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A. Halanay

Department of Mathematics and Informatics, University Politehnica of Bucharest
313 Splaiul Independentei, 060042 Bucharest, Romania
and Institute of mathematics Simion Stoilow of the Romanian Academy

L. Pandolfi

Dipartimento di Scienze Matematiche "Giuseppe Luigi Lagrange", Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino, Italy

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Abstract

Heat equations with memory of Gurtin-Pipkin type (i.e. Eq. (1) with $\alpha = 0$) have controllability properties which strongly resemble those of the wave equation. Instead, recent counterexamples show that when $\alpha > 0$ the control properties do not parallel those of the (memoryless) heat equation, in the sense that there are square integrable initial conditions which cannot be controlled to zero. The proof of this fact, in previous papers, consists in the construction of two quite special examples of systems with memory which cannot be controlled to zero. Here we prove that lack of controllability holds in general, for every smooth memory kernel $M(t)$.

*A. Halanay: halanay@mathem.pub.ro; L. Pandolfi: luciano.pandolfi@polito.it The research of the first author is partially supported by Romanian CNCS Grant PN-II-ID-PCE-2011-3-0211. The research of the second author fits the plans of INDAM-CNR and of the project "Groupement de Recherche en Contrôle des EDP entre la France et l'Italie (CONEDP)".

1 Introduction

The following integro-differential equation is often used to model thermal systems with memory, see [1, 8, 16]:

$$w_t = \alpha w_{xx} + \int_0^t M(t-s) w_{xx}(x,s)\,\mathrm{d}\,s\,, \qquad w(x,0) = \xi(x),. \tag{1}$$

Here $w = w(x,t)$. The variable x belongs to an interval which we normalize to $(0,\pi)$. The time $t = 0$ is the time after which a boundary control f is applied to the system, i.e. we assume the boundary conditions

$$w(0,t) = f(t)\,, \quad w(\pi,t) = 0 \quad t > 0\,.$$

Note that we implicitly assume that the system is at rest for negative times, $w(t) = 0$ if $t < 0$.

The number α is nonnegative. If α is zero then we get a model proposed by Gurtin and Pipkin in [13]. The controllability, when $\alpha = 0$, has been studied in several paper, see references below. So, here we explicitly assume

$$\alpha > 0$$

and we call Eq. (1) the (CGM) model (after Colemann and Gurtin).

It appears that (CGM) has been rarely studied from the control point of view. Our goal in this paper is to understand whether the point $\xi_0 = 0$ can be hit at time $T > 0$, as in the case for the memoryless heat equation, i.e. the special case of (CGM) obtained when $M(t) \equiv 0$.

The precise definition of controllability requires that we specify the properties of the solutions. The following results are proved in Section 2, where the definition of "solution" can be found:

Theorem 1 *Let $M(t) \in C^1(0,+\infty)$. For every $f \in L^2(0,T)$ and for every initial condition $\xi \in L^2(0,\pi)$ there exists a unique solution $w(\cdot,T) = w^{f,\xi}(\cdot,T) \in L^2(0,T;L^2(0,\pi))$.*

The solution is not continuous in time (see, when $M = 0$, the example in [20, p. 217]), unless $f(t)$ is smooth. So, pointwise computation of $w(\cdot,t)$ in $L^2(0,\pi)$ is meaningless in general. However, let A be the operator in $L^2(0,\pi)$:

$$\mathrm{dom}\, A = H^2(0,\pi) \cap H_0^1(0,\pi)\,, \qquad A\phi = \phi''\,. \tag{2}$$

Note that A^{-1} exists and it is bounded. Then we have:

Corollary 1 *Let $M(t) \in C^1(0,+\infty)$. For every function $f \in L^2(0,T)$ and for every initial condition $\xi \in L^2(0,\pi)$, the function $t \mapsto A^{-1} w^{f,\xi}(\cdot,t)$ is continuous from $[0,+\infty)$ to $L^2(0,\pi)$.*

Thanks to this result, the following definition makes sense:

Definition 2 *We say that the initial condition ξ is controllable to* 0 *at time T if there exists $f \in L^2(0,T)$ such that $A^{-1}w^{f,\xi}(\cdot;T) = 0$.*

We say that (CGM) is null controllable at time T if for every $\xi \in L^2(0,\pi)$ there exists $f \in L^2(0,T)$ such that $A^{-1}w^{f,\xi}(\cdot;T) = 0$.

In the memoryless case, $M(t) \equiv 0$, the system is null controllable at *any* time $T > 0$. When $M(t) \neq 0$ but $M(t) = 0$ for $0 \leq t \leq T_0$ then Eq. (1) for $t \leq T_0$ coincide with the memoryless heat equation $w_t = \alpha w_{xx}$ and any initial condition can be controlled to 0 at any time $T < T_0$. Keeping this fact in mind, our main result is:

Theorem 2 *Let $\alpha > 0$ and let $M(t) \in C^1(0,T)$, not identically zero. Let T be any time such that $R(T) \neq 0$, where $R(t)$ is the resolvent kernel of $M(t)$.*

There exist initial data ξ which cannot be controlled to 0 *at time T.*

1.1 Comments and references

Under smoothness assumption on the kernel $M(t)$, when $\alpha = 0$ and $M(0) > 0$, Eq. (1) can be seen as a perturbation of the wave equation and its properties resemble those of the wave equation. In particular, the solutions belong to $C(0,+\infty; L^2(0,\pi))$ for every $f \in L^2_{\rm loc}(0,+\infty)$ and every initial condition $\xi \in L^2(0,\pi)$. Furthermore, there exists T such that the reachable set

$$\left\{ w^{f,0}(\cdot,T)\,, \quad f \in L^2(0,T) \right\}$$

is equal to $L^2(0,\pi)$. Several different techniques have been used in the proof, but the basic idea is always to compare with the wave equation, see [2, 17, 23, 24, 26, 28]. Furthermore, the infimum of the control times is the same as that for the (memoryless) wave equation (see [4, 10, 17, 25, 27]).

Instead, when $\alpha > 0$ the properties of Eq. (1) strongly resemble those of the standard, memoryless, heat equation in spite that it is not possible to control an initial condition to be identically zero for every $t > T$, where T is a preassigned time, see [15]. So, it is a natural conjecture that the controllability properties of system (1) with $\alpha > 0$ should be similar to those of the (memoryless) heat equation. Along this line of thought, it was proved in [5] that, for a suitable class of completely monotonic kernels, the reachable states at every time $T > 0$ are dense in $L^2(0,\pi)$ and this supports the conjecture that every initial condition $\xi \in L^2(0,\pi)$ can be controlled to hit the target $\xi_0(x) \equiv 0$ at a certain time T, of course without remaining equal to zero in the future, due to the negative results in [15]. This conjecture was disproved in [12, 14, 29]. These papers show that *there exist* kernels $M(t)$ which are even of class C^∞, and such that for every $T > 0$ there exist initial data which cannot be controlled to hit 0. The proofs in these papers exibits particular counterexamples to controllability. The goal of this paper is the proof that in the presence of memory, i.e. for *every* smooth kernel $M(t)$ not identically zero, there exist initial conditions which cannot be controlled to zero, as stated in Theorem 2.

We mention that the papers [21, 7] proves controllability for the system studied in [29] (a generalized telegraph equation) if the control is distributed and if the subset on which the control acts is not constant in time.

2 Preliminaries

The number α has to be positive and so, changing the time scale, i.e. replacing $w(x,t)$ with $w(x,t/\alpha)$, we can assume

$$\alpha = 1\,.$$

This transformation changes $M(t)$ to $M(t/\alpha)$ which is renamed $M(t)$.

We present a transformation which simplifies the computations in this paper. We consider a Volterra integral equation on $t \geq 0$

$$y(t) + \int_0^t M(t-s)y(s)\,\mathrm{d}\,s = f(t)\,.$$

It is known (see [11, Ch. 2]) that it is uniquely solvable for every square integrable $f(t)$, and that the solution is given by

$$y(t) = f(t) - \int_0^t R(t-s)f(s)\,\mathrm{d}\,s\,.$$

The function $R(t)$, the resolvent kernel of $M(t)$, solves

$$R(t) = M(t) - \int_0^t M(t-s)R(s)\,\mathrm{d}\,s\,.$$

We apply formally this transformation, "solving" Eq. (1) with respect to the "unknown" w_{xx}. We get

$$w_t = w_{xx} + \int_0^t R(t-s)w_s(s)\,\mathrm{d}\,s\,.$$

Integrating by parts we get

$$w_t = w_{xx} + aw(t) + \int_0^t L(t-s)w(s)\,\mathrm{d}\,s - R(t)\xi\,, \qquad w(0) = \xi\,. \tag{3}$$

Here,

$$a = R(0) = M(0)\,, \qquad L(t) = R'(t)\,.$$

By definition, a solution of Eq. (1) is a solution of the Volterra integro-differential equation (3) (solutions can be defined in several different but equivalent ways).

We recall that the operator A in (2) is a selfadjoint operator with compact resolvent, which generates a holomorphic semigroup e^{At}.

We introduce the following transformation $D \in \mathcal{L}(\mathbb{R}; L^2(0,\pi))$:

$$u(x) = (Dr)(x) = \frac{(\pi - x)r}{\pi} \text{ so that } u \text{ solves } \begin{cases} u_{xx} = 0 \text{ in } (0,\pi) \\ u(0) = r\,,\ u(\pi) = 0\,. \end{cases}$$

A known fact (see [18, p. 180]) is the following:

Theorem 3 *Let $f \in L^2(0,T)$, $g \in L^2(0,T;L^2(0,\pi))$ and $\xi \in L^2(0,\pi)$. The solution of the heat equation*

$$\theta_t = \theta_{xx} + g\,, \qquad \theta(x,0) = \xi(x)\,, \quad \theta(0,t) = f(t)\,,\ \theta(\pi,t) = 0$$

is given by

$$\theta(\cdot,t) = \theta^{f,\xi,g}(\cdot,t) = e^{At}\xi + \int_0^t e^{A(t-s)} g(s)\,\mathrm{d}\,s - A\int_0^t e^{A(t-s)} Df(s)\,\mathrm{d}\,s\,. \quad (4)$$

The solution is unique in $L^2_{\mathrm{loc}}(0,+\infty;L^2(0,\pi))$ and $A^{-1}\theta(\cdot,t) \in C(0,+\infty;L^2(0,\pi))$. Furthermore, if $\xi = 0$ then $\theta(\cdot,t) \in L^2_{\mathrm{loc}}(0,+\infty,H^{1/2}(0,\pi))$.

We apply formula (4) to (3) with

$$g(t) = aw(t) + \int_0^t L(t-s)w(s)\,\mathrm{d}\,s - R(t)\xi$$

and we find the following Volterra integral equation for $w(x,t)$:

$$\begin{aligned} & w(x,t) - \int_0^t e^{A(t-s)}\left[aw(s) + \int_0^s L(s-r)w(r)\,\mathrm{d}\,r\right]\mathrm{d}\,s \\ & = \left\{e^{At}\xi - \int_0^t e^{A(t-s)}R(s)\xi\,\mathrm{d}\,s\right\} - A\int_0^t e^{A(t-s)}Df(s)\,\mathrm{d}\,s \end{aligned} \quad (5)$$

Theorem 1 and Corollary 1 follow from this formula, thanks to the properties of the solutions of the (memoryless) heat equation stated in Theorem 3.

See [19] for the theory of Volterra integral and integro-differential equations in Banach spaces, and [6] for further information on the semigroup approach to boundary value problems for parabolic equations.

2.1 Projection of the system on the eigenspaces

The previous results allows us to project system (3) on the eigenvectors of the operator A. Let

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx\,, \quad n \in \mathbb{N}\,.$$

So, $\{\phi_n\}$ is an orthonormal basis of $L^2(0,\pi)$, whose elements are eigenvectors of the operator A in (2):

$$\phi_n'' = -n^2\phi_n\,, \qquad \phi_n(0)=0\,,\ \phi_n(\pi)=0\,.$$

Let

$$w_n(t) = \int_0^\pi w(x,t)\phi_n(x)\,\mathrm{d}\,x \qquad \xi_n = \int_0^\pi \xi(x)\phi_n(x)\,\mathrm{d}\,x\,.$$

Then $w_n(t)$ solves

$$w_n'(t) = (a-n^2)w_n + \int_0^t L(t-s)w_n(s)\,\mathrm{d}\,s - R(t)\xi_n + ng(t)\,, \qquad g(t) = \sqrt{\frac{2}{\pi}}f(t)$$

and

$$w(x,t) = \sum \phi_n(x)w_n(t)\,. \tag{6}$$

We introduce

$$\mu_n^2 = n^2 - a$$

(we have $\mu_n > 0$ for large n) so that

$$w_n(t) - \int_0^t e^{-\mu_n^2(t-\tau)}\int_0^\tau L(\tau-s)w_n(s)\,\mathrm{d}\,s\ \mathrm{d}\,\tau$$
$$= \left(e^{-\mu_n^2 t} - \int_0^t e^{-\mu_n^2(t-s)}R(s)\,\mathrm{d}\,s\right)\xi_n + \int_0^t \left(ne^{-\mu_n^2(t-s)}\right)g(s)\,\mathrm{d}\,s\,. \tag{7}$$

Let $T>0$. We define a transformation $\mathcal{L}$ in $L^2(0,T;L^2(0,\pi))$, as follows:

$$\mathcal{L}\left(\sum \phi_n(x)h_n(t)\right) = \sum \phi_n(x)\left(\mathcal{L}_n h_n\right)(t)$$

where

$$\left(\mathcal{L}_n h\right)(t) = \int_0^t e^{-\mu_n^2(t-s)}\int_0^s L(s-r)h(r)\,\mathrm{d}\,r\ \mathrm{d}\,s\,.$$

Then we have

$$(I-\mathcal{L})\,w = \sum \phi_n(x)\left\{\left(e^{-\mu_n^2 t} - \int_0^t e^{-\mu_n^2(t-s)}R(s)\,\mathrm{d}\,s\right)\xi_n\right.$$
$$\left. + \int_0^t \left(ne^{-\mu_n^2(t-s)}\right)g(s)\,\mathrm{d}\,s\right\}\,. \tag{8}$$

We prove:

Lemma 3 *The transformation $\mathcal{L}$ in $L^2(0,T;L^2(0,\pi))$ is linear and continuous. The transformation $(I-\mathcal{L})$ is invertible and its inverse is continuous.*

Proof. Linearity is clear. We prove the continuity of $\mathcal{L}$, using the fact that $\{\phi_n\}$ is an orthonormal basis of $L^2(0,\pi)$. This implies that

$$\left\|\left(\sum h_n(t)\phi_n(x)\right)\right\|^2_{L^2(0,T;L^2(0,\pi))} = \sum \int_0^T |h_n(t)|^2\,\mathrm{d}\,t\,.$$

Then we have:

$$\int_0^T |(\mathcal{L}_n h)(t)|^2 \, \mathrm{d}\, t = \int_0^T \left| \int_0^t e^{-\mu_n^2 (t-s)} \int_0^s L(s-r)h(r)\, \mathrm{d}\, r \; \mathrm{d}\, s \right|^2 \, \mathrm{d}\, t$$
$$\leq T^2 \left(\int_0^T e^{-2\mu_n^2 s} \, \mathrm{d}\, s \right) \left(\int_0^T L^2(s)\, \mathrm{d}\, s \right) \left(\int_0^T h^2(r)\, \mathrm{d}\, r \right) \, \mathrm{d}\, s$$
$$\leq C \int_0^T |h(s)|^2 \, \mathrm{d}\, s\,.$$

We can chose the constant C independent of n thanks to the fact that $\mu_n^2 > 0$ for large n. So, we have

$$\left\| \mathcal{L} \left(\sum h_n(t)\phi_n(x) \right) \right\|^2_{L^2(0,T;L^2(0,\pi))} = \int_0^T \int_0^{,\pi} \left| \sum (\mathcal{L}_n h_n)(t)\phi_n(x) \right|^2 \, \mathrm{d}\, x \; \mathrm{d}\, t$$
$$= \int_0^T \sum |(\mathcal{L}_n h_n)(t)|^2 \, \mathrm{d}\, t \leq C \sum \int_0^T |h_n(s)|^2 \, \mathrm{d}\, s$$
$$= C \left\| \left(\sum h_n(t)\phi_n(x) \right) \right\|^2_{L^2(0,T;L^2(0,\pi))}\,.$$

This proves continuity of the transformation $\mathcal{L}$ and so also of $I - \mathcal{L}$. In order to prove that this last transformation has a bounded inverse, we exibit explicitly its inverse.

To compute the inverse, we must solve, for every $k(x,t) = \sum \phi_n(x)k_n(t)$,

$$(I - \mathcal{L}) \left(\sum \phi_n(x) f_n(t) \right) = k(x,t) = \sum \phi_n(x)k_n(t)$$

i.e.

$$\sum \phi_n \left\{ f_n(t) - \int_0^t f_n(\tau) \int_0^{t-\tau} L(t-\tau-s)e^{-\mu_n^2 s} \, \mathrm{d}\, s \; \mathrm{d}\, \tau \right\} = \sum \phi_n(x)k_n(t)\,.$$

We introduce $H_n(t)$, the resolvent kernels of

$$Z_n(t) = -\int_0^t L(t-s)e^{-\mu_n^2 s} \, \mathrm{d}\, s\,. \tag{9}$$

Then we must choose

$$f_n(t) = k_n(t) - \int_0^t H_n(t-s)k_n(s)\, \mathrm{d}\, s$$

and so

$$(I - \mathcal{L})^{-1} \sum \phi_n(x)k_n(t) = \sum \phi_n(x) \left\{ k_n(t) - \int_0^t H_n(t-s)k_n(s)\, \mathrm{d}\, s \right\}\,.$$

Continuity of this transformation is seen as above, using the fact that $\mu_n^2 > 0$ for large n, so that $|Z_n(t)| \le M/\mu_n^2$ (for large n) where $M = M_T$. So, Gronwall inequality applied to

$$|H_n(t)| \le |Z_n(t)| + \int_0^t |Z_n(s)| \cdot |H_n(s)| \,\mathrm{d}\, s$$

gives

$$|H_n(t)| \le \frac{M}{\mu_n^2}\,, \qquad M = M_T\,. \tag{10}$$

Continuity now follows as above. ∎

Using (8) we find that

$$\begin{aligned}
&w(x,t) = (I-\mathcal{L})^{-1}\sum \phi_n(x)\left\{\left(e^{-\mu_n^2 t} - \int_0^t e^{-\mu_n^2(t-s)}R(s)\,\mathrm{d}\,s\right)\xi_n\right.\\
&\left. - \int_0^t e^{-\mu_n^2(t-s)}g_n(s)\,\mathrm{d}\,s\right\} = \sum \phi_n(x)\left\{-\left[\int_0^t e^{-\mu_n^2(t-s)}g_n(s)\,\mathrm{d}\,s\right.\right.\\
&\left. + \int_0^t H_n(t-\tau)\int_0^\tau e^{-\mu_n^2(\tau-s)}g_n(s)\,\mathrm{d}\,s\;\mathrm{d}\,\tau\right]\\
&+\left[e^{-\mu_n^2 t} - \int_0^t e^{-\mu_n^2(t-s)}R(s)\,\mathrm{d}\,s\right.\\
&\left.\left. - \int_0^t H_n(t-\tau)\left(e^{-\mu_n^2\tau} - \int_0^\tau e^{-\mu_n^2(\tau-s)}R(s)\,\mathrm{d}\,s\right)\,\mathrm{d}\,\tau\right]\xi_n\right\}
\end{aligned} \tag{11}$$

Now we recall the definition of controllability at time T and we can state:

Theorem 4 *Equation (1) is controllable to 0 at time T if for every sequence $\{\xi_n\} \in l^2$ there exists a function $g \in L^2(0,T)$ which solves the following moment problem:*

$$\begin{aligned}
&\left[\int_0^T \left(n e^{-\mu_n^2(T-s)}\right) g(s)\,\mathrm{d}\,s\right.\\
&\left. - \int_0^T H_n(T-\tau)\int_0^\tau \left(n e^{-\mu_n^2(\tau-s)}\right) g(s)\,\mathrm{d}\,s\;\mathrm{d}\,\tau\right]\\
&= -\left[e^{-\mu_n^2 T} - \int_0^T e^{-\mu_n^2(T-s)}R(s)\,\mathrm{d}\,s\right.\\
&\left. - \int_0^T H_n(T-\tau)\left(e^{-\mu_n^2\tau} - \int_0^\tau e^{-\mu_n^2(\tau-s)}R(s)\,\mathrm{d}\,s\right)\,\mathrm{d}\,\tau\right]\xi_n\,.
\end{aligned} \tag{12}$$

The proof of Theorem 2 is then reduced to the proof that this moment problem is not solvable.

3 The proof of Theorem 2

Let N_0 be such that

$$n \geq N_0 \implies \mu_n^2 > 0\,.$$

We shall consider the moment problem in Theorem 4 only for the indices $n \geq N_0$ and we shall prove that it can't be solved.

We first examine the right hand side of (12). We recall that $H_n(t)$ is the resolvent kernel of $Z_n(t)$ in (9) so that the following equality holds:

$$H_n(t) = -\int_0^t L(t-s)e^{-\mu_n^2 s}\,\mathrm{d}\,s + \int_0^t \left[\int_0^{t-\tau} L(t-\tau-s)e^{-\mu_n^2 s}\,\mathrm{d}\,s\right] H_n(\tau)\,\mathrm{d}\,\tau$$

The function $L(t)$ is bounded on $[0,T]$ for every $T>0$ and $\mu_n^2>0$, so, using Gronwall inequality, there exists C (which depends on T but not on n) such that

$$|H_n(t)| \leq C\frac{1}{\mu_n^2}$$

(a fact already used in the proof of Lemma 3).

We fix T such that $R(T)\neq 0$. On every compact interval, using boundedness of $M'(t)$ hence of $R'(t)$, we have:

$$\int_0^T R(s)e^{-\mu_n^2(T-s)}\,\mathrm{d}\,s = \frac{1}{\mu_n^2}\left(R(T) - e^{-\mu_n^2 T}R(0) - \int_0^T e^{-\mu_n^2(T-s)}R'(s)\,\mathrm{d}\,s\right)\,,$$

$$\left|\int_0^T e^{-\mu_n^2(T-s)}R'(s)\,\mathrm{d}\,s\right| \leq \frac{\text{const}}{\mu_n^2}\,,$$

$$\left|\int_0^T H_n(T-\tau)\int_0^\tau e^{\mu_n^2(\tau-s)}R(s)\,\mathrm{d}\,s\ \mathrm{d}\,\tau\right| \leq \frac{\text{const}}{\mu_n^4}\,,$$

$$\left|\int_0^T H_n(T-\tau)e^{-\mu_n^2\tau}\,\mathrm{d}\,\tau\right| \leq \left(\int_0^T e^{-\mu_n^2\tau}\,\mathrm{d}\,\tau\right)\sup_{[0,T]}|H_n(t)| \leq \frac{\text{const}}{\mu_n^4}$$

(inequality (10) is used in the last row).

So, we have also

$$\left|\int_0^T H_n(T-\tau)\left(e^{-\mu_n^2\tau} + \int_0^\tau e^{\mu_n^2(\tau-s)}R(s)\,\mathrm{d}\,s\right)\,\mathrm{d}\,\tau\right| \leq \frac{\text{const}}{\mu_n^4}\,.$$

Let

$$
\begin{aligned}
d_n &= \left[e^{-\mu_n^2 T} - \int_0^T e^{-\mu_n^2 (T-s)} R(s) \,\mathrm{d}\, s \right. \\
&\left. - \int_0^T H_n(T-\tau) \left(e^{-\mu_n^2 \tau} - \int_0^\tau e^{-\mu_n^2(\tau - s)} R(s)\,\mathrm{d}\, s \right) \,\mathrm{d}\,\tau \right] \xi_n \\
&= \left[e^{-\mu_n^2 T} - \frac{1}{\mu_n^2} \left(R(T) - e^{-\mu_n^2 T} R(0) - \int_0^T e^{-\mu_n^2 (T-s)} R'(s)\,\mathrm{d}\, s \right) \right. \\
&\left. - \int_0^T H_n(T-\tau) \left(e^{-\mu_n^2 \tau} - \int_0^\tau e^{-\mu_n^2(\tau - s)} R(s)\,\mathrm{d}\, s \right) \,\mathrm{d}\,\tau \right] \xi_n \,.
\end{aligned}
$$

Using the existence of C such that

$$
\mu_n^2 e^{-\mu_n^2 T} < \frac{C}{\mu_n^2}
$$

the previous equalities, with $R(T) \neq 0$, give

$$
\mu_n^2 d_n = \left(-R(T) + \frac{M_n}{\mu_n^2} \right) \xi_n
$$

where $\{M_n\}$ is a *bounded* sequence. Hence, we get:

Lemma 4 *Let $R(T) \neq 0$. There exists $N > N_0$ with the following property: for every $\{c_n\} \in l^2([N, +\infty))$ the equation in $l^2([N, +\infty))$*

$$
\mu_n^2 d_n = \left(-R(T) + \frac{M_n}{\mu_n^2} \right) \xi_n = c_n
$$

admits a solution $\{\xi_n\} \in l^2([N, +\infty))$.

We go back to the moment problem (12) for $n \geq N$. If equation (1) is controllable to 0 at time T, then the moment problem

$$
\begin{aligned}
&\left[\int_0^T \left(n\mu_n^2 e^{-\mu_n^2 (T-s)} \right) g(s)\,\mathrm{d}\, s \right. \\
&\left. - \int_0^T H_n(T-\tau) \int_0^\tau \left(n\mu_n^2 e^{-\mu_n^2 (\tau-s)} \right) g(s)\,\mathrm{d}\, s \,\mathrm{d}\,\tau \right] = c_n
\end{aligned}
$$

is solvable *for every* sequence $\{c_n\} \in l^2 = l^2(N, +\infty)$. We exchange the order of integration and we rewrite this equalities as

$$
\int_0^T E_n(s) g(T-s)\,\mathrm{d}\, s = c_n \,, \qquad n \geq N \tag{13}
$$

where
$$E_n(s) = n\mu_n^2 \left[e^{-\mu_n^2 s} - \int_0^s e^{-\mu_n^2 (s-\tau)} H_n(\tau)\,\mathrm{d}\,\tau \right]$$
We recall from [3, Theorem I.2.1] that if the moment problem (13) is solvable for every l^2-sequence $\{c_n\}$ $(n \geq N)$ then the sequence $\{E_n(t)\}$ admits a *bounded* biorthogonal sequence $\{\chi_n(t)\}$ in $L^2(0,T)$; i.e. if and only if there exists a *bounded* sequence $\{\chi_n(t)\}$ in $L^2(0,T)$ such that
$$\int_0^T E_n(t)\chi_k(t)\,\mathrm{d}\,t = \delta_{n,k} = \begin{cases} 1 & \text{if} \quad n = k \\ 0 & \text{if} \quad n \neq k\,. \end{cases}$$

We are going to prove that this sequence does not exist, relying on known properties of the (memoryless) heat equation (for a positive result on the same lines, for Schroedinger equation, see [22]). We proceed in two steps: the first step computes "explicitly" $H_n(t)$. The second step, using this expression of $H_n(t)$, shows that *a bounded sequence* $\{\chi_n(t)\}$ *does not exist, i.e. the moment problem is not solvable.*

We proceed with the proof.

Step 1: a formula for $H_n(t)$. Here we find a formula for $H_n(t)$, for every *fixed* index n. So, for clarity, the fixed index n is not indicated in the computations and $H_n(t)$ (any fixed n) is denoted $H(t)$. Analogously, μ_n^2, with n fixed, is indicated as μ^2. Furthermore, we use $\star$ to denote the convolution,
$$f \star g = (f \star g)(t) = \int_0^t f(t-s)g(s)\,\mathrm{d}\,s\,.$$
We shall use the commutativity and the associativity of the convolution:
$$f \star g = g \star f\,, \qquad f \star (g \star h) = f \star (g \star h)\,.$$

The convolution of a function with itself is denoted as follows:
$$f^{\star 1} = f\,, \qquad f^{\star 2} = f \star f\,, \quad f^{\star k} = f \star f^{\star (k-1)}\,.$$
We introduce
$$e_k(t) = \frac{t^k}{k!} e^{-\mu^2 t} \quad \text{so that} \quad e_0 \star e_k = e_{k+1}\,.$$

By definition, $H(t)$ is the resolvent kernel of
$$Z(t) = -\int_0^t L(t-s)e^{-\mu^2 s}\,\mathrm{d}\,s = -L \star e_0\,.$$

We shall use:

Lemma 5 *Let $G(t)$ be any (integrable) function and $\tilde{G} = G \star e_k$. Then,*
$$Z \star \tilde{G} = e_{k+1} \star (-L \star G)$$

In fact:

$$Z \star \tilde{G} = (-L \star e_0) \star (G \star e_k) = (e_0 \star e_k) \star (-L \star G) = e_{k+1} \star (-L \star G)\,.$$

The previous lemma shows that

$$Z^{\star k} = (-1)^k L^{\star k} \star e_{k-1}\,.$$

The known formula of the resolvent ([11, p. 36]) gives

$$\begin{aligned} H(t) &= \sum_{k=1}^{+\infty} (-1)^{k-1} Z^{\star k} = -\sum_{k=1}^{+\infty} L^{\star k} \star e_{k-1} = \\ &= -\int_0^t \left(\sum_{k=1}^{+\infty} L^{\star k}(t-s) \frac{s^{k-1}}{(k-1)!} \right) e^{-\mu^2 s} \,\mathrm{d}\, s\,. \end{aligned} \tag{14}$$

The series converges uniformly since the following holds:

$$|L(t)| < M \quad 0 \le t \le T \implies |L^{\star k}(t)| \le \frac{T^k M^k}{k!} \quad 0 \le t \le T\,.$$

Step 2: the bounded biorthogonal sequence does not exist. We reintroduce dependence on the index n. So

$$e_k(t) = \frac{t^k}{k!} e^{-\mu_n^2 t}\,.$$

We go back to the moment problem (13). We prove that it is not solvable as follows: we prove that if the sequence $\{E_n(x,t)\}$ admits a biorthogonal sequence $\{\chi_k(t)\}$, then this sequence cannot be bounded. So, let

$$\delta_{n,k} = (n\mu_n^2) \left[\int_0^T \chi_k(t) \left(e^{-\mu_n^2 t} - \int_0^t H_n(t-\tau) e^{-\mu_n^2 \tau} \,\mathrm{d}\,\tau \right) \mathrm{d}\, t \right]. \tag{15}$$

We have, using (14):

$$\begin{aligned} &\int_0^t H_n(t-\tau) e^{-\mu_n^2 \tau} \,\mathrm{d}\,\tau = e_0 \star H_n = -e_0 \star \left(\sum_{k=1}^{+\infty} L^{(\star k)} \star e_{k-1} \right) = -\sum_{k=1}^{+\infty} L^{(\star k)} \star e_k \\ &= -\int_0^t \left[\sum_{k=1}^{+\infty} L^{(\star k)}(t-s) \frac{s^k}{k!} \right] e^{-\mu_n^2 s} \,\mathrm{d}\, s = \int_0^t G(t,s) e^{-\mu_n^2 s} \,\mathrm{d}\, s\,. \end{aligned}$$

Note that $G(t,s)$ does not depend on n and equality (15) can be written as

$$\delta_{n,k} = \int_0^T \left(n\mu_n^2 e^{-\mu_n^2 r} \right) \left[\chi_k(r) - \int_r^T G(s,r) \chi_k(s) \,\mathrm{d}\, s \right] \mathrm{d}\, r\,. \tag{16}$$

If $\{\chi_k(t)\}$ is bounded in $L^2(0,T)$ then the sequence of the functions in the bracket is a *bounded* biorthogonal sequence of $\left\{n\mu_n^2 e^{-\mu_n^2 t}\right\}$. We proved in [14] that for every $T>0$ the sequence $\{\mu_n^2\lambda_n e^{-\mu_n^2 t}\}$ does not admit any bounded biorthogonal sequence in $L^2(0,T)$ and so $\{\chi_k(t)\}$ *cannot be bounded. This completes the proof of Theorem 2.*

For completeness, we sketch the proof of the absence of bounded biorthogonal sequences (see [14] for additional details):

Lemma 6 *Any sequence $\{\Psi_n(t)\}$ which is biorthogonal to $\{\mu_n^2\lambda_n e^{-\mu_n^2 t}\}$ in $L^2(0,T)$ is unbounded.*

Proof. Let e_n be the function $e^{-\mu_n^2 t}$ in $L^2(0,\infty)$ and denote by e_n^T its restriction to $(0,T)$.

$$E(\infty)=\operatorname{cl\,span}\{e_n\}\subseteq L^2(0,\infty)\,,\qquad E(T)=\operatorname{cl\,span}\{e_n^T\}\subseteq L^2(0,T)\,.$$

$E(\infty)$ is a proper subspace of $L^2(0,\infty)$ (Müntz Theorem, see [30]). Let P_T : $L^2(0,\infty)\to L^2(0,T)$ be the operator $P_T f=f|_{(0,T)}$. The operator P_T is an isomorphism between $E(\infty)$ and $E(T)$ (see [30, formula (9.a) p. 55]).

Suppose that $\{\tilde\psi_n\}$ is biorthogonal to $\{e_n^T\}$ in $L^2(0,T)$. We prove that the sequence $\{\tilde\psi_n\}$ is *exponentially unbounded.*

Let ψ_n be the orthogonal projection of $\tilde\psi_n$ on $E(T)$. Then, $\{\psi_n\}$ is biorthogonal to $\{e_n^T\}$ too and

$$\|\psi_n\|_{L^2(0,T)}\le\|\tilde\psi_n\|_{L^2(0,T)}\,.$$

We have ($(\cdot,\cdot)$ is the inner product in the indicated spaces)

$$\delta_{jn}=(\psi_j,e_n^T)_{L^2(0,T)}=(\psi_j,e_n^T)_{E(T)}=(\psi_j,P_Te_n)_{E(T)}=(P_T^*\psi_j,e_n)_{E(\infty)}\,.$$

Hence $\{P_T^*\psi_n\}$ is biorthogonal to $\{e_n\}$ and furthermore $\varphi_n=P_T^*\psi_n\in E(\infty)$ since $P_T\in\mathcal{L}(E(\infty),E(T))$. Hence, $\{\varphi_n\}$ is the biorthogonal sequence of $\{e_n\}$ whose L^2-norm is minimal.

Using [9, Lemma 3.1] we have:

$$||\varphi_n||_{L^2(0,\infty)}=\frac{2}{n^2}e^{[\pi+O(1)]n},\quad n\to\infty\,. \tag{17}$$

Since $P_T^*\in\mathcal{L}(E(T),E(\infty))$ is boundedly invertible, there exist *positive numbers* m and M such that for every n we have

$$m\|\psi_n\|_{L^2(0,T)}\le\|P_T^*\psi_n\|_{L^2(0,+\infty)}\le M\|\psi_n\|_{L^2(0,T)}$$

since $P_T^*\psi_n=\varphi_n$. It follows that

$$\|\tilde\psi_n\|_{L^2(0,T)}\ge\|\psi_n\|_{L^2(0,T)}\ge\frac{1}{M}\|\varphi_n\|_{L^2(0,\infty)}\quad\forall n\,. \tag{18}$$

So, *any* biorthogonal sequence of $\{e^{-\mu_n^2 t}\}$ in $L^2(0,+\infty)$ is *exponentially* unbounded and from (18) we see that *any* biorthogonal sequence of $\{e^{-\mu_n^2 t}\}_{n\ge N_T}$ in $L^2(0,T)$ is *exponentially* unbounded too.

Let us go back to the sequence $\{\Psi_n(t)\}$. This sequence cannot be bounded. Otherwise, the sequence $\{\mu_n^2\lambda_n\Psi_k(t)\}$ is a biorthogonal sequence to $\{e^{-\mu_n^2 t}\}$ such that

$$\|\mu_n^2\lambda_n\Psi_k(t)\|_{L^2(0,T)} \le C\mu_n^2\lambda_n \le Cn^3$$

a contradiction to (17) and (18). ∎

This result can be applied to the sequence $\{\Psi_n(t)\}$,

$$\Psi_n(t) = \left[\chi_n(t) - \int_t^T G(s,t)\chi_n(s)\,\mathrm{d}\,s\right]$$

which appears in (16). Lemma 6 shows that *this sequence is unbounded,* as we wished to prove.

Remark 7 *Instead of a time T in which $R(T) \ne 0$ we might have used a time T at which $R^{(k)}(T) \ne 0$ and $R^{(m)}(T) = 0$ for $m < k$, but this does not change the content of Theorem 2 in an essential way.*

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