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Abstract

The problem of aggregating multi-agent preference orderings has received considerable attention in many fields of research, such as *multi-criteria decision aiding* and *social choice theory*; nevertheless, the case in which the agents' importance is expressed in the form of a rank-ordering, instead of a set of weights, has not been much debated. The aim of this article is to present a novel algorithm – denominated as "Ordered Paired-Comparisons Algorithm" (OPCA), which addresses this decision-making problem in a relatively simple and practical way. The OPCA is organized into three main phases: (i) turning multi-agent preference orderings into sets of paired comparisons, (ii) synthesizing the paired-comparison sets, and (iii) constructing a fused (or consensus) ordering. Particularly interesting is phase two, which introduces a new aggregation process based on a priority sequence, obtained from the agents' importance rank-ordering. A detailed description of the new algorithm is supported by practical examples.

Keywords: Decision making, Multiple rank-ordered agents, Preference ordering, Fusion, Ordinal semi-democratic, Paired comparison, OPCA.

1. Introduction

A general decision-making problem is that of aggregating multi-agent preference orderings of different alternatives into a single fused ordering. Assume that M decision-making agents[1] (D_1 to D_M) formulate preference orderings among n alternatives of interest (a, b, c, d, etc.). The objective is to aggregate the M agents' orderings into a single fused ordering, which should reflect them as much as possible; for this reason, the fused ordering can also be defined as *consensus* or *compromise* ordering [1, 2]. This aggregation should also take into account the agents' importance, which is not necessarily equal for all of them.

This decision-making problem is very diffused in a variety of real-life contexts, ranging from *multi-criteria decision aiding* [3] to *social choice theory* [4, 5]. Some of the reasons for this diffusion are that: (i) preference orderings are probably the most intuitive and effective way to represent preference judgments of alternatives [6], and (ii) they do not require a common scale – neither

[1] By a decision-making agent we will consider any of a wide variety of different types of entities. Examples could be human beings, individual criteria in a multi-criteria decision process, software based intelligent agents on the Internet, etc..

numeric, linguistic or ordinal – to be shared by the interacting agents [7].

The literature includes a variety of algorithms or aggregation techniques, which can be generally divided in two categories [8]: (i) methods in which all agents have the same importance [9, 10, 11], and (ii) methods in which agents have recognised attributes and/or privileged positions of power, represented by weights [3, 12, 13, 14].

Regarding the second category of methods, in some practical contexts weights are not available and/or their definition can be arbitrary and controversial. For example, weights are often imposed according to political strategies; e.g., the scientific committee of a competitive examination for promotion of faculty members may (arbitrarily) decide that scientific publications will account for 40% of the total performance, research projects for 20%, teaching activity for 30%, etc.. Although the literature provides several techniques for guiding weight quantification – for example, the AHP procedure [15, 16], the method proposed in [17], or that in [18] – they are often neglected in practice, probably because of their complexity.

For these contexts, the problem of weight assignment is partially overcome by expressing the agents' importance in the form of a rank-ordering – such as $D_1 > (D_2 \sim D_3) > \ldots > D_M$ – instead of a set of weights defined on a *cardinal* scale. In fact, the formulation of such a rank-ordering is certainly simpler and more intuitive than that of a set of weights, especially when the agent importance prioritization is uncertain [6].

This paper will focus on this specific problem, which can be denominated as *ordinal semi-democratic*; the adjective *semi-democratic* indicates that agents do not necessarily have the same importance, while *ordinal* indicates that their rank is defined by a crude ordering. This makes the set of the possible solutions relatively wide, since they may range between the two extreme situations of *dictatorship* – in which the resulting fused ordering basically reflects the preference ordering by the most important agent (dictator) – and *democracy* – where the agents' preference orderings are considered as equi-important.

The ordinal semi-democratic decision-making problem is intriguing for two features: (i) the way the preference orderings are compared, and (ii) the way they are synthesized into a fused ordering, which should also reflect the agents' importance rank-ordering. Despite the adaptability to a large number of practical contexts, this specific problem has received little attention in the literature. Yager [7] proposed an algorithm, hereafter abbreviated as YA (which stands for Yager's Algorithm), which addresses the problem in a relatively simple, fast and automatable way. Unfortunately, this algorithm has some limitations: (i) it is applicable to linear preference orderings only, with neither incomparabilities nor omissions of the alternatives [19], (ii) the resulting fused ordering may sometimes not reflect the preference ordering for the majority of agents [20], and (iii) the fused ordering is determined neglecting an important part of the information available [21]. These limitations will be clarified in the next sections.

The objective of this paper is to introduce a new algorithm, denominated as "Ordered Paired-Comparisons Algorithm" (hereafter abbreviated as OPCA), able to overcome the YA's limitations. The main features of this algorithm are that (i) agents' preference orderings are decomposed into sets of paired comparisons of the alternatives, and (ii) the different importance of agents determines a different priority sequence when comparing and synthesizing these sets into a fused ordering.

The remainder of the paper is organized into three sections. Sect. 2 recalls the YA in detail. Sect. 3 illustrates the OPCA. The description of both algorithms is supported by practical examples. Sect. 4 presents a structured comparison of the two algorithms, aimed at highlighting the advantages of the OPCA with respect to the YA. The concluding section summarizes the original contributions of this paper and its practical implications, limitations and suggestions for future research.

2. Basics of the Yager's Algorithm (YA)

In this section we recall the YA. For a more rigorous description, we refer the reader to the original contribution by Yager [7].

The algorithm can be schematized in the following three basic phases, which are described individually in Sects. 2.1 to 2.3:

- construction and reorganization of preference vectors;
- definition of the reading sequence;
- construction of the fused ordering.

2.1 Construction and reorganization of preference vectors

The YA is applicable to (non-strict[2]) *linear* orderings only. The goal of this phase is building preference vectors based on the preference orderings by the agents. For each agent's vector, we place the alternatives as they appear in the ordering, with the most preferred one(s) in the top positions. If at any point $p > 1$ alternatives are tied (i.e., indifferent), we place them in the same element and then place the null set ("Null") in the next $p - 1$ lower positions. For example, when considering three alternatives (a, b and c) with the ordering $(a \sim b) > c$, the resulting vector will conventionally be $[\{a \sim b\}, \text{Null}, \{c\}]^T$. By adopting this convention, the number (n) of elements of a vector will coincide with the number of alternatives of interest.

Considering four fictitious agents (D_1 to D_4), with four relevant orderings of six alternatives (a, b, c, d, e and f), and assuming a certain (linear) rank-ordering between agents (i.e., $D_4 > (D_2 \sim D_3) > D_1$), the resulting preference vectors can be constructed as shown in Tab. 1. For simplicity, vectors will be denominated as the relevant agents (i.e., D_i). Each vector element can be associated with an indicator (j) depicting the position/level of the element, in the preference vector.

[2] The adjective "non-strict" means that these orderings allow the relationship of indifference ("~") between alternatives. For simplicity, the adjective will be omitted hereafter.

Tab. 1. Construction of preference vectors related to the orderings by four fictitious agents (D_1 to D_4).

Agents		D_1	D_2	D_3	D_4
Preference orderings		$b > a > (d \sim e) > f > c$	$c > b > (a \sim d \sim e) > f$	$b > (a \sim c) > f > (d \sim e)$	$a > c > b > d > e > f$
	j	Elements	Elements	Elements	Elements
	6	$\{b\}$	$\{c\}$	$\{b\}$	$\{a\}$
	5	$\{a\}$	$\{b\}$	$\{a, c\}$	$\{c\}$
Preference vectors	4	$\{d, e\}$	$\{a, d, e\}$	Null	$\{b\}$
	3	Null	Null	$\{f\}$	$\{d\}$
	2	$\{f\}$	Null	$\{d, e\}$	$\{e\}$
	1	$\{c\}$	$\{f\}$	Null	$\{f\}$

$n = 6$ total alternatives are considered: a, b, c, d, e and f.
The agents' importance ordering is $D_4 > (D_2 \sim D_3) > D_1$.

Next, preference vectors are transformed into "reorganized" vectors, conventionally denominated as D_i^*. This transformation consists in (i) sorting the D_i vectors decreasingly with respect to the agents' importance and (ii) aggregating those with indifferent importance (e.g., D_2 and D_3 in the example) into a single vector. This aggregation is performed through a level-by-level union of the vector elements, where alternatives in elements with the same (j-th) position are considered as indifferent. The resulting D_i^* vectors will therefore have a strictly decreasing importance ordering. Going back to the example in Tab. 1, the four vectors (D_1 and D_4) are turned into three reorganised vectors (D_1^* to D_3^*, see Tab. 2). It can be noted that D_2^* – given by the aggregation of two vectors with equal importance (i.e., D_2 and D_3) – contains two occurrences for each alternative. Of course, the total number of "reorganized" vectors will be smaller than or equal to the number (M) of initial preference orderings (3 against 4 in the example presented).

2.2 Definition of the reading sequence

This phase defines a sequence for reading the elements of the D_i^* vectors, according to the following pseudo-code:

1. Initialise the sequence number to $S = 0$.
2. Consider the elements with lowest position, by setting $j = 1$.
3. Consider the most important D_i^* vector, by setting $i = 1$.
4. Set $S = S + 1$.
5. Associate the element of interest with the sequence number S.
6. If i is lower than the total number of D_i^* vectors, then:
7. Set $i = i + 1$.
8. Consider the element with position j, related to the i-th D_i^* vector.
9. Go To Step 4.
10. End If.
11. If $j < n$ (i.e., total number of alternatives), then:
12. Set $j = j + 1$.
13. Go To Step 3.

14. End If.

15. End.

The sequence defines a *bottom-up* level-by-level reading of vector elements. The first elements read are those with lowest position (j = 1). When considering elements with the same (j-th) position, priority is given to the vectors from agents of greater importance. After having read all the elements with (j-th) position, we move up to the (j+1)-th position, repeating the reading sequence. Tab. 2 reports the sequence numbers (S) associated with each element of the reorganized vectors in the example presented.

Tab. 2. Reorganized vectors (D_i^*) related to the four preference vectors in Tab. 1 and relevant sequence numbers (S).

Agents		D_1^* (D_4)		D_2^* ($D_2 \sim D_3$)		D_3^* (D_1)	
	j	S	Elements	S	Elements	S	Elements
Reorganized vectors	6	16	$\{a\}$	17	$\{b, c\}$	18	$\{b\}$
	5	13	$\{c\}$	14	$\{a, b, c\}$	15	$\{a\}$
	4	10	$\{b\}$	11	$\{a, d, e\}$	12	$\{d, e\}$
	3	7	$\{d\}$	8	$\{f\}$	9	Null
	2	4	$\{e\}$	5	$\{d, e\}$	6	$\{f\}$
	1	1	$\{f\}$	2	$\{f\}$	3	$\{c\}$

2.3 Construction of the fused ordering

This third phase is aimed at determining a fused ordering through a gradual selection of the alternatives. The following pseudo-code illustrates the algorithm for constructing the fused ordering:

1. Initialise the gradual ordering to “Null”.
2. Initialise S = 1.
3. Consider the element with sequence number S.
4. If the element is not “Null”, then:
5. Identify the alternative(s) in the element of interest.
6. If all these alternatives are not yet present in the gradual ordering, then:
7. Include the alternative(s) not yet present at the top of the gradual ordering. Tied alternatives should be considered as indifferent (~).
8. If the gradual ordering includes all the (n) alternatives, then:
9. Go to Step 15.
10. End If.
11. End If.
12. End If.
13. Increment $S = S + 1$.
14. Go to Step 3.
15. The final fused ordering is given by the gradual ordering.
16. End.

The YA can be classified as an AND-ing type as for an alternative to be in a higher positions of the fused ordering, it should be in a higher position for any of the individual orderings (i.e., AND relationship). Reversing the perspective, a generic alternative is excluded from the higher positions of the fused ordering when it is in a lower position in (at least) one of the individual preference orderings.

Applying the algorithm to the vectors in Tab. 2, the resulting fused ordering is $a > b > d > e > c > f$. Tab. 3 shows the gradual construction of the fused ordering; the first two columns report the S value of the element of interest and the alternative(s) that it contains, while the last two report the alternatives not yet included in the gradual ordering and the gradual ordering itself.

Tab. 3. Step-by-step construction of the fused ordering when applying the YA to the example in Tab. 1.

Step (S)	Element	Residual alternatives	Gradual ordering
0	-	$\{a, b, c, d, e, f\}$	Null
1	$\{f\}$	$\{a, b, c, d, e\}$	f
2	$\{f\}$	$\{a, b, c, d, e\}$	f
3	$\{c\}$	$\{a, b, d, e\}$	$c > f$
4	$\{e\}$	$\{a, b, d\}$	$e > c > f$
5	$\{d, e\}$	$\{a, b\}$	$d > e > c > f$
6	$\{f\}$	$\{a, b\}$	$d > e > c > f$
4	$\{d\}$	$\{a, b\}$	$d > e > c > f$
8	$\{f\}$	$\{a, b\}$	$d > e > c > f$
9	Null	$\{a, b\}$	$d > e > c > f$
10	$\{b\}$	$\{a\}$	$b > d > e > c > f$
11	$\{a, d, e\}$	Null	$a > b > d > e > c > f$
End	-	-	-

3. Ordered Paired-Comparisons Algorithm (OPCA)

This section introduces the OPCA, supporting the description with a practical example. Sect. 3.1 focuses on the input data admitted by this algorithm, while the remaining three subsections (i.e., Sects. 3.2 to 3.4) illustrate the three basic phases of this algorithm (see the scheme in Fig. 1):

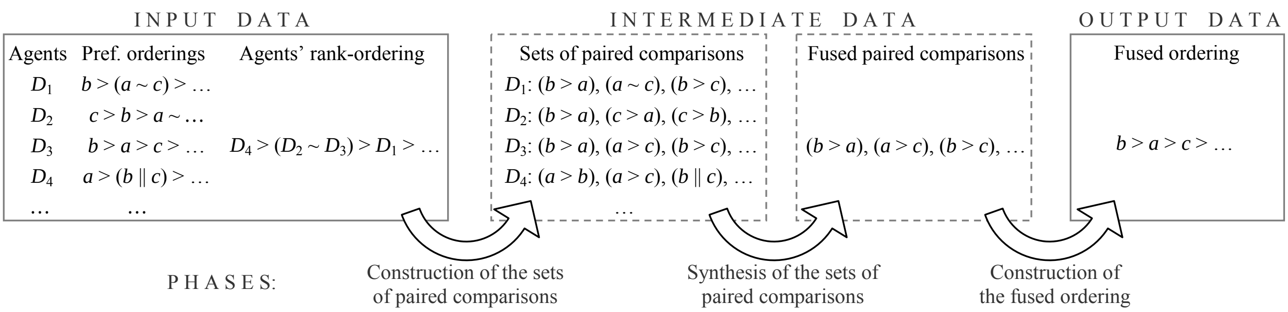


Fig. 1. Characteristic phases of the OPCA.

- Construction of the sets of paired comparisons;
- Synthesis of the sets of paired comparisons;
- Construction of the fused ordering.

The second phase is the core of the algorithm and relies on an original aggregation process, in which the agents' importance determines a priority sequence for synthesizing the sets of paired comparisons.

3.1 OPCA input data

As explained in Sect. 2.1, the YA is applicable to *linear* orderings only, where no alternatives are omitted and any two alternatives are comparable. The authors believe that – to fit a relatively large amount of practical contexts – the general ordinal semi-democratic decision-making problem should admit orderings in which some alternatives are omitted and/or incomparable with each other. According to the Mathematics' Order theory, such orderings are classified as *partial* [19] and can be diagrammed as graphs with branches, which determine different possible paths from the element(s) at the top to that one(s) at the bottom. If two alternatives are not comparable, there exists no direct path from the first to the second one (or viceversa).

By admitting partial preference orderings, agents would not be forced to include dubious alternatives in their preference orderings or to make dubious comparisons. For the purpose of example, let us consider the preference orderings illustrated in Fig. 2; the alternatives in this fictitious decision-making problem are *a*, *b*, *c*, *d*, *e* and *f*, and the agents' importance ordering is assumed to be $D_4 > (D_2 \sim D_3) > D_1$. It can be noticed that the partial ordering by agent D_1 includes two possible paths (A and B); the alternatives positioned along path A (i.e., *a*, *d*, and *e*) are not comparable with that one positioned along path B (i.e., *f*). Also, orderings do not necessarily include all the alternatives; e.g., alternatives *a* and *e* are omitted by D_2, while *f* is omitted by D_4.

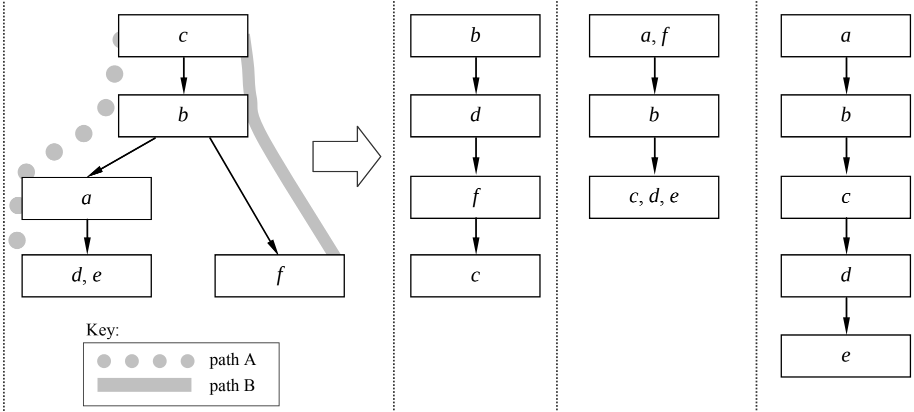
Agent	D_1	D_2	D_3	D_4
Type of preference ordering	partial	linear	linear	linear
Ordering (analytic form)	$c > b > \{[a > (d \sim e)] \parallel f\}$	$b > d > f > c$	$(a \sim f) > b > (c \sim d \sim e)$	$a > b > c > d > e$
(graphic form)				
Alternatives of interest	$\{a, b, c, d, e, f\}$	$\{b, c, d, f\}$	$\{a, b, c, d, e, f\}$	$\{a, b, c, d, e\}$
Omitted alternatives	None	$\{a, e\}$	None	$\{f\}$
Incomparable alternatives	(a, d and e) with f	None	None	None

Fig. 2. Graphical representation of the preference orderings by four fictitious agents (D_1 to D_4). The alternatives in the decision-making problem are *a*, *b*, *c*, *d*, *e* and *f*. The agents' rank-ordering is assumed to be $D_4 > (D_2 \sim D_3) > D_1$. Symbols ">", "~" and "||" respectively depict the strict preference, indifference and incomparability relationship.

3.2 Construction of the sets of paired comparisons

At this stage, agents' preference orderings are turned into corresponding sets of paired comparison relationships. Since the decision-making problem in Fig. 2 includes $n = 6$ alternatives, there will be $C_2^6 = 15$ total paired comparisons (see the first column in Tab. 4). For each paired comparison there are three possible relationships: "strict preference" (">"), "indifference" ("~") and "incomparability ("||").

The decision of using paired-comparison relationships is motivated by several reasons:

1. They allow to express the preference between two alternatives in a natural and intuitive way;
2. They represent a practical expedient for splitting the agents' preference orderings into elementary elements (i.e., the paired-comparison relationships) and facilitating the comparison of the orderings.
3. They can be derived from both linear or partial preference orderings.
4. They could also be derived from agents' judgements expressed in other forms (e.g., measurements/evaluations on ordinal/interval/ratio scales), as long as they admit relationships of ordering among the alternatives.

Chen et al. [22] have recently suggested an algorithm that, similarly to the OPCA, uses paired comparisons for synthesising the preference orderings. This technique is not directly comparable to the OPCA as it requires the agents' importance hierarchy to be defined by a set of weights and not by a (linear) rank-ordering.

Tab. 4. Decomposition of the preference orderings from Fig. 2 into sets of paired-comparison relationships.

Agent		D_1	D_2	D_3	D_4
Preference orderings		$c > b > \{[a > (d \sim e)] \parallel f\}$	$b > d > f > c$	$(a \sim f) > b > (c \sim d \sim e)$	$a > b > c > d > e$
Alternatives of interest		$\{a, b, c, d, e, f\}$	$\{b, c, d, f\}$	$\{a, b, c, d, e, f\}$	$\{a, b, c, d, e\}$
Omitted alternatives		None	$\{a, e\}$	None	$\{f\}$
Incomparable alternatives		(a, d and e) with f	None	None	None
Paired-comparison relationships	(a, b)	$b > a$	$a \parallel b$	$a > b$	$a > b$
	(a, c)	$c > a$	$a \parallel c$	$a > c$	$a > c$
	(a, d)	$a > d$	$a \parallel d$	$a > d$	$a > d$
	(a, e)	$a > e$	$a \parallel e$	$a > e$	$a > e$
	(a, f)	$a \parallel f$	$a \parallel f$	$a \sim f$	$a \parallel f$
	(b, c)	$c > b$	$b > c$	$b > c$	$b > c$
	(b, d)	$b > d$	$b > d$	$b > d$	$b > d$
	(b, e)	$b > e$	$b \parallel e$	$b > e$	$b > e$
	(b, f)	$b > f$	$b > f$	$f > b$	$b \parallel f$
	(c, d)	$c > d$	$d > c$	$c \sim d$	$c > d$
	(c, e)	$c > e$	$c \parallel e$	$c \sim e$	$c > e$
	(c, f)	$c > f$	$f > c$	$f > c$	$c \parallel f$
	(d, e)	$d \sim e$	$d \parallel e$	$d \sim e$	$d > e$
	(d, f)	$d \parallel f$	$d > f$	$f > d$	$d \parallel f$
	(e, f)	$e \parallel f$	$e \parallel f$	$f > e$	$e \parallel f$

The symbols ">", "~", and "||" respectively stand for "strictly preferred to", "indifferent to" and "incomparable to".
The agents' importance ordering is $D_4 > (D_2 \sim D_3) > D_1$.

3.2 Synthesis of the sets of paired comparisons

The goal of this phase is to synthesize the agents' sets of paired comparisons into a single set of fused[3] paired comparisons. The synthesis process is based on several steps. First, for a generic paired comparison, each agent assigns a vote to the alternatives, according to the scoring system in Tab. 5. The vote is assigned to the preferred alternative or, in the case alternatives are tied, it is split equally between them (i.e., 0.5 and 0.5). If two alternatives are incomparable for one agent, that agent will be excluded from voting. Among the possible scoring systems, the proposed one seems relatively natural and intuitive.

Tab. 5. Possible paired-comparison relationships between two generic alternatives (*a* and *b*) and corresponding scores.

Relationship		Score assigned to the alternatives	
		a	*b*
$a > b$	*a* preferred to *b*	1	0
$b > a$	*b* preferred to *a*	0	1
$a \sim b$	*a* indifferent to *b*	0.5	0.5
$a \parallel b$	*a* incomparable to *b*	*N/A*	*N/A*

Next, for a generic paired comparison, the following parameters are determined:

m i.e., the number of voting agents, excluding those for which the two alternatives of interest are incomparable. Obviously, m will coincide with the total score given by the voting agents and it will be smaller than or equal to M.

q i.e., a *quorum* threshold. If $m \geq q$, the synthesis process is performed as explained later in this section; if $m < q$, the synthesis process is aborted, determining a fused paired-comparison relationship of incomparability ("||").

t i.e., a preference threshold used in the synthesis process; later in this section we will explain how to determine suitable values of t.

The use of q is for preventing dubious fused paired-comparison relationships, in situations where agents find it difficult to compare the two alternatives of interest. For example, regarding the paired comparison (a, f) or (e, f), one agent only (out of four) is able to compare these pairs of alternatives. In these situations, it seems reasonable that the synthesis results in an incomparability relationship, when m does not reach the threshold q.

We conventionally set:

$$q = M / 2, \qquad (1)$$

although we are aware that this threshold may be varied depending on the required "degree of prudence".

Tab. 6(a) contains the vote assignment related to the paired comparisons in Tab. 4.

[3] The adjective "fused" indicates that each paired-comparison relationship represents the fusion between the homologous paired-comparison relationships, obtained from the preference orderings.

Tab. 6. (a) Assignment of the scores to the paired-comparison relationships. (b) Process for synthesising the individual sets of paired comparisons into a single one. For each paired comparison, we report the turn-by-turn score and the corresponding cumulative score (labelled as "CUM").

(a)

Paired-comparison relationships and scores	Sets of paired-comparison relationships D_1	D_2	D_3	D_4
(a, b)	$b > a$	$a \parallel b$	$a > b$	$a > b$
a	0	N/A	1	1
b	1	N/A	0	0
(a, c)	$c > a$	$a \parallel c$	$a > c$	$a > c$
a	0	N/A	1	1
c	1	N/A	0	0
(a, d)	$a > d$	$a \parallel d$	$a > d$	$a > d$
a	1	N/A	1	1
c	0	N/A	0	0
(a, e)	$a > e$	$a \parallel e$	$a > e$	$a > e$
a	1	N/A	1	1
e	0	N/A	0	0
(a, f)	$a \parallel f$	$a \parallel f$	$a \sim f$	$a \parallel f$
a	N/A	N/A	0.5	N/A
f	N/A	N/A	0.5	N/A
(b, c)	$c > b$	$b > c$	$b > c$	$b > c$
b	0	1	1	1
c	1	0	0	0
(b, d)	$b > d$	$b > d$	$b > d$	$b > d$
b	1	1	1	1
d	0	0	0	0
(b, e)	$b > e$	$b \parallel e$	$b > e$	$b > e$
b	1	N/A	1	1
e	0	N/A	0	0
(b, f)	$b > f$	$b > f$	$f > b$	$b \parallel f$
b	1	1	0	N/A
f	0	0	1	N/A
(c, d)	$c > d$	$d > c$	$c \sim d$	$c > d$
c	1	0	0.5	1
d	0	1	0.5	0
(c, e)	$c > e$	$c \parallel e$	$c \sim e$	$c > e$
c	1	N/A	0.5	1
e	0	N/A	0.5	0
(c, f)	$c > f$	$f > c$	$f > c$	$c \parallel f$
c	1	0	0	N/A
f	0	1	1	N/A
(d, e)	$d \sim e$	$d \parallel e$	$d \sim e$	$d > e$
d	0.5	N/A	0.5	1
e	0.5	N/A	0.5	0
(d, f)	$d \parallel f$	$d > f$	$f > d$	$d \parallel f$
d	N/A	1	0	N/A
f	N/A	0	1	N/A
(e, f)	$e \parallel f$	$e \parallel f$	$f > e$	$e \parallel f$
e	N/A	N/A	0	N/A
f	N/A	N/A	1	N/A

(b)

Paired-comparison relationships and scores	Turn 1 (D_4)		Turn 2 ($D_2 \sim D_3$)			Turn 3 (D_1)		m	q	t	Fused paired comparisons
(a, b)	$a > b$	CUM	$a \parallel b$	$a > b$	CUM	$b > a$	CUM				
a	1	1	N/A	1	2	0	2	3	2	1.25	$a > b$
b	0	0	N/A	0	0	1	1				
(a, c)	$a > c$	CUM	$a \parallel c$	$a > c$	CUM	$c > a$	CUM				
a	1	1	N/A	1	2	0	2	3	2	1.25	$a > c$
c	0	0	N/A	0	0	1	1				
(a, d)	$a > d$	CUM	$a \parallel d$	$a > d$	CUM	$a > d$	CUM				
a	1	1	N/A	1	2	1	3	3	2	1.25	$a > d$
d	0	0	N/A	0	0	0	0				
(a, e)	$a > e$	CUM	$a \parallel e$	$a > e$	CUM	$a > e$	CUM				
a	1	1	N/A	1	2	1	3	3	2	1.25	$a > e$
e	0	0	N/A	0	0	0	0				
(a, f)	$a \parallel f$	CUM	$a \parallel f$	$a \sim f$	CUM	$a \parallel f$	CUM				
a	N/A	0	N/A	0.5	0.5	N/A	0.5	1	2	$0.75^{(*)}$	$a \parallel f$
f	N/A	0	N/A	0.5	0.5	N/A	0.5				
(b, c)	$b > c$	CUM	$b > c$	$b > c$	CUM	$c > b$	CUM				
b	1	1	1	1	3	0	3	4	2	1.5	$b > c$
c	0	0	0	0	0	1	1				
(b, d)	$b > d$	CUM	$b > d$	$b > d$	CUM	$b > d$	CUM				
b	1	1	1	1	3	1	4	4	2	1.5	$b > d$
d	0	0	0	0	0	0	0				
(b, e)	$b > e$	CUM	$b \parallel e$	$b > e$	CUM	$b > e$	CUM				
b	1	1	N/A	1	2	1	3	3	2	1.25	$b > e$
e	0	0	N/A	0	0	0	0				
(b, f)	$b \parallel f$	CUM	$b > f$	$f > b$	CUM	$b > f$	CUM				
b	N/A	0	1	0	1	1	2	3	2	1.25	$b > f$
f	N/A	0	0	1	0	0	0				
(c, d)	$c > d$	CUM	$d > c$	$c \sim d$	CUM	$c > d$	CUM				
c	1	1	0	0.5	1.5	1	2.5	4	2	1.5	$c \sim d$
d	0	0	1	0.5	1.5	0	1.5				
(c, e)	$c > e$	CUM	$c \parallel e$	$c \sim e$	CUM	$c > e$	CUM				
c	1	1	N/A	0.5	1.5	1	2.5	3	2	1.25	$c > e$
e	0	0	N/A	0.5	0.5	0	0.5				
(c, f)	$c \parallel f$	CUM	$f > c$	$f > c$	CUM	$c > f$	CUM				
c	N/A	0	0	0	0	1	1	3	2	1.25	$f > c$
f	N/A	0	1	1	2	0	2				
(d, e)	$d > e$	CUM	$d \parallel e$	$d \sim e$	CUM	$d \sim e$	CUM				
d	1	1	N/A	0.5	1.5	0.5	2	3	2	1.25	$d > e$
e	0	0	N/A	0.5	0.5	0.5	1				
(d, f)	$d \parallel f$	CUM	$d > f$	$f > d$	CUM	$d \parallel f$	CUM				
d	N/A	0	1	0	1	N/A	1	2	2	1	$d \sim f$
f	N/A	0	0	1	1	N/A	1				
(e, f)	$e \parallel f$	CUM	$e \parallel f$	$f > e$	CUM	$e \parallel f$	CUM				
e	N/A	0	N/A	0	0	N/A	0	1	2	$0.75^{(*)}$	$e \parallel f$
f	N/A	0	N/A	1	1	N/A	1				

The agents' importance ordering is $D_4 > (D_2 \sim D_3) > D_1$.
The parameter m represents the number of agents formulating a paired-comparison relationships different from "$\parallel$"; q is the quorum threshold, which is determined by using Eq. 1; t is the preference threshold, which is conventionally set to the t^* value, determined according to Eq. 2 (shown later in this section).
(*) for the paired comparisons (a, f) and (e, f), the condition $m \geq q$ is not satisfied, therefore the result of the synthesis process is an incomparability relationship ("$\parallel$").

Agents' votes are reorganized according to two criteria: (i) agents are sorted decreasingly with respect to their importance rank-ordering, and (ii) for equi-important agents (such as D_2 e D_3, in the example in Fig. 2), the corresponding scores are aggregated. Tab. 6(b) contains the reorganized votes related to the paired comparisons in Tab. 4.

In the case $m \geq q$, i.e., when the quorum threshold is reached, the synthesis process is performed as follows. For each paired comparison (a, b), the agents' votes are examined gradually, proceeding in descending order with respect to their relative importance. If two (or more) agents have indifferent importance, their votes are examined in the same turn (e.g., the votes by D_2 and D_3 are both examined in Turn 2, see Tab. 6(b)). Of course, the total number of turns will be smaller than or equal to the number (M) of agents (3 against 4 in the example presented). Simplifying, the synthesis process establishes that the first alternative whose cumulative score ("CUM") reaches a preference threshold t (that we will focus on later) is the preferred one. The following pseudo-code illustrates the synthesis of the agents' paired comparisons into a fused one:

```
Initialise the cumulative scores of a and b (CUM(a) and CUM(b) respectively) to 0.
If m < q (quorum is not reached), then:
    The resulting fused paired comparison is a || b
Else If m ≥ q (quorum is reached), then:
    Set the value of the preference threshold t.
    For each (i-th) turn:
        CUM(a) = CUM(a) + the score of a in that turn.
        CUM(b) = CUM(b) + the score of b in that turn.
        If CUM(a) ≥ t OR CUM(b) ≥ t, then:
            Exit For.
        End If.
    End For.
    If CUM(a) ≥ t AND CUM(b) < t, then:
        The resulting fused paired comparison is a > b.
    Else If CUM(a) < t AND CUM(b) ≥ t, then:
        The resulting fused paired comparison is b > a.
    Else If CUM(a) ≥ t AND CUM(b) ≥ t, then:
        If CUM(a) = CUM(b), then:
            The resulting fused paired comparison is a ~ b.
        Else If CUM(a) > CUM(b), then:
            The resulting fused paired comparison is a > b.
        Else If CUM(a) < CUM(b), then:
            The resulting fused paired comparison is b > a.
        End If.
    Else If CUM(a) < t AND CUM(b) < t, then:
        The resulting fused paired comparison is a || b.
    End If.
End If.
End.
```

More precisely, the fused paired comparison may consists of three possible relationships:

1. *Indifference*. It occurs when both the alternatives reach t at the same turn and, until that turn, they have had the same score. For example, considering the paired comparison (c, d) in the example in Tab. 6, the resulting relationship is $c \sim d$, since the two alternatives both reach t (i.e., 1.5) in turn 2 and have the same cumulative score at that moment.

2. *Strict preference* of one alternative with respect to the other one (i.e., $a > b$ or $b > a$). It occurs when (i) one alternative reaches t in a certain turn while the other one not, or (ii) both the alternatives reach t in a certain turn (and possibly exceed it), but the cumulative score ("CUM") until that turn is higher for one alternative (i.e., the preferred one) with respect to the other one. For example, considering the paired comparison (a, b) in the example in Tab. 6, the resulting relationship is $a > b$, since the alternative a reaches t in turn 2, whilst b not.
3. *Incomparability* ($a \parallel b$). It can occur in two different cases: (i) when $m < q$ (i.e., when the quorum threshold is not reached by the number of agents involved in the synthesis process, i.e., those formulating paired-comparison relationships different from "||") and (ii) when $m \geq q$ but none of the two alternatives reaches t, even though all agents have assigned their vote. The latter situation can occur when using relatively high values of t.

In the proposed synthesis approach, the different "voting power" of agents determines a different priority order when expressing their (unitary) vote. This is the most important difference with respect to other approaches in which the agent vote is weighted and there is no priority order when voting [3, 12, 13, 14].

Let us now focus the attention on the rationale for choosing a suitable t value. As a first consideration, when $m/2 < t \leq m$, the result of the synthesis is not affected by the agents' voting order. In other words, when $t \in \left]m/2, m\right]$, the criterion for selecting one alternative over another – in a generic paired comparison (a, b) – degenerates into that of the majority, regardless of the agents' voting order. For instance, if a has a total score larger than or equal to t, the synthesis will certainly result in $a > b$, since there will be no voting sequence for which b can reach t before a. Let us consider the example in Tab. 7, in which the votes by four agents should be synthesized into a fused paired comparison (a, b). Three of all the possible agents' rank-orderings are considered. When $t \in \left]m/2 = 2, m = 4\right]$, the result of the synthesis does not depend on the voting sequence (it is always $a \parallel b$ in this case). For this reason, $\left]m/2, m\right]$ can be classified as range of *no-voting-order-effect* (see Fig. 3).

Tab. 7. Example of synthesis of paired-comparison judgements by four agents. The synthesis result is calculated by varying the agents' rank-ordering and the t value.

	Paired-comparison relationships			
	D_1	D_2	D_3	D_4
(a, b)	$a > b$	$b > a$	$b > a$	$a > b$
a	1	0	0	1
b	0	1	1	0

Agents' rank orderings	Fused paired comparison		
	$t = 1$	$1 < t \leq m/2$	$m/2 < t \leq m$
1: $D_1 > D_2 > D_3 > D_4$	$a > b$	$b > a$	$a \parallel b$
2: $D_3 > D_2 > D_1 > D_4$	$b > a$	$b > a$	$a \parallel b$
3: $(D_1 \sim D_3) > D_4 > D_2$	$a \sim b$	$a > b$	$a \parallel b$
4: $D_1 \sim D_2 \sim D_3 \sim D_4$	$a \sim b$	$a \sim b$	$a \parallel b$
N: …	?	?	$a \parallel b$

In this case, $M = 4$.

We remark that, as t increases within this range, the required level of agreement between agents increases. The extreme case is $t = m$, corresponding to a *unanimity* situation where the fused paired comparison will always result into an incomparability relationship, except when all the *m* agents involved share the same preference or indifference relationship.

On the other hand, in the case $1 \le t \le m/2$, the voting order of agents may affect the result of the synthesis process. For the purpose of example, Tab. 7 shows that, when $t = 1$ or $1 < t \le m/2$, results may change depending on the voting order. Since the agents' voting order may affect the result of the synthesis, $[1, m/2]$ may be denominated as range of *voting-order-effect*. There are two extreme cases: (i) $t = 1$, which denotes the *dictatorship* situation, in which the fused paired comparison coincides with that of the most important agent, and (ii) $t = m/2$, which denotes the borderline situation with respect to the *no-voting-order-effect* range.

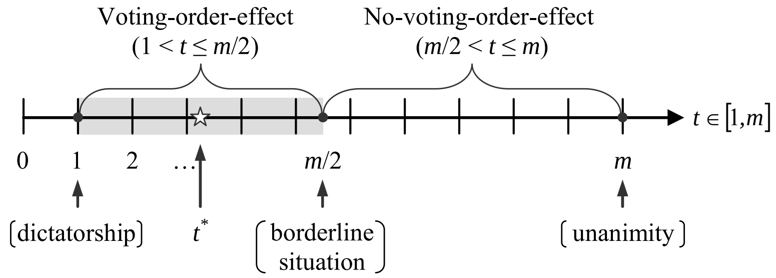


Fig. 3. Schematic subdivision of the range of variability of the *t* threshold. The parameter *m* depicts the number of agents involved in the synthesis process.

Based on the previous considerations, a reasonable value of *t* can be that in the middle of the *voting-order-effect* range (see the representation in Fig. 3, where it is denoted by a star), i.e.:

$$t = t^* = \frac{1 + m/2}{2}. \tag{2}$$

The t^* value is purely conventional and a different value – as long as included in the *voting-order-effect* range – could lead to slightly different synthesis results. The *t* values reported in the semi-last column of Tab. 6(b) are obtained applying Eq. 2; the last column contains the resulting fused paired-comparison relationships.

3.3 Construction of the fused ordering

This stage aims at summarizing the set of fused paired-comparison relationships into a fused ordering. This problem is not new in the scientific literature and a variety of techniques have been proposed over the years. The oldest technique is probably that by Condorcet [9, 11, 23], while in recent decades several approaches based on the concept of "graph kernel" (borrowed from *graph theory*) have gained a certain popularity [3, 24].

Without going into the pros and cons of the different techniques, we suggest to use the so-called "Paired Comparison Chart" (PCC) by Dym et al. [25]. Despite some limitations, this technique is relatively simple and effective.

Let us briefly illustrate it, considering the set of fused paired comparisons resulting from Tab. 6(b). Comparisons are reported in a matrix, as illustrated in Tab. 8. For a generic paired comparison, we assign a score to the alternatives according to the same scoring system in Tab. 5: in the case of strict preference, the preferred alternative earns one point; in the case of indifference, the point is divided equally between the two alternatives (i.e., 0.5 and 0.5), while in the case of incomparability, it is not assigned to any alternative (and the field is conventionally filled with "*N/A*").

Next, the sum of the scores of each alternative is determined ("Σ" in Tab. 8). In the classical PCC procedure – which does not admit incomparability relationships between alternatives – the most preferred alternatives are those with higher Σ values. However, the use of Σ values would penalize alternatives that cannot be compared to some others. For example, the alternatives a and b in Tab. 8 have the same Σ value; however, while a is preferred to the totality of the alternatives (except f, due to the incomparability relationship $a \parallel f$), b loses the comparison with a.

This distortion can be avoided by normalizing the Σ values, dividing them by the relevant C values, i.e., the numbers of "usable" paired comparisons for which the alternatives of interest do not result in any incomparability relationship. For example, in Tab. 8 the alternative f has $C = 3$, since it is only comparable to b, c and d. By changing the perspective, the C value represents the maximum score achievable by a certain alternative, considering the usable paired comparisons.

Having said that, the most preferred alternatives are those with a higher "Σ/C" score (see the column labelled with "Σ/C" in Tab. 8). In the case exemplified in Tab. 8, the fused ordering is $a > b > f > d > c > e$.

Tab. 8. Example of Paired Comparison Chart related to the fused paired comparisons in the last column of Tab. 6(b). The fused ordering is constructed according to the normalized score (in the column labelled with "Σ/C").

	a	*b*	*c*	*d*	*e*	*f*	Σ	C	Σ/C
a	-	1	1	1	1	*N/A*	4	4	1
b	0	-	1	1	1	1	4	5	0.8
c	0	0	-	0.5	1	0	1.5	5	0.3
d	0	0	0.5	-	1	0.5	2	5	0.4
e	0	0	0	0	-	*N/A*	0	4	0
f	*N/A*	0	1	0.5	*N/A*	-	1.5	3	0.5

Resulting fused ordering:
$a(1) > b(0.8) > f(0.5) > d(0.4) > c(0.3) > e(0)$

Σ is the total score of each alternative; C is the number of paired comparisons usable for determining the total score of each alternative.

A "prudence threshold" (Q) can be introduced to prevent the alternatives which are difficult to compare with other ones from being dubiously included in the fused ordering: when the C value of one alternative does not reach Q, the alternative is excluded from the fused ordering. The threshold Q can be conventionally set to $Q = n/2$, n being the total number of alternatives of the decision-making problem. According to this convention, alternatives that are not comparable with (at least) the majority of the remaining ones (i.e., alternatives for which $C < Q$) are preventively excluded from the fused ordering. In the example in Tab. 8, all the alternatives meet the minimum requirement $C \geq Q = 3$ (since $n = 6$), therefore it is not necessary to exclude any alternative.

A particular feature of the PCC technique is that it always returns a fused ordering, also when the fused paired comparisons do not satisfy the property of transitivity [9]. On the other hand, it can be demonstrated that this technique may violate the Arrow's axiom of *independence of irrelevant alternatives* [8]; however, Dym et al. [25] showed that the negative consequences of this feature are not crucial. For specific situations, other more sophisticated techniques can be used to obtain a fused ordering [3].

4. Structured comparison between YA and OPCA

This section presents a structured comparison of the YA and the OPCA, from two perspectives: that of the three *ad hoc*[4] criteria of *consistency*, *efficiency* and *versatility*, and that of some popular axioms borrowed from *social choice theory* [8]; a short description is presented in Tab. 9.

Tab. 9. *Ad hoc* criteria and *social choice theory* axioms for comparing the YA and the OPCA.

Perspective		Description
(a) *Ad hoc* criteria	*Consistency*	The fused ordering should reflect the preference orderings for the majority of agents, especially the most important ones. A practical way to check this is to observe the "compatibility", at the level of individual paired comparisons, between the fused ordering and the agents' preference orderings. We say that a relationship in the fused ordering is consistent if it holds in the majority of the agents' preference orderings.
	Efficiency	Algorithm's ability to use the information contained in the individual preference orderings. For instance, an algorithm that focuses on the lower part of the preference orderings only, or one that ignores the preference orderings of certain agents, cannot be considered as very efficient.
	Versatility	The versatility criterion can be related to two different aspects: (i) the algorithm's ability to adapt to a variety of input data and (ii) the ability to adapt to a democratic case, i.e., when all agents are equi-important. Both these peculiarities make one algorithm potentially applicable to a great amount of practical contexts.
(b) Axioms from *social choice theory*	*Idempotency*	If all of the preference vectors are the same, the resulting fused ordering is this one.
	Monotonicity	If any agent modifies his or her preference ordering by promoting a certain alternative, then the fused ordering should respond only by promoting that same alternative or not changing, never by placing it lower than before.
	Non-dictatorship	The algorithm should account for the wishes of multiple agents. It cannot simply mimic the preference ordering of a single agent (dictator).
	Unrestricted domain or *universality*	For any set of individual agent preference orderings, the algorithm should yield a unique and complete ranking of the alternatives.
	Independence of irrelevant alternatives	The preference between x and y should depend only on the individual preferences between x and y. In other words, if one alternative is removed, then the algorithm should still create the same ordering of the remaining alternatives.
	Non-imposition or *citizen sovereignty*	Every ranking of the alternatives should be possible as outcome.
	Pareto-efficiency	If every voter prefers a certain alternative over another, then the fused ordering must prefer this alternative over the other too.

Considering the *consistency* criterion, the YA is somehow weak, as noted by Wang [20] and Franceschini et al. [21]. This aspect is evident when analysing the YA's results to the decision-making problem in Tab. 1: the paired-comparison relationships from the fused ordering and those from the individual preference orderings have several inconsistencies; e.g., for agents D_2, D_3 and D_4 (which represents the majority and, by the way, are all more important than D_1), $c > d$ and $c > e$, while these relationships are reversed in the fused ordering. Among the fifteen overall paired comparisons, four – i.e., more than 25%! – look inconsistent or dubious (see Tab. 10). These

[4] The adjective "*ad hoc*" indicates that these criteria were specifically defined for (i) facilitating the comparison between the YA and the OPCA, and (ii) highlighting the advantages of the latter algorithm with respect to the former one.

inconsistencies are due to the YA's logic of selection of the alternatives, which is rather drastic as the occurrence of one alternative in a low position – even for a single preference ordering – can determine a very low position in the fused ordering. E.g., in the example in Tab. 1, *c* is in the penultimate position of the fused ordering as it was relegated by D_1 at the bottom of the preference ordering.

Applying the OPCA to the same decision-making problem in Tab. 1, we obtain a fused ordering (i.e., $c > b > a > d > e > f$), which is significantly better than that one produced by the YA, since all the paired comparisons – except one – seem consistent (see Tab. 10). This is not an isolated coincidence, but depends on the fact that the phase two of the OPCA tends to maximize the consistency between the fused ordering and the preference orderings. Also, the OPCA's fused ordering seems more reasonable than the YA's, since *c* makes up four positions, consistently with its relatively high rank position in the majority of the agents' preference orderings. Tab. A.1 and Tab A.1 (in appendix) respectively illustrate the construction of the fused paired comparisons and the corresponding PCC.

Tab. 10. Comparison between the YA and the OPCA in terms of consistency between the agents' preference orderings and the fused ordering, at the level of paired comparisons. This comparison is based on the YA's and OPCA's results to the decision-making problem in Tab. 1.

Paired comparison	Relationship in the preference orderings				Relationship in the fused ordering	
	D_4	D_2	D_3	D_1	YA	OPCA
(a, b)	$a > b$	$b > a$	$b > a$	$b > a$	$a > b$	$b > a$
(a, c)	$a > c$	$c > a$	$a \sim c$	$a > c$	$a > c$	$c > a$
(a, d)	$a > d$	$a \sim d$	$a > d$	$a > d$	$a > d$	$a > d$
(a, e)	$a > e$	$a \sim e$	$a > e$	$a > e$	$a > e$	$a > e$
(a, f)	$a > f$	$a > f$	$a > f$	$a > f$	$a > f$	$a > f$
(b, c)	$c > b$	$c > b$	$b > c$	$b > c$	$b > c$	$c > b$
(b, d)	$b > d$	$b > d$	$b > d$	$b > d$	$b > d$	$b > d$
(b, e)	$b > e$	$b > e$	$b > e$	$b > e$	$b > e$	$b > e$
(b, f)	$b > f$	$b > f$	$b > f$	$b > f$	$b > f$	$b > f$
(c, d)	$c > d$	$c > d$	$c > d$	$d > c$	$d > c$	$c > d$
(c, e)	$c > e$	$c > e$	$c > e$	$e > c$	$e > c$	$c > e$
(c, f)	$c > f$	$c > f$	$c > f$	$f > c$	$c > f$	$c > f$
(d, e)	$d > e$	$d \sim e$	$d \sim e$	$d \sim e$	$d > e$	$d > e$
(d, f)	$d > f$	$d > f$	$f > d$	$d > f$	$d > f$	$d > f$
(e, f)	$e > f$	$e > f$	$f > e$	$e > f$	$e > f$	$e > f$

The fused orderings obtained by the YA and OPCA are respectively $a>b>d>e>c>f$ and $c>b>a>d>e>f$.
The cells highlighted in grey depict the inconsistent or dubious results.

For the purpose of further example, let us analyse the consistency of the OPCA's solution to the decision-making problem exemplified in Fig. 2. The fused ordering seems to reflect the agents' (partial and linear) preference orderings quite well, since almost the totality of the paired-comparison relationships are consistent (see Tab. A.3, in appendix).

Considering the *efficiency* criterion, the YA does not performs very well since it tends to overlook the upper positions of the preference orderings; e.g., in the example proposed in Tab. 1 the fused ordering is determined after having read just eleven out of eighteen total elements; in particular, the two top levels of the preference vectors have been totally ignored (see Tab. 3).

The OPCA is significantly more efficient in this sense, since the fused ordering is determined considering the totality of the elements in the preference orderings, not only those in the lower positions.

The *versatility* criterion can be analyzed from the following two angles: (i) the ability to adapt to a variety of input data and (ii) the ability to adapt to a democratic case. Regarding the first aspect, the YA is applicable to *linear* preference orderings only, while the OPCA is significantly more versatile, since it admits *partial* preference orderings, with omitted and/or incomparable alternatives. Regarding the second aspect, it can be shown that the YA may lose its effectiveness in the case of *democracy*. For example, let us assume that all the four preference orderings in Tab. 1 are equi-important. The individual orderings would be merged into a single reorganized vector (in Tab. A.4(a), in appendix) and the reading sequence of the vector elements would be trivial: i.e., from the bottom to the top. The resulting fused ordering would be $(a \sim b) > (d \sim e) > (c \sim f)$, which lacks in discrimination power, since it contains nothing less than three relationships of indifference (for six total alternatives); see the step-by-step construction in Tab. A.4(b) (in appendix).

As regards the OPCA, in the case of democracy the equi-important agents would express their vote in a single turn. Tab. A.5 (in appendix) reports the synthesis process and the resulting fused paired comparisons when using $t = t^* = 1.5$ (since $m = 4$ for each paired comparison). Next, the set of fused paired comparisons is turned into a fused ordering, through the PCC method (see the matrix in Tab. A.5(b), in appendix). The resulting fused ordering would be $b > a > c > d > e > f$, which seems to have an acceptable discrimination power (i.e., it contains no indifference relationships).

Tab. 11 presents a concise comparison between the YA and the OPCA, from the point of view of the three *ad hoc* criteria examined so far and some popular axioms from the *social choice theory*, showing whether they are met (or not) by the YA and the OPCA. The mathematical proof is left to the reader. Despite their substantial differences, the two algorithms meet the same axioms.

5. Conclusions

This paper presented the OPCA, a new algorithm for aggregating preference orderings into a fused one. Similarly to the YA, the OPCA can be applied to specific problems in which the agents' importance is expressed in the form of a rank-ordering. The OPCA has three main advantages with respect to the YA: (i) it is more consistent, since the fused ordering better reflects the multi-agent preference orderings, (ii) it is more efficient in using the information available, and (iii) it is more versatile, since it admits partial preference orderings (with omitted and/or incomparable alternatives) and can be applied effectively even when agents are equi-important. Also, it is intuitive and easily automatable.

Tab. 11. Concise comparison between the YA and the OPCA, on the basis of (a) the *ad hoc* criteria and (b) the axioms from *social choice theory*, illustrated in Tab. 9. The symbols "✓" and "×"respectively indicate the axioms

satisfied or not by the two algorithms.

	Criterion	YA	OPCA
(a) *Ad hoc* criteria	*Consistency*	Limited, due to the (rather drastic) mechanism for selecting the alternatives in the fused ordering.	Good, thanks to the new logic, which is based on paired-comparison data derived from the preference orderings.
	Efficiency	Limited, as the fused ordering is determined after having read a relatively small portion of the vector elements, overlooking the upper positions.	Good, since the OPCA better uses the information contained in the preference orderings.
	Versatility	Limited, as (i) it admits *linear* preference orderings only and (ii) it does not perform well in the case of equi-important agents.	Good, as (i) it admits *partial* preference orderings, with omitted and/or incomparable alternatives, and (ii) it performs quite well even in the case of equi-importance agents.
(b) *Social choice theory* axioms	*Idempotency*	✓	✓(*)
	Monotonicity	✓	✓
	Non-dictatorship	✓	✓
	Unrestricted domain or *universality*	✓	✓
	Independence of irrelevant alternatives	✗	✗
	Non-imposition or *citizen sovereignty*	✓	✓
	Pareto-efficiency	✓	✓

(*) Since the fused ordering is linear, the *idempotency* axiom is satisfied in the case agents' preference orderings are linear.

The most original part of the OPCA is the aggregation of the agents' sets of paired comparisons into a single one. Contrary to other approaches, in this stage, the different "voting power" of agents determines a different voting priority, and not a different weight of the vote.

The choice of the t value makes it possible to switch with a continuity from the situation of *dictatorship* (when $t = 1$) to that of *democracy* (when $t > m/2$). For synthesising the set of fused paired comparisons into a fused ordering, we suggested to use the PCC method because of its simplicity and effectiveness [25]. The fact remains that it can be replaced by more sophisticated approaches from the existing literature.

Future research go in two directions: (i) sensitivity analysis of the robustness of the OPCA with respect to small variations in the preference orderings and/or in the t, q and Q thresholds, (ii) application of the algorithm to various decision-making frameworks.

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Appendix

See the following tables.

Tab. A.1. Application of the OPCA to the decision-making problem in Tab. 1: (a) decomposition of the preference orderings in Tab. 1 into sets of paired-comparison relationships and assignment of the scores to the alternatives; (b) process for synthesising the individual sets of paired comparisons into a single one. For each paired comparison, we report the turn-by-turn score and the corresponding cumulative score (labelled as "CUM"); in this example, the threshold t was conventionally set to $t = t^* = 1.5$, since $m = 4$ for each paired comparison.

(a)

Paired-comparison relationships and scores	Sets of paired-comparison relationships D_1	D_2	D_3	D_4
(a, b)	$b > a$	$b > a$	$b > a$	$a > b$
a	0	0	0	1
b	1	1	1	0
(a, c)	$a > c$	$c > a$	$a \sim c$	$a > c$
a	1	0	0.5	1
c	0	1	0.5	0
(a, d)	$a > d$	$a \sim d$	$a > d$	$a > d$
a	1	0.5	1	1
d	0	0.5	0	0
(a, e)	$a > e$	$a \sim e$	$a > e$	$a > e$
a	1	0.5	1	1
e	0	0.5	0	0
(a, f)	$a > f$	$a > f$	$a > f$	$a > f$
a	1	1	1	1
f	0	0	0	0
(b, c)	$b > c$	$c > b$	$b > c$	$c > b$
b	1	0	1	0
c	0	1	0	1
(b, d)	$b > d$	$b > d$	$b > d$	$b > d$
b	1	1	1	1
d	0	0	0	0
(b, e)	$b > e$	$b > e$	$b > e$	$b > e$
b	1	1	1	1
e	0	0	0	0
(b, f)	$b > f$	$b > f$	$b > f$	$b > f$
b	1	1	1	1
f	0	0	0	0
(c, d)	$d > c$	$c > d$	$c > d$	$c > d$
c	0	1	1	1
d	1	0	0	0
(c, e)	$e > c$	$c > e$	$c > e$	$c > e$
c	0	1	1	1
e	1	0	0	0
(c, f)	$f > c$	$c > f$	$c > f$	$c > f$
c	0	1	1	1
f	1	0	0	0
(d, e)	$d \sim e$	$d \sim e$	$d \sim e$	$d > e$
d	0.5	0.5	0.5	1
e	0.5	0.5	0.5	0
(d, f)	$d > f$	$d > f$	$f > d$	$d > f$
d	1	1	0	1
f	0	0	1	0
(e, f)	$e > f$	$e > f$	$f > e$	$e > f$
e	1	1	0	1
f	0	0	1	0

(b)

Paired-comparison relationships and scores	Turn 1 (D_4)		Turn 2 ($D_2 \sim D_3$)			Turn 3 (D_1)		Fused paired comparisons ($t^* = 1.5$)
(a, b)	$a > b$	CUM	$b > a$	$b > a$	CUM	$b > a$	CUM	
a	1	1	0	0	1	0	1	$b > a$
b	0	0	1	1	2	1	3	
(a, c)	$a > c$	CUM	$c > a$	$a \sim c$	CUM	$a > c$	CUM	
a	1	1	0	0.5	1.5	1	2.5	$a \sim c$
c	0	0	1	0.5	1.5	0	1.5	
(a, d)	$a > d$	CUM	$a \sim d$	$a > d$	CUM	$a > d$	CUM	
a	1	1	0.5	1	2.5	1	3.5	$a > d$
d	0	0	0.5	0	0.5	0	0.5	
(a, e)	$a > e$	CUM	$a \sim e$	$a > e$	CUM	$a > e$	CUM	
a	1	1	0.5	1	2.5	1	3.5	$a > e$
e	0	0	0.5	0	0.5	0	0.5	
(a, f)	$a > f$	CUM	$a > f$	$a > f$	CUM	$a > f$	CUM	
a	1	1	1	1	3	1	4	$a > f$
f	0	0	0	0	0	0	0	
(b, c)	$c > b$	CUM	$c > b$	$b > c$	CUM	$b > c$	CUM	
b	0	0	0	1	1	1	2	$c > b$
c	1	1	1	0	2	0	2	
(b, d)	$b > d$	CUM	$b > d$	$b > d$	CUM	$b > d$	CUM	
b	1	1	1	1	3	1	4	$b > d$
d	0	0	0	0	0	0	0	
(b, e)	$b > e$	CUM	$b > e$	$b > e$	CUM	$b > e$	CUM	
b	1	1	1	1	3	1	4	$b > e$
e	0	0	0	0	0	0	0	
(b, f)	$b > f$	CUM	$b > f$	$b > f$	CUM	$b > f$	CUM	
b	1	1	1	1	3	1	4	$b > f$
f	0	0	0	0	0	0	0	
(c, d)	$c > d$	CUM	$c > d$	$c > d$	CUM	$d > c$	CUM	
c	1	1	1	1	3	0	3	$c > d$
d	0	0	0	0	0	1	1	
(c, e)	$c > e$	CUM	$c > e$	$c > e$	CUM	$e > c$	CUM	
c	1	1	1	1	3	0	3	$c > e$
e	0	0	0	0	0	1	1	
(c, f)	$c > f$	CUM	$c > f$	$c > f$	CUM	$f > c$	CUM	
c	1	1	1	1	3	0	3	$c > f$
f	0	0	0	0	0	1	1	
(d, e)	$d > e$	CUM	$d \sim e$	$d \sim e$	CUM	$d \sim e$	CUM	
d	1	1	0.5	0.5	2	0.5	2.5	$d > e$
e	0	0	0.5	0.5	1	0.5	1.5	
(d, f)	$d > f$	CUM	$d > f$	$f > d$	CUM	$d > f$	CUM	
d	1	1	1	0	2	1	3	$d > f$
f	0	0	0	1	1	0	1	
(e, f)	$e > f$	CUM	$e > f$	$f > e$	CUM	$e > f$	CUM	
e	1	1	1	0	2	1	3	$e > f$
f	0	0	0	1	1	0	1	

The agents' importance ordering is $D_4 > (D_2 \sim D_3) > D_1$.

Tab. A.2. Paired Comparison Chart related to the set of fused paired comparisons resulting from Tab. A.1(b). A fused ordering is constructed based on the total score earned by alternatives (in the column labelled with "Σ"). The Σ values do not need to be normalized, because of the lack of relationships of incomparability between the alternatives.

	a	*b*	*c*	*d*	*e*	*f*	Σ
a	-	0	0.5	1	1	1	3.5
b	1	-	0	1	1	1	4
c	0.5	1	-	1	1	1	4.5
d	0	0	0	-	1	1	2
e	0	0	0	0	-	1	1
f	0.5	0	0	0	0	-	0.5

Resulting fused ordering:
$c(4.5) > b(4) > a(3.5) > d(2) > e(1) > f(0.5)$

Tab. A.3. Consistency analysis of the OPCA's solution to the decision-making problem in Fig. 2. The compatibility of the fused ordering with the agents' preference orderings is evaluated at the level of paired-comparison relationships.

Paired comparison	Relationship in the preference orderings D_4	D_2	D_3	D_1	Relationship in the fused ordering
a, *b*	$a > b$	$a \parallel b$	$a > b$	$b > a$	$a > b$
a, *c*	$a > c$	$a \parallel c$	$a > c$	$c > a$	$a > c$
a, *d*	$a > d$	$a \parallel d$	$a > d$	$a > d$	$a > d$
a, *e*	$a > e$	$a \parallel e$	$a > e$	$a > e$	$a > e$
a, *f*	$a \parallel f$	$a \parallel f$	$a \sim f$	$a \parallel f$	$a \parallel f$
b, *c*	$b > c$	$b > c$	$b > c$	$c > b$	$b > c$
b, *d*	$b > d$	$b > d$	$b > d$	$b > d$	$b > d$
b, *e*	$b > e$	$b \parallel e$	$b > e$	$b > e$	$b > e$
b, *f*	$b \parallel f$	$b > f$	$f > b$	$b > f$	$b > f$
c, *d*	$c > d$	$d > c$	$c \sim d$	$c > d$	$c \sim d$
c, *e*	$c > e$	$c \parallel e$	$c \sim e$	$c > e$	$c > e$
c, *f*	$c \parallel f$	$f > c$	$f > c$	$c > f$	$f > c$
d, *e*	$d > e$	$d \parallel e$	$d \sim e$	$d \sim e$	$d > e$
d, *f*	$d \parallel f$	$d > f$	$f > d$	$d \parallel f$	$d \sim f$
e, *f*	$e \parallel f$	$e \parallel f$	$f > e$	$e \parallel f$	$e \parallel f$

The fused orderings obtained by the OPCA is $a>b>f>d>c>e$.
The cells highlighted in grey depict dubious paired-comparison relationships from the fused ordering.

Tab. A.4. Application of the YA to the four preference orderings in Tab. 1, assuming that agents are equi-important (i.e., $D_1 \sim D_2 \sim D_3 \sim D_4$): (a) single reorganized vector; (b) step-by-step construction of the fused ordering.

(a)

D_1^* ($D_1 \sim D_2 \sim D_3 \sim D_4$)		
j	*S*	Elements
6	6	{*a*, (2)*b*, *c*, *f*}
5	5	{(2)*a*, *b*, (2)*c*}
4	4	{*a*, *b*, (2)*d*, (2)*e*}
3	3	{*d*, *f*}
2	2	{*d*, (2)*e*, *f*}
1	1	{*c*, (2)*f*}

(b)

Step (*S*)	Element	Residual alternatives	Gradual ordering
0	-	{*a*, *b*, *c*, *d*, *e*, *f*}	Null
1	{*c*, (2)*f*}	{*a*, *b*, *d*, *e*}	$c \sim f$
2	{*d*, (2)*e*, *f*}	{*a*, *b*}	$(d \sim e) > (c \sim f)$
3	{*d*, *f*}	{*a*, *b*}	$(d \sim e) > (c \sim f)$
4	{*a*, *b*, (2)*d*, (2)*e*}	Null	$(a \sim b) > (d \sim e) > (c \sim f)$
End	-	-	-

Tab. A.5. Application of the OPCA to the four preference orderings in Tab. 1, assuming that agents are equi-important (i.e., $D_1 \sim D_2 \sim D_3 \sim D_4$): (a) process for synthesising the sets of paired comparisons into a set of fused paired comparisons (t was conventionally set to $t^* = 1.5$, since $m = 4$ for each paired comparison); (b) Paired Comparison Chart related to the set of fused paired comparisons.

(a)

		Turn 1 ($D_1 \sim D_2 \sim D_3 \sim D_4$)				Fused paired comparisons ($t^* = 1.5$)
Paired-comparison relationships and scores	(a, b)	$a > b$	$b > a$	$b > a$	$b > a$	
	a	1	0	0	0	$b > a$
	b	0	1	1	1	
	(a, c)	$a > c$	$c > a$	$a \sim c$	$a > c$	
	a	1	0	0.5	1	$a > c$
	c	0	1	0.5	0	
	(a, d)	$a > d$	$a \sim d$	$a > d$	$a > d$	
	a	1	0.5	1	1	$a > d$
	c	0	0.5	0	0	
	(a, e)	$a > e$	$a \sim e$	$a > e$	$a > e$	
	a	1	0.5	1	1	$a > e$
	e	0	0.5	0	0	
	(a, f)	$a > f$	$a > f$	$a > f$	$a > f$	
	a	1	1	1	1	$a > f$
	f	0	0	0	0	
	(b, c)	$c > b$	$c > b$	$b > c$	$b > c$	
	b	0	0	1	1	$b \sim c$
	c	1	1	0	0	
	(b, d)	$b > d$	$b > d$	$b > d$	$b > d$	
	b	1	1	1	1	$b > d$
	d	0	0	0	0	
	(b, e)	$b > e$	$b > e$	$b > e$	$b > e$	
	b	1	1	1	1	$b > e$
	e	0	0	0	0	
	(b, f)	$b > f$	$b > f$	$b > f$	$b > f$	
	b	1	1	1	1	$b > f$
	f	0	0	0	0	
	(c, d)	$c > d$	$c > d$	$c > d$	$d > c$	
	c	1	1	1	0	$c > d$
	d	0	0	0	1	
	(c, e)	$c > e$	$c > e$	$c > e$	$e > c$	
	c	1	1	1	0	$c > e$
	e	0	0	0	1	
	(c, f)	$c > f$	$c > f$	$c > f$	$f > c$	
	c	1	1	1	0	$c > f$
	f	0	0	0	1	
	(d, e)	$d > e$	$d \sim e$	$d \sim e$	$d \sim e$	
	d	1	0.5	0.5	0.5	$d > e$
	e	0	0.5	0.5	0.5	
	(d, f	$d > f$	$d > f$	$f > d$	$d > f$	
	d	1	1	0	1	$d > f$
	f	0	0	1	0	
	(e, f)	$e > f$	$e > f$	$f > e$	$e > f$	
	e	1	1	0	1	$e > f$
	f	0	0	1	0	

(b)

	a	b	c	d	e	f	Σ
a	-	0	1	1	1	1	4
b	1	-	0.5	1	1	1	4.5
c	0	0.5	-	1	1	1	3.5
d	0	0	0	-	1	1	2
e	0	0	0	0	-	1	1
f	0	0	0	0	0	-	0

Fused ordering:
$b(4.5)>a(4)>c(3.5)>d(2)>e(1)>f(0)$