 too, by our assumption. Therefore

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} |u - u_0|^2 = 0,$$

and *c*) is proved. \square

Remark 3.3.3. From Lemma 3.3.3 and remark 3.3.1 we deduce that the sequence $\{F_\varepsilon + I_{X_\varepsilon}\}_\varepsilon$ is equicoercive with respect to the strong topology of $L^2(\Omega)$. Moreover, again by Lemma 3.3.3, we can deduce that if F_ε Γ -converges to F in $L^2(\Omega)$, then $F(u) = +\infty$ whenever $u \neq 0$ in $\Omega \setminus K$.

We will use the following technical lemma, whose proof is classical:

Lemma 3.3.4. Let $A \subseteq \mathbb{R}^d$ be an open bounded set, with Lipschitz boundary, and set

$A^\delta = \{x \in A : \text{dist}(x, \partial A) > \delta\}$, then there exists a constant $c > 0$ such that, for every $u \in H_0^1(A)$, we have

$$\int_{A \setminus A^\delta} |u|^2 dx \leq C\delta^2 \int_{A \setminus A^\delta} |\nabla u|^2 dx$$

Now we can finally state the Γ -convergence result.

Theorem 3.3.1. Let F_ε be defined by (3.9). Then, for any $u \in \mathbb{L}^2(\Omega)$, one has

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u) = F(u) = \begin{cases} \int_{\Omega} f^{hom}(\nabla u) dx, & u \in H_0^1(A) \\ +\infty & \text{otherwise.} \end{cases}$$

in the strong topology of $\mathbb{L}^2(\Omega)$, with $f^{hom} : \mathbb{R}^d \rightarrow [0, +\infty]$ defined by

$$f^{hom}(\xi) = \inf \left\{ \int_Y |\xi + \nabla u|^2 dx : u \in H_{per}^1(\mathbb{R}^d) \right\}. \quad (3.12)$$

Proof. We consider separately the Γ -lim inf (1.1) and Γ -lim sup (1.2) inequalities.

Step 1. Let u_ε be a sequence in $\mathbb{L}^2(\Omega_\varepsilon)$ strongly converging to a function u . We have to prove that $F(u) \leq \liminf F_\varepsilon(u_\varepsilon)$; so, without loss of generality, we can suppose that $\liminf F_\varepsilon(u_\varepsilon) < +\infty$. By lemma 3.3.3, we have $u \in H^1(\Omega)$, with $u = 0$ in $\Omega \setminus K$, and the convergence $T_\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$. Therefore we conclude that $F(u) < +\infty$.

Now, by proposition 3.6 in [1], we know that

$$F(u) \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx.$$

Hence, we conclude that

$$\begin{aligned} \liminf_{\varepsilon} F_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} |u_\varepsilon|^2 \geq \\ \liminf_{\varepsilon} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 &\geq F(u) = \int_{\Omega} f^{\text{hom}}(\nabla u) dx. \end{aligned}$$

Step 2. We have to show that for any $u \in H_0^1(A)$ (if $u \in \mathbb{L}^2(\Omega) \setminus H_0^1(A)$ the result is trivial) there exists a sequence $u_\varepsilon \in H_0^1(\Omega_\varepsilon; \partial\Omega)$, with $u_\varepsilon \rightarrow u$ in $\mathbb{L}^2(\Omega)$, such that

$$F(u) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon).$$

Let consider a function $u \in H_0^1(A)$ and the zero extension \tilde{u} of u out of A , defined as

$$\tilde{u}(x) = \begin{cases} u(x) & x \in A \\ 0 & x \in \Omega \setminus A, \end{cases}$$

so that $\tilde{u} \in H_0^1(\Omega)$. Using the result in proposition 3.6 in [1], we can find a sequence $u_\varepsilon \in H^1(A \cap \Omega_\varepsilon) \cap L^2(A)$, with $u_\varepsilon \rightarrow \tilde{u}$ in $\mathbb{L}^2(\Omega)$, such that

$$\limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon \cap A} |\nabla u_\varepsilon|^2 dx \leq \int_A f^{\text{hom}}(\nabla u) dx = \int_{\Omega} f^{\text{hom}}(\nabla \tilde{u}) dx. \quad (3.13)$$

To construct our recovery sequence we fix constant $\delta > 0$ and a set $A^\delta = \{x \in A : \text{dist}(x, \partial A) > \delta\}$. Then we consider a cut off function $\varphi \in \mathcal{C}_0^\infty(A)$, with $0 \leq \varphi \leq 1$, $\text{spt}(\varphi) \subseteq A$, $\varphi = 1$ in A^δ , $|\nabla \varphi| < c/\delta$, and we take a new sequence defined as

$$v_\varepsilon(x) = \varphi(x)u_\varepsilon(x) = \begin{cases} u_\varepsilon(x) & x \in A^\delta \\ \varphi(x)u_\varepsilon(x) & x \in A \setminus A^\delta \\ 0 & x \in \Omega \setminus A, \end{cases}$$

so that $v_\varepsilon \in H_\varepsilon$. We will use the following algebraic inequality:

$$|a + b|^2 \leq (1 + \eta)|a|^2 + \left(1 + \frac{1}{\eta}\right)|b|^2, \quad (3.14)$$

for all $a, b, \eta \in \mathbb{R}$, with $\eta > 0$.

For our sequence v_ε , since $\text{spt}(v_\varepsilon) \subseteq A = \dot{K}$, we have

$$\begin{aligned} F_\varepsilon(v_\varepsilon) &= \int_{(\Omega_\varepsilon \cap A)} |\nabla v_\varepsilon|^2 + \int_{(\Sigma_\varepsilon \setminus K) \cap A} |v_\varepsilon|^2 = \\ &= \int_{\Omega_\varepsilon \cap A^\delta} |\nabla u_\varepsilon|^2 + \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\nabla \varphi u_\varepsilon + \varphi \nabla u_\varepsilon|^2. \end{aligned}$$

Consider the second term and use equation (3.14) and the regularity of φ , so that $|\nabla \varphi| \leq 1/\delta$:

$$\begin{aligned} &\int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\nabla \varphi u_\varepsilon + \varphi \nabla u_\varepsilon|^2 \leq (1 + \eta) \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\varphi \nabla u_\varepsilon|^2 + \\ &+ \left(1 + \frac{1}{\eta}\right) \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\nabla \varphi u_\varepsilon|^2 \leq (1 + \eta) \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\nabla u_\varepsilon|^2 + \\ &+ \left(1 + \frac{1}{\eta}\right) \frac{1}{\delta^2} \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} 2(|u_\varepsilon - \tilde{u}|^2 + |\tilde{u}|^2). \end{aligned}$$

Note that, by (3.13), $\int_{\Omega_\varepsilon \cap A} |\nabla u_\varepsilon|^2 \leq c$, so that

$$(1 + \eta) \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\varphi \nabla u_\varepsilon|^2 \leq \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\nabla u_\varepsilon|^2 + \eta c.$$

Now, by the convergence $u_\varepsilon \rightarrow \tilde{u}$, we have $1/\delta^2 \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |u_\varepsilon - \tilde{u}|^2 = o(1)$, as $\varepsilon \rightarrow 0$, with δ fixed, and, by lemma 3.3.4, being $u \in H_0^1(A)$, and $u = \tilde{u}$ in $\Omega_\varepsilon \cap (A \setminus A^\delta)$,

$$\frac{1}{\delta^2} \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |u|^2 \leq c \frac{1}{\delta^2} \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\nabla u|^2 \leq c \int_{A \setminus A^\delta} |\nabla u|^2,$$

that tends to 0 as $\delta \rightarrow 0$. Hence we have, for any $\delta > 0$, $\eta > 0$,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) &\leq \limsup_{\varepsilon \rightarrow 0} \left[\int_{\Omega_\varepsilon \cap A^\delta} |\nabla u_\varepsilon|^2 + \int_{\Omega_\varepsilon \cap (A \setminus A^\delta)} |\nabla u_\varepsilon|^2 + \eta c \right. \\ &\quad \left. + \left(1 + \frac{1}{\eta}\right) \int_{A \setminus A^\delta} |\nabla u|^2 \right] \leq . \\ &\leq \left[\limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon \cap A} |\nabla u_\varepsilon|^2 \right] + \eta c + \left(1 + \frac{1}{\eta}\right) \int_{A \setminus A^\delta} |\nabla u|^2. \end{aligned}$$

Taking the limit first as $\delta \rightarrow 0$ and then as $\eta \rightarrow 0$, we have

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon \cap A} |\nabla u_\varepsilon|^2,$$

so that, using (3.13), we finally get

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(v_\varepsilon) \leq \int_A f^{\text{hom}}(\nabla u)^2 = F(u)$$

□

Corollary 3.3.1. *Consider $\lambda^1 = \min_{u \in L^2(\Omega)} (F(u) + I_X(u))$. Then we have*

$$\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon^1 = \lambda^1 \quad (3.15)$$

Proof. In theorem 3.3.1 we have proved that $F_\varepsilon \xrightarrow{\Gamma(L^2(\Omega))} F$. From lemma 3.3.1, it follows that

$$F_\varepsilon + I_{X_\varepsilon} \xrightarrow{\Gamma(L^2(\Omega))} F + I_X.$$

From remark 3.3.3 we know that $(F_\varepsilon + I_{X_\varepsilon})_\varepsilon$ is equicoercive. So, from theorem 1.1.1, we obtain immediately (3.15). □

As a consequence of the gamma convergence result, we can consider the differential equation associated to the Euler equation defined by (3.12): this means that our limit homogenized problem will be

$$\begin{cases} -\operatorname{div}(a^{\text{hom}} \nabla u) = |Y| \lambda u & u \in A \\ u = 0 & u \in \partial A, \end{cases} \quad (3.16)$$

where $a^{\text{hom}} \xi \xi = f^{\text{hom}}(\xi)$.

Remark 3.3.4. *It will be useful in the sequel to underline the relationship between equation (3.12) and the associated problem on the periodicity perforated cell Y : the solution w_ξ of the minimum problem defined by $f^{\text{hom}}(\xi)$ is in fact of type $w_\xi = \xi \cdot \chi$, where χ is the vector whose components solve the equation*

$$\begin{cases} \Delta \chi^i(x) = 0 & x \in Y \\ \frac{\partial \chi}{\partial \nu}(x) \cdot \nu^i = 0 & x \in \Sigma^0 \\ \chi \in H_{\text{per}}^1(Y) \end{cases} \quad (3.17)$$

3.4 Convergence of eigenvalues and eigenfunctions.

In this section we will show the proof of theorem 3.1.1, we will consider the limit of λ_ε^j and u_ε^j , for any $j \in \mathbb{N}$, as ε goes to 0, proving that for any λ , eigenvalue of the limit problem (3.16), there exists λ_ε^j , eigenvalue of the problem on the perforated domain (3.3), such that

$$|\lambda - \lambda_\varepsilon^j| \xrightarrow{\varepsilon \rightarrow 0} 0,$$

therefore, in theorem 3.4.2, we will state an estimate for this convergence.

First we want to associate to the problem (3.3) a linear operator K_ε , whose spectrum will be related to the eigenvalues of (3.3). So let consider the embedding operator

$$J_\varepsilon : H_\varepsilon \rightarrow L^2(\Omega_\varepsilon).$$

Being Ω bounded and with sufficiently regular boundary, we know that J_ε is compact.

Now take the operator

$$\begin{aligned} \tilde{K}_\varepsilon : L^2(\Omega_\varepsilon) &\rightarrow H_\varepsilon \\ f &\mapsto \tilde{K}_\varepsilon f, \end{aligned} \quad (3.18)$$

where $\tilde{K}_\varepsilon f$ is the unique solution of the problem

$$\begin{cases} -\Delta u^\varepsilon(x) = f, & x \in \Omega_\varepsilon, \\ \nabla u^\varepsilon(x) \cdot n_\varepsilon = -q(x)u^\varepsilon(x), & x \in \Sigma_\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Omega, \end{cases} \quad (3.19)$$

that is, in weak formulation, the function $u_\varepsilon \in H_\varepsilon$ satisfying

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla v + \int_{\Sigma_\varepsilon \setminus K} u_\varepsilon v = \int_{\Omega_\varepsilon} f v, \quad (3.20)$$

for any $v \in H_\varepsilon$.

Note that, by Lax-Milgram theorem, at $\varepsilon > 0$ fixed, for any $f \in L^2(\Omega_\varepsilon)$, there exists a unique $u_\varepsilon \in H_\varepsilon$ solving problem (3.19) or, equivalently, (3.20), so that \tilde{K}_ε is well defined. We will consider the operator

$$K_\varepsilon : H_\varepsilon \rightarrow H_\varepsilon, \quad K_\varepsilon = \tilde{K}_\varepsilon \cdot J_\varepsilon. \quad (3.21)$$

Lemma 3.4.1. *The operator $K_\varepsilon : H_\varepsilon \rightarrow H_\varepsilon$ is positive, linear, compact and self-adjoint.*

Proof. The proof of the linearity and compactness of \tilde{K}_ε is classical, so it is the fact that \tilde{K}_ε is self-adjoint and positive, see for example [37]. Being J_ε the compact embedding operator, we simply get the thesis by composition. \square

By lemma 3.4.1 and the general spectral theory, we have that the spectrum of the operator K_ε is made by a sequence of positive eigenvalues converging to zero:

$$+\infty > \mu_\varepsilon^1 \geq \mu_\varepsilon^2 \geq \dots \geq \mu_\varepsilon^j \geq \dots > 0.$$

Now observe that, if μ_ε^j is an eigenvalue for K_ε , i.e. there exists $u_\varepsilon^j \in H_\varepsilon$ such that $K_\varepsilon u_\varepsilon^j = \mu_\varepsilon^j u_\varepsilon^j$, then

$$\begin{cases} -\Delta u_\varepsilon^j(x) = \frac{1}{\mu_\varepsilon^j} u_\varepsilon^j(x), & x \in \Omega_\varepsilon, \\ \nabla u_\varepsilon^j(x) \cdot n_\varepsilon = -q(x) u_\varepsilon^j(x), & x \in \Sigma_\varepsilon, \\ u_\varepsilon(x)^j = 0, & x \in \partial\Omega, \end{cases} \quad (3.22)$$

that is $\lambda_\varepsilon^j = \frac{1}{\mu_\varepsilon^j}$ is an eigenvalue of problem (3.3), hence we have that

$$0 < \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \dots \leq \lambda_\varepsilon^j \leq \dots + \infty.$$

This proves the first part of theorem 3.1.1.

Let us prove that λ_ε^1 is simple.

Proof. First we show that if u_ε^1 is an eigenfunction associated to λ_ε^1 , then u_ε^1 doesn't change sign. To do this assume the contrary. Then $u_\varepsilon^+ = \max\{u_\varepsilon^1, 0\}$ and $u_\varepsilon^- = \min\{u_\varepsilon^1, 0\}$ are non-trivial functions. Furthermore u_ε^+ and u_ε^- are in H_ε , so, by the variational characterization of the first eigenvalue,

$$\lambda_\varepsilon^1 \leq \frac{\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^+|^2 + \int_{\Sigma_\varepsilon} q |u_\varepsilon^+|^2}{\int_{\Omega_\varepsilon} |u_\varepsilon^+|^2}, \quad \lambda_\varepsilon^1 \leq \frac{\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^-|^2 + \int_{\Sigma_\varepsilon} q |u_\varepsilon^-|^2}{\int_{\Omega_\varepsilon} |u_\varepsilon^-|^2}. \quad (3.23)$$

Summing up these inequalities one has

$$\lambda_\varepsilon^1 \left(\int_{\Omega_\varepsilon} |u_\varepsilon^+|^2 + \int_{\Omega_\varepsilon} |u_\varepsilon^-|^2 \right) \leq \int_{\Omega_\varepsilon} |\nabla u_\varepsilon^+|^2 + |\nabla u_\varepsilon^-|^2 + \int_{\Sigma_\varepsilon} q (|u_\varepsilon^+|^2 + |u_\varepsilon^-|^2),$$

and, by the fact that $u_\varepsilon^+ u_\varepsilon^- = 0$, we get

$$\lambda_\varepsilon^1 \left(\int_{\Omega_\varepsilon} |u_\varepsilon^1|^2 \right) \leq \int_{\Omega_\varepsilon} |\nabla u_\varepsilon^1|^2 + \int_{\Sigma_\varepsilon} q |u_\varepsilon^1|^2.$$

But u_ε^1 is an eigenfunction associated to λ_ε^1 , hence this last inequality is actually an equality, and so are equations (3.23). Then we have that u_ε^+ is a non negative solution of the equation $-\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon$, with Neumann conditions on Σ_ε and Dirichlet on $\partial\Omega$, that is zero at $\partial\Omega$ and it vanishes in the interior of Ω also: this contradicts the maximum principle, see [37], proposition IX.30.

Now assume that there exist two different and linearly independent eigenfunctions u_ε^1 and v_ε^1 associated to λ_ε^1 ; then, taking $c = \left(\int_{\Omega_\varepsilon} v_\varepsilon^1\right)^{-1} \left(\int_{\Omega_\varepsilon} u_\varepsilon^1\right)$, we have that $u_\varepsilon^1 - cv_\varepsilon^1$ is an eigenfunction too, with

$$\int_{\Omega_\varepsilon} u_\varepsilon^1 - cv_\varepsilon^1 = 0;$$

therefore $u_\varepsilon^1 - cv_\varepsilon^1$ changes sign, and this contradicts our previous argument. \square

This concludes the proof of theorem 3.1.1. Before studying the behavior of eigenvalues, as $\varepsilon \rightarrow 0$, we present the following statement

Lemma 3.4.2. *For any $j \in \mathbb{N}$, there exist two positive constants c_j and c , independent from ε , and a constant $\varepsilon_0 > 0$, such that*

$$c \leq \lambda_\varepsilon^j \leq c_j \quad \forall \varepsilon < \varepsilon_0 \quad (3.24)$$

Proof.

Lower bound.

By theorem 3.1.1, it suffices to prove the inequality $c \leq \lambda_\varepsilon^j$, for λ_ε^j , being $\lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \dots \leq \lambda_\varepsilon^j$. By corollary 3.3.1, we have that $\lim_{\varepsilon \rightarrow 0} \lambda_\varepsilon^1 = \lambda_1 > 0$, then by the theorem of permanence of sign, there exists ε_0 such that, for any $\varepsilon < \varepsilon_0$, we have $\lambda_\varepsilon^1 > 0$, i.e. there exists $c > 0$ such that $\lambda_\varepsilon^1 \geq c$.

Upper bound.

Let us take $\varphi_i \in C_0^\infty(A)$, for $i = 1, \dots, j$, a set of non-zero functions with disjoint supports. We extend by zero out of A , obtaining $\varphi_i \in C_0^\infty(\Omega)$. Since these functions are orthogonal in H_ε , there is a non-trivial linear combination $\psi_\varepsilon = \gamma_\varepsilon^1 \varphi_1 + \dots + \gamma_\varepsilon^j \varphi_j$ such that

$$(\psi_\varepsilon, u_\varepsilon^1)_{H_\varepsilon} = \dots = (\psi_\varepsilon, u_\varepsilon^{j-1})_{H_\varepsilon} = 0.$$

Then ψ_ε is a competitor for the minimum problem defined by λ_ε^j , so that

$$\lambda_\varepsilon^j \leq \frac{\int_{\Omega_\varepsilon} |\nabla \psi_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} |\psi_\varepsilon|^2}{\int_{\Omega_\varepsilon} |\psi_\varepsilon|^2} = \frac{\sum_{i=0}^{j-1} (\gamma_\varepsilon^i)^2 \left(\int_{\Omega_\varepsilon} |\nabla \varphi_i|^2 + \int_{\Sigma_\varepsilon \setminus K} |\varphi_i|^2 \right)}{\sum_{i=0}^{j-1} (\gamma_\varepsilon^i)^2 \int_{\Omega_\varepsilon} \varphi_i^2}.$$

Now, being

$$c'_j = \sup_{0 \leq i \leq j-1} \int_{\Omega_\varepsilon} |\nabla \varphi_i|^2, \quad c''_j = \sup_{0 \leq i \leq j-1} \int_{\Omega_\varepsilon} |\varphi_i|^2,$$

one has

$$\lambda_\varepsilon^j \leq \frac{c'_j}{c''_j} = c_j.$$

\square

Before stating the first result, concerning the convergence of eigenvalues and eigenspaces, we want to present, for the reader convenience, the definition of a particular type of convergence, that we will use in theorem 3.4.1.

Definition 3.4.1. *Let $\{S_j\}_j$ be a sequence of convex subsets of a reflexive Banach space X . We say that $\{S_j\}_j$ Mosco-converges to the set S , writing*

$$S_j \xrightarrow{M} S,$$

if the following relation is satisfied:

$$w - \limsup_{j \rightarrow +\infty} S_j = S = s - \liminf_{j \rightarrow +\infty} S_j. \quad (3.25)$$

By $w - \limsup_j S_j$ we denote the set of $x \in X$ for which there exists a sequence $x_j \rightharpoonup x$ weakly and such that $x_j \in S_j$ frequently, i.e. for infinitely many indices $j \in \mathbb{N}$. By $s - \liminf_j S_j$ we mean the set of $x \in X$ for which there exists a sequence $x_j \rightarrow x$ strongly and such that $x_j \in S_j$ definitively.

Remark 3.4.1. *To show (3.25) it suffices to prove*

$$w - \limsup_{j \rightarrow +\infty} S_j \subseteq S \subseteq s - \liminf_{j \rightarrow +\infty} S_j, \quad (3.26)$$

in fact the following relation is always satisfied:

$$s - \liminf_{j \rightarrow +\infty} S_j \subseteq w - \limsup_{j \rightarrow +\infty} S_j.$$

We will use the Urysohn property for convex sets, that we recall here without proof, see for example [40]:

Property 3.4.1. *Let $\{S_j\}_j$, S be a sequence of convex subsets of a reflexive Banach space X . Then $S_j \xrightarrow{M} S$ if and only if for every subsequence S_{j_k} there exists a further subsequence $S_{j_{k_l}}$ that Mosco converges to S .*

Before showing our first result on the convergence of eigenvalues and eigenspaces, we state a classical property of Gamma convergence:

Lemma 3.4.3. *Let $\{F_\varepsilon\}_\varepsilon$ be a sequence of functional defined in $L^2(\Omega)$ such that*

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon = F$$

in the strong topology of $L^2(\Omega)$. Let $\{v_\varepsilon\}_\varepsilon$ be a sequence of functions in $L^2(\Omega)$ such that $v_\varepsilon \rightarrow v$ strongly in $L^2(\Omega)$ as ε tends to zero, and $\{\lambda_\varepsilon\}_\varepsilon$ a sequence of real numbers converging to λ , as $\varepsilon \rightarrow 0$. Hence

$$F_\varepsilon(u) + \lambda_\varepsilon \int_{\Omega_\varepsilon} v_\varepsilon u \xrightarrow{\varepsilon \rightarrow 0} F(u) + |Y|\lambda \int_{\Omega} vu. \quad (3.27)$$

Proof. The proof follows immediately from the weak-strong convergence

$$\lambda_\varepsilon \int_{\Omega_\varepsilon} v_\varepsilon u = \lambda_\varepsilon \int_{\Omega} v_\varepsilon u \chi_{\Omega_\varepsilon} \rightarrow \lambda |Y| \int_{\Omega} v u.$$

□

Corollary 3.4.1. *Consider the functional F_ε described in (3.9), and its Γ -limit F ; define the new functional*

$$G_\varepsilon(v) = F_\varepsilon(v) - 2\lambda_\varepsilon \int_{\Omega_\varepsilon} u_\varepsilon v, \quad (3.28)$$

where $\lambda_\varepsilon \rightarrow \lambda$ and $u_\varepsilon \rightarrow u$ strongly in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$. Then

$$G_\varepsilon(v) \xrightarrow{\Gamma} G(v) = F(v) - 2\lambda |Y| \int_{\Omega} uv. \quad (3.29)$$

Proof.

Γ -lim inf.

Let v be in the domain of G , i.e. in $H_0^1(A)$, the domain of F , and consider its zero extension out of A in $H_0^1(\Omega)$; Let $v_\varepsilon \rightarrow v$ strongly in $L^2(\Omega)$, with $G_\varepsilon(v_\varepsilon) < C$. This means that v_ε is bounded in H_ε and, by lemma 3.3.3 we have

$$T_\varepsilon v_\varepsilon \rightarrow v$$

weakly in $H^1(\Omega)$, up to subsequence, and, by Rellich theorem, strongly in $L^2(\Omega)$. Since $F_\varepsilon \xrightarrow{\Gamma} F$, we have

$$\liminf_{\varepsilon} F_\varepsilon(v_\varepsilon) \geq F(v),$$

and, by equation (3.27),

$$\liminf_{\varepsilon} G_\varepsilon(v_\varepsilon) = \liminf_{\varepsilon} F_\varepsilon(v_\varepsilon) - 2\lambda_\varepsilon \int_{\Omega_\varepsilon} u_\varepsilon v_\varepsilon \geq F(v) - 2|Y|\lambda \int_{\Omega} uv = G(v).$$

Γ -lim sup.

The Gamma limsup inequality follows directly from the Γ -convergence of F_ε to F and from equation (3.27).

□

We can now prove the following

Theorem 3.4.1. *Let $(\lambda_\varepsilon^j, u_\varepsilon^j)$ and (λ^j, w^j) be the eigenpairs of problems (3.3) and (3.16), respectively. Then*

- 1) $\lambda_\varepsilon^j \rightarrow \lambda^j$ as $\varepsilon \rightarrow 0$, for every $j \in \mathbb{N}$;

2) if λ^j has multiplicity m_j and $\lambda^j = \lambda^{j+1} = \dots = \lambda^{j+m_j-1}$, and we set

$$S_\varepsilon^j = \text{span} [T_\varepsilon u_\varepsilon^j, \dots, T_\varepsilon u_\varepsilon^{j+m_j-1}], \quad S^j = \text{span} [u^j, \dots, u^{j+m_j-1}], \quad (3.30)$$

then

$$S_\varepsilon^j \xrightarrow[\varepsilon \rightarrow 0]{M} S^j$$

in $L^2(\Omega)$, for every $j \in \mathbb{N}$.

Remark 3.4.2. In definition 3.4.1 we used a discrete index $j \in \mathbb{N}$ with $j \rightarrow \infty$, but in theorem 3.4.1 we have the real index $\varepsilon \rightarrow 0$. This means that the Mosco convergence of S_ε^j is actually the convergence over any subsequence $\varepsilon_h \rightarrow 0$, as $h \in \mathbb{N}$ tends to infinity. We simply write $S_\varepsilon^j \xrightarrow[\varepsilon \rightarrow 0]{M} S^j$ to have a more suitable notation for the reader.

Proof. Let us start with the convergence of eigenvalues. Fix $j \in \mathbb{N}$; by lemma 3.4.2, λ_ε^j is equibounded with respect to ε , then there exists a subsequence $\{\varepsilon_h\}_{h \in \mathbb{N}}$ such that $\varepsilon_h \rightarrow 0$ as $h \rightarrow \infty$ and

$$\lambda_{\varepsilon_h}^j \xrightarrow[h \rightarrow \infty]{} \bar{\lambda}^j.$$

To simplify the notation, as we did in remark 3.4.2, we will simply say that

$$\lambda_\varepsilon^j \xrightarrow[\varepsilon \rightarrow 0]{} \bar{\lambda}^j,$$

knowing that the convergence is up to subsequence.

Being u_ε^j eigenfunction for problem (3.3), we have also $F_\varepsilon(u_\varepsilon^j) < C$, so that there exists $\bar{u}^j \in H_0^1(A)$ such that $T_\varepsilon u_\varepsilon^j \rightarrow \bar{u}^j$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$, up to subsequence; moreover, by lemma 3.3.3, \bar{u}^j is zero out of A . We want to show that $(\bar{\lambda}^j, \bar{u}^j)$ is eigenpair of problem (3.16).

Since u_ε^j is a solution of problem (3.3), with $\lambda_\varepsilon = \lambda_\varepsilon^j$, it realizes the minimum of the Euler equation, that is the minimum of functional G_ε in corollary 3.4.1:

$$\min_{u \in H_\varepsilon} \left(\int_{\Omega_\varepsilon} \nabla u_\varepsilon^j \nabla u + \int_{\Sigma_\varepsilon \setminus K} u_\varepsilon^j u - 2\lambda_\varepsilon^j \int_{\Omega_\varepsilon} u_\varepsilon^j u \right) = \min_{u \in L^2(\Omega)} G_\varepsilon(u) = G_\varepsilon(u_\varepsilon^j).$$

Now, by corollary 3.4.1, we have $G_\varepsilon \xrightarrow{\Gamma} G$, hence, using theorem 1.1.1, one has

$$G(\bar{u}^j) = \min_{u \in L^2(\Omega)} G(u) = \lim_{\varepsilon \rightarrow 0} \min_{u \in L^2(\Omega)} G_\varepsilon(u) = \lim_{\varepsilon \rightarrow 0} G_\varepsilon(u_\varepsilon^j),$$

and, considering G as the Euler equation associated to the homogenized problem (3.16), we claim that $(\bar{\lambda}^j, \bar{u}^j)$ is an eigenpair.

To complete the proof we proceed as follows:

Step 1 We show that the set of the limit eigenvalues coincides with the set of eigenvalues of the homogenized problem, i.e.

$$\{\bar{\lambda}^j, j \in \mathbb{N}\} = \{\lambda^j, j \in \mathbb{N}\}. \quad (3.31)$$

Step 2 We show that, for any $j \in \mathbb{N}$, one has $\bar{\lambda}^j = \lambda^j$.

Step 3 We show the Mosco convergence using the Urysohn property 3.4.1.

Step 1.

In the first part of the proof we showed that $\{\bar{\lambda}^j, j \in \mathbb{N}\} \subseteq \{\lambda^j, j \in \mathbb{N}\}$; now we want to prove the opposite inclusion. By theorem 3.1.1, we have

$$0 < \lambda_\varepsilon^1 \leq \lambda_\varepsilon^2 \leq \dots \leq \lambda_\varepsilon^j \leq \dots + \infty,$$

so that

$$0 < \bar{\lambda}^1 \leq \bar{\lambda}^2 \leq \dots \leq \bar{\lambda}^j \leq \dots$$

On the other side we have that

$$\lambda_\varepsilon^j = \min_{\substack{u \in H_\varepsilon, \\ (u, u_\varepsilon^i) = 0 \\ i=1, \dots, j-1}} \frac{\int_{\Omega_\varepsilon} |\nabla u|^2 + \int_{\Sigma_\varepsilon \setminus K} u^2}{\int_{\Omega_\varepsilon} u^2} \geq \min_{\substack{u \in H_\varepsilon, \\ (u, u_\varepsilon^i) = 0 \\ i=1, \dots, j-1}} \frac{\int_{\Omega_\varepsilon} |\nabla u|^2}{\int_{\Omega_\varepsilon} u^2} = \mu_\varepsilon^j,$$

where μ_ε^j is an eigenvalue of the homogeneous Neumann problem on the perforated domain Ω_ε , i.e. the problem (2.8). In section 2.2, we showed the convergence of μ_ε^j :

$$0 < \mu_\varepsilon^1 \leq \mu_\varepsilon^2 \leq \dots \leq \mu_\varepsilon^j \leq \dots < +\infty,$$

and

$$\mu_\varepsilon^j \xrightarrow{\varepsilon \rightarrow 0} \mu^j,$$

with μ^j eigenvalue of the corresponding homogenized problem (2.14). This implies the inequality

$$\bar{\lambda}^j \geq \mu^j \xrightarrow{j \rightarrow \infty} +\infty.$$

Hence we get

$$0 < \bar{\lambda}^1 \leq \bar{\lambda}^2 \leq \dots \leq \bar{\lambda}^j \leq \dots \rightarrow \infty. \quad (3.32)$$

Now assume by contradiction that there exists $\lambda \in \{\lambda^j, j \in \mathbb{N}\}$, such that $\lambda \notin \{\bar{\lambda}^j, j \in \mathbb{N}\}$. From (3.32) we have that there exists $m \in \mathbb{N}$ such that $\lambda < \bar{\lambda}^{m+1}$. Now we want to construct a sequence $w_\varepsilon \in H_\varepsilon$ such that

i)

$$(w_\varepsilon, u_\varepsilon^j)_{H_\varepsilon} = 0, \quad j = 1, \dots, m$$

where u_ε^i are eigenfunctions of problem (3.3);

ii)

$$\frac{\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} w_\varepsilon^2}{\int_{\Omega_\varepsilon} w_\varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} \lambda.$$

Observe that *i*) and *ii*) implies the contradiction: we have, by *i*), that w_ε is a competitor for the minimum problem defined by $\lambda_\varepsilon^{m+1}$, then

$$\bar{\lambda}^{m+1} \xleftarrow{\varepsilon \rightarrow 0} \lambda_\varepsilon^{m+1} \leq \frac{\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} w_\varepsilon^2}{\int_{\Omega_\varepsilon} w_\varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} \lambda,$$

that contradicts the assumption $\lambda < \bar{\lambda}^{m+1}$.

To construct such a sequence let consider u as the solution of problem (3.16), i.e.

$$-\operatorname{div}(a^{\operatorname{hom}} \nabla u) = \lambda |Y| u,$$

with

$$\int_{\Omega} u^2 = 1,$$

and let v_ε be the solution of

$$\begin{cases} -\Delta v^\varepsilon = \lambda u, & x \in \Omega_\varepsilon, \\ \nabla v^\varepsilon \cdot n_\varepsilon = -q v^\varepsilon, & x \in \Sigma_\varepsilon, \\ v^\varepsilon = 0. & x \in \partial\Omega, \end{cases}$$

By lemma 3.3.3 we have $T_\varepsilon v_\varepsilon \rightarrow u$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. Hence define

$$w_\varepsilon = v_\varepsilon - \sum_{i=1}^m (v_\varepsilon, u_\varepsilon^i)_{H_\varepsilon} u_\varepsilon^i.$$

Then w_ε satisfies *i*), indeed

$$(w_\varepsilon, u_\varepsilon^j)_{H_\varepsilon} = (v_\varepsilon, u_\varepsilon^j) - \sum_{i=1}^m (v_\varepsilon, u_\varepsilon^i) (u_\varepsilon^j, u_\varepsilon^i) = 0.$$

To show *ii*) observe that $v_\varepsilon \rightarrow u$ and $u_\varepsilon^i \rightarrow \bar{u}^i$, as we showed in the first part of the proof; so that, being $\lambda \notin \{\bar{\lambda}^j, j \in \mathbb{N}\}$, we must have

$$(v_\varepsilon, u_\varepsilon^i)_{H_\varepsilon} = \int_{\Omega_\varepsilon} \nabla v_\varepsilon \nabla u_\varepsilon^i \rightarrow |Y| \int_{\Omega} \nabla u \nabla \bar{u}^i = 0, \quad i = 1, \dots, m, \quad (3.33)$$

because u and \bar{u}^j are associated to different eigenvalues λ and $\bar{\lambda}^j$. Hence

$$\frac{\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} w_\varepsilon^2}{\int_{\Omega_\varepsilon} w_\varepsilon^2} = \frac{\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} v_\varepsilon^2 + \sum_{i=1}^m (v_\varepsilon, u_\varepsilon^i)_{H_\varepsilon}^2 \left(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon^i|^2 + \int_{\Sigma_\varepsilon \setminus K} |u_\varepsilon^i|^2 \right) + R_1(\varepsilon)}{\int_{\Omega_\varepsilon} v_\varepsilon^2 + \sum_{i=1}^m (v_\varepsilon, u_\varepsilon^i)_{H_\varepsilon}^2 \int_{\Omega_\varepsilon} |u_\varepsilon^i|^2 + R_2(\varepsilon)},$$

where $R_1(\varepsilon)$ and $R_2(\varepsilon)$ are the mixed products coming from the squares $|\nabla w_\varepsilon|^2$ and $|w_\varepsilon|^2$, so that $R_1 \rightarrow 0$ and $R_2 \rightarrow 0$, as $\varepsilon \rightarrow 0$, by equation (3.33). Therefore we get *ii*) applying equation (3.33) again:

$$\frac{\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} w_\varepsilon^2}{\int_{\Omega_\varepsilon} w_\varepsilon^2} = \frac{\int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} v_\varepsilon^2 + o_\varepsilon(1)}{\int_{\Omega_\varepsilon} v_\varepsilon^2 + o_\varepsilon(1)} \xrightarrow{\varepsilon \rightarrow 0} \lambda.$$

Step 2.

It suffices to prove that $\bar{m}(\lambda) = m(\lambda)$, for any λ eigenvalue of the homogenized problem (3.16), where $\bar{m}(\lambda)$ and $m(\lambda)$ are defined by

$$\bar{m}(\lambda) = \#\{j : \lambda = \bar{\lambda}^j\} \quad m(\lambda) = \#\{j : \lambda^j = \lambda\}.$$

We want to prove the two inequalities

- i) $\bar{m}(\lambda) \leq m(\lambda)$;
- ii) $\bar{m}(\lambda) \geq m(\lambda)$.

Let E_λ be the eigenspace associated to λ and \bar{E}_λ the eigenspace associated to any $\bar{\lambda}^j$, for all j such that $\bar{\lambda}^j = \lambda$; then we have

$$\bar{E}_\lambda \subseteq E_\lambda.$$

Being \bar{u}^j and u^j , $j \in \mathbb{N}$ orthonormal basis of $L^2(\Omega)$, we get

$$\bar{m}(\lambda) = \dim \bar{E}_\lambda \leq \dim E_\lambda = m(\lambda).$$

To prove the other inequality assume by contradiction that $\bar{m}(\lambda) < m(\lambda)$. By equation (3.31) and (3.32), there exists $M \in \mathbb{N}$ such that

$$\bar{\lambda}^M = \lambda < \bar{\lambda}^{M+1}. \quad (3.34)$$

Let k be the first index such that $\bar{\lambda}^k = \bar{\lambda}^M = \lambda$, so that $\bar{u}^k, \dots, \bar{u}^M$ are the corresponding eigenfunctions described in the first part of the proof. By the assumption $\bar{m}(\lambda) < m(\lambda)$ there exists a solution $u \in H_0^1(\Omega)$ of the homogenized equation

$$-\operatorname{div}(a^{\operatorname{hom}} \nabla u) = \lambda |Y| u,$$

such that $(u, \bar{u}^i) = 0$ for $i = k, \dots, M$. Moreover, being $\bar{u}^1, \dots, \bar{u}^{k-1}$ associated to eigenvalues $\bar{\lambda}^i \neq \lambda$, we also have $(u, \bar{u}^i) = 0$ for $i = 1, \dots, k-1$. Let v_ε be the solution of problem

$$\begin{cases} -\Delta v^\varepsilon = \lambda_\varepsilon u, & x \in \Omega_\varepsilon, \\ \nabla v^\varepsilon \cdot n_\varepsilon = -q v^\varepsilon, & x \in \Sigma_\varepsilon, \\ v^\varepsilon = 0. & x \in \partial\Omega, \end{cases}$$

define, as in step 1, the sequence

$$w_\varepsilon = v_\varepsilon - \sum_{i=1}^M (v_\varepsilon, u_\varepsilon^i)_{H_\varepsilon} u_\varepsilon^i,$$

where u_ε^i are eigenfunctions of problem (3.3). Hence equation (3.33) holds and so the properties

i)

$$(w_\varepsilon, u_\varepsilon^j)_{H_\varepsilon} = 0, \quad j = 1, \dots, M$$

ii)

$$\frac{\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} w_\varepsilon^2}{\int_{\Omega_\varepsilon} w_\varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} \lambda.$$

Then, using the variational characterization of $\lambda_\varepsilon^{M+1}$, one has

$$\lambda_\varepsilon^{M+1} \leq \frac{\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 + \int_{\Sigma_\varepsilon \setminus K} w_\varepsilon^2}{\int_{\Omega_\varepsilon} w_\varepsilon^2}$$

and, taking the limit as $\varepsilon \rightarrow 0$,

$$\bar{\lambda}^{M+1} \leq \lambda,$$

which contradicts (3.34). Moreover, by the fact that $\bar{\lambda}^j = \lambda_j$ for any $j \in \mathbb{N}$, we can say that the limit $\lambda_{\varepsilon_h}^j \xrightarrow{h \rightarrow \infty} \bar{\lambda}^j$ does not depend on the particular subsequence, therefore

$$\lambda_\varepsilon^j \rightarrow \lambda^j.$$

Step 3.

Let us consider, for any $j \in \mathbb{N}$, a subsequence $u_{\varepsilon_h}^j$ of solutions of problem (3.3) such that $T_{\varepsilon_h} u_{\varepsilon_h}^j \rightarrow \bar{u}^j$ as $h \rightarrow +\infty$. By the first part of the proof we know that such a sequence exists and \bar{u}^j is a solution of the homogenized problem (3.16). Set

$$\bar{S}^j = \text{span}[\bar{u}^j, \dots, \bar{u}^{j+m_j-1}].$$

To prove the Mosco convergence we consider

$$S' = s - \liminf_{h \rightarrow +\infty} S_{\varepsilon_h}^j \quad S'' = w - \limsup_{h \rightarrow +\infty} S_{\varepsilon_h}^j.$$

By the Urysohn property 3.4.1, the Mosco convergence is independent from the particular subsequence, then we will simply consider ε instead of ε_h . By remark 3.4.1 we have to show

- 1) $S'' \subseteq \overline{S}^j$,
- 2) $\overline{S}^j \subseteq S'$,
- 3) $\overline{S}^j = S^j$.

Consider $v \in S''$, i.e. there exists a sequence $\{v_\varepsilon\}_\varepsilon \subseteq H_0^1(\Omega)$ with $v_\varepsilon \rightarrow v$ strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$, and $v_\varepsilon \in S_\varepsilon^j$ frequently. This means that there exist suitable constants c_ε^i such that

$$v_\varepsilon = \sum_{i=0}^{m_j-1} c_\varepsilon^i T_\varepsilon u_\varepsilon^{j+i}.$$

Since $\{v_\varepsilon\}_\varepsilon$ is a bounded sequence in $L^2(\Omega)$, one has

$$\sum_{i=0}^{m_j-1} (c_\varepsilon^i)^2 < +\infty,$$

hence, up to subsequence, for any $i = 0, \dots, m_j - 1$, there exists c^i such that

$$\lim_{\varepsilon \rightarrow 0} c_\varepsilon^i \rightarrow c^i.$$

Therefore

$$v = \sum_{i=0}^{m_j-1} c^i \overline{u}^{j+i} \in \overline{S}^j,$$

so that 1) is proved.

Consider now $v \in \overline{S}^j$; this means that

$$v = \sum_{i=0}^{m-1} c^i \overline{u}^{j+i}.$$

If we set

$$v_\varepsilon = \sum_{i=0}^{m-1} c^i T_\varepsilon (u_\varepsilon^{j+i})$$

then $v_\varepsilon \in S_\varepsilon^j$ and $v_\varepsilon \rightarrow v$, strongly in $L^2(\Omega)$, i.e. $v \in S'$ and 2) is proved.

To prove 3) observe that \bar{S}^j , by definition and by (3.31), is a subspace of S^j , and, being \bar{u}^j , $j \in \mathbb{N}$, an orthonormal basis of $L^2(\Omega)$, one has

$$\dim \bar{S}^j = j = \dim S^j,$$

so that we get 3). \square

Theorem 3.4.1 shows that any eigenvalue of the homogenized problem (3.16) is the limit, as $\varepsilon \rightarrow 0$, of the corresponding eigenvalue of the problem (3.3), in the perforated domain, and the same is for any eigenspace, in the sense of Mosco. Our last result gives the rate of this convergence. In order to obtain it, we will use many technical tools, that we formulate in the sequel.

Lemma 3.4.4. *Let H be a Hilbert separable space and $A : H \rightarrow H$ a linear compact self-adjoint operator. Suppose that there exist two real numbers μ , α and a vector $u \in H$, with $\|u\|_H = 1$ and*

$$\|Au - \mu u\|_H < \alpha.$$

Then there is an eigenvalue μ_j of the operator A , such that

$$i) \quad |\mu_j - \mu| < \alpha;$$

ii) for any $d > \alpha$ there exists a vector \tilde{u} in the eigenspace associated to eigenvalues $\mu_k \in [\mu_j - d, \mu_j + d]$, with $\|\tilde{u}\|_H = 1$, such that

$$\|u - \tilde{u}\|_H < \frac{2\alpha}{d}.$$

This lemma is often known in the literature as Višik lemma; for the proof see for example [41].

We will use the following trace type inequality, whose proof follows from the classical Poincaré-Wirtinger and trace inequalities, see [21, 22]:

Lemma 3.4.5. *For any $u \in H_\varepsilon$ one has*

$$\int_{\Sigma_\varepsilon} |u|^2 \leq c \left(\varepsilon^{-1} \int_{\Omega_\varepsilon} |u|^2 + \varepsilon \int_{\Omega_\varepsilon} |\nabla u|^2 \right). \quad (3.35)$$

Using this lemma 3.4.5 we can easily prove the following

Property 3.4.2. *For any $u, v \in H_\varepsilon$, define the norm*

$$\|u\|_\varepsilon^2 = \int_{\Omega_\varepsilon} |\nabla u|^2 + \int_{\Sigma_\varepsilon \setminus K} |u|^2, \quad (3.36)$$

coming from the scalar product

$$a_\varepsilon(u, v) = \int_{\Omega_\varepsilon} \nabla u \nabla v + \int_{\Sigma_\varepsilon \setminus K} uv. \quad (3.37)$$

Hence there exists a constant $c \in \mathbb{R}$ such that

$$\|u\|_{H_\varepsilon} \leq \|u\|_\varepsilon \leq c\varepsilon^{\frac{1}{2}} \|u\|_{H_\varepsilon}. \quad (3.38)$$

We finally state the last preliminary tool; see for example [25] for the proof.

Lemma 3.4.6. *Let $\chi \in L^\infty_{per}(Y)$ be such that*

$$\int_Y \chi(y) dy = 0.$$

There exists a constant $c > 0$ such that, for any $u, v \in H_0^1(\Omega)$,

$$\left| \int_\Omega \chi\left(\frac{x}{\varepsilon}\right) uv dx \right| \leq c\varepsilon \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \quad (3.39)$$

Now we can state the result on the rate of convergence of eigenvalues and correspondent eigenfunctions (considered with their multiplicity).

Theorem 3.4.2. *Let λ^j , $j \in \mathbb{N}$, be an eigenvalue of problem (3.16) of multiplicity m_j :*

$$\lambda^{j-1} < \lambda^j = \lambda^{j+1} = \dots = \lambda^{j+m_j-1} < \lambda^{j+m_j}.$$

Let $(\lambda_\varepsilon^j, u_\varepsilon^j)_j$ be the eigenpairs of problem (3.3) on the perforated domain. Then there exist orthogonal matrix $M_\varepsilon \in \mathcal{M}^{m_j \times m_j}$ and constants ε_j, C_j such that, for any $\varepsilon < \varepsilon_j$,

$$\|U_\varepsilon^{j+l-1} - \sum_{k=1}^{m_j} M_\varepsilon^{lk} u_\varepsilon^{j+k-1}\|_{H_\varepsilon} \leq C_j \sqrt{\varepsilon}, \quad l = 1, \dots, m_j, \quad (3.40)$$

$$\|u^{j+l-1} - \sum_{k=1}^{m_j} M_\varepsilon^{lk} T_\varepsilon u_\varepsilon^{j+k-1}\|_{L^2(\Omega)} \leq C_j \sqrt{\varepsilon}, \quad l = 1, \dots, m_j, \quad (3.41)$$

with

$$U_\varepsilon^j(x) = u^j(x) + \varepsilon \chi\left(\frac{x}{\varepsilon}\right) \nabla u^j(x), \quad (3.42)$$

here χ is a solution of the cell problem (3.17).

Remark 3.4.3. *Observe that the function $\sum_{k=1}^{m_j} M_\varepsilon^{lk} u_\varepsilon^{j+k-1}$ in (3.41) belongs to the set S_ε^j , defined in (3.30); this means, being $u^{j+l-1} \in S^j$, that S_ε^j converges to S^j in the sense of Mosco, moreover, the rate of this convergence is $\sqrt{\varepsilon}$.*

Proof. The proof, that follows from lemma 3.4.4, will be obtained through these three steps:

Step 1. We prove the following fundamental estimate, that involves the operator K_ε , defined in (3.21)

$$\|K_\varepsilon U_\varepsilon^j - \frac{1}{\lambda^j} U_\varepsilon^j\|_{H_\varepsilon} \leq c^j \sqrt{\varepsilon}, \quad (3.43)$$

in the simpler hypothesis of $u^j \in C_0^\infty(A)$.

Step 2. Applying lemma 3.4.4, we prove (3.40) and (3.41), and discuss the case of λ^j of multiplicity m_j .

Step 3. We generalized the proof for $u^j \in H_0^1(\Omega)$.

Step 1.

Here we assume that $\partial A \in C^{2,\alpha}$, for $\alpha > 0$, so that $u^j \in C^2(\bar{A})$ and, by hypothesis $u^j \in C_0^\infty(A)$, we get $U_\varepsilon^j \in H_0^1(A)$. We have, using definition (3.37),

$$\begin{aligned} \|K_\varepsilon U_\varepsilon^j - \frac{1}{\lambda^j} U_\varepsilon^j\|_\varepsilon &= \sup_{\substack{\varphi \in H_\varepsilon \\ \|\varphi\|_\varepsilon=1}} a_\varepsilon \left(K_\varepsilon U_\varepsilon^j - \frac{1}{\lambda^j} U_\varepsilon^j, \varphi \right) = \\ &= \sup_{\substack{\varphi \in H_\varepsilon \\ \|\varphi\|_\varepsilon=1}} \left[a_\varepsilon (K_\varepsilon U_\varepsilon^j, \varphi) - a_\varepsilon \left(\frac{1}{\lambda^j} U_\varepsilon^j, \varphi \right) \right]. \end{aligned}$$

Now, being $K_\varepsilon U_\varepsilon^j$ the solution of problem (3.19), with $f = U_\varepsilon^j$, we have

$$a_\varepsilon (K_\varepsilon U_\varepsilon^j, \varphi) = \int_{\Omega_\varepsilon} U_\varepsilon^j \varphi,$$

and

$$a_\varepsilon \left(\frac{1}{\lambda^j} U_\varepsilon^j, \varphi \right) = \frac{1}{\lambda^j} \left(\int_{\Omega_\varepsilon} \nabla U_\varepsilon^j \nabla \varphi + \int_{\Sigma_\varepsilon \setminus K} U_\varepsilon^j \varphi \right) = \frac{1}{\lambda^j} \int_{\Omega_\varepsilon} \nabla U_\varepsilon^j \nabla \varphi,$$

because $u^j \in C_0^\infty(A)$.

Hence, for the first term we have

$$\begin{aligned} \int_{\Omega_\varepsilon} U_\varepsilon^j \varphi &= \int_{\Omega_\varepsilon} (u^j + \varepsilon \chi_\varepsilon \nabla u^j) \varphi = \int_{\Omega_\varepsilon \cap A} u^j \varphi + \varepsilon \int_{\Omega_\varepsilon} \chi_\varepsilon \nabla u^j \varphi \leq \\ &\leq \int_{\Omega_\varepsilon \cap A} u^j \varphi + \varepsilon C \|\varphi\|_{L^2(\Omega_\varepsilon)}, \end{aligned}$$

in fact, having $\nabla u^j \in C_0^\infty(A) \subseteq L^\infty(\Omega_\varepsilon)$, $\chi_\varepsilon \in H_\varepsilon$ and $\varphi \in H_\varepsilon$, and using Cauchy-Schwarz inequality,

$$\int_{\Omega_\varepsilon} \chi_\varepsilon \nabla u^j \varphi \leq \|\nabla u^j\|_{L^\infty} \|\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)} \|\varphi\|_{L^2(\Omega_\varepsilon)} < C \|\varphi\|_{L^2(\Omega_\varepsilon)}.$$

Therefore, by inequality (3.38), we get

$$\int_{\Omega_\varepsilon} U_\varepsilon^j \varphi \leq \int_{\Omega_\varepsilon \cap A} u^j \varphi + \varepsilon^{1/2} c_1 \|\varphi\|_\varepsilon.$$

On the other hand, for the second term, using same estimate of the first one, we have

$$\begin{aligned} -\frac{1}{\lambda^j} \int_{\Omega_\varepsilon} \nabla U_\varepsilon^j \nabla \varphi &= -\frac{1}{\lambda^j} \int_{\Omega_\varepsilon \cap A} \left(\nabla u^j + \varepsilon \nabla \chi \left(\frac{x}{\varepsilon} \right) \frac{1}{\varepsilon} \nabla u^j \right) \nabla \varphi + \\ -\frac{1}{\lambda^j} \int_{\Omega_\varepsilon} \varepsilon \chi_\varepsilon D^2 u^j \nabla \varphi &\leq -\frac{1}{\lambda^j} \int_{\Omega_\varepsilon \cap A} (\nabla u^j + \nabla_y \chi(y) \nabla u^j) \nabla \varphi - \frac{C}{\lambda^j} \varepsilon \|\varphi\|_{H_\varepsilon} \leq \\ &\leq -\frac{1}{\lambda^j} \int_{\Omega_\varepsilon \cap A} (\nabla u^j + \nabla_y \chi(y) \nabla u^j) \nabla \varphi - \frac{c_2}{\lambda^j} \varepsilon^{1/2} \|\varphi\|_\varepsilon. \end{aligned}$$

It remains to estimate these two terms:

$$\int_{\Omega_\varepsilon \cap A} u^j \varphi, \quad \int_{\Omega_\varepsilon \cap A} (\nabla u^j + \nabla_y \chi(y) \nabla u^j) \nabla \varphi.$$

Let us start with the first one:

$$\int_{\Omega_\varepsilon \cap A} u^j \varphi = |Y| \int_A u^j T_\varepsilon \varphi + \int_A (\chi_{\Omega_\varepsilon} - |Y|) u^j T_\varepsilon \varphi.$$

Now consider the function $h \in L_{\text{per}}^\infty(Y)$, $h_\varepsilon = \chi_{\Omega_\varepsilon} - |Y|$: we have $\int_Y h(y) dy = 0$, $u^j \in C_0^\infty(A) \subseteq H_0^1(A)$ and $T_\varepsilon \varphi \in H^1(\Omega)$, by the definition of the extension operator, so that we can use lemma 3.4.6, getting

$$\int_A (\chi_{\Omega_\varepsilon} - |Y|) u^j T_\varepsilon \varphi \leq C \varepsilon \|u^j\|_{H_0^1(A)} \|T_\varepsilon \varphi\|_{H_0^1(\Omega)}.$$

Hence, using the continuity of the extension operator and equation (3.38), one has

$$\int_A (\chi_{\Omega_\varepsilon} - |Y|) u^j T_\varepsilon \varphi \leq c_3 \varepsilon^{1/2} \|\varphi\|_\varepsilon.$$

For the second term:

$$\begin{aligned} -\frac{1}{\lambda^j} \int_{\Omega_\varepsilon \cap A} (\nabla u^j + \nabla_y \chi(y) \nabla u^j) \nabla \varphi &= -\frac{1}{\lambda^j} \int_A a^{\text{hom}} \nabla u^j \nabla T_\varepsilon \varphi + \\ -\frac{1}{\lambda^j} \int_A [\chi_{\Omega_\varepsilon} (\nabla u^j + \nabla_y \chi_\varepsilon \cdot \nabla u^j) - a^{\text{hom}} \nabla u^j] \nabla T_\varepsilon \varphi. \end{aligned}$$

We can consider, as before, the periodic function

$$h \left(\frac{x}{\varepsilon} \right) = \chi_{\Omega_\varepsilon} (1 + \nabla_y \chi_\varepsilon \cdot \nabla u^j - a^{\text{hom}}),$$

and, using lemma 3.4.6 and the continuity of the extension operator,

$$\begin{aligned} \left| \int_A [\chi_{\Omega_\varepsilon} (\nabla u^j + \nabla_y \chi_\varepsilon \cdot \nabla u^j) - a^{\text{hom}} \nabla u^j] \nabla T_\varepsilon \varphi \right| &= \left| \int_A h_\varepsilon \nabla u^j \nabla T_\varepsilon \varphi \right| \leq \\ &\leq C\varepsilon \|u^j\|_{H_0^1(A)} \|T_\varepsilon \varphi\|_{H_0^1(\Omega)} \leq c_4 \varepsilon^{1/2} \|\varphi\|_\varepsilon. \end{aligned}$$

Therefore, putting together all these estimates, we get

$$\begin{aligned} \|K_\varepsilon U_\varepsilon^j - \frac{1}{\lambda^j} U_\varepsilon^j\|_\varepsilon &= \int_{\Omega_\varepsilon \cap A} u^j \varphi - \frac{1}{\lambda^j} \int_{\Omega_\varepsilon \cap A} (\nabla u^j + \nabla_y \chi(y) \nabla u^j) \nabla \varphi + \\ &\quad + \varepsilon^{1/2} c_1 \|\varphi\|_\varepsilon - \varepsilon^{1/2} \frac{c_2}{\lambda^j} \|\varphi\|_\varepsilon \leq \\ &\leq |Y| \int_A u^j T_\varepsilon \varphi - \frac{1}{\lambda^j} \int_A a^{\text{hom}} \nabla u^j \nabla T_\varepsilon \varphi + \\ &\quad + \varepsilon^{1/2} c_1 \|\varphi\|_\varepsilon - \varepsilon^{1/2} \frac{c_2}{\lambda^j} \|\varphi\|_\varepsilon \varepsilon^{1/2} c_3 \|\varphi\|_\varepsilon - \varepsilon^{1/2} \frac{c_4}{\lambda^j} \|\varphi\|_\varepsilon \end{aligned}$$

Note that, being u^j the solution of the homogenized problem (3.16), one has

$$|Y| \int_A u^j T_\varepsilon \varphi - \frac{1}{\lambda^j} \int_A a^{\text{hom}} \nabla u^j \nabla T_\varepsilon \varphi = 0,$$

and we finally get

$$\|K_\varepsilon U_\varepsilon^j - \frac{1}{\lambda^j} U_\varepsilon^j\|_{H_\varepsilon} \leq \|K_\varepsilon U_\varepsilon^j - \frac{1}{\lambda^j} U_\varepsilon^j\|_\varepsilon \leq c^j \sqrt{\varepsilon}$$

Step 2.

To apply lemma 3.4.4, we need to use a normalized function: $\|U_\varepsilon^j\|_{H_\varepsilon} = 1$. In our hypothesis we have $u^j, \nabla u^j \in C_0^\infty(A)$ and $\chi_\varepsilon \in H_0^1(\Omega_\varepsilon)$, so that

$$\|U_\varepsilon^j - u^j\|_{H_\varepsilon} \leq C\varepsilon \tag{3.44}$$

and, being $u^j \neq 0$, we must have $\|U_\varepsilon^j\|_{H_\varepsilon} \geq \alpha > 0$. So we can use the normalized function, naming it again U_ε^j :

$$U_\varepsilon^j = \frac{U_\varepsilon^j}{\|U_\varepsilon^j\|_{H_\varepsilon}},$$

getting

$$\|K_\varepsilon U_\varepsilon^j - \frac{1}{\lambda^j} U_\varepsilon^j\|_{H_\varepsilon} \leq c^j \sqrt{\varepsilon} \frac{1}{\|U_\varepsilon^j\|_{H_\varepsilon}} \leq \frac{c^j}{\alpha} \sqrt{\varepsilon}.$$

Now we can apply lemma 3.4.4 to the linear continuous compact and self-adjoint operator K_ε , with $\mu = (\lambda^j)^{-1}$, $\alpha = c^j/\alpha\sqrt{\varepsilon}$: then there exists $(\lambda_\varepsilon^j)^{-1}$, eigenvalue of K_ε , such that

$$\left| \frac{1}{\lambda_\varepsilon^j} - \frac{1}{\lambda^j} \right| \leq \frac{c^j}{\alpha} \sqrt{\varepsilon},$$

moreover, for any $d > 0$, there exists a normalized function \tilde{u}_ε in the eigenspace associated to eigenvalues in the interval $[\lambda_\varepsilon^j - d, \lambda_\varepsilon^j + d]$, such that

$$\|U_\varepsilon^j - \tilde{u}_\varepsilon\|_{H_\varepsilon} \leq 2 \frac{c^j \sqrt{\varepsilon}}{\alpha d},$$

that is equation (3.40) in an implicit form. In order to understand better the convergence of eigenfunctions in the case of multiple eigenvalues, suppose to have λ^j of multiplicity m_j , as in our hypothesis:

$$\lambda^{j-1} < \lambda^j = \lambda^{j+1} = \dots = \lambda^{j+m_j-1} < \lambda^{j+m_j},$$

and set

$$d_j = \min \left(\frac{1}{\lambda^{j-1}} - \frac{1}{\lambda^j}, \frac{1}{\lambda^j} - \frac{1}{\lambda^{j+m_j}} \right)$$

$$\Lambda^j = \left(\frac{1}{\lambda^j} - d_j, \frac{1}{\lambda^j} + d_j \right),$$

then $1/\lambda_\varepsilon^i \in \Lambda^j$ if and only if $j \leq i \leq j + m_j - 1$. For any of these λ^i we construct the function $U_\varepsilon^{j+i}(x) = u^{j+i}(x) + \varepsilon \chi(\frac{x}{\varepsilon}) \nabla u^{j+i}(x)$ and, repeating *step 1*, we get

$$\|K_\varepsilon U_\varepsilon^{j+i} - \frac{1}{\lambda^{j+i}} U_\varepsilon^{j+i}\|_{H_\varepsilon} \leq \frac{c^{j+i}}{\alpha^{j+i}} \sqrt{\varepsilon}, \quad j \leq i \leq j + m_j - 1.$$

Hence, by lemma 3.4.4, there exists an eigenfunction in the eigenspace associated to eigenvalues in the interval Λ^j , i.e. there exists a matrix $M_\varepsilon \in \mathcal{M}^{m_j \times m_j}$ and eigenfunction u_ε^{j+i} associated to $\lambda_\varepsilon^{j+i}$, with $1/\lambda_\varepsilon^{j+i} \in \Lambda^j$, such that

$$\|U_\varepsilon^{j+i} - \sum_{l=0}^{m_j-1} M_\varepsilon^{il} u_\varepsilon^{j+l}\|_{H_\varepsilon} \leq 2 \frac{c^{j+i}}{\alpha^{j+i} d_j} \sqrt{\varepsilon} = C^{j+i} \sqrt{\varepsilon}, \quad j \leq i \leq j + m_j - 1,$$

that is equation (3.40); in order to derive equation (3.41) we simply note that, for any $j \in \mathbb{N}$, being $\chi \in H_\varepsilon$ and $u^j \in C_0^\infty(A)$,

$$\|U_\varepsilon^j - u^j\|_{H_\varepsilon} = \varepsilon \|\chi_\varepsilon \nabla u^j\|_{H_\varepsilon} \leq C\varepsilon.$$

Step 3.

We want to generalize to the case $u^j \in H_0^1(A)$; this means that ∇u^j could not be zero in ∂A , making U_ε^j not in $H_0^1(A)$ and inequality (3.44) holds true just in the $L^2(\Omega)$ norm. Consider ψ_ε a family of smooth functions in $C_0^\infty(A)$ such that $0 \leq \psi_\varepsilon \leq 1$

$$\psi_\varepsilon = \begin{cases} 1 & \text{if } x \in A, d(x, \partial A) > 2\varepsilon \\ 0 & \text{if } x \in \Omega \setminus A, \end{cases} \quad (3.45)$$

and $\|\nabla\psi_\varepsilon\|_\infty \leq 2/\varepsilon$. Then take, for any $j \in \mathbb{N}$,

$$\tilde{U}_\varepsilon^j = u^j + \varepsilon\psi_\varepsilon\chi_\varepsilon\nabla u^j,$$

so that $\tilde{U}_\varepsilon^j \in H_0^1(A)$.

The following estimates hold true

$$\|\tilde{U}_\varepsilon^j - U_\varepsilon^j\|_{L^2(\Omega)} \leq C\varepsilon^{3/2},$$

$$\|\tilde{U}_\varepsilon^j - U_\varepsilon^j\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}.$$

Hence, repeating the proof of *Step 1*, using \tilde{U}_ε^j instead of U_ε^j , and these last estimates, we get the thesis in the general case. \square

Chapter 4

Homogenization of oscillating constraint problems: the general Cartesian case

In this chapter we define an oscillating constraint problem in the cartesian case, for vector functions; we will distinguish three different situation: the first and the second can be treated with the standard almost periodic homogenization theory, see section 1.2, the third, that is the most interesting and original case, will be treat in next chapter, only in the scalar case.

4.1 Problem setup

We want to study the homogenization of the following functional

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega} |\nabla u|^2 dx & u \in H^1(\Omega; V_\varepsilon) \\ +\infty & \text{otherwise} \end{cases} \quad (4.1)$$

Here we consider a sufficiently regular set $\Omega \subset \mathbb{R}^n$ and a function u , actually depending on the small parameter ε , taking values on an oscillating constraint V_ε . More precisely we take

$$\begin{aligned} u_\varepsilon : \Omega \in \mathbb{R}^n &\rightarrow V_\varepsilon \subset \mathbb{R}^{m+m'} \\ x &\mapsto u_\varepsilon(x) = (u_1^\varepsilon(x), u_2^\varepsilon(x)) \end{aligned} \quad (4.2)$$

where $u_2^\varepsilon(x) = \varphi_\varepsilon(u_1^\varepsilon(x))$; the function $\varphi_\varepsilon = \delta_\varepsilon \varphi\left(\frac{y}{\varepsilon}\right)$ is defined by φ , that is $Y = (0, 1)^m$ -periodic, and takes values from \mathbb{R}^m to $\mathbb{R}^{m'}$. In the sequel we will specify the regularity and hypothesis on this function, that will be different in each cases. Moreover φ_ε satisfies the condition $\varphi_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$;

hence our oscillating manifold V_ε , in some sense, is converging to the space $V = \mathbb{R}^m \times \{0\} = \mathbb{R}^m$.

Remark 4.1.1. *To simplify the notation, in the sequel we will name $u_1^\varepsilon = u_\varepsilon$ and $u_2^\varepsilon = \varphi_\varepsilon(u_\varepsilon)$, getting*

$$u_\varepsilon(x) = (u_\varepsilon(x), \varphi_\varepsilon(u_\varepsilon(x))).$$

For the functional F_ε we have

$$F_\varepsilon(u_\varepsilon) = \int_{\Omega} |\nabla u_\varepsilon|^2 dx = \int_{\Omega} \left(|\nabla u_\varepsilon|^2 + \left(\frac{\delta}{\varepsilon}\right)^2 |D_y \varphi(u_\varepsilon) \nabla u_\varepsilon|^2 \right) dx \quad (4.3)$$

where $y = u/\varepsilon$.

Let us define the matrix $A(y) = D_y \varphi(y)$, so that F_ε becomes

$$F_\varepsilon(u_\varepsilon) = \int_{\Omega} |\nabla u_\varepsilon|^2 dx = \int_{\Omega} \left(|\nabla u_\varepsilon|^2 + \left(\frac{\delta}{\varepsilon}\right)^2 \left| A\left(\frac{u_\varepsilon}{\varepsilon}\right) \nabla u_\varepsilon \right|^2 \right) dx.$$

Observing the second component of this functional, seems to be reasonable to study three different cases.

- I. $\delta/\varepsilon \rightarrow 0$, then the second term will disappear and we will study the standard homogenization of the first one.
- II. $\delta/\varepsilon \rightarrow c$, we will use some standard homogenization formula for almost periodic functionals.
- III. $\delta/\varepsilon \rightarrow +\infty$, this is the most interesting case, in which we can't apply the classical homogenization theory and, intuitively, the Γ -limit will be finite only in some particular directions: near the level curves of φ .

We consider the first two cases in the following sections of this chapter; the last and most difficult case will be pointed out in the next chapter.

Remark 4.1.2. *In order to consider homogenization and Γ -convergence of our functional, we have to fix a reasonable topology. Here, differently from most classical examples, the small parameter ε is not in the domain of the function u , but in its codomain; so we can't use the standard $L^2(\Omega; V_\varepsilon)$ strong or weak topologies. We will use a topology τ defined as follow:*

Definition 4.1.1. *Let u_ε be a sequence of functions defined as in (4.2), $u_\varepsilon = (u_1^\varepsilon, u_2^\varepsilon = \varphi_\varepsilon(u_1^\varepsilon))$. We say that $u_\varepsilon \rightarrow u = (u_1, 0)$ as $\varepsilon \rightarrow 0$ if and only if $u_1^\varepsilon \rightarrow u_1$ in the strong topology of $L^2(\Omega, \mathbb{R}^m)$.*

4.2 The case $\delta/\varepsilon \rightarrow 0$: direct computation of the Γ -limit

Theorem 4.2.1. *Let φ be a function in $W^{1,\infty}(\mathbb{R}^m; \mathbb{R}^{m'})$. For the functional F_ε , defined in (4.1), the following property holds*

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) = F(u) = \begin{cases} \int_{\Omega} |\nabla u|^2 dx & u \in H^1(\Omega, V) \\ +\infty & \text{otherwise} \end{cases}$$

in the topology τ defined in 4.1.1; here we use the same notation as before, so that $u = (u_1, 0)$.

Proof. Let us consider the two conditions of Γ -convergence separately.

- i) We take a sequence $u_\varepsilon \in H^1(\Omega, V_\varepsilon)$ converging with respect the topology τ to a function $u = (u_1, 0)$, i.e. $u_1^\varepsilon \rightarrow u_1$ in $L^2(\Omega, \mathbb{R}^m)$. We prove that

$$F(u) \leq \liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon). \quad (4.4)$$

To do this we impose, possibly passing to a subsequence, that exists the $\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) < +\infty$; otherwise (4.4) is trivial. By the definition of F_ε one has

$$\lambda > F_\varepsilon(u_\varepsilon) \geq \int_{\Omega} |\nabla u_\varepsilon|^2 = \|\nabla u_\varepsilon\|_{H^1(\Omega)}$$

Then, up to subsequence, we have the weak convergence $u_1^\varepsilon \rightharpoonup u_1$ in $H^1(\Omega, V)$. Hence we get

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_1^\varepsilon|^2 dx = \int_{\Omega} |\nabla u_1|^2 dx = F(u).$$

- ii) Let $u = (u_1, 0)$ be a function in $H^1(\Omega, V)$. We have to find a sequence $u_\varepsilon \in H^1(\Omega, V_\varepsilon)$, converging to u in the topology τ , such that

$$F(u) \geq \limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \quad (4.5)$$

We can take the sequence $u_\varepsilon = (u_1, \delta\varphi(u_1/\varepsilon)) \in H^1(\Omega, V_\varepsilon)$, therefore, by the hypothesis on the regularity of φ , one has

$$\int_{\Omega} |D_y\varphi(y) \cdot \nabla u_1|^2 dx \leq \int_{\Omega} |D_y\varphi(y)|^2 |\nabla u_1|^2 dx$$

$$\leq C \int_{\Omega} |\nabla u_1|^2 dx < +\infty.$$

Now, using the hypothesis $\delta/\varepsilon \rightarrow 0$, we claim (4.5):

$$\lim_{\varepsilon \rightarrow 0} F_{\varepsilon}(u_{\varepsilon}) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla u_1|^2 dx + \left(\frac{\delta}{\varepsilon}\right)^2 \int_{\Omega} |D_y \varphi(u_1) \nabla u_1|^2 dx = F(u).$$

□

4.3 The case $\delta/\varepsilon \rightarrow c$: almost periodic homogenization theory

Differently from the previous case, it can be more difficult to compute the Γ -limit directly, as we did in section 4.2. In this second case we will use theorem 1.2.2; to apply this general classic result, we will prove conditions *i*) and *ii*) on the density of our functional F_{ε} . Here we consider the same hypothesis on the function φ , as in the previous section: we will take it in $W^{1,\infty}(\mathbb{R}^m; \mathbb{R}^{m'})$.

Let us start from a simpler case in which the ratio δ/ε is constant and equal c , that is its limit as $\varepsilon \rightarrow 0$. Consider the function

$$f_c : \mathbb{R}^m \times M^{m \times n} \rightarrow \mathbb{R} \\ (s, \xi) \mapsto |\xi|^2 + c^2 |D\varphi(s)|^2.$$

We define the functional $F_{\varepsilon}(u) = \int_{\Omega} f_c\left(\frac{u}{\varepsilon}, \nabla u\right) dx$. We want to show that f_c satisfies properties *i*) and *ii*) of theorem 1.2.2. First we can make the expression of f_c more explicit, considering its second term, evaluated on $\left(\frac{u}{\varepsilon}, \nabla u\right)$:

$$\left| D\delta\varphi\left(\frac{u}{\varepsilon}\right) \right|^2 = \sum_{i=1}^n \sum_{j=1}^{m'} \left| D_{x_i} \delta\varphi^j\left(\frac{u}{\varepsilon}\right) \right|^2 = \frac{\delta^2}{\varepsilon^2} \sum_{i=1}^n \sum_{j=1}^{m'} \left| \sum_{k=1}^m D_{y_k} \varphi^j\left(\frac{u}{\varepsilon}\right) D_{x_i} u^k \right|^2.$$

Now use the same notation as before, calling $A_k^j = D_{y_k} \varphi^j$, define the matrix $\xi_i^k := D_{x_i} u^k$, and substitute them in the previous formula, recalling that $\delta/\varepsilon = c$:

$$\left| D\delta\varphi\left(\frac{u}{\varepsilon}\right) \right|^2 = c^2 \sum_{i=1}^n \sum_{j=1}^{m'} \left| \sum_{k=1}^m \xi_i^k A_k^j \right|^2.$$

We finally get the following formula for f

$$f(s, \xi) = |\xi|^2 + c^2 \sum_{i=1}^n \sum_{j=1}^{m'} \left| \sum_{k=1}^m \xi_i^k A_k^j(s) \right|^2.$$

Observing that, by the Cauchy-Schwarz inequality and the regularity of u and φ , one has

$$\left| \sum_{k=1}^m \xi_i^k A_k^j(s) \right|^2 \leq |\xi_i|^2 |A^j|^2 < +\infty$$

we have the first condition

$$|\xi|^2 \leq f(s, \xi) \leq (1 + M)|\xi|^2 \leq \beta(1 + |\xi|)^2$$

where $\beta = 1 + M$ and M satisfies $c^2 |A^j|^2 \leq c^2 \sup |D\varphi|^2 = M$.

For the second one we recall that a function f is u.a.p. if it is the uniform limit of a sequence of trigonometric polynomials. Our function f_c is the following one

$$\begin{aligned} f_c(s, \xi) &= |\xi|^2 + c^2 |A(s)\xi|^2 = |\xi|^2 + c^2 \sum_{i=1}^n \sum_{j=1}^{m'} \left(\sum_{k=1}^m \xi_i^k A_k^j(s) \right)^2 = \\ &= |\xi|^2 + c^2 \sum_{i=1}^n \sum_{j=1}^{m'} \left(\sum_{k=1}^m \xi_i^k A_k^j(s) \right) \left(\sum_{h=1}^m \xi_i^h A_h^j(s) \right) = \\ &= |\xi|^2 + c^2 \sum_{i,j,k,h} \xi_i^k \xi_i^h A_k^j(s) A_h^j(s) = |\xi|^2 + c^2 \sum_{i,j,k,h} a_{ij}^{hk}(s) \xi_i^h \xi_j^k \end{aligned}$$

with $a(s)$, Y -periodic, by our assumptions on φ . Then f_c is actually a periodic function with respect the variable $s \in \mathbb{R}^m$ and, trivially, u.a.p..

Therefore f_c satisfies both *i*) and *ii*) of theorem 1.2.2 and we can state the homogenization result:

Theorem 4.3.1. *There exists a quasi convex function $f_c^{hom} : M^{m \times n} \rightarrow \mathbb{R}$ such that for every $u \in H^1(\Omega; V)$ the limit*

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \int_{\Omega} f_c \left(\frac{u}{\varepsilon}, \nabla u \right) dx = \int_{\Omega} f_c^{hom}(\nabla u) dx$$

exists, and the function f_c^{hom} satisfies the asymptotic homogenization formula

$$\begin{aligned} f_c^{hom}(\xi) &= \lim_{T \rightarrow +\infty} \inf \left\{ \frac{1}{T^n} \int_{(0,T)^n} f_c(u + \xi x, \nabla u + \xi) dx \right. \\ &\quad \left. u \in H_0^1((0, T)^n; V) \right\} \end{aligned}$$

for all $\xi \in M^{m \times n}$.

Now consider the general case $\delta/\varepsilon \rightarrow c$, that is, for sufficiently small ε , $|\delta/\varepsilon - c| < \eta$, for all $\eta > 0$, hence we have

$$(c - \eta)^2 \leq \left(\frac{\delta}{\varepsilon}\right)^2 \leq (c + \eta)^2.$$

In this case we take the function $f_\varepsilon(s, \xi) = |\xi|^2 + (\delta/\varepsilon)^2 |D\varphi(s)|^2$, that defines the integral $F_\varepsilon = \int_\Omega f_\varepsilon\left(\frac{u}{\varepsilon}, \nabla u\right) dx$. Note that

$$f_\varepsilon(s, \xi) = |\xi|^2 + \left(\frac{\delta}{\varepsilon}\right)^2 |D\varphi(s)|^2 \leq \left(\frac{c + \eta}{c}\right)^2 (|\xi|^2 + c^2 |D\varphi(s)|^2),$$

hence we have

$$\left(\frac{c - \eta}{c}\right)^2 f_c(s, \xi) \leq f_\varepsilon(s, \xi) \leq \left(\frac{c + \eta}{c}\right)^2 f_c(s, \xi), \quad (4.6)$$

and the same for integrals:

$$\left(\frac{c - \eta}{c}\right)^2 F_\varepsilon^c \leq F_\varepsilon \leq \left(\frac{c + \eta}{c}\right)^2 F_\varepsilon^c.$$

Now, for F_ε^c and f_c theorem 4.3.1 holds, therefore, taking the limit as $\eta \rightarrow 0$ in (4.6) we claim the same result for f_ε and F_ε : there exists a quasi convex function $f^{\text{hom}} : M^{m \times n} \rightarrow \mathbb{R}$ such that, for every $u \in H^1(\Omega; V)$,

$$\Gamma(L^2)\text{-}\lim_{\varepsilon \rightarrow 0} \int_\Omega f_\varepsilon\left(\frac{u}{\varepsilon}, \nabla u\right) dx = \int_\Omega f^{\text{hom}}(\nabla u) dx,$$

with

$$f^{\text{hom}}(\xi) = \lim_{T \rightarrow +\infty} \inf \left\{ \frac{1}{T^n} \int_{(0, T)^n} f_c(u + \xi x, \nabla u + \xi) dx \right. \\ \left. u \in H_0^1((0, T)^n; V) \right\}.$$

Chapter 5

Homogenization of scalar oscillating constraint problems: the case $\delta/\varepsilon \rightarrow +\infty$

Being interested in the geometric properties of the oscillating constraint V_ε , we will prove the homogenization result only for curves taking values in the oscillating constraint converging to \mathbb{R}^m ; the vectorial case is much different and it involves notions as quasi-convexity, poly-convexity and rank-1-convexity.

So now we have $F_\varepsilon : L^2([0, 1], V_\varepsilon) \rightarrow [0, +\infty]$ as in equation (4.1); $\varphi : \mathbb{R}^m \rightarrow \mathbb{R}$ is 1-periodic, with $\varphi_\varepsilon(y) = \delta\varphi(y/\varepsilon)$; the oscillating constraint is $V_\varepsilon = \{(y, z) \in \mathbb{R}^{m+1} : z = \varphi_\varepsilon(y)\}$, that converges to \mathbb{R}^m and we assume $\delta/\varepsilon \rightarrow +\infty$. We will consider F_ε in the following unconstrained form:

$$F_\varepsilon(u_\varepsilon) = \int_0^1 \left(|u'_\varepsilon|^2 + \left(\frac{\delta}{\varepsilon}\right)^2 |D_y\varphi(u_\varepsilon)u'_\varepsilon|^2 \right) dt. \quad (5.1)$$

Here it is clear that the second term of F_ε tends to $+\infty$ when $\varepsilon \rightarrow 0$; this fact forces the minima of F_ε to stay where the gradient of φ is zero, i.e. in a level set of φ . In this chapter we will assume that the regularity for the constraint is $\varphi \in \mathcal{C}^1([0, 1])$.

We define $\forall w \in \mathbb{R}^m, z \in \text{Im}(\varphi) \subset \mathbb{R}, c > 0$, the minima problems

$$\psi_T^{z,c}(w) = \frac{1}{T} \min \left\{ \int_0^T |u'|^2 : \begin{array}{l} |u(0)| \leq \sqrt{m}, |u(T) - Tw| \leq \sqrt{m}, \\ |\varphi(u) - z| \leq c \end{array} \right\}. \quad (5.2)$$

$$\psi_T^z(w) = \frac{1}{T} \min \left\{ \int_0^T |u'|^2 : \begin{array}{l} |u(0)| \leq \sqrt{m}, |u(T) - Tw| \leq \sqrt{m}, \\ \varphi(u) = z \end{array} \right\}. \quad (5.3)$$

Remark 5.0.1. *Boundary conditions in definitions (5.2) and (5.3) cannot be taken as $u(0) = 0$ and $u(T) = Tw$, because the domain of $\psi_T^{z,c}$ or ψ_T^z will reduce to vectors $w \in \mathbb{R}^m$ satisfying the constraint $\varphi(Tw) = z$ or $|\varphi(Tw) - z| \leq c$, and the definition makes sense only if $0 \in \{\varphi = z\}$, for some $z \in \mathbb{R}$, for ψ_T^z , or if the distance of 0 and Tw from the set $\{\varphi = z\}$ is less than c , for $\psi_T^{z,c}$. Such a condition will reduce too much the domain of $\psi_T^{z,c}$ and ψ_T^z , so that it has to be relaxed, as in definitions (5.2) and (5.3), asking the boundary values of u to be near to the origin and the point Tw .*

We finally define

$$\psi_{\text{hom}}^z(w) = \lim_{T \rightarrow \infty} \psi_T^z(w), \quad (5.4)$$

and

$$\psi_{\text{hom}}(w) = \left(\min_{z \in \text{Im}(\varphi)} \psi_{\text{hom}}^z \right)^{**} (w). \quad (5.5)$$

Before showing our main result, we state some preliminary lemmata and properties that we will use in the proof. Let us start with two geometric hypotheses for the function φ or, equivalently, for the functional $\psi_T^{z,c}$, and for its level sets:

Hypothesis 5.0.1. *For any constraint function $\varphi \in C^1$, we will assume that there exists a continuous function $\omega(c)$, with $\omega(c) \rightarrow 0$ as $c \rightarrow 0$, such that, for any $T > 0$, $w \in \mathbb{R}^m$, $z \in \text{Im}(\varphi)$, one of the two following conditions is satisfied:*

1.

$$\psi_T^{z,c}(w) = +\infty; \quad (5.6)$$

2.

$$\psi_T^{z,c}(w) \geq (1 - \omega(c)) \psi_T^{z'}(w') - \frac{k(c)}{T}, \quad (5.7)$$

for suitable $z' \in \text{Im}(\varphi)$, $w' \in \mathbb{R}^m$ and $k(c) \in \mathbb{R}$ such that $|z - z'| \leq c$, $\|w - w'\| \leq \sqrt{m}/T$ and $k(c)$ is independent of z .

Hypothesis 5.0.2. *Let us fix $z \in \text{Im}(\varphi)$ and $w \in \mathbb{R}^m$ such that $\psi_T^z(w) < +\infty$, for any $T > 0$. Hence, for any $x, y \in \{\varphi = z\}$, there exist a constant $C \in \mathbb{R}$ and a path $\gamma : [0, 1] \rightarrow \mathbb{R}^m$, with $\gamma(0) = x$, $\gamma(1) = y$ and $\varphi(\gamma(t)) = z$, such that*

$$l(\gamma) = \int_0^1 \sqrt{1 + |\gamma'(t)|^2} dt \leq C \|x - y\|.$$

Remark 5.0.2. Note that we can prove hypothesis 5.0.2 assuming that if $\{\varphi = z\}$ has a connected unbounded component, then it is the union of C^1 sets.

5.1 Proof of the Homogenization result

Using the geometric hypotheses stated above, we can prove the existence of the homogenization formula:

Lemma 5.1.1. Let φ satisfy 5.0.1, 5.0.2 and the following hypothesis:

H) for any $z \in \text{Im}(\varphi)$ one of the following two conditions holds true for the set $\{\varphi = z\}$:

- i) it is made by a unique connected component unbounded, so that it "connects" \mathbb{R}^m .
- ii) it is made by infinitely many bounded connected components.

Then, for any $w \in \mathbb{R}^m$, for any $z \in \text{Im}(\varphi)$, the limit

$$\lim_{T \rightarrow +\infty} \min \frac{1}{T} \left\{ \int_0^T |u'(t)|^2 dt : |u(0)| \leq \sqrt{m}, |u(T) - Tw| \leq \sqrt{m}, \right. \\ \left. \varphi(u) = z \right\}$$

exists.

Proof. Let us fix two constants $S \gg T > 0$, a vector $w \in \mathbb{R}^m$ and $z \in \text{Im}(\varphi)$, such that $\psi_T^z(w) < +\infty$. Then there exists a minimizer for $\psi_T^z(w)$, that is $v_T : [0, T] \rightarrow \mathbb{R}^m$, with $|v_T(0)| \leq \sqrt{m}$, $|v_T(T) - Tw| \leq \sqrt{m}$ and $\varphi(v_T(t)) = z$. We want to construct a competitor for $\psi_S^z(w)$ by a patchwork procedure, using v_T . In what follows we consider $[Tw]$ as the integer part component-wise of $Tw \in \mathbb{R}^m$, and $[Tw] + 1 = ([Tw_1] + 1, \dots, [Tw_m] + 1)$.

Hence let consider two curves

$$\gamma : [0, 1] \rightarrow \mathbb{R}^m, \quad \gamma(0) = v_T(T), \quad \gamma(1) = v_T(0) + [Tw] + 1, \\ \varphi(\gamma(t)) = z$$

and

$$\gamma_1 : [0, 1] \rightarrow \mathbb{R}^m, \quad \gamma_1(0) = v_T(T) + \left[\frac{S}{[T+1]} - 1 \right] ([Tw] + 1), \\ \gamma_1(1) = v_T(0) + \left[\frac{S}{[T+1]} \right] ([Tw] + 1), \quad \varphi(\gamma_1(t)) = z.$$

Note that, by hypothesis 5.0.2, such curves exist and, moreover, we have

$$\begin{aligned} l(\gamma) &\leq \tilde{C}|v_T(0) + [Tw] + 1 - v_T(T)| \leq C, \\ l(\gamma_1) &\leq \tilde{C}_1 \left| v_T(T) + \left[\frac{S}{[T+1]} - 1 \right] ([Tw] + 1) - v_T(0) + \right. \\ &\quad \left. - \left[\frac{S}{[T+1]} \right] ([Tw] + 1) \right| \leq C_1, \end{aligned}$$

with C and C_1 depending on the dimension m . We can also assume that γ and γ_1 have constant velocity. Now define the function

$$\begin{aligned} \tilde{v}_T(t) &: \left[0, \left[\frac{S}{[T+1]} \right] (T+1) + 1 \right] \rightarrow \mathbb{R}^m \\ \tilde{v}_T(t) &= \begin{cases} v_T(t - k(T+1)) + k([Tw] + 1) & k(T+1) \leq t \leq k(T+1) + T, \\ \gamma(t - k(T+1) - T) + k([Tw] + 1) & k(T+1) + T \leq t \\ & \leq (k+1)(T+1), \\ \gamma_1(t - \left[\frac{S}{[T+1]} \right] (T+1)) & \left[\frac{S}{[T+1]} \right] (T+1) \leq t \\ & \leq \left[\frac{S}{[T+1]} \right] (T+1) + 1, \end{cases} \end{aligned} \tag{5.8}$$

for $k = 0, \dots, \left[\frac{S}{[T+1]} - 1 \right]$. We rescale \tilde{v}_T on the interval $[0, S]$, getting $v_{T,S} : [0, S] \rightarrow \mathbb{R}^m$, with

$$v_{T,S}(t) = \tilde{v}_T \left(\frac{1}{S} \left(\left[\frac{S}{[T+1]} \right] (T+1) + 1 \right) t \right).$$

Observe that one has, for any $T > 0$ fixed,

$$K(S, T) = \frac{1}{S} \left(\left[\frac{S}{[T+1]} \right] (T+1) + 1 \right) \xrightarrow{S \rightarrow \infty} 1.$$

By construction we have $|v_{T,S}(0)| = |v_T(0)| \leq \sqrt{m}$, $|v_{T,S}(S) - Sw| = |v_T(0) + \left[\frac{S}{[T+1]} \right] ([Tw] + 1) - Sw| \leq \sqrt{m}$, and $\varphi(v_{T,S}(t)) = z$, by the periodicity of φ , therefore, using change of variable $s = tK(S, T)$,

$$\begin{aligned} \psi_S^z(w) &\leq \frac{1}{S} \int_0^S |v'_{S,T}(t)|^2 dt = \frac{K(S, T)}{S} \int_0^{SK(S, T)} |\tilde{v}'(s)|^2 ds = \\ &= \frac{K(S, T)}{S} \left[\frac{S}{[T+1]} \right] \left(\int_0^T |v'_T(t)|^2 dt + \int_0^1 |\gamma'(t)|^2 dt \right) + \\ &+ \frac{K(S, T)}{S} \int_0^1 |\gamma'_1(t)|^2 dt \leq \frac{K(S, T)}{S} \frac{S}{T} (T\psi_T^z(w) + (l(\gamma))^2) + \end{aligned}$$

$$+\frac{K(S,T)}{S}(l(\gamma_1))^2 \leq K(S,T) \left(\psi_T^z(w) + \frac{C^2}{T} + \frac{C_1^2}{S} \right).$$

Now, taking \limsup as $S \rightarrow \infty$ and after \liminf as $T \rightarrow \infty$, we get

$$\limsup_{S \rightarrow \infty} \psi_S^z(w) \leq \liminf_{T \rightarrow \infty} \psi_T^z(w),$$

so that the limit exists. \square

For the function $\psi_{\text{hom}}(w)$ the following property holds:

Property 5.1.1. ψ_{hom} is a 2-homogeneous function, i.e. for any $\lambda \in \mathbb{R}$ and $w \in \mathbb{R}^2$ one has

$$\psi_{\text{hom}}(\lambda w) = \lambda^2 \psi_{\text{hom}}(w)$$

Proof. In order to simplify the notation we only consider the case $\lambda > 0$: the general case holds true, but it needs some different notation for the proof. Consider $z \in \text{Im}(\varphi)$, $w \in \mathbb{R}^m$, such that $\psi_{T/\lambda}(w) < +\infty$; let u be as a solution of the minimum problem defined by $\psi_{T/\lambda}(\lambda w)$; we have $|u(0)| \leq \sqrt{m}$, $|u(T/\lambda) - Tw| \leq \sqrt{m}$ and

$$\psi_{T/\lambda}^z(\lambda w) = \frac{\lambda}{T} \int_0^{T/\lambda} |u'(t)|^2 dt = \frac{\lambda}{T} \int_0^T \left| u' \left(\frac{s}{\lambda} \right) \right|^2 \frac{ds}{\lambda};$$

hence, taking $v(s) = u(s/\lambda)$, one has $|v(0)| = |u(0)| \leq \sqrt{m}$, $|v(T) - Tw| = |u(T/\lambda) - Tw| \leq \sqrt{m}$, and

$$\psi_{T/\lambda}^z(\lambda w) = \lambda^2 \frac{1}{T} \int_0^T |v'(s)|^2 ds = \lambda^2 \psi_T^z(w).$$

Taking the minimum over $z \in \text{Im}(\varphi)$ and then the convex envelope, and considering the limit as $T \rightarrow \infty$, we get the thesis. \square

Remark 5.1.1. We will also use the fact that ψ_{hom} is local Lipschitz in its domain: in fact observe that, if $\psi_{\text{hom}}(w) < +\infty$, i.e. $w \in \text{dom}\psi$, there exist $T_0 > 0$ and $\eta > 0$ such that, for any $T > T_0$, we can find a curve $u : [0, T] \rightarrow \mathbb{R}^m$ and a level $z \in \text{Im}(\varphi)$, such that $|u(0)| < \sqrt{m}$, $|u(T) - Tw| < \sqrt{m}$, $\varphi(u) = z$, and $\psi_{\text{hom}}(w) \leq 1/T \int_0^T |u'|^2 dt + \eta$. Now u is a curve in the level set $\{\varphi = z\}$, joining two points at distance $|u(T) - u(0)| \leq |Tw| + 2\sqrt{m}$, hence, by hypothesis 5.0.2, there exists a constant C such that

$$(l(u))^2 \leq C(|Tw| + 2\sqrt{m})^2.$$

Therefore, taking u with constant velocity, one has

$$\begin{aligned} \psi_{\text{hom}}(w) &\leq \psi_{\text{hom}}^z(w) \leq \eta + \psi_T^z(w) \leq \\ &\leq \eta + \frac{1}{T} \int_0^T |u'|^2 dt \leq \eta + \left(\frac{l(u)}{T} \right)^2 \leq \eta + C^2 \left(\frac{|Tw| + 2\sqrt{m}}{T} \right)^2, \end{aligned}$$

so that, taking the limit as $T \rightarrow +\infty$ and $\eta \rightarrow 0$, we get

$$\psi_{hom}(w) \leq C^2 |w|^2.$$

Hence we can apply lemma 1.1.3, getting that there exists $c_{hom}(\sigma) \in \mathbb{R}$ such that

$$|\psi_{hom}(\xi_1) - \psi_{hom}(\xi_2)| \leq c_{hom}(\sigma)(1 + |\xi_1| + |\xi_2|)|\xi_1 - \xi_2|, \quad (5.9)$$

for any $\xi_1, \xi_2 \in \mathbb{R}^m$ such that $\psi_{hom}(\xi_i) < +\infty$ and $\text{dist}(\xi_i, \partial \text{dom}\psi) > \sigma$.

The last preliminary result is related to the $L^2([0, 1])$ and $H^1([0, 1])$ approximation by piecewise affine functions:

Lemma 5.1.2. *Let $\{u_\varepsilon\}_{\varepsilon>0} \subseteq H^1([0, 1])$ be a sequence converging to $u \in H^1([0, 1])$ in the strong topology of $L^2([0, 1])$, as $\varepsilon \rightarrow 0$. Suppose that $\{u_\varepsilon\}_{\varepsilon>0}$ is equibounded in $H^1([0, 1])$, that is $\|u_\varepsilon\|_{H^1} < \lambda$, for any $\varepsilon > 0$. Consider $M, J^M \in \mathbb{N}$ and a partition $\{t_i\}_{i=0, \dots, J^M}$ of the interval $[0, 1]$, with $|t_i - t_{i-1}| < 1/M$.*

Let $v_{\varepsilon, M}$ be the piecewise affine function defined on index $i = 1, \dots, J^M$, satisfying

$$v_{\varepsilon, M}(t_i) = u_\varepsilon(t_i), \quad v'_{\varepsilon, M}(t)|_{[t_{i-1}, t_i]} = \frac{u_\varepsilon(t_i) - u_\varepsilon(t_{i-1})}{t_i - t_{i-1}}, \quad i = 1, \dots, J^M.$$

Then we have

i) *there exists a function $v_M \in H^1([0, 1])$, depending on the parameter M , such that*

$$v_{\varepsilon, M} \xrightarrow{\varepsilon \rightarrow 0} v_M \quad (5.10)$$

strongly in $L^2([0, 1])$ and weakly in $H^1([0, 1])$;

ii)

$$v_M \xrightarrow{M \rightarrow \infty} u \quad (5.11)$$

strongly in $L^2([0, 1])$ and weakly in $H^1([0, 1])$.

Proof. Observe that, by construction, we have

$$\int_0^1 |v_{\varepsilon, M} - u_\varepsilon|^2 dt = \sum_{i \notin I_\varepsilon} \int_{t_{i-1}}^{t_i} |v_{\varepsilon, M} - u_\varepsilon|^2 + \sum_{i \in I_\varepsilon} \int_{t_{i-1}}^{t_i} |v_{\varepsilon, M} - u_\varepsilon|^2.$$

The function $v_{\varepsilon, M} - u_\varepsilon$ vanishes in t_i , for $i = 1, \dots, J^M$, then we can use Poincaré inequality, getting

$$\sum_{i=1}^{J^M} \int_{t_{i-1}}^{t_i} |v_{\varepsilon, M} - u_\varepsilon|^2 dt \leq \sum_{i=1}^{J^M} c |\Delta t_i|^2 \int_{t_{i-1}}^{t_i} |v'_{\varepsilon, M} - u'_\varepsilon|^2 dt \leq$$

$$\leq \sum_{i=1}^{J^M} \frac{2c}{M^2} \left(\int_{t_{i-1}}^{t_i} |v'_{\varepsilon, M}|^2 + \int_{t_{i-1}}^{t_i} |u'_\varepsilon|^2 \right) = \frac{2c}{M^2} \left(\int_0^1 |v'_{\varepsilon, M}|^2 + \int_0^1 |u'_\varepsilon|^2 \right).$$

At the same time, being $u_\varepsilon \in H^1([0, 1])$ absolutely continuous, we have

$$u_\varepsilon(t_i) - u_\varepsilon(t_{i-1}) = \int_{t_{i-1}}^{t_i} u'_\varepsilon dt \Rightarrow |u_\varepsilon(t_i) - u_\varepsilon(t_{i-1})|^2 = \left(\int_{t_{i-1}}^{t_i} u'_\varepsilon dt \right)^2,$$

then, using Cauchy-Schwarz inequality,

$$|u_\varepsilon(t_i) - u_\varepsilon(t_{i-1})|^2 \leq \int_{t_{i-1}}^{t_i} 1 dt \int_{t_{i-1}}^{t_i} |u'_\varepsilon|^2 dt = \Delta t_i \int_{t_{i-1}}^{t_i} |u'_\varepsilon|^2 dt.$$

Hence

$$\int_0^1 |v'_{\varepsilon, M}|^2 dt = \sum_{i=1}^{J^M} \frac{|u_\varepsilon(t_i) - u_\varepsilon(t_{i-1})|^2}{\Delta t_i} \leq \int_0^1 |u'_\varepsilon|^2 dt. \quad (5.12)$$

Therefore one has

$$\int_0^1 |v_{\varepsilon, M} - u_\varepsilon|^2 dt \leq \frac{2c}{M^2} \left(\int_0^1 |v'_{\varepsilon, M}|^2 + \int_0^1 |u'_\varepsilon|^2 \right) \leq \frac{4c}{M^2} \lambda, \quad (5.13)$$

being $\|u_\varepsilon\|_{H^1} < \lambda$. Then we have

$$\|v_{\varepsilon, M}\|_{L^2} \leq \|v_{\varepsilon, M} - u_\varepsilon\|_{L^2} + \|u_\varepsilon\|_{L^2} \leq \frac{4c}{M^2} \lambda + \lambda,$$

and, by equation (5.12), and by hypothesis of equiboundedness of $\{u_\varepsilon\}_\varepsilon$,

$$\|v_{\varepsilon, M}\|_{H^1} < \lambda, \quad \forall \varepsilon > 0.$$

Hence there exists a function $v_M \in H^1([0, 1])$ such that $v_{\varepsilon, M} \rightharpoonup v_M$ weakly in $H^1([0, 1])$, and, by Rellich theorem, strongly in $L^2([0, 1])$, as ε tends to zero. Part *i*) is then proved.

Consider now *ii*): we have, by hypothesis, part *i*) and equation (5.13)

$$\int_0^1 |v_M - u|^2 = \lim_{\varepsilon \rightarrow 0} \int_0^1 |v_{\varepsilon, M} - u_\varepsilon|^2 \leq \frac{4c}{M^2} \lambda.$$

Hence, taking the limit as $M \rightarrow +\infty$,

$$\lim_{M \rightarrow +\infty} \int_0^1 |v_M - u|^2 \leq \lim_{M \rightarrow +\infty} \frac{4c}{M^2} \lambda = 0,$$

so that $v_M \rightarrow u$ strongly in L^2 . For the weak convergence consider $\varphi \in H^1([0, 1])$; one has, using part *i*), hypothesis and Cauchy-Schwarz inequality,

$$\begin{aligned} \int_0^1 |v'_M - u'| \varphi' &= \lim_{\varepsilon \rightarrow 0} \int_0^1 |v'_{\varepsilon, M} - u'_\varepsilon| \varphi' \leq \\ &\leq \lim_{\varepsilon \rightarrow 0} \|\varphi\|_{H^1} \int_0^1 |v'_{\varepsilon, M} - u'_\varepsilon|^2 \leq \|\varphi\|_{H^1} \frac{4c}{M^2} \lambda. \end{aligned}$$

Therefore, taking the limit as $M \rightarrow +\infty$, we get *ii*). \square

Remark 5.1.2. *Lemma 5.1.2 holds true inverting the role of ε and M , such that the limit of piecewise affine functions $v_{\varepsilon, M}$ does not depend on the order of $\varepsilon \rightarrow 0$ and $M \rightarrow \infty$. We state the result in this way in order to use it in the proof of Γ -liminf in theorem 5.1.1, where the order of the limits is exactly $\varepsilon \rightarrow 0$ before $M \rightarrow \infty$.*

Now we can finally state our main result:

Theorem 5.1.1. *Let F_ε be defined in (5.1), such that the constraint φ is $C^1(\mathbb{R}^m)$ and it satisfies hypothesis H) of lemma 5.1.1; let hypothesis 5.0.1 be satisfied; then, for any $u \in L^2([0, 1]; \mathbb{R}^m)$ one has*

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u) = F(u) = \begin{cases} \int_0^1 \psi_{hom}(u') dt & u \in H^1([0, 1]) \\ +\infty & \text{otherwise,} \end{cases}$$

in the strong topology of $L^2([0, 1])$.

Proof.

Γ -lim inf

We want to prove that, for any sequence u_ε converging in the strong topology of $L^2([0, 1])$ to a function $u \in L^2([0, 1])$, as $\varepsilon \rightarrow 0$, one has

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq F(u). \quad (5.14)$$

First of all observe that we can assume, without loss of generality, that

$$F_\varepsilon(u_\varepsilon) \leq \lambda < +\infty \quad \forall \varepsilon > 0, \quad (5.15)$$

otherwise the Γ -lim inf inequality (5.14) is trivial. By the equiboundedness of F_ε , we also deduce that $\int_0^1 |u'_\varepsilon|^2 dt < \lambda$, i.e. $\|u_\varepsilon\|_{H^1} < \lambda$, for any $\varepsilon > 0$; hence, up to subsequence, $u_\varepsilon \rightharpoonup u$ weakly in $H^1([0, 1])$ too.

Let us take four positive constants ε , N , M , K ; by hypothesis on the function φ , there exist two constant $a, b \in \mathbb{R}$, such that $Im(\varphi) \subseteq [a, b]$: so we can divide the image of φ in $N + 1$ values $\{a = z_0, z_1, \dots, z_N = b\}$, with

$$|z_j - z_{j-1}| = \frac{1}{N}, \quad \forall j = 1, \dots, N.$$

Now we want to split the domain of u_ε too. Consider the following set

$$\tau_\varepsilon^N = \{t \in [0, 1] : \varphi_\varepsilon(u_\varepsilon(t)) \in \{z_0, z_1, \dots, z_N\}\}$$

and set

$$t_0 = 0, \\ t_i = \min \{t \in \tau_\varepsilon^N : t > t_{i-1}, \varphi_\varepsilon(u_\varepsilon(t)) \neq \varphi_\varepsilon(u_\varepsilon(t_{i-1}))\}. \quad (5.16)$$

Remark 5.1.3. *Observe that, by definition, t_i actually depends on ε and N ; we simply name them t_i just to simplify the notation.*

Note that the set τ_ε^N is compact, being an inverse image of a compact set, by the continuous function φ , so that the existence of the first instant t_1 is guaranteed; at every step we evaluate the minimum always of a compact set that is $\tau_\varepsilon^{N,i} = \{t \in [0, 1] : \varphi_\varepsilon(u_\varepsilon(t)) \in \{z_0, z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_N\}\}$, for some $i \in \mathbb{N}$. Therefore the set of t_i , $i \in \mathbb{N}$, is well defined.

With this division of the domain of u_ε we want to highlight the instants t_i where $\varphi_\varepsilon(u_\varepsilon)$ reaches the levels z_0, z_1, \dots, z_N , so that, for two consecutive t_i , there exists $j = 0, \dots, N$ such that

$$|\varphi_\varepsilon(u_\varepsilon(t_i)) - \varphi_\varepsilon(u_\varepsilon(t_{i-1}))| = |z_j - z_{j\pm 1}| = \frac{1}{N}.$$

The procedure described to find every t_i ends when we reach the point $t_i = 1$; we cannot tell a priori that the number of t_i is finite, for ε and N fixed, so we present the following

Lemma 5.1.3. *For any $N > 0$, there exists ε_0 such that $\forall \varepsilon < \varepsilon_0$, the number of $\{t_i\}_i$ defined in (5.16) is finite, i.e.*

$$J_\varepsilon^N = \#\{t_i\}_i < +\infty$$

Proof. By equation (5.15), for any $i = 0, \dots, J_\varepsilon^N$, using Jensen's inequality, one has that

$$\begin{aligned} \lambda > F_\varepsilon(u_\varepsilon) &\geq \delta^2 \int_{t_{i-1}}^{t_i} |\varphi_\varepsilon(u_\varepsilon)'|^2 dt \geq \frac{\delta^2}{t_i - t_{i-1}} (\varphi_\varepsilon(u_\varepsilon(t_i)) - \varphi_\varepsilon(u_\varepsilon(t_{i-1})))^2 = \\ &= \frac{\delta^2}{t_i - t_{i-1}} \frac{1}{N^2}. \end{aligned}$$

Define $\Delta t_i = t_i - t_{i-1}$ and note that, by our hypothesis, one has $\delta \rightarrow 0$ and $\delta/\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, so that there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon < \varepsilon_0$ one has $\delta \geq \varepsilon$. Hence we get

$$\lambda > \frac{\varepsilon^2}{\Delta t_i} \frac{1}{N^2}.$$

Then, for any $i = 0, \dots, J_\varepsilon^N$, the minimum distance between two consecutive instants t_i is a fixed constant:

$$\Delta t_i \geq \frac{\varepsilon^2}{\lambda N^2},$$

and, being $t_i \in [0, 1]$ bounded, we must have

$$J_\varepsilon^N \leq \frac{\lambda N^2}{\varepsilon^2} < +\infty.$$

□

Thanks to this lemma we now have that $\{t_i\}_{i \leq J_\varepsilon^N}$ is a partition of $[0, 1]$.

We can show another property that we will use in the sequel:

Lemma 5.1.4. *For any $K > 0$, there exists a constant $\varepsilon_1 > 0$ such that, for $\varepsilon < \varepsilon_1$ we have*

$$\Delta t_i \geq K\varepsilon^2, \quad i = 1, \dots, J_\varepsilon^N. \quad (5.17)$$

Proof. Let us suppose the contrary: there exists a constant $K > 0$ such that for any ε_1 we can find $\varepsilon < \varepsilon_1$ such that $\Delta t_i < K\varepsilon^2$ for some $i = 1, \dots, J_\varepsilon^N$. Then, let i be an index such that $\Delta t_i < K\varepsilon^2$; by (5.15) and Jensen's inequality,

$$\lambda > \frac{\delta^2}{\Delta t_i} \frac{1}{N^2} \geq \frac{\delta^2}{\varepsilon^2} \frac{1}{KN^2}.$$

This is absurd because, for $\varepsilon \rightarrow 0$, the last term tends to infinity. □

Lemma 5.1.4 gives a lower bound for the interval of the partition t_i , $i = 0, \dots, J_\varepsilon^N$; but, on the other side, this partition should not be thin enough to our claim: if we consider u_ε such that $\varphi_\varepsilon(u_\varepsilon(t)) = z_j$ for some $j = 0, \dots, N$ and for any $t \in [0, 1]$, then $J_\varepsilon^N = 1$ and we will have $\Delta t_1 = 1$. To avoid such a situation we introduce a new partition \bar{t}_i , for $i = 1, \dots, \bar{J}_\varepsilon^{N,M}$, actually depending on M too, such that

$$\bar{\Delta} t_i = \bar{t}_i - \bar{t}_{i-1} < \frac{1}{M}.$$

This partition is obtained from t_i , possibly adding other instants, to satisfy the condition $\bar{\Delta} t_i < 1/M$. For example, if we have $\varphi_\varepsilon(u_\varepsilon(t_{i-1})) = z_j$ and $\varphi_\varepsilon(u_\varepsilon(t_i)) = z_{j\pm 1}$, for some i, j , but $t_i - t_{i-1} > 1/M$, then we will add enough instants in $[t_{i-1}, t_i]$ in order to get $\bar{\Delta} t_i < 1/M$. Note that $\{\bar{t}_i\}_i$ is still made by a finite number of instants, by lemma 5.1.3 indeed there exists ε_0 such that, for any $\varepsilon < \varepsilon_0$, we have

$$\bar{J}_\varepsilon^{N,M} \leq \frac{\lambda N^2}{\varepsilon^2} M.$$

With this new partition we lost the information about $\varphi_\varepsilon(u_\varepsilon(\bar{t}_i))$, but we still know, by the continuity of φ and, more precisely, by the medium values theorem, that

$$|\varphi_\varepsilon(u_\varepsilon(\bar{t}_{i-1})) - \varphi_\varepsilon(u_\varepsilon(\bar{t}_i))| < \frac{2}{N}; \quad (5.18)$$

and, repeating the proof of lemma 5.1.4 to $\{\bar{t}_i\}_i$, we have also

$$K\varepsilon^2 < \overline{\Delta t}_i < \frac{1}{M}, \quad \forall \varepsilon < \varepsilon_1. \quad (5.19)$$

Now we want to define a type of intervals that we will not use in the computation of $F_\varepsilon(u_\varepsilon)$: consider the set of indices

$$I_\varepsilon^K = \{i \leq J_\varepsilon^{N,M} : |\bar{t}_i - \bar{t}_{i-1}| < K\varepsilon\}$$

and the respective set of intervals $B_\varepsilon^K = \cup_{i \in I_\varepsilon^K} [\bar{t}_{i-1}, \bar{t}_i] \subset [0, 1]$. Observe that, by boundedness of $F_\varepsilon(u_\varepsilon)$, one has

$$\lambda \geq \sum_{i \in I_\varepsilon^K} \int_{\bar{t}_{i-1}}^{\bar{t}_i} |\varphi'|^2 dt \geq \sum_{i \in I_\varepsilon^K} \frac{\delta^2}{N^2 |\bar{t}_i - \bar{t}_{i-1}|} \geq |I_\varepsilon^K| \frac{\delta^2}{N^2 K \varepsilon},$$

so that

$$|I_\varepsilon^K| \leq \frac{\lambda N^2 K \varepsilon}{\delta^2} \Rightarrow |B_\varepsilon^K| \leq \left(\frac{\varepsilon}{\delta}\right)^2 \lambda N^2 K^2 \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (5.20)$$

Hence, for the set $(0, 1) \setminus B_\varepsilon^K$ we have

$$K\varepsilon \leq \overline{\Delta t}_i < \frac{1}{M}, \quad \forall i \notin I_\varepsilon^K. \quad (5.21)$$

Now we can evaluate F_ε on u_ε using our partition:

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &= \int_0^1 |u'_\varepsilon|^2 dt + \delta^2 \int_0^1 |\varphi_\varepsilon(u_\varepsilon)'|^2 dt \geq \int_0^1 |u'_\varepsilon|^2 dt = \sum_{i=1}^{J_\varepsilon^{N,M}} \int_{\bar{t}_{i-1}}^{\bar{t}_i} |u'_\varepsilon|^2 dt = \\ &= \sum_{i \notin I_\varepsilon^K} \int_{\bar{t}_{i-1}}^{\bar{t}_i} |u'_\varepsilon|^2 dt + \sum_{i \in I_\varepsilon^K} \int_{\bar{t}_{i-1}}^{\bar{t}_i} |u'_\varepsilon|^2 dt \geq \sum_{i \notin I_\varepsilon^K} \int_{\bar{t}_{i-1}}^{\bar{t}_i} |u'_\varepsilon|^2 dt. \end{aligned}$$

Then we use the change of variable $s = (t - \bar{t}_{i-1})/\varepsilon$, naming $v_\varepsilon(s) = (u_\varepsilon(\varepsilon s + \bar{t}_{i-1}))/\varepsilon - [u_\varepsilon(\bar{t}_{i-1})/\varepsilon]$, so that $v'_\varepsilon(s) = u'_\varepsilon(\varepsilon s + \bar{t}_{i-1})$, getting

$$F_\varepsilon(u_\varepsilon) \geq \sum_{i \notin I_\varepsilon^K} \varepsilon \int_0^{\frac{\overline{\Delta t}_i}{\varepsilon}} |v'_\varepsilon(s)|^2 ds = \sum_{i \notin I_\varepsilon^K} \frac{\varepsilon}{\overline{\Delta t}_i} \overline{\Delta t}_i \int_0^{\frac{\overline{\Delta t}_i}{\varepsilon}} |v'_\varepsilon(s)|^2 ds.$$

Now let us call

$$T_\varepsilon^i = \frac{\overline{\Delta t}_i}{\varepsilon}, \quad w_\varepsilon^i = \frac{u_\varepsilon(\bar{t}_i) - u_\varepsilon(\bar{t}_{i-1})}{\overline{\Delta t}_i}.$$

Hence we have

$$|v_\varepsilon(0)| = \left| \frac{u_\varepsilon(\bar{t}_{i-1})}{\varepsilon} - \left[\frac{u_\varepsilon(\bar{t}_{i-1})}{\varepsilon} \right] \right| \leq \sqrt{m},$$

$$|v_\varepsilon(T_\varepsilon^i) - T_\varepsilon^i w_\varepsilon^i| = \left| \frac{u_\varepsilon(\bar{t}_{i-1})}{\varepsilon} - \left[\frac{u_\varepsilon(\bar{t}_{i-1})}{\varepsilon} \right] \right| \leq \sqrt{m},$$

and, by (5.18) and the periodicity of φ , for a suitable $i = 0, \dots, N$,

$$\begin{aligned} |\varphi(v_\varepsilon(s)) - z_i| &= \left| \varphi \left(\frac{u_\varepsilon(s\varepsilon + \bar{t}_{i-1})}{\varepsilon} - \left[\frac{u_\varepsilon(\bar{t}_{i-1})}{\varepsilon} \right] \right) - z_i \right| = \\ &= |\varphi_\varepsilon(u_\varepsilon(s\varepsilon + \bar{t}_{i-1})) - z_i| \leq \frac{2}{N}, \quad \forall s \in [0, T_\varepsilon^i]. \end{aligned}$$

Then, using definition (5.2), one has

$$F_\varepsilon(u_\varepsilon) \geq \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i \psi_{T_\varepsilon^i}^{z_i, \frac{2}{N}}(w_\varepsilon^i)$$

Now we want to use hypothesis 5.0.1, to replace $\psi_{T_\varepsilon^i}^{z_i, \frac{2}{N}}$, with $\psi_{T_\varepsilon^i}^{z'_i}$ where we have the stronger constraint $\varphi(v_\varepsilon) = z'_i$. We have that, assuming $\psi_{T_\varepsilon^i}^{z_i, \frac{2}{N}}(w_\varepsilon^i) < \infty$, for any $i \notin I_\varepsilon^K$ and a fixed $N > 0$, there exists z'_i and \bar{w}_ε^i such that

$$\psi_{T_\varepsilon^i}^{z_i, \frac{2}{N}}(w_\varepsilon^i) \geq \left(1 - \omega \left(\frac{2}{N} \right) \right) \psi_{T_\varepsilon^i}^{z'_i}(\bar{w}_\varepsilon^i) - \frac{k(N)}{T_\varepsilon^i}$$

and

$$|w_\varepsilon^i - \bar{w}_\varepsilon^i| \leq \frac{\sqrt{m}}{T_\varepsilon^i}.$$

Hence we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\geq \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i \psi_{T_\varepsilon^i}^{z_i, \frac{2}{N}}(w_\varepsilon^i) \geq \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i \left(\left(1 - \omega \left(\frac{2}{N} \right) \right) \psi_{T_\varepsilon^i}^{z'_i}(\bar{w}_\varepsilon^i) - \frac{k(N)}{T_\varepsilon^i} \right) \\ &\geq \left(1 - \omega \left(\frac{2}{N} \right) \right) \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i \psi_{T_\varepsilon^i}^{z'_i}(\bar{w}_\varepsilon^i) - \frac{k(N)}{T_\varepsilon^i} \end{aligned}$$

and

$$|w_\varepsilon^i - \bar{w}_\varepsilon^i| \leq \frac{\sqrt{m}}{T_\varepsilon^i}, \quad i \notin I_\varepsilon^K. \quad (5.22)$$

By lemma 5.1.1, we can state that, for any $\alpha > 0$, there exists $T_0 > 0$ and $K > 0$, such that, by equation (5.21), $T_\varepsilon^i = \frac{\Delta t_i}{\varepsilon} > K > T_0$, being $i \notin I_\varepsilon^K$, and

$$\psi_{T_\varepsilon^i}^{z'_i}(\bar{w}_\varepsilon^i) > \psi_{\text{hom}}^{z'_i}(\bar{w}_\varepsilon^i) - \alpha.$$

Then, being ψ_{hom} the minimum over z of ψ_{hom}^z , we have that

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\geq \left(1 - \omega\left(\frac{2}{N}\right)\right) \sum_{i \notin I_N^K} \bar{\Delta}t_i \psi_{\text{hom}}^{z'_i}(\bar{w}_\varepsilon^i) - \alpha \left(1 - \omega\left(\frac{2}{N}\right)\right) - \frac{k(N)}{K} \geq \\ &\geq \left(1 - \omega\left(\frac{2}{N}\right)\right) \sum_{i \notin I_N^K} \bar{\Delta}t_i \psi_{\text{hom}}(\bar{w}_\varepsilon^i) - \alpha \left(1 - \omega\left(\frac{2}{N}\right)\right) - \frac{k(N)}{K}. \end{aligned}$$

By remark 5.1.1 we know that ψ_{hom} is a local lipschitz function: there exists a constant $c_{\text{hom}} > 0$ such that

$$\begin{aligned} \psi_{\text{hom}}(\bar{w}_\varepsilon^i) &= [\psi_{\text{hom}}(\bar{w}_\varepsilon^i) - \psi_{\text{hom}}(w_\varepsilon^i)] + \psi_{\text{hom}}(w_\varepsilon^i) \geq \\ &\geq -c_{\text{hom}}(1 + |\bar{w}_\varepsilon^i| + |w_\varepsilon^i|)|\bar{w}_\varepsilon^i - w_\varepsilon^i| + \psi_{\text{hom}}(w_\varepsilon^i). \end{aligned}$$

Applying this last inequality to F_ε we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\geq \left(1 - \omega\left(\frac{2}{N}\right)\right) \sum_{i \notin I_N^K} \bar{\Delta}t_i \psi_{\text{hom}}(w_\varepsilon^i) - \alpha \left(1 - \omega\left(\frac{2}{N}\right)\right) - \frac{k(N)}{K} + \\ &\quad - c_{\text{hom}} \left(1 - \omega\left(\frac{2}{N}\right)\right) \sum_{i \notin I_\varepsilon^K} \bar{\Delta}t_i (1 + |\bar{w}_\varepsilon^i| + |w_\varepsilon^i|) |\bar{w}_\varepsilon^i - w_\varepsilon^i|. \end{aligned}$$

For the second term, using the discrete version of Cauchy-Schwarz inequality, we have

$$\begin{aligned} &c_{\text{hom}} \left(1 - \omega\left(\frac{2}{N}\right)\right) \sum_{i \notin I_\varepsilon^K} (\bar{\Delta}t_i)^{\frac{1}{2}} (1 + |\bar{w}_\varepsilon^i| + |w_\varepsilon^i|) (\bar{\Delta}t_i)^{\frac{1}{2}} |\bar{w}_\varepsilon^i - w_\varepsilon^i| \leq \\ &\leq c_{\text{hom}} \left(1 - \omega\left(\frac{2}{N}\right)\right) \left(\sum_{i \notin I_\varepsilon^K} \bar{\Delta}t_i (1 + |\bar{w}_\varepsilon^i| + |w_\varepsilon^i|)^2 \right)^{\frac{1}{2}} \left(\sum_{i \notin I_\varepsilon^K} \bar{\Delta}t_i |\bar{w}_\varepsilon^i - w_\varepsilon^i|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Now, the second factor is bounded because, using equation (5.22), we have

$$\left(\sum_{i \notin I_\varepsilon^K} \bar{\Delta}t_i |\bar{w}_\varepsilon^i - w_\varepsilon^i|^2 \right)^{\frac{1}{2}} \leq \sqrt{m} \left(\sum_{i \notin I_\varepsilon^K} \bar{\Delta}t_i \frac{\varepsilon^2}{\Delta t_i} \right)^{\frac{1}{2}} \leq$$

$$\leq \sqrt{m} \left(\sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i \frac{1}{K^2} \right)^{\frac{1}{2}} \leq \frac{\sqrt{m}}{K}.$$

The first factor is bounded too, indeed

$$\begin{aligned} & \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i |w_\varepsilon^i|^2 \leq \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i \frac{1}{\overline{\Delta t}_i^2} (u_\varepsilon(\bar{t}_i) - u_\varepsilon(\bar{t}_{i-1}))^2 \leq \\ & \leq \sum_{i \notin I_\varepsilon^K} \frac{1}{\overline{\Delta t}_i} \left(\int_{\bar{t}_{i-1}}^{\bar{t}_i} u'_\varepsilon dt \right)^2 \leq \sum_{i \notin I_\varepsilon^K} \frac{1}{\overline{\Delta t}_i} \left(\int_{\bar{t}_{i-1}}^{\bar{t}_i} 1^2 dt \right) \left(\int_{\bar{t}_{i-1}}^{\bar{t}_i} |u'_\varepsilon|^2 dt \right) \leq \\ & \sum_{i \notin I_\varepsilon^K} \left(\int_{\bar{t}_{i-1}}^{\bar{t}_i} |u'_\varepsilon|^2 dt \right) \leq \int_0^1 |u'_\varepsilon|^2 dt < \lambda. \end{aligned}$$

Hence, being $\overline{\Delta t}_i > K\varepsilon$, for $i \notin I_\varepsilon^K$, and using inequality (5.22), one has

$$\begin{aligned} & \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i (1 + |\bar{w}_\varepsilon^i| + |w_\varepsilon^i|)^2 \leq \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i \left(C_1 + C_2 |w_\varepsilon^i|^2 + C_3 \frac{m}{(T_\varepsilon^i)^2} \right) \leq \\ & \leq C_1 + C_2 \lambda + C_3 m \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i \frac{\varepsilon^2}{K^2 \varepsilon^2} \leq C' + \frac{C''}{K^2}. \end{aligned}$$

Therefore we get

$$\begin{aligned} & c_{\text{hom}} \left(1 - \omega \left(\frac{2}{N} \right) \right) \sum_{i \notin I_\varepsilon^K} \overline{\Delta t}_i (1 + |\bar{w}_\varepsilon^i| + |w_\varepsilon^i|) |\bar{w}_\varepsilon^i - w_\varepsilon^i| \leq \\ & \leq c_{\text{hom}} \left(1 - \omega \left(\frac{2}{N} \right) \right) \left(C' + \frac{C''}{K^2} \right)^{\frac{1}{2}} \frac{\sqrt{m}}{K} = \left(1 - \omega \left(\frac{2}{N} \right) \right) \frac{c(K)}{K}, \end{aligned}$$

with $c(K) \rightarrow 0$ as $K \rightarrow +\infty$.

Define now, in the whole partition $\{\bar{t}_i\}_{i \leq J_\varepsilon^{N,M}}$, a sequence of piecewise affine functions $\{l_{\varepsilon,M,N}\}_{\varepsilon>0, M>0, N>0}$ approximating u_ε , with derivative equal w_ε^i , i.e. set, for any $i = 1, \dots, \bar{J}_\varepsilon^{N,M}$

$$l_{\varepsilon,M,N}(\bar{t}_i) = u_\varepsilon(\bar{t}_i), \quad l'_{\varepsilon,M,N}(t)|_{[\bar{t}_{i-1}, \bar{t}_i]} = w_\varepsilon^i = \frac{u_\varepsilon(\bar{t}_i) - u_\varepsilon(\bar{t}_{i-1})}{\overline{\Delta t}_i}.$$

Then, by lemma 5.1.2, we have

i) there exists $l_{M,N} \in H^1([0, 1])$ such that

$$l_{\varepsilon,M,N} \rightarrow l_{M,N}$$

strongly in $L^2([0, 1])$ and weakly in $H^1([0, 1])$, as $\varepsilon \rightarrow 0$.

ii)

$$l_{M,N} \rightarrow u$$

strongly in $L^2([0, 1])$ and weakly in $H^1([0, 1])$, as $M, N \rightarrow +\infty$.

Using $l_{\varepsilon, M, N}$, we want to construct a piecewise affine function, without using indices in I_ε^K : let consider the function

$$v_{\varepsilon, M, N, K} = \begin{cases} u_\varepsilon(\bar{t}_{i-1}) + w_\varepsilon^i(t - \bar{t}_{i-1}) & t \in [\bar{t}_{i-1}, \bar{t}_i], i \notin I_\varepsilon^K \\ u_\varepsilon(\bar{t}_{i-1}) & t \in [\bar{t}_{i-1}, \bar{t}_i], i \in I_\varepsilon^K. \end{cases} \quad (5.23)$$

Hence $v_{\varepsilon, M, N, K}$ has a countable quantity of discontinuities (jumps), in the points $t = \bar{t}_i$, for $i \in I_\varepsilon^K$. We can construct, by translation of $v_{\varepsilon, M, N, K}$ in these points, the continuous piecewise function $l_{\varepsilon, K, M, N} \in H^1([0, 1])$, that satisfies

$$l_{\varepsilon, K, M, N}(t)' = \begin{cases} w_\varepsilon^i & t \in [\bar{t}_{i-1}, \bar{t}_i], i \notin I_\varepsilon^K, \\ 0 & t \in [\bar{t}_{i-1}, \bar{t}_i], i \in I_\varepsilon^K. \end{cases} \quad (5.24)$$

For such a function one has

$$\int_0^1 |l_{\varepsilon, K, M, N}(t)'|^2 = \sum_{i \notin I_\varepsilon^K} \int_{\bar{t}_{i-1}}^{\bar{t}_i} |w_\varepsilon^i|^2 dt = \sum_{i \notin I_\varepsilon^K} \Delta \bar{t}_i |w_\varepsilon^i|^2 < \lambda.$$

Hence there exists $l_{K, M, N} \in H^1([0, 1])$ such that $l_{\varepsilon, K, M, N} \rightarrow l_{K, M, N}$ as $\varepsilon \rightarrow 0$, weakly in H^1 and strongly in L^2 . On the other side, by equation (5.20) we have

$$\|l_{M, N} - l_{K, M, N}\|_\infty \leq C \|l_{M, N} - l_{K, M, N}\|_{H^1} = \lim_{\varepsilon \rightarrow 0} C \|l_{\varepsilon, M, N} - l_{\varepsilon, K, M, N}\|_{H^1} =$$

$$\begin{aligned} &= \lim_{\varepsilon \rightarrow 0} C \int_{B_\varepsilon^K} |l'_{\varepsilon, M, N}|^2 \leq \lim_{\varepsilon \rightarrow 0} C \sqrt{\|l_{\varepsilon, M, N}\|_{H^1}} \sqrt{|B_\varepsilon^K|} \leq \\ &\leq \sqrt{\lambda} C \lim_{\varepsilon \rightarrow 0} \sqrt{\lambda} N K \left(\frac{\varepsilon}{\delta}\right) = 0, \end{aligned}$$

so that $l_{\varepsilon, K, M, N} \rightarrow l_{M, N}$ as $\varepsilon \rightarrow 0$, weakly in H^1 and strongly in L^2 .

Now observe that, being $l'_{\varepsilon, K, M, N} = 0$ for $t \in [\bar{t}_{i-1}, \bar{t}_i]$ and $i \in I_\varepsilon^K$, one has

$$\begin{aligned} \sum_{i \notin I_\varepsilon^K} \Delta \bar{t}_i \psi_{\text{hom}}(w_\varepsilon^i) &= \sum_{i \notin I_\varepsilon^K} \int_{\bar{t}_{i-1}}^{\bar{t}_i} \psi_{\text{hom}}(l'_{\varepsilon, K, M, N}) dt = \\ &= \int_{(0,1) \setminus B_\varepsilon^K} \psi_{\text{hom}}(l'_{\varepsilon, K, M, N}) dt = \int_0^1 \psi_{\text{hom}}(l'_{\varepsilon, K, M, N}) dt \end{aligned}$$

Therefore, using lower semicontinuity of ψ_{hom} , one has

$$\begin{aligned}
\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) &\geq \liminf_{\varepsilon \rightarrow 0} \left[\left(1 - \omega\left(\frac{2}{N}\right)\right) \int_0^1 \psi_{\text{hom}}(l'_{\varepsilon,K,M,N}) + \right. \\
&\quad \left. - \left(\alpha + \frac{c(K)}{K}\right) \left(1 - \omega\left(\frac{2}{N}\right)\right) - \frac{k(N)}{K} \right] \geq \\
&\geq \left(1 - \omega\left(\frac{2}{N}\right)\right) \int_0^1 \psi_{\text{hom}}(l'_{M,N}) + \\
&\quad - \left(\alpha + \frac{c(K)}{K}\right) \left(1 - \omega\left(\frac{2}{N}\right)\right) - \frac{k(N)}{K}.
\end{aligned}$$

Now we can take the limit as K tends to infinity, getting

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \left(1 - \omega\left(\frac{2}{N}\right)\right) \int_0^1 \psi_{\text{hom}}(l'_{M,N}) - \alpha \left(1 - \omega\left(\frac{2}{N}\right)\right)$$

Finally we can consider the limit as M, N go to infinity, using again the lower semicontinuity of ψ_{hom} and knowing that $\omega(2/N)$ tends to zero as $N \rightarrow \infty$, getting

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \int_0^1 \psi_{\text{hom}}(u') dt - \alpha,$$

and, by the arbitrariness of constant α , taking the limit as $\alpha \rightarrow 0$, we get the thesis:

$$\liminf_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \geq \int_0^1 \psi_{\text{hom}}(u') dt = F(u).$$

Γ- lim sup

Consider a function $u \in H^1([0, 1])$, we want to find a sequence $u_\varepsilon \in H^1([0, 1])$ such that

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \leq F(u). \tag{5.25}$$

First of all note that, by construction, ψ_{hom} is a convex, lower semi continuous and coercive function, then we can apply the Caratheodory theorem, see [53] for details: for any fixed $w \in \mathbb{R}^m$ there exists a convex combination of vectors $w_i \in \mathbb{R}^m$, with coefficient $\lambda_i \in \mathbb{R}$, for $i = 1, \dots, m+1$, such that

$$\sum_{i=1}^{m+1} \lambda_i = 1, \quad \sum_{i=1}^{m+1} \lambda_i w_i = w, \tag{5.26}$$

and

$$\psi_{\text{hom}}(w) = \sum_{i=1}^{m+1} \lambda_i \psi_{\text{hom}}^{z_i}(w_i). \quad (5.27)$$

We can prove (5.25) for a linear function u and extend this result by density to the whole $H^1([0, 1])$. So take $w \in \mathbb{R}^m$ and consider the function $u(t) = tw$, for $t \in [0, 1]$. By Caratheodory theorem we can find $\lambda_i \in \mathbb{R}$ and $w_i \in \mathbb{R}^m$ satisfying (5.26) and (5.27). Let us fix three positive constants ε , T and K , such that $T \gg K$. For any vector $\lambda_i w_i$ there exists a function $u_i : [0, \lambda_i T] \rightarrow \mathbb{R}^m$ that is a solution for the minimum problem

$$\begin{aligned} \psi_{\lambda_i T}^{z_i}(w_i) &= \frac{1}{\lambda_i T} \min \left\{ \int_0^{\lambda_i T} |v'|^2 dt \mid v(0) - 0 \mid < \sqrt{m}, \right. \\ &\quad \left. |v(\lambda_i T) - \lambda_i T w_i| < \sqrt{m}, \varphi_\varepsilon(v) = z_i \right\}. \end{aligned}$$

We want to construct a recovery sequence by a pacthwork procedure using u_i , $i = 1, \dots, m+1$. Let consider the function $\tilde{u} : [0, \sum_{i=1}^{m+1} \lambda_i T + mK] = [0, T + mK] \rightarrow \mathbb{R}^m$ defined as

$$\tilde{u}(t) = \begin{cases} u_j \left(t - \left(\sum_{i=1}^{j-1} \lambda_i T + (j-1)K \right) \right) & \sum_{i=1}^{j-1} \lambda_i T + (j-1)K \leq t \leq \\ & \leq \sum_{i=1}^j \lambda_i T + (j-1)K \\ & j = 1, \dots, m+1 \\ \gamma_j \left(t - \left(\sum_{i=1}^j \lambda_i T + (j-1)K \right) \right) & \sum_{i=1}^j \lambda_i T + (j-1)K \leq t \leq \\ & \leq \sum_{i=1}^j \lambda_i T + jK \\ & j = 1, \dots, m \end{cases} \quad (5.28)$$

where $\gamma_j : [0, K] \rightarrow \mathbb{R}^m$ are the line segments from $\gamma_j(0) = u_j(\lambda_j T)$ to $\gamma_j(K) = u_{j+1}(0) + \sum_{i=1}^j (\lambda_i T w_i + 1)$, for $j = 1, \dots, m$ and γ_j with constant velocity. Note that, by boundary conditions on u_j , one has

$$l(\gamma_j) \leq 2\sqrt{m} \Rightarrow |\gamma_j'| \leq \frac{2\sqrt{m}}{K}. \quad (5.29)$$

Then we define $u : [0, T] \rightarrow \mathbb{R}^m$ as

$$u(t) = \tilde{u} \left(\frac{1}{T} \left(\sum_{i=1}^{m+1} \lambda_i T + mK \right) t \right). \quad (5.30)$$

We have that, being $T \gg K$,

$$G(K, T) = \frac{1}{T} \left(\sum_{i=1}^{m+1} \lambda_i T + mK \right) = \frac{1}{T} (T + mK) \xrightarrow{K \rightarrow \infty} 1.$$

Now we consider the function $\tilde{u}_\varepsilon : [0, \varepsilon T] \rightarrow \mathbb{R}^m$, $\tilde{u}_\varepsilon(t) = \varepsilon u(t/\varepsilon)$, and we define $\bar{u}_\varepsilon : [0, \frac{\varepsilon T}{[\varepsilon T]+1}] \rightarrow \mathbb{R}^m$ as

$$\begin{aligned} \bar{u}_\varepsilon(t) &= \tilde{u}_\varepsilon(t - j\varepsilon T), & j\varepsilon T \leq t \leq (j+1)\varepsilon T, \\ & & j = 0, \dots, \frac{1}{[\varepsilon T]+1} - 1. \end{aligned} \quad (5.31)$$

Finally consider our recovery sequence $u_\varepsilon : [0, 1] \rightarrow \mathbb{R}^m$, with $u_\varepsilon(t) = \bar{u}_\varepsilon\left(\frac{\varepsilon T}{[\varepsilon T]+1}t\right)$. For such a sequence one has

$$\begin{aligned} & \int_0^1 |u'_\varepsilon(t)|^2 dt = \frac{[\varepsilon T]+1}{\varepsilon T} \int_0^{\frac{\varepsilon T}{[\varepsilon T]+1}} \left| u'_\varepsilon\left(\frac{s([\varepsilon T]+1)}{\varepsilon T}\right) \right|^2 ds = \\ &= \frac{\varepsilon T}{[\varepsilon T]+1} \int_0^{\frac{\varepsilon T}{[\varepsilon T]+1}} |\bar{u}'_\varepsilon(s)|^2 ds = \frac{\varepsilon T}{[\varepsilon T]+1} \left(\frac{1}{[\varepsilon T]+1}\right) \int_0^{\varepsilon T} |\tilde{u}'_\varepsilon(s)|^2 ds = \\ &= \frac{\varepsilon T}{([\varepsilon T]+1)^2} \int_0^T |\tilde{u}'_\varepsilon(\varepsilon y)|^2 \varepsilon dy = \left(\frac{\varepsilon T}{[\varepsilon T]+1}\right)^2 \frac{1}{T} \int_0^T |u'(y)|^2 dy = \\ &= \left(\frac{\varepsilon T}{[\varepsilon T]+1}\right)^2 \frac{1}{T} \int_0^{T+mK} \left| u'\left(\frac{t}{G(K,T)}\right) \right|^2 \frac{dt}{G(K,T)} = \\ &= \left(\frac{\varepsilon T}{[\varepsilon T]+1}\right)^2 G(K,T) \left(\sum_{i=1}^{m+1} \frac{1}{\lambda_i T} \lambda_i \int_0^{\lambda_i T} |u'_i(t)|^2 dt + \sum_{i=1}^m \frac{1}{T} \int_0^K |\gamma'(t)|^2 \right) \leq \\ &\leq \left(\frac{\varepsilon T}{[\varepsilon T]+1}\right)^2 G(K,T) \left(\sum_{i=1}^{m+1} \lambda_i \psi_{\lambda_i T}^{z_i}(w_i) + \frac{1}{T} \sum_{i=1}^m \frac{4m}{K} \right). \end{aligned}$$

At the same time we have, making the changes of variables used above,

$$\left(\frac{\delta}{\varepsilon}\right)^2 \int_0^1 \left| \varphi\left(\frac{u_\varepsilon(t)}{\varepsilon}\right) \right|^2 dt \leq \left(\frac{\varepsilon T}{[\varepsilon T]+1}\right)^2 G(K,T) \frac{1}{T} |\nabla \varphi| \left(\frac{\delta}{\varepsilon}\right)^2 \sum_{i=1}^m \frac{4m}{K},$$

because the only non zero contribute of u_ε is made by the curves γ_j , being $\varphi(u_j) = z_j$, so that $\nabla \varphi(u_j) = 0$. Therefore

$$F_\varepsilon(u_\varepsilon) \leq \left(\frac{\varepsilon T}{[\varepsilon T]+1}\right)^2 G(K,T) \left(\sum_{i=1}^{m+1} \lambda_i \psi_{\lambda_i T}^{z_i}(w_i) + \frac{4m^2}{TK} + \left(\frac{\delta}{\varepsilon}\right)^2 \frac{4m^2 C}{KT} \right).$$

Now, consider for example $T = \delta^{1/3}/\varepsilon$ and $K = \delta^{2/3}/\varepsilon$, such that condition $T \gg K$ is satisfied, then we get

$$\left(\frac{\delta}{\varepsilon}\right)^2 \frac{4m^2 C}{KT} = \left(\frac{\delta}{\varepsilon}\right)^2 \frac{4m^2 C \varepsilon^2}{\delta} = \delta 4m^2 C \rightarrow 0,$$

$$\frac{1}{T} \frac{4m^2}{K} = 4m^2 \frac{\varepsilon^2}{\delta} \rightarrow 0.$$

Hence we can consider the limit as $K \rightarrow \infty$ (and $T \gg K \rightarrow \infty$): being

$$\left(\frac{\varepsilon T}{[\varepsilon T] + 1} \right)^2 \rightarrow 1, \quad G(K, T) \rightarrow 1$$

one has

$$F_\varepsilon(u_\varepsilon) \leq \sum_{i=1}^{m+1} \lambda_i \psi_{\text{hom}}^{z_i}(w_i) + 4m^2 \frac{\varepsilon^2}{\delta} + \delta 4m^2 C.$$

Taking the lim sup as $\varepsilon \rightarrow 0$ we finally get

$$\limsup_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon) \leq \psi_{\text{hom}}(w) = \int_0^1 \psi_{\text{hom}}(u') dt = F(u).$$

□

5.2 Examples

Example 5.2.1. Let us consider the following constraint function: $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\varphi(x, y) = \sin(2\pi x) \sin(2\pi y).$$

We want to show that the homogenized function of the oscillating constrained problem associated to φ is the squared l_1 norm: $\psi : \mathbb{R}^2 \rightarrow [0, +\infty)$,

$$\psi_{\text{hom}}(w) = (|w_1| + |w_2|)^2$$

First of all note that we have two types of level sets for the function φ : the level $\{\varphi = 0\}$ that has only one connected component unbounded, the lattice with vertices in \mathbb{Z}^2 , and the level $\{\varphi = c\}$, with $c \neq 0$, that is made by infinitely many connected component, all bounded. Moreover the set $\{\varphi = 0\}$ is made by union of \mathcal{C}^1 sets and it can be proved that it satisfies hypothesis 5.0.1.

Hence the homogenization formula for this problem is the following

$$\psi_{\text{hom}}(w) = \left(\min_z \left(\lim_{T \rightarrow \infty} \psi_T^z(w) \right) \right)^{**},$$

where

$$\psi_T^z(w) = \frac{1}{T} \min \left\{ \int_0^T |u'|^2 dt : |u(0)| < \sqrt{2}, |u(T) - Tw| < \sqrt{2}, \varphi(u) = z \right\} \quad (5.32)$$

In this particular case, we will show that it is simpler:

Lemma 5.2.1. *For any $z \in \text{Im}(\varphi)$, $z \neq 0$, we have*

$$\psi_{\text{hom}}^z(w) = +\infty, \quad \forall w \neq 0$$

Proof. Let $T > 0$ and $z \neq 0$ be fixed and suppose that exists $w \in \mathbb{R}^2$ such that $\psi_{\text{hom}}^z(w) < \infty$. By definition $\psi_{\text{hom}}^z(w)$ is a minimum between curves that satisfy these conditions:

$$|u(0)| < \sqrt{2}, \quad |u(T) - Tw| < \sqrt{2}, \quad \varphi(u) = z.$$

Being $z \neq 0$, we know that level $\{\varphi = z\}$ is made by many bounded connected components, and, by condition $|u(0)| < \sqrt{2}$, the initial point $u(0)$ has to be in one of these bounded components inside the ball $B_{\sqrt{2}}(0)$; moreover u has always to stay in this component, so that $|u(T)| < \sqrt{2}$. Hence, by condition $|Tw - u(T)| < \sqrt{2}$, we get

$$|Tw| < |u(T)| + \sqrt{2} < 2\sqrt{2}.$$

This is true for any $T > 0$, then

$$\lim_{T \rightarrow \infty} |Tw| < 2\sqrt{2}.$$

This is absurd, unless $w = 0$ □

Using this last lemma, we can say that

$$\psi_{\text{hom}}(w) = \psi_{\text{hom}}^0(w).$$

Now we want to prove that

$$\psi_{\text{hom}}^0(w) \leq (|w_1| + |w_2|)^2 = \|w\|_{l^1}^2. \quad (5.33)$$

First of all we name the vector $T\tilde{w} = u(T) - u(0)$. By definition we have $T\tilde{w} = u(T) - Tw + Tw - u(0)$ and $Tw = T\tilde{w} + u(0) + (Tw - u(T))$, so that

$$|T\tilde{w}| \leq |u(T) - Tw| + |Tw| + |u(0)| < 2\sqrt{2} + |Tw|. \quad (5.34)$$

$$|Tw| \leq |T\tilde{w}| + |u(0)| + |Tw - u(T)| < \sqrt{2} + |T\tilde{w}|,$$

so that

$$|T\tilde{w}| \geq |Tw| - 2\sqrt{2}. \quad (5.35)$$

Let $u : [0, T] \rightarrow \mathbb{R}^2$ be a curve with constant velocity, such that $|u(0)| < \sqrt{2}$, $|u(T) - Tw| < \sqrt{2}$, $\varphi(u) = 0$. Moreover suppose that the velocity of u is $|u'| = \frac{\|T\tilde{w}\|_{l^1}}{T}$; this makes sense because u lies on the lattice $\varphi = 0$, so that the length of u is bigger or equal that $\|T\tilde{w}\|_{l^1}$. Being $\psi_{\text{hom}}^0(w)$ the minimum on curves of this type, we have, using (5.34)

$$\begin{aligned} \psi_{\text{hom}}^0(w) &\leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u'|^2 dt \leq \lim_{T \rightarrow \infty} \frac{1}{T} T \|T\tilde{w}\|_{l^1}^2 \leq \\ &\lim_{T \rightarrow \infty} \left(\frac{2\sqrt{2}}{T} + \|w\|_{l^1}^2 \right) = (|w_1| + |w_2|)^2. \end{aligned}$$

To prove the other inequality, take a general curve $u : [0, T] \rightarrow \mathbb{R}^2$ satisfying same conditions $|u(0)| < \sqrt{2}$, $|u(T) - Tw| < \sqrt{2}$, $\varphi(u) = 0$. Hence, using the change of variable $s = t/T$, the function $v(s) = u(sT)$ and Jensen inequality, one has

$$\int_0^T |u'|^2 dt = \int_0^1 |u'(sT)|^2 T ds = \frac{1}{T} \int_0^1 |v'(s)|^2 ds \geq \frac{1}{T} \left(\int_0^1 |v'| ds \right)^2.$$

Note that the last term is the length of the curve v , lying in the lattice $\varphi = 0$, i.e. bigger or equal than $\|T\tilde{w}\|_{l^1}^2$. Therefore, using (5.35),

$$\int_0^T |u'|^2 dt \geq \frac{1}{T} \|T\tilde{w}\|_{l^1}^2 \geq \frac{1}{T} (\|Tw\|_{l^1} - 2\sqrt{2})^2.$$

Now we take the infimum on curves u of this type and then the limit for $T \rightarrow \infty$, getting, by definition of $\psi_{\text{hom}}^0(w)$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \min_u \int_0^T |u'|^2 dt = \psi_{\text{hom}}^0(w) \geq \lim_{T \rightarrow \infty} \frac{1}{T^2} (\|Tw\|_{l^1} - 2\sqrt{2})^2 = \|w\|_{l^1}^2.$$

Example 5.2.2. In this second example we consider the oscillating constraint problem defined by $\varphi(x, y) = d((x, y), \mathbb{Z}^2)$, i.e. the distance from the points with integer coordinates. We can show that the corresponding homogenized function is the following norm, defined on the whole \mathbb{R}^2 :

$$\psi(w) = \left(\|w\|_\infty \frac{\pi}{2} \right)^2 = \left(\max\{w_1, w_2\} \frac{\pi}{2} \right)^2.$$

For such a φ , as in the previous example, we have only one level set made by a unique connected component and unbounded, that is the set

$$\{(x, y) \in \mathbb{R}^2 : \varphi(x, y) = 1/2\};$$

for any other value $c \in [0, \sqrt{2}/2]$, the corresponding level set is made by infinitely many bounded connected components.

So, first of all, we can prove, exactly in the same way used for lemma 5.2.1, the following result:

Lemma 5.2.2. *For any $z \neq 1/2$, one has*

$$\psi_{hom}^z(w) = +\infty, \quad \forall w \neq 0.$$

As a consequence of this lemma we know that $\psi_{hom}(w) = \psi_{hom}^{1/2}(w)$. Before proceeding as before, by showing the two inequality for $\psi_{hom}^{1/2}$, we have to show what is the short way to reach a point $w \in \mathbb{R}^2$, passing through the level set $\varphi = 1/2$. Let u be a curve defined on $[0, T]$ with values into \mathbb{R}^2 satisfying the following conditions:

$$|u(0)| < \sqrt{2}, \quad |u(T) - Tw| < \sqrt{2} \quad \varphi(u) = 1/2.$$

We call \tilde{u} the same curve as u , but starting from a point $\tilde{u}(0)$, that is the nearest point from $u(0)$ with one of the two components belonging to \mathbb{Z} , and ending in $\tilde{u}(T)$, defined in the same way. Note that, by construction, the distance between $u(0)$ and $\tilde{u}(0)$ is smaller then the arc that holds them, that is at most a quart of the circumference, so we have

$$|u(0) - \tilde{u}(0)| < \frac{1}{2} \frac{\pi}{4} = \frac{\pi}{8}, \quad |u(T) - \tilde{u}(T)| < \frac{\pi}{8}. \quad (5.36)$$

We introduce \tilde{u} in order to find the lenght of u in a easy way, knowing that, when $T \rightarrow \infty$, $|u - \tilde{u}| \rightarrow 0$.

Proposition 5.2.1. *Let u and \tilde{u} be defined as before. Then we have*

$$\int_0^T \tilde{u}' dt \geq |\tilde{u}(T) - \tilde{u}(0)|_\infty \frac{\pi}{2}$$

Consider now a curve u satisfying boundary conditions of $\psi_T^{1/2}(w)$, with constant velocity. We identify $\tilde{u}(0)$ and $\tilde{u}(T)$ as before. Let $T\tilde{w}$ be the vector holding $\tilde{u}(0)$ with $\tilde{u}(T)$; by property 5.2.1, this minimum length will be $|T\tilde{w}|_\infty \pi/2$; hence, using conditions (5.36), we have

$$u'(t) = \frac{|T\tilde{w}|_\infty \pi/2 \pm \pi/4}{T},$$

where $\pi/4$ is the maximum difference between u and \tilde{u} . See the following figure for example:

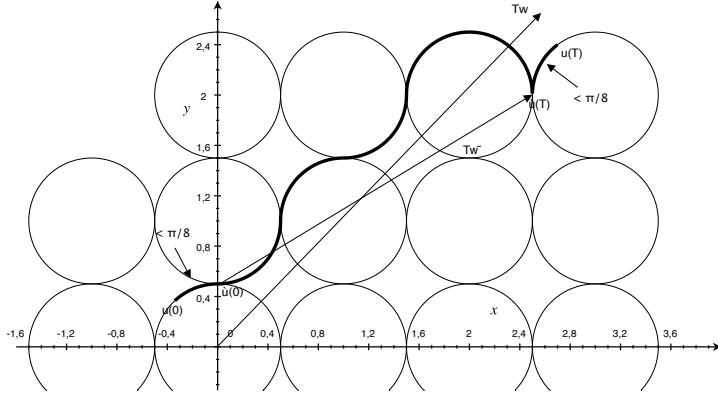


Figure 5.1: An example of a constrained curve with minimum length.

Therefore

$$\psi_{\text{hom}}(w) \leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u'|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} T (|\tilde{w}|_\infty \pi/2 \pm \pi/4T)^2.$$

Now note that, by our hypothesis, one has

$$T\tilde{w} = -\tilde{u}(0) + Tw + (\tilde{u}(T) - Tw),$$

$$|\tilde{u}(0)| \leq |\tilde{u}(0) - u(0)| + |u(0)| \leq \frac{\pi}{8} + \sqrt{2},$$

$$|\tilde{u}(T) - Tw| \leq |\tilde{u}(T) - u(T)| + |u(T) - Tw| \leq \frac{\pi}{8} + \sqrt{2},$$

so that

$$|T\tilde{w}| \leq |Tw| + \frac{\pi}{4} + 2\sqrt{2},$$

hence

$$\psi_{\text{hom}}(w) \leq \lim_{T \rightarrow \infty} \left(|w|_\infty \frac{\pi}{2} + \frac{(\frac{\pi}{8} + \sqrt{2})\pi/2}{T} \pm \frac{\pi}{4T} \right)^2 = \left(|w|_\infty \frac{\pi}{2} \right)^2.$$

For the other inequality let us take a general curve satisfying the conditions defined by $\psi_T^{1/2}$: then, using change of variable $s = t/T$ and Jensen's inequality, with the same notation as in the previous examples, we have

$$\int_0^T |u'|^2 dt = \frac{1}{T} \int_0^1 |v'|^2 ds \geq \frac{1}{T} \left(\int_0^1 |v'| ds \right)^2.$$

Now by property 5.2.1, we know that the length of v is bigger than (as $T \rightarrow \infty$) $|T\tilde{w}|_\infty \pi/2$, so that

$$\int_0^T |u'|^2 dt \geq \frac{1}{T} \left(|T\tilde{w}|_\infty \frac{\pi}{2} \pm \frac{\pi}{4T} \right)^2.$$

Furthermore we have

$$Tw = \tilde{u}(0) + T\tilde{w} + (Tw - \tilde{u}(T)) \Rightarrow |Tw| \leq \sqrt{2} + |T\tilde{w}| + \sqrt{2} + \frac{\pi}{8},$$

so that

$$|T\tilde{w}| \geq |Tw| - \left(2\sqrt{2} + \frac{\pi}{8} \right) = |Tw| - c.$$

Hence

$$\int_0^T |u'|^2 dt \geq \frac{1}{T} \left(|Tw|_\infty \frac{\pi}{2} \pm \frac{\pi}{4T} - c \right)^2.$$

This inequality holds for any curve u satisfying $\psi_T^{1/2}$ conditions and for any $T > 0$, so we can take the infimum over u and the limit as $T \rightarrow \infty$, getting

$$\begin{aligned} \psi_{\text{hom}}(w) &= \min_u \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |u'|^2 \geq \lim_{T \rightarrow \infty} \frac{1}{T^2} \left(|Tw|_\infty \frac{\pi}{2} \pm \frac{\pi}{4T} - c \right)^2 = \\ &= \left(|w|_\infty \frac{\pi}{2} \right)^2 \end{aligned}$$

Example 5.2.3. We consider an example for curves in \mathbb{R}^3 . In order to get the constraint function φ we can take the network

$$\mathcal{L} = \{(x, y, z) \in \mathbb{R}^3 : x = i, y = j, z = k, \forall i, j, k \in \mathbb{Z}\},$$

as Figure 5.2 shows, and define $\varphi(x, y, z) = d^2((x, y, z), \mathcal{L})$.

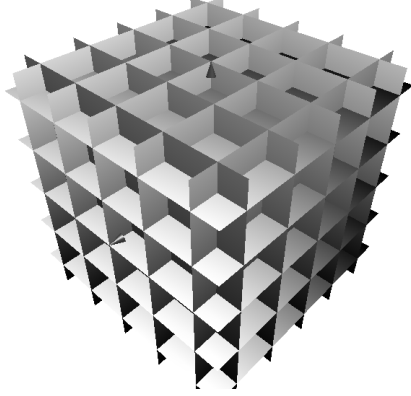


Figure 5.2: The network $\mathcal{L} = \{(x, y, z) \in \mathbb{R}^3 : x = i, y = j, z = k, \forall i, j, k \in \mathbb{Z}\}$.

Hence there exists only one level set of φ unbounded and connected: the network $\mathcal{L} = \{\varphi = 0\}$, and it satisfies trivially the hypothesis 5.0.2, being a union of \mathcal{C}^1 sets. Hypothesis 5.0.1 is more tricky to verify: if we take $z \neq 0$ and $c < |z|$ then $\psi_T^{z,c}(w) = +\infty$; so one has to prove that for c sufficiently small we have

$$\psi_T^{0,c}(w) \geq (1 + o_c(1))\psi_T^0(w') - \frac{k(c)}{T},$$

for any $w, w' \in \mathbb{R}^3$, with $w' = w + o_T(1)$.

Assuming that the geometric stability of $\{\varphi = 0\}$ is satisfied, we have

$$\psi_{\text{hom}}(w) = \lim_{T \rightarrow \infty} \psi_T^0(w),$$

where $\psi_T^0(w)$ measures the minimal length of a curve from 0 to Tw , lying in the level set $\{\varphi = 0\}$. In order to compute such a metric we can consider the parallelepiped with edges $x = [w_1], y = [w_2], z = [w_3]$, so that its faces belong to the network \mathcal{L} . Hence the minimal curve will stay in the plane $y = 0$ until a point $(x, 0, t)$, with $0 \leq t \leq z$, and in the plane $x = [w_1]$, until the point $[w] = (x, y, z)$. Therefore its length will be

$$\min_{0 \leq t \leq z} f(t), \quad f(t) = \left(\sqrt{x^2 + t^2} + \sqrt{y^2 + (z - t)^2} \right).$$

The minimum of $f(t)$ is reached for $t = zx/(y + z)$ and it is

$$l(x, y, z) = \sqrt{(|x| + |y|)^2 + z^2}.$$

Now we have to find the minimum of $l(x, y, z)$ on the permutations of x, y, z : observe that

$$\begin{aligned} |y| \leq |z| &\Rightarrow (|x| + |y|)^2 + z^2 \leq (|x| + |z|)^2 + y^2 \\ |x| \leq |y| &\Rightarrow (|z| + |y|)^2 + x^2 \leq (|x| + |y|)^2 + z^2, \end{aligned}$$

so that we have

$$\psi_{\text{hom}}(w) = (\min\{|x|, |y|, |z|\})^2 + (|x| + |y| + |z| - \min\{|x|, |y|, |z|\})^2,$$

that is the euclidean norm for the minimal component of w added to the l_1 norm of the other two components.

Example 5.2.4. As a second example of curves in \mathbb{R}^3 consider the following constraint function

$$\varphi(x, y, z) = d^2((x, y, z), \mathbb{Z}^3).$$

Observe that this is the natural generalization in \mathbb{R}^3 of example 5.2.2.

Hence the level set $\mathcal{L} = \{\varphi = 1/4\}$, Figure 5.3, has a unique unbounded connected component, that satisfies hypothesis 5.0.2 and 5.0.1.

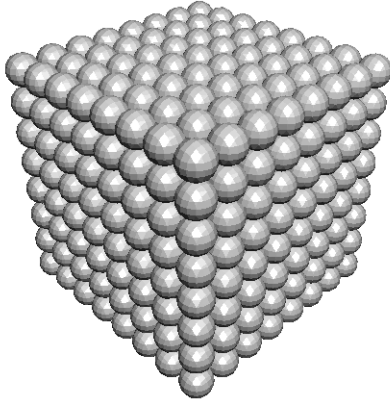


Figure 5.3: The network $\mathcal{L} \subseteq \mathbb{R}^3$.

In order to find an upper bound for $\psi_{\text{hom}}(w)$, we can use a similar argument of example 5.2.3: consider the parallelepiped of edges $x = [w_1]$, $y = [w_2]$ and $z = [w_3]$: the length of a curve u_ε , connecting 0 and w , with the strict constraint $\varphi(u_\varepsilon) = 1/4$, is less or equal to the one of a curve v_ε , from 0 to a point $([w_1], 0, t)$, with $0 \leq t \leq [w_3]$, lying in the plane $y = 0$, and from $([w_1], 0, t)$ to $[w]$, in the plane $x = [w_1]$. Hence we can exploit the result of the example 5.2.2 in these two planes, so that we have

$$l(v_\varepsilon) = \min_{0 \leq t \leq z} \frac{\pi}{2} (\|(x, t)\|_\infty + \|(z - t), y)\|_\infty),$$

where $\|(\xi_1, \xi_2)\|_\infty = \max(|\xi_1|, |\xi_2|)$ is the l_∞ norm of the vector $\xi \in \mathbb{R}^2$.

Note that if $z < x + y$ then $\|(x, t)\|_\infty + \|(z - t), y)\|_\infty = |x| + |y|$, while if $z \geq x + y$ then $\|(x, t)\|_\infty + \|(z - t), y)\|_\infty = |z|$. Therefore

$$l(v_\varepsilon) = \frac{\pi}{2} \min(\max(|x| + |y|, |z|), \max(|x| + |z|, |y|), \max(|z| + |y|, |x|)).$$

Now we have

$$|x| \leq |y| \Leftrightarrow \max(|z| + |y|, |x|) \leq \max(|x| + |z|, |y|),$$

$$|y| \leq |z| \Leftrightarrow \max(|x| + |z|, |y|) \leq \max(|x| + |y|, |z|).$$

This means that

$$\|(x, y, z)\|_\infty = |x| \wedge |z| \geq |y| \Rightarrow l(v_\varepsilon) = \frac{\pi}{2} \|y, x + z\|_\infty,$$

$$\|(x, y, z)\|_\infty = |x| \wedge |y| \geq |z| \Rightarrow l(v_\varepsilon) = \frac{\pi}{2} \|z, x + y\|_\infty,$$

$$\|(x, y, z)\|_\infty = |y| \wedge |z| \geq |x| \Rightarrow l(v_\varepsilon) = \frac{\pi}{2} \|x, y + z\|_\infty,$$

$$\|(x, y, z)\|_\infty = |y| \wedge |x| \geq |z| \Rightarrow l(v_\varepsilon) = \frac{\pi}{2} \|z, x + z\|_\infty,$$

$$\|(x, y, z)\|_\infty = |z| \wedge |y| \geq |x| \Rightarrow l(v_\varepsilon) = \frac{\pi}{2} \|x, y + z\|_\infty,$$

$$\|(x, y, z)\|_\infty = |z| \wedge |x| \geq |y| \Rightarrow l(v_\varepsilon) = \frac{\pi}{2} \|y, x + z\|_\infty.$$

Hence we get

$$l(v_\varepsilon) = \frac{\pi}{2} \|(\min(|x|, |y|, |z|), \|(x, y, z)\|_1 - \min(|x|, |y|, |z|))\|_\infty,$$

and the upper bound

$$\begin{aligned} \psi_{\text{hom}}(w) &\leq \psi_{\text{hom}}^{1/4}(w) \leq \\ &\leq \left(\frac{\pi}{2} \|(\min(|x|, |y|, |z|), \|(x, y, z)\|_1 - \min(|x|, |y|, |z|))\|_\infty \right)^2. \end{aligned}$$

Chapter 6

Density of \mathbb{R}^2 oscillating constraint problems in Finsler metrics

In chapter 5, we stated that the existence of at least one level set of φ containing an unbounded connected component is necessary for the boundedness of ψ_{hom} ; hence, for at least one $z \in \text{Im}(\varphi)$, the level set $\{\varphi_\varepsilon = z\}$ is a periodic unbounded and connected ε -network over \mathbb{R}^m , that represents the “allowed” zones for curves u_ε in F_ε . This means that ψ_{hom} measures the distance between the origin and the point w , not with the euclidean norm, but with the length of a curve that microscopically lies in the lattice defined by the constraint. Hence the following inequality is satisfied

$$\psi_{\text{hom}}(w) \geq |w|^2,$$

with equality reached by the trivial case $\varphi = 0$. Moreover, being ψ_{hom} two-homogeneous, it is also symmetric:

$$\psi_{\text{hom}}(-w) = \psi_{\text{hom}}(w).$$

Therefore ψ_{hom} is a convex, two-homogeneous and symmetric function, controlled from below by the euclidean norm.

This shows that the Γ -limit of an oscillating constraint problem, for curves with values in \mathbb{R}^2 , is a metric. In this chapter we characterized metrics defined by an oscillating surface on \mathbb{R}^3 , i.e. the function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$: more precisely we show that they are dense in Finsler metrics, with respect to Γ -convergence.

By a symmetric Finsler metric in \mathbb{R}^2 , controlled from below by the Euclidean metric, we mean a function $\psi : \mathbb{R}^2 \rightarrow [0, +\infty]$ such that

- i) ψ is 2-homogeneous: $\psi(\lambda w) = \lambda^2 \psi(w)$, $\forall w \in \mathbb{R}^2$, $\forall \lambda \in \mathbb{R}$;

ii) ψ is convex;

iii) $\forall w \in \mathbb{R}^2$, we have $\psi(w) \geq |w|^2$;

Observe that from *i*) one has $\psi(w) = \psi(-w)$, $\forall w \in \mathbb{R}^2$. Observe that, differently from the case treated in [17], we don't ask the boundedness of ψ from above: this allows us to treat cases of metrics whose domain is not the whole \mathbb{R}^2 .

By the hypothesis on ψ , we know that its domain, i.e. the set where ψ is finite, has to be a convex cone in \mathbb{R}^2 symmetric with respect to the origin and centered at $(0, 0)$, hence it is a subspace of \mathbb{R}^2 . So, if $\text{dom}(\psi) \neq \{0\}$, we might have two different cases:

1. $\text{dom} \psi$ is a line through the origin, i.e. a subspace of dimension one (\mathbb{R}), so that ψ is finite only in one direction and we have

$$\sup_{|w|=1} \psi(w) = +\infty;$$

2. $\text{dom} \psi$ is the whole \mathbb{R}^2 , so that we have

$$\max_{|w|=1} \psi(w) = M < +\infty.$$

It is clear that this distinction can't be applied to the situation of metrics defined on \mathbb{R}^n , $n > 2$, that, in general, will contain more cases.

In both cases, we want to prove that for any $\eta > 0$ and ψ satisfying conditions *i*), *ii*) and *iii*), there exists a periodic function $\varphi_\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$, defining the oscillating constraint and the functional F_ε , in the way we will see in the sequel, such that the homogenized function ψ_η of the Γ -limit

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u) = \int_0^1 \psi_\eta(u') dt$$

satisfies the inequality

$$|\psi_\eta(w) - \psi(w)| \leq \eta |w|^2. \quad (6.1)$$

Let us first recall what an oscillating constraint problem is, in the case of curves taking values in \mathbb{R}^2 : as in equation (4.1), we will consider the following functional

$$F_\varepsilon(v) = \begin{cases} \int_\Omega |\nabla v|^2 dx & v \in H^1(\Omega; V_\varepsilon) \\ +\infty & \text{otherwise} \end{cases} \quad (6.2)$$

where Ω is an open and bounded subset of \mathbb{R} , for example $(0, 1)$, and the function v , actually depending on the small parameter ε , is a curve taking values on an oscillating constraint V_ε . More precisely we take

$$\begin{aligned} v_\varepsilon : \Omega \in \mathbb{R} &\rightarrow V_\varepsilon \subset \mathbb{R}^3 \\ t &\mapsto v_\varepsilon(t) = (v_1^\varepsilon(t), v_2^\varepsilon(t)) \end{aligned} \quad (6.3)$$

where $v_2^\varepsilon(t) = \varphi_\varepsilon(v_1^\varepsilon(t))$, $v_1^\varepsilon(t)$ is a curve in \mathbb{R}^2 , $\varphi_\varepsilon(y) = \delta_\varepsilon \varphi\left(\frac{y}{\varepsilon}\right)$ is defined by the function $\varphi(y)$, that is $(0, 1)^2$ -periodic, and takes values from \mathbb{R}^2 to \mathbb{R} . Moreover we assume that $\delta_\varepsilon \rightarrow 0$ and $\delta_\varepsilon/\varepsilon \rightarrow \infty$, as $\varepsilon \rightarrow 0$; hence our oscillating manifold V_ε , in some sense, is converging to the space $V = \mathbb{R}^2 \times \{0\} = \mathbb{R}^2$.

Remark 6.0.1. *To simplify the notation, as in equation (4.3), in the sequel we will separate the two component of the curve v : the free component v_1 , taking values in \mathbb{R}^2 , that we will name simply u , and the constrained one, defined by $\varphi_\varepsilon(u)$, so that we will have*

$$v_1(t) = u(t), \quad v_2(t) = \varphi_\varepsilon(u(t)).$$

In this way, we can express the functional F_ε in the following equivalent unconstrained form: $F_\varepsilon : L^2(\Omega; \mathbb{R}^2) \rightarrow [0, +\infty]$

$$F_\varepsilon(u) = \int_{\Omega} |\nabla v_\varepsilon|^2 dx = \int_{\Omega} \left(|u'_\varepsilon|^2 + \left(\frac{\delta}{\varepsilon}\right)^2 |D_y \varphi(u_\varepsilon) u'_\varepsilon|^2 \right) dx \quad (6.4)$$

where $y = u_\varepsilon/\varepsilon$.

Before treating the two different cases of the 2-homogeneous convex function ψ , we show in the following lemma that F_ε is invariant under reparametrization:

Lemma 6.0.3. *Let u_ε be a minimizer for F_ε , then there exists a curve $v_\varepsilon \in H^1([0, 1])$, with constant velocity $v'_\varepsilon = c$, such that*

$$\min_{u \in H^1([0, 1])} F_\varepsilon(u) = F_\varepsilon(u_\varepsilon) = F_\varepsilon(v_\varepsilon).$$

Proof. Consider the curve $u_\varepsilon : [0, 1] \rightarrow \mathbb{R}^2$ and set $\varphi_\varepsilon(z) = \varphi\left(\frac{z}{\varepsilon}\right)$. Then our functional reads

$$F_\varepsilon(u_\varepsilon) = \int_0^1 \left(|u'_\varepsilon|^2 + \delta^2 |D\varphi_\varepsilon(u_\varepsilon) \cdot u'_\varepsilon|^2 \right) dt.$$

Now think on a monotone $\mathcal{C}^1([0, 1])$ function $g : [0, 1] \rightarrow [0, 1]$, with $g(0) = 0$, $g(1) = 1$, and consider new curve $v_\varepsilon = u_\varepsilon \circ g(t) = u_\varepsilon(g(t))$, that is a reparametrization of u_ε . Then the functional becomes

$$F_\varepsilon(v_\varepsilon) = \int_0^1 \left(|u'_\varepsilon(g(t))g'(t)|^2 + \delta^2 |D\varphi_\varepsilon(u_\varepsilon(g(t))) \cdot u'_\varepsilon(g(t))g'(t)|^2 \right) dt =$$

$$= \int_0^1 (|u'_\varepsilon(g(t))|^2 + \delta^2 |D\varphi_\varepsilon(u_\varepsilon(g(t))) \cdot u'_\varepsilon(g(t))|^2) (g'(t))^2 dt.$$

Now use the change of variable $s = g(t)$, $ds = g'(t)dt$:

$$F_\varepsilon(v_\varepsilon) = \int_{g(0)}^{g(1)} (|u'_\varepsilon(s)|^2 + \delta^2 |D\varphi_\varepsilon(u_\varepsilon(s)) \cdot u'_\varepsilon(s)|^2) g'(g^{-1}(s)) ds.$$

We can write $g'(g^{-1}(s))$, using the formula for the derivative of the inverse function, as

$$g'(g^{-1}(s)) = \frac{1}{(g^{-1})'(g(t))} = \frac{1}{(g^{-1})'(s)},$$

then we get

$$F_\varepsilon(v_\varepsilon) = \int_0^1 (|v_\varepsilon|^2 + \delta^2 |D\varphi_\varepsilon \cdot v'_\varepsilon|^2) \frac{1}{(g^{-1})'(s)} ds.$$

Since we have to find the minimum of this functional, we are interested in the optimal reparametrization g , with the constraint $g(0) = 0$, $g(1) = 1$, that is $\int_0^1 g'(t) dt = 1$, that minimizes F_ε . In general the problem is to find a function D that realizes

$$\min \left\{ \int_0^1 \frac{N}{D} : \int_0^1 D = 1 \right\}, \quad (6.5)$$

where, for us, $N = |v_\varepsilon|^2 + \delta^2 |\nabla\varphi_\varepsilon \cdot v'_\varepsilon|^2$ and $D = (g^{-1})'(s) = (g'(t))^{-1}$.

We have, by Cauchy Schwarz inequality,

$$\int_0^1 \frac{N}{D} = \left(\int_0^1 \frac{N}{D} \right) \left(\int_0^1 D \right) \geq \left(\int_0^1 \sqrt{\frac{N}{D}} \sqrt{D} \right)^2 = \left(\int_0^1 \sqrt{N} \right)^2.$$

Now observe that this lower bound can be reached when $D = C\sqrt{N}$, where the constant C is, by the constraint $\int D = 1$, $C = (\int \sqrt{N})^{-1}$; so, putting this optimal D in our original functional, we get

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\geq \int_0^1 \frac{N}{C\sqrt{N}} = \frac{1}{C} \int_0^1 \sqrt{N} = \left(\int_0^1 \sqrt{N} \right)^2 = \\ &= \left(\int_0^1 \sqrt{|u_\varepsilon|^2 + \delta^2 |D\varphi_\varepsilon \cdot u'_\varepsilon|^2} \right)^2. \end{aligned}$$

To conclude the proof note that:

$$F_\varepsilon(u_\varepsilon) \geq \left(\int_0^1 \sqrt{1 + \delta^2 \left| D\varphi_\varepsilon \cdot \frac{u'_\varepsilon}{|u'_\varepsilon|^2} \right|^2} |u'_\varepsilon| dt \right)^2 =$$

$$= \left(\int_{\Gamma} \sqrt{1 + \delta^2 |D\varphi_\varepsilon \cdot t_\varepsilon|^2} dl \right)^2,$$

where Γ is the image of the curve u_ε , dl is its arc length and t_ε its tangent versor. Hence it is clear that the minimum of F_ε doesn't depend on the velocity of u_ε , but only on the image Γ . \square

6.1 Degenerate Finsler metrics

As we said before, the domain of ψ is a vector space of dimension 1. It is not restrictive to assume that $\text{dom}(\psi) = \{(w_1, w_2) \in \mathbb{R}^2 : w_2 = 0\}$: all the other cases can be obtained from this simply by a change of basis of \mathbb{R}^2 . Since the only 2-homogeneous function in one variable is quadratic, then there exists a constant $k > 0$ such that $\psi(w) = k|w|^2$ for any $w \in \text{dom}(\psi)$, and ψ is uniquely determined by k , that is its value at any $w \in \mathbb{R}^2$, $|w| = 1$. Now we want to construct a periodic function φ , defining the oscillating constraint, such that the density function of the Γ -limit of this problem is the quadratic function ψ .

Consider any 1-periodic smooth function $g : \mathbb{R} \rightarrow \mathbb{R}$ and take $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$, with $\varphi(x, y) = \sin(2\pi(x - g(y)))$. The level sets $\{\varphi = z\}$ are defined by the equation $x - g(y) = c$, for a suitable $c \in \mathbb{R}$. We can represent them as the graph of the function $x = g(y) + c$, so that they are all the same line type graphs, horizontally translated by c . Hence $\psi_T^z(w)$ is independent of z , for any T, w . In fact if Z_1 and Z_2 are the sets of admissible functions in (5.3), for z_1, z_2, T, w fixed, then, for any $u = (u_1, u_2) \in Z_1$, there exists $c \in \mathbb{R}$ such that $(u_1 + c, u_2) \in Z_2$ and viceversa. We can then choose $z = 0$ in equation (5.3), obtaining, from equation (5.5),

$$\psi_{\text{hom}}(w) = \lim_{T \rightarrow +\infty} \psi_T^0(w)$$

To find the constant k , we consider the homogenized function in any vector w of the domain of ψ , with $|w| = 1$:

$$\psi(1) = \lim_{T \rightarrow \infty} \frac{1}{T} \min \left\{ \int_0^T |u'|^2 : |u(0)| < \sqrt{2}, |u(T) - Tw| < \sqrt{2}, \varphi(u) = 0 \right\}.$$

It can be proved dealing by hand that the limit exists and, as we showed in chapter 5, it coincides with the length of the curve u that satisfies $\varphi = 0$. Hence we have

$$\psi(1) = k = \frac{1}{T} \left(\int_0^T |u'| dt \right)^2 = \left(\int_0^1 \sqrt{1 + g'(s)} ds \right)^2.$$

In this first case we can reach a $\psi(w) = k|w|^2$ satisfying i), ii), iii), with $\text{dom}(\psi) = \mathbb{R}$, as the Γ -limit of the constraint problem defined by $\varphi(x, y) =$

$\sin(x-g(y))$, by taking the periodic function g such that $\left(\int_0^1 \sqrt{1+g'(s)} ds\right)^2 = k$.

Example 6.1.1. An example of a degenerate Finsler metric is the following norm

$$\psi(w) = \begin{cases} |w|^2 & \text{if } w = (0, w_2) \\ +\infty & \text{otherwise.} \end{cases}$$

We will show that this norm can be obtained as the homogenized function of the problem defined by a constraint $\varphi(x, y) = \varphi(x)$ periodic and not constant; for example we can take $\varphi(x, y) = \sin(2\pi x)$.

The level set $\{\varphi = z\}$, with $z \in [-1, 1]$, are made by pairs of vertical lines $x = \frac{\arcsin z}{2\pi} + k$ and $x = \frac{\pi - \arcsin z}{2\pi} + k$, with $k \in \mathbb{Z}$.

First of all note that the domain of ψ_{hom}^z is the same that the one of ψ :

Lemma 6.1.1. For any $z \in [-1, 1]$,

$$\text{dom}(\psi_{\text{hom}}^z) = \{w = (w_1, w_2) \in \mathbb{R}^2 : w_1 = 0\}.$$

Proof. Let $z \in [-1, 1]$ and $T > 0$ be fixed. By conditions $|u(0)| < \sqrt{2}$ and $\varphi(u) = z$, in definition (5.32), we can deduce that $u(0)$ has to stay in the level set near the origin, i.e in the ball $B_{\sqrt{2}}(0)$:

$$u(0) = \left(\frac{\arcsin z}{2\pi}, u_2(0)\right) \vee u(0) = \left(\frac{\arcsin z}{2\pi} \pm 1, u_2(0)\right).$$

This means that $u(T)$ will be in the same line of $u(0)$, so that

$$u(T) = u(0) = \left(\frac{\arcsin z}{2\pi}, u_2(T)\right) \vee u(T) = \left(\frac{\arcsin z}{2\pi} \pm 1, u_2(T)\right).$$

Therefore, if we consider the second condition of definition (5.32),

$$\sqrt{2} > |u(T) - Tw|^2 \geq \left|Tw_1 - \left(\frac{\arcsin z}{2\pi} \pm 1\right)\right|^2 + |Tw_2 - u_2(T)|^2 \geq$$

$$\left|Tw_1 - \left(\frac{\arcsin z}{2\pi} \pm 1\right)\right|^2 \geq |Tw_1|^2 - \left|\frac{\arcsin z}{2\pi} \pm 1\right|^2 \geq |Tw_1|^2 - \sqrt{2}.$$

Hence, for any $T > 0$, we have that

$$|Tw_1|^2 < 2\sqrt{2}.$$

Getting the limit for $T \rightarrow \infty$, we get $w_1 = 0$. □

So now we can consider only vectors w in the line $w_1 = 0$; first we show that, for these vectors, $\psi_{\text{hom}}^z(w) \leq |w_2|^2$. Fix $z \in [-1, 1]$ and $T > 0$; consider a curve $u : [0, T] \rightarrow \mathbb{R}^2$ in the space

$$X_w^{T,z} = \left\{ u : [0, T] \rightarrow \mathbb{R}^2 : |u(0)| < \sqrt{2}, |u(T) - Tw| < \sqrt{2}, \varphi(u) = z \right\},$$

with constant velocity. Let us call $T\tilde{w} = u(T) - u(0)$, i.e. the length of u ; we have

$$|T\tilde{w}| = |u(T) - Tw + Tw - u(0)| \leq |u(T) - Tw| + |Tw| + |u(0)| < 2\sqrt{2} + |Tw|.$$

By definition (5.32) one has

$$\begin{aligned} \psi_T^z(w) &\leq \frac{1}{T} \int_0^T |u'|^2 dt = \frac{1}{T} \int_0^T \left| \frac{T\tilde{w}}{T} \right|^2 dt = \\ &= \frac{1}{T} T |\tilde{w}|^2 \leq |w|^2 + \frac{2\sqrt{2}}{T} = |w_2|^2 + \frac{2\sqrt{2}}{T}. \end{aligned}$$

Taking the limit $T \rightarrow \infty$ we get

$$\psi_{\text{hom}}^z(w) \leq |w_2|^2.$$

For the other inequality consider a general function $u \in X_w^{T,z}$; hence, following the same procedure of example 5.2.1

$$\begin{aligned} \int_0^T |u'|^2 dt &= \int_0^1 |u'(sT)| T ds = \frac{1}{T} \int_0^1 |v'(s)| ds \geq \frac{1}{T} \left(\int_0^1 |v'| ds \right)^2 \geq \\ &= \frac{1}{T} |T\tilde{w}|^2 > \frac{1}{T} (|Tw| - 2\sqrt{2})^2. \end{aligned}$$

Multiplying by $1/T$, taking the minimum on $X_w^{T,z}$ and then the limit for $T \rightarrow \infty$, we get

$$\psi_{\text{hom}}^z(w) \geq |w_2|^2.$$

By these two inequality we know that $\psi_{\text{hom}}^z(w) = |w_2|^2$; this result is independent by z , therefore

$$\psi_{\text{hom}}(w) = \left(\min_z \psi_{\text{hom}}^z(w) \right)^{**} = \left(\min_z |w_2|^2 \right)^{**} = |w_2|^2.$$

Remark 6.1.1. Note that in this example $\sup_{|w|=1} \psi(w) = +\infty$, so that the domain of ψ is a line in \mathbb{R}^2 . We can adapt to this particular case, the general procedure described above: for us $\varphi(x, y) = \sin(2\pi x - g(y)) = \sin(2\pi x)$, i.e. $g(y) = 0$. Then, using the same notation, we have

$$k = \left(\int_0^1 \sqrt{1 + g'(s)} ds \right)^2 = 1.$$

Therefore, in the domain $\{w \in \mathbb{R}^2 : w_1 = 0\}$,

$$\psi(w) = k|w|^2 = |w|^2 = |w_2|^2$$

6.2 Non-degenerate Finsler metrics

We want to construct a function φ , more precisely its unique connected level set, such that $\psi_\eta = \psi_{\text{hom}}$ associated to φ , satisfies equation (6.1).

Now ψ is defined in all directions of \mathbb{R}^2 , but, by the 2-homogeneity, we will consider only the set $\{w \in \mathbb{R}^2 : \psi(w) = 1\}$, or, more precisely, the convex set

$$C_\psi = \{w \in \mathbb{R}^2 : \psi(w) \leq 1\}.$$

Note that, by condition $\psi(w) \geq |w|^2$, we have that $C_\psi \subseteq B_1(0)$, and, by the symmetry, C_ψ is centered in the origin.

Being interested on a density result, we can approximate this convex set by a polygon of $2N$ vertices, by symmetry, $\pm V_1, \dots, \pm V_N$, whose directions are $\pm\nu_1, \dots, \pm\nu_N$. Always by density, we can also assume that these vertices are rational, i.e. for each $i = 1, \dots, N$ there exists a point $z_i \in \mathbb{Z}^2$ and $t_i \in \mathbb{R}$, such that $t_i V_i = z_i$; for example we can refer to the following figure:

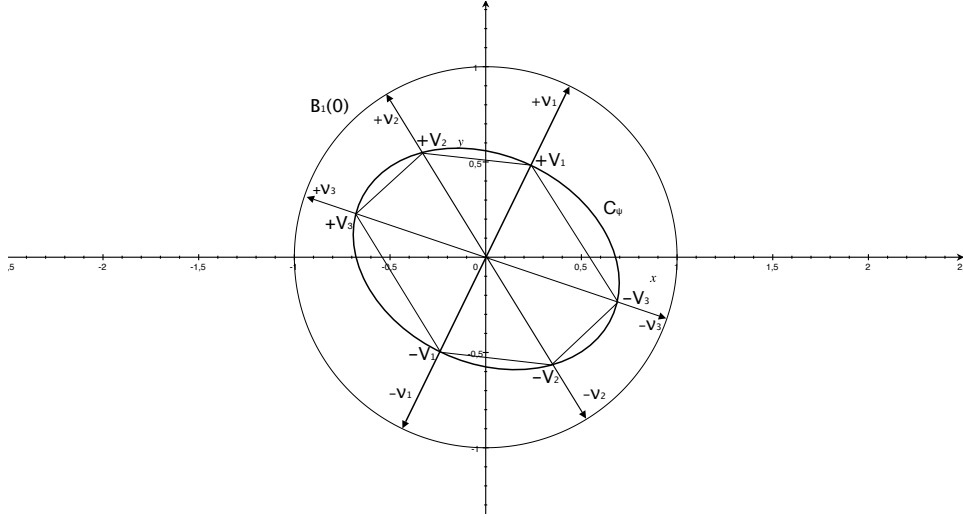


Figure 6.1: The set C_ψ and its polygonal approximation with directions ν_i .

Remark 6.2.1. *We want to use the homogenization formula of theorem 5.1.1 for the constraint φ ; in order to do this all hypothesis of this theorem have to be fulfilled. For now we assume that theorem 5.1.1 holds true, and we will verify all hypothesis when the function φ will be constructed. Hence let us assume that ψ_{hom} is the one defined in equation (5.5).*

The fact that ψ is finite in \mathbb{R}^2 means that there exists at least one level set of φ with one connected component, unbounded and “connecting” \mathbb{R}^2 : indeed if $\max_{|w|=1} \psi(w) = M < +\infty$, there exists $z \in \text{Im}(\varphi)$ such that $\psi_{\text{hom}}^z(w) < +\infty$ for any $w \in \mathbb{R}^2$; this means that there exists T_0 such that,

for $T > T_0$, Tw must be reached by the set $\{\varphi = z\}$. More precisely, if we consider the definition of ψ_{hom}^z , we have the condition $|u(T) - Tw| < \sqrt{2}$; assuming $\varphi \in \mathcal{C}^1$,

$$|\varphi(u(T)) - \varphi(Tw)| \leq L_\varphi |u(T) - Tw| < L_\varphi \sqrt{2},$$

so that the distance of Tw from the set $\{\varphi = z\}$ have to be less than a fixed constant. This means, by the arbitrariness of $w \in \mathbb{R}^2$ and taking the limit as $T \rightarrow \infty$, that $\{\varphi = z\}$ must have at least one connected component unbounded.

Moreover, by remark 6.2.1 the homogenized function is defined by the minimum formula

$$\psi_{\text{hom}}(w) = \left(\min_{z \in \text{Im}(\varphi)} \psi_{\text{hom}}^z \right)^{**} (w), \quad (6.6)$$

so we will construct φ such that this minimum is reached at the level $z = 0$, the unique level set of φ unbounded and connecting \mathbb{R}^2 .

Let Q be the periodicity square for all directions ν_i , i.e. the square of edge the least common multiple $\tau = \text{lcm}(t_1, \dots, t_N)$ with one of the vertices in the origin and taken as the torus. Now we construct the zero level set of φ . Inside Q , starting from the origin, we draw the lines in directions ν_i ; if we consider Q as the torus, we will repeat these lines inside Q then, by construction, they will cut each other in many segments of a certain length L_i^j , for $i = 1, \dots, N$, $j = 1, \dots, M$, for some $M \in \mathbb{N}$, making the set L . See for example next figure:

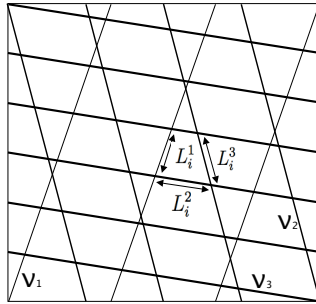


Figure 6.2: The Q square with lines in directions ν_i and segments L_i^j .

We can synthetically describe a point in L as $t_i \mathbb{Z}^2 + \nu_i \mathbb{R}$, indeed $t_i \mathbb{Z}^2$ is a point of Q , where the lines in direction ν_i passes through, and $\nu_i \mathbb{R}$ is nothing but the line itself.

Observe that, by the 2-homogeneity, we have

$$\psi(V_i) = 1 = \psi\left(\frac{V_i}{|V_i|}|V_i|\right) = |V_i|^2\psi(\nu_i) \Rightarrow \psi(\nu_i) = |V_i|^{-2},$$

so that, the more $|V_i|$ is near to 0, the more $\psi(\nu_i)$ is bigger. This means intuitively that, if a vertex V_i is inside of the unit ball, i.e. $|V_i| < 1$, the metric associated to ψ will be bigger than the euclidean one, in direction ν_i . Hence, if we think at the previous case, in which ψ was the length of the curve passing through the set $\varphi = 0$, in order to obtain a metric bigger than the euclidean norm, we have to stretch the lines of $\varphi = 0$, i.e. the lines of L in directions ν_i , so that the curve passing through them will be longer, obtaining $\psi(\nu_i) > 1$.

Following this idea we modify the length of any segments L_i^j , changing them in curves such that their length will be $L_i^j\sqrt{\psi(\nu_i)}$, as the following figure shows

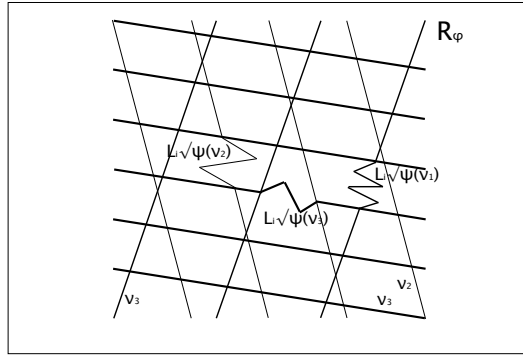


Figure 6.3: The construction of the set R_φ , with three segments modified.

The set of all these modified lines inside Q , extended by periodicity, will represent the level set of φ connecting \mathbb{R}^2 , i.e., by our assumption, the set $\{\varphi = 0\}$, that we name R_φ . For example we can take as φ the squared distance from R_φ . Hence observe that, for such a constraint, the following hypothesis are fulfilled:

H1) $\varphi \in C^1([0, 1])$;

H2) for any $z \in Im(\varphi)$ one of the following two conditions holds true for the set $\{\varphi = z\}$:

- i) it is made by a unique connected component unbounded, so that it "connects" \mathbb{R}^2 .
- ii) it is made by infinitely many bounded connected component.

The proof of *H2*) can be obtained arguing as in Example 5.2.1. Hence, by theorem 5.1.1 the following homogenization result holds true:

$$\Gamma\text{-}\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u) = F(u) = \int_0^1 \psi_{\text{hom}}(u') dt,$$

in the strong topology of L^2 , with ψ_{hom} defined by equations (5.3), (5.4) and (5.5), so that we are allowed to use formula (6.6).

Note that for any $i = 1, \dots, N$

$$\psi_{\text{hom}}(\nu_i) = \lim_{T \rightarrow \infty} \frac{1}{T} \min \left\{ \int_0^T |u'|^2, |u(0)| \leq \sqrt{2}, |u(T) - T\nu_i| \leq \sqrt{2}, u \in R_\varphi \right\}.$$

Now take a function $u_i : [0, T] \rightarrow \mathbb{R}^2$ that satisfies $\varphi(u_i) = 0$, i.e. passes through the lines of R_φ in direction ν_i with constant velocity, and with $|u_i(0)| \leq \sqrt{2}$, $|u_i(T) - T\nu_i| \leq \sqrt{2}$. Observe that, by periodicity of R_φ , the space covered by u_i is at least

$$\sqrt{\psi(\nu_i)} [T + 1]$$

and, by construction, it will be a competitor for the minimum problem defined by $\psi_{\text{hom}}(\nu_i)$, hence

$$\psi_{\text{hom}}(\nu_i) \leq \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T |u'|^2 dt = \lim_{T \rightarrow +\infty} \psi(\nu_i) \frac{1}{T^2} [T + 1]^2 = \psi(\nu_i).$$

By convexity we can extend the result for any $w \in \mathbb{R}^2$, obtaining

$$\psi_{\text{hom}}(w) \leq \psi(w).$$

To get the other inequality let us consider $w \in \mathbb{R}^2$; by construction of R_φ , there exists $T > 0$ such that $Tw \in \{\varphi = 0\}$, and, by periodicity, $(T + k)w \in \{\varphi = 0\}$ too, for any $k \in \mathbb{N}$. Now take a function $v \in H^1([0, T])$, with $v(0) = 0$, $v(T) = Tw$ and $\varphi(v) = 0$, that realizes the minimum defined by $\psi_{\text{hom}}^0 = \psi_{\text{hom}}$. Observe that, by lemma 6.0.3, we can assume, without loss of generality, that $v' = c$. we call λ_i the vector sum of the segments of v in the direction ν_i (without the modification made by $\sqrt{\psi(\nu_i)}$).

Remark 6.2.2. *We cannot exclude a priori the case where u repeats some lines in the same direction ν_i but with opposite sign; in this case λ_i will not consider these two portions of space. Therefore, in general, the space covered by v , will be greater or equal then $\sum_{i=1}^N \lambda_i \sqrt{\psi(\nu_i)}$.*

Note that, by construction, we have

$$\sum_{i=1}^N \lambda_i \nu_i = Tw.$$

Hence, for T sufficiently big,

$$\begin{aligned} T\psi_{\text{hom}}(w) &= \int_0^T |v'|^2 = \int_0^T c^2 = Tc^2 \geq T \left(\frac{\sum_{i=1}^N \lambda_i \sqrt{\psi(\nu_i)}}{T} \right)^2 = \\ &= \frac{1}{T} \left(\sum_{i=1}^N \lambda_i \sqrt{\psi(\nu_i)} \right)^2 \frac{\left(\sum_{i=1}^N \lambda_i \right)^2}{\left(\sum_{i=1}^N \lambda_i \right)^2} = \frac{\left(\sum_{i=1}^N \lambda_i \right)^2}{T} \left(\sum_{i=1}^N \frac{\lambda_i}{\sum_{i=1}^N \lambda_i} \sqrt{\psi(\nu_i)} \right)^2; \end{aligned}$$

the coefficients in the sum are a convex combination, then, by convexity of $\sqrt{\psi}$ and after by the 2-homogeneity of ψ , we get

$$\begin{aligned} T\psi_{\text{hom}}(w) &\geq \frac{\left(\sum_{i=1}^N \lambda_i \right)^2}{T} \left(\sqrt{\psi \left(\sum_{i=1}^N \frac{\lambda_i}{\sum_{i=1}^N \lambda_i} \nu_i \right)} \right)^2 = \\ &= \frac{\left(\sum_{i=1}^N \lambda_i \right)^2}{T} \frac{1}{\left(\sum_{i=1}^N \lambda_i \right)^2} \psi \left(\sum_{i=1}^N \lambda_i \nu_i \right) = \frac{1}{T} \psi(Tw) = T\psi(w). \end{aligned}$$

Therefore, by these two inequalities, we get

$$\psi_{\text{hom}}(w) = \psi(w),$$

that means that the convex and 2-homogeneous function ψ is the homogenized function of the oscillating constraint problem defined by a function φ , having R_φ as the level set $\{\varphi = 0\}$. The density result is obtained considering the density of polygons in convex sets of \mathbb{R}^2 .

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