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Analysis of composite plates by unified formulation - cell based smoothed finite element method and field consistent elements

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Abstract

In this article, we combine the Carrera's Unified Formulation (CUF) [8, 6] and cell based smoothed finite element method [26] for studying the static bending and the free vibration of thin and thick laminated plates. A 4-noded quadrilateral element based on field consistency requirement is used for this study to suppress the shear locking phenomenon. The combination of cell based smoothed finite element method and field consistent approach with CUF allows a very accurate prediction of field variables. The accuracy and efficiency of the proposed approach are demonstrated through numerical experiments.

1 Introduction

With the rapid development of engineering, there is an increasing demand for new materials which suits the harsh working environment without losing its mechanical, thermal or electrical properties. Engineered materials such as the composite materials are used in the construction of aeronautical and aerospace vehicles, as well as civil and mechanical structures. This is attributed to their excellent strength-to and stiffness-to-weight ratios and their possibility of tailoring the properties in optimizing the structural response. However, the analysis of such structures is a complex task, compared with conventional single layer metallic structures. This is because of the exhibition of coupling among membrane, torsion and bending strains; weak transverse shear rigidities; and discontinuity of the mechanical characteristics along the thickness of the laminates. For these reasons, in recent years, there is a great deal of interest in research for accurately modelling and simulating the characteristics of composite structures through different higher-order displacement functions for two-dimensional theories. This is because the two dimensional theories lead to less expensive models compared to three-dimensional one. In this context, the applications of analytical/numerical methods based on various 2D higher-order theories for static and dynamic analyses of rectangular laminates have recently attracted the attention of several researchers.

Various structural theories proposed for evaluating the characteristics of composite laminates under different loading situations have been reviewed by [34, 27, 18] and recently by Khanda *et al.,* [20]. In general, three different approaches have been used to study laminated composite structures: single layer theories, discrete layer theories and mixed plate theory. In the single layer theory approach, layers in laminated composites are assumed to be one equivalent single layer (ESL), whereas in the discrete layer theory approach, each layer is considered in the analysis. Although the discrete layer theories provide very accurate prediction of the displacements and the stresses, increasing the number of layers increases the number of unknowns. This can be prohibitively costly and significantly increase the computational time [44]. To overcome the above limitation, zig-zag models developed by Murukami [28] can satisfy the transverse shear stresses continuity conditions at the interfaces. Moreover, the number of unknowns are independent of the number of layers. Reddy and Robbins [38] presented a review of various equivalent-single-layer and layerwise laminated plate theories and their finite element models.

Recently, some researchers have attempted to combine the single layer theories and the discrete layer theory to overcome the limitations of each one. Carrera [8, 29, 6] derived a series of axiomatic approaches, coined as 'Carrera Unified Formulation' (CUF) for the gen-

eral description of two-dimensional formulations for multilayered plates and shells. With the unified formulation it is possible to implement in a single software a series of hierarchical formulations, thus affording a systematic assessment of different theories, ranging from simple ESL models up to higher order layerwise descriptions. This formulation is a valuable tool for gaining a deep insight into the complex mechanics of laminated structures. Demasi [10, 11, 12, 13, 14] presented mixed plate theories based on the Generalized Unified Formulation (GUF).

The CUF has been implemented in the finite element method [8, 29] and more recently in the meshless methods based upon collocation with radial basis function [16]. Although FEM provides a general and systematic technique for constructing basis functions, a number of difficulties still exists in the development of plate elements based on shear deformation theories, one of which is the shear locking phenomena. Different techniques by which the locking phenomena can be suppressed include: (a) retain the original interpolations and use optimal integration rule [17]; (b) assumed natural strain method [1, 41] and (c) enhanced assumed strain method [40]. Recently, Carrera *et al.,* [7] employed 4-noded mixed interpolation of tensorial components (MITC) technique for multilayered plate elements.

By incorporating the strain smoothing technique into the finite element method (FEM), Liu *et al.,* [26] have formulated a series of smoothed finite element methods (SFEM), named as cell-based SFEM (CS-FEM) [32, 3], node-based SFEM [25], edge-based SFEM [24], face-based SFEM [31] and α-FEM [23]. Nguyen-Xuan *et al.,* [33] employed CS-FEM for Mindlin-Reissner plates. The curvature at each point is obtained by a non-local approximation via smoothing function. From the numerical study presented, it was concluded that the CS-FEM technique is robust, computationally inexpensive, free of locking and importantly insensitive to mesh distortions. In [2], CS-FEM has been combined with the extended FEM to study moving boundary problems.

In this study, a $\mathcal{C}^o$ 4-noded quadrilateral element is employed to study the static bending and free vibration of laminated composites. The plate kinematics is based on Carrera Unified Formulation (CUF) and a sinusoidal shear deformation theory is used to approximate the displacements. A CS-FEM with field consistent shear flexible consistency approach is employed to study the response of laminated composites. The influence of various parameters, viz., the thickness of the plate, the fiber orientation, the ply lay up and the material properties on the response of the laminated composite plates is numerically studied.

The paper is organized as follows. Section 2 presents an overview of the Unified Formulation, the finite element discretization and the cell-based smoothing technique for implementation of the CUF. A discussion on computing the fundamental nuclei is also given.

The present formulation is compared with results available in the literature and the numerical results are presented in Section 3, bringing out the influence of various parameters on the static bending and the natural frequencies, followed by concluding remarks in the last section.

2 Carrera Unified Formulation

2.1 Basis of CUF

Let us consider a laminated plate composed of perfectly bonded layers with coordinates x, y along the in-plane directions and z along the thickness direction of the whole plate, while z_k is the thickness of the k^{th} layer. The CUF is a useful tool to implement a large number of two-dimensional models with the starting point is the description at the layer level. By following the axiomatic modelling approach, the displacements $\mathbf{u}(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$ are written according to the general expansion as:

$$\mathbf{u}(x, y, z) = \sum_{\tau=0}^{N} F_\tau(z)\mathbf{u}_\tau(x, y) \tag{1}$$

where $F(z)$ are known functions to model the thickness distribution of the unknowns, N is the order of the expansion assumed for the through-thickness behaviour. By varying the free parameter N, a *hierarchic* series of two-dimensional models can be obtained. The strains are related to the displacement field via the geometrical relations:

$$\begin{aligned} \epsilon_{pG} &= \left[\begin{array}{ccc} \boldsymbol{\varepsilon_{xx}} & \boldsymbol{\varepsilon_{yy}} & \boldsymbol{\gamma_{xy}} \end{array} \right]^{\mathrm{T}} = \mathbf{D}_p \mathbf{u} \\ \epsilon_{nG} &= \left[\begin{array}{ccc} \boldsymbol{\gamma_{xz}} & \boldsymbol{\gamma_{yz}} & \boldsymbol{\varepsilon_{zz}} \end{array} \right]^{\mathrm{T}} = \left(\mathbf{D}_{np} + \mathbf{D}_{nz}\right) \mathbf{u} \end{aligned} \tag{2}$$

where $\mathbf{D}_p, \mathbf{D}_{np}$ and $\mathbf{D}_{nz}$ are differential operators given by:

$$\mathbf{D}_p = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix}, \quad \mathbf{D}_{np} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix},$$
$$\mathbf{D}_{nz} = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix}. \tag{3}$$

The 3D constitutive equations are given as:

$$\begin{aligned} \boldsymbol{\sigma}_{pC} &= \mathbf{C}_{pp}\epsilon_{pG} + \mathbf{C}_{pn}\epsilon_{nG} \\ \boldsymbol{\sigma}_{nC} &= \mathbf{C}_{np}\epsilon_{pG} + \mathbf{C}_{nn}\epsilon_{nG} \end{aligned} \tag{4}$$

with

$$\mathbf{C}_{pp} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \quad \mathbf{C}_{pn} = \begin{bmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ 0 & 0 & C_{36} \end{bmatrix}$$
$$\mathbf{C}_{np} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{13} & C_{23} & C_{36} \end{bmatrix} \quad \mathbf{C}_{nn} = \begin{bmatrix} C_{55} & C_{45} & 0 \\ C_{45} & C_{44} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \tag{5}$$

The *Principle of Virtual Displacements* (PVD) in case of multilayered plate subjected to mechanical loads is written as:

$$\sum_{k=1}^{N_k} \int_{\Omega_k} \int_{A_k} \left\{ (\delta\epsilon_{pG}^k)^{\mathrm{T}} \boldsymbol{\sigma}_{pC}^k + (\delta\epsilon_{nG}^k)^{\mathrm{T}} \boldsymbol{\sigma}_{nC}^k \right\} \, \mathrm{d}\Omega_k \, \mathrm{d}z = \sum_{k=1}^{N_k} \int_{\Omega_k} \int_{A_k} \rho^k \delta \mathbf{u}_s^{k^{\mathrm{T}}} \ddot{\mathbf{u}}^k \, \mathrm{d}\Omega_k \, \mathrm{d}z + \sum_{k=1}^{N_k} \delta \mathbf{L}_e^k \tag{6}$$

where ρ^k is the mass density of the k^{th} layer, Ω_k, A_k are the integration domain in (x, y) and the z direction, respectively, k indicates the layer and the subscripts G and C indicate the geometrical and the constitutive equations, respectively. Upon substituting the geometric relations (Equation (2)), the constitutive relations (Equation (4)) and the unified formulation into the variational PVD statement, we have:

$$\begin{aligned} \int_{\Omega_k} \int_{A_k} \Big\{ \left(\mathbf{D}_p^k F_s \delta \mathbf{u}_s^k \right)^{\mathrm{T}} \left\{ \mathbf{C}_{pp}^k \mathbf{D}_p^k F_\tau \mathbf{u}_\tau^k + \mathbf{C}_{pn}^k (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k) F_\tau \mathbf{u}_\tau^k \right\} + \\ \left[(\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k) f_x \delta \mathbf{u}_s^k \right)^{\mathrm{T}} (\mathbf{C}_{np}^k \mathbf{D}_p^k F_\tau \mathbf{u}_\tau^k + \mathbf{C}_{nn}^k (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k) F_\tau \mathbf{u}_\tau^k) \big] \Big\} \, \mathrm{d}\Omega_k \, \mathrm{d}z = \\ \sum_{k=1}^{N_k} \int_{\Omega_k} \int_{A_k} \rho^k \delta \mathbf{u}_s^{k^{\mathrm{T}}} \ddot{\mathbf{u}}^k \, \mathrm{d}\Omega_k \, \mathrm{d}z + \sum_{k=1}^{N_k} \delta \mathbf{L}_e^k \end{aligned} \tag{7}$$

After integration by parts, the governing equations for the plate in the mechanical case are obtained:

$$\delta \mathbf{u}_s^{k^{\mathrm{T}}} : \mathbf{K}_{uu}^{k\tau s} \mathbf{u}_\tau^k = \mathbf{P}_{u\tau}^k \tag{8}$$

and in the case of free vibrations, we have:

$$\delta \mathbf{u}_s^{k^{\mathrm{T}}} : \mathbf{K}_{uu}^{k\tau s} \mathbf{u}_\tau^k = \mathbf{M}^{k\tau s} \ddot{\mathbf{u}}_\tau^k \tag{9}$$

where the fundamental nucleus $\mathbf{K}_{uu}^{k\tau s}$ is obtained as:

$$\mathbf{K}_{uu}^{k\tau s} = \left[(-\mathbf{D}_p^k)^{\mathrm{T}} (\mathbf{C}_{pp}^k \mathbf{D}_p^k + \mathbf{C}_{pn}^k (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}) + (-\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k)^{\mathrm{T}} (\mathbf{C}_{np}^k \mathbf{D}_p^k + \mathbf{C}_{nn}^k (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k)) \right] F_\tau F_s \tag{10}$$

and $\mathbf{M}^{k\tau s}$ is the fundamental nucleus for the inertial term given by:

$$M_{ij}^{k\tau s} = \begin{cases} \rho^k F_\tau F_s & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{11}$$

and the corresponding Newmann type boundary conditions on Γ^k are:

$$\Pi_d^{k\tau s} \mathbf{u}_\tau^k = \Pi_d^{k\tau s} \overline{\mathbf{u}}_\tau^k \tag{12}$$

where

$$\Pi_d^{k\tau s} = \left[(\mathbf{I}_p^k)^{\mathrm{T}} \left(\mathbf{C}_{pp}^k \mathbf{D}_p^k + \mathbf{C}_{pn}^k (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz}) \right) + (\mathbf{I}_{np}^k)^{\mathrm{T}} \left(\mathbf{C}_{np}^k \mathbf{D}_p^k + \mathbf{C}_{nn} (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k) \right) \right] \tag{13}$$

where $\mathbf{I}_p^k$ and $\mathbf{I}_{pn}^k$ depend on the boundary, given by:

$$\mathbf{I}_p^k = \begin{bmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ n_y & n_x & 0 \end{bmatrix}, \quad \mathbf{I}_{np}^k = \begin{bmatrix} 0 & 0 & n_x \\ 0 & 0 & n_y \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

where n_x, n_y are the normals and $\mathbf{P}_{u\tau}^k$ are variationally consistent loads with applied pressure. For more detailed derivation and for explicit form of the fundamental nuclei, interested readers are referred to [8, 29].

2.2 Element description

The plate element employed in this study is a $\mathcal{C}^0$ continuous element and according to the isoparametric description, the components of each displacement unknown $\mathbf{u}_\tau$ are expressed as:

$$\mathbf{u}_\tau = N_I \mathbf{q}_{\tau I}, \quad I = 1, 2, \cdots, N_n \tag{15}$$

where N_I are the standard finite element shape functions. By introducing the unified formulation for the displacements, given by Equation (15) into the strain-displacement relations (see Equation (2)), we have:

$$\begin{aligned} \epsilon_{pG}^k &= \mathbf{D}_p^k (F_\tau \mathbf{u}_\tau^k) = \mathbf{D}_p^k (F_\tau N_I) \mathbf{q}_{\tau I}^k \\ \epsilon_{nG}^k &= (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k)(F_\tau \mathbf{u}_\tau^k) = \mathbf{D}_{n\Omega}^k (F_\tau N_I) \mathbf{q}_{\tau I}^k + F_{\tau,z} N_I \mathbf{q}_{\tau I}^k \end{aligned} \tag{16}$$

Upon substituting, Equations (15) and 16 into Equations (10) and 11, we can compute the stiffness and the mass matrix of the system. The formulation is implemented in MATLAB and the solution to the static bending is computed by Gauss-elimination algorithm and the solution to free vibration problem is computed from a standard eigenvalue algorithm.

2.3 Overview of the strain smoothing method

In the following, the strain smoothing method is briefly discussed. For more detailed discussion, readers are referred to the literature and the references therein [32, 3]. The strain smoothing method (SSM) was proposed in [9], where the strain is written as the divergence of a spatial average of the standard (compatible) strain field - i.e., symmetric gradient of the displacement field. Elements are divided into subcells, as shown in Figure (1). The strain field, $\tilde{\varepsilon}^h_{ij}$ used to compute the stiffness matrix is computed by a weighted average of the standard strain field ε^h_{ij}. At a point $\mathbf{x}_C$ in an element Ω^h, the smoothed strain field is given by:

$$\tilde{\varepsilon}^h_{ij} = \int_{\Omega^h} \varepsilon^h_{ij}(\mathbf{x}) \Phi(\mathbf{x} - \mathbf{x}_C) \, \mathrm{d}\mathbf{x} \tag{17}$$

where Φ is a smoothing function that generally satisfies the following properties:

$$\Phi \geq 0 \quad \text{and} \quad \int_{\Omega^h} \Phi(\mathbf{x}) \, \mathrm{d}\mathbf{x} = 1 \tag{18}$$

One possible choice of Φ is given by

$$\Phi = \begin{cases} \frac{1}{A_C} & \mathbf{x}_C \in \Omega_C \\ 0 & \mathbf{x}_C \notin \Omega_C \end{cases}$$

where A_C is the area of the subcell. This process of smoothing the gradient field is called as *'cell-based smoothed finite element method'* (CS-FEM) in the literature [32, 3].

2.4 In-plane strain

In this article, we apply smoothing technique to the in-plane strains by a divergence estimation via a spatial averaging of the strain fields as explained earlier. In other words, the domain integrals are transformed into boundary integrals. This smoothing technique avoids the evaluation of the derivatives of the shape functions and thus avoiding the iso-parametric mapping. The smoothing is performed over arbitrary smoothing cell, Ω_C, illustrated in Figure (1) with boundary $\Gamma_C = \bigcup_{b=1}^{nb} \Gamma^b_C$, where Γ^b_C is the boundary segment of Ω_C and nb is the total number of edges of each smoothing cell. The relation between the smoothed in-plane strain and the nodal displacements is given by:

$$\tilde{\epsilon}^k_{pG} = \tilde{\mathbf{D}}^k_{pC} \mathbf{q}^k_{\tau I} \tag{19}$$

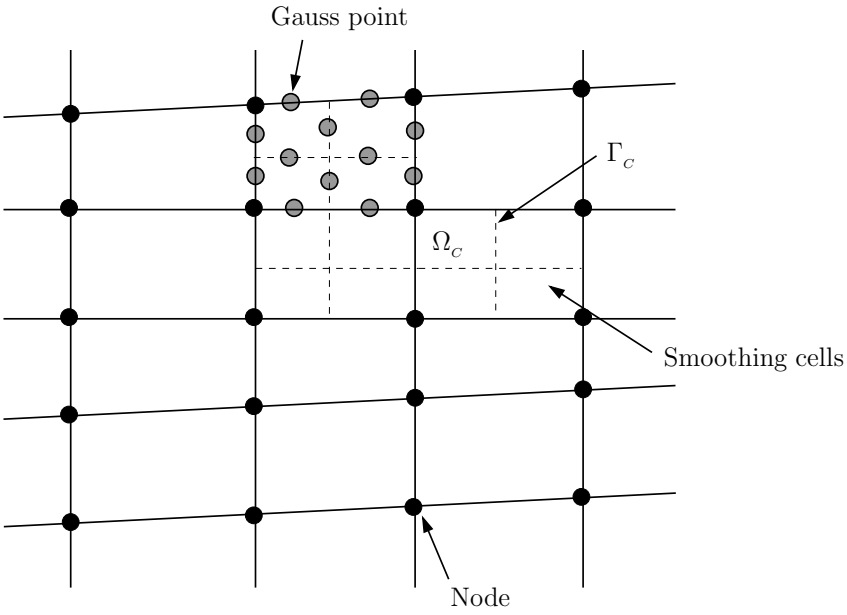

Figure 1: Example of finite element meshes and smoothing cells. The integration is performed along the boundary of each subcell. The smoothing cells, Ω_C and Γ_C is the boundary of the subcell.

where $\tilde{\mathbf{D}}_{pC}^{k}$ is the smoothed strain-displacement matrix, given by:

$$\tilde{\mathbf{D}}_{pC}^{k} = \frac{1}{A_C} \int_{\Gamma_C} F_\tau N_I \begin{bmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ n_y & n_x & 0 \end{bmatrix} \, \mathrm{d}\Gamma \tag{20}$$

where n_x and n_y are the normals to the edge of the smoothing cell. Nguyen-Xuan *et al.,* [32] did a systematic study on the influence of the number of subcells on the performance of Reissner-Mindlin plates. It was concluded that elements with one subcell exhibits two zero energy modes, while elements with two, three and four subcells maintain sufficient rank and no zero-energy modes. In this study, we use four subcells per element.

2.5 Shear locking

If the interpolation functions given for a QUAD-4 are used directly to interpolate the unknown displacement fields in deriving the shear strains $(\gamma_{xz}, \gamma_{yz})$ and the membrane strains (ϵ_{pG}), the element will lock and show oscillations in the shear and the membrane stresses. The oscillations are due to the fact that the derivative functions of the out-of plate displacement do not match that of the rotations in the shear strain definition. To alleviate the

locking phenomenon, the terms corresponding to the derivative of the out-of plate displacement must be consistent with the rotation terms. In this study, field redistributed shape functions are used to alleviate the shear locking [30]. The field consistency requires that the transverse shear strains and the membrane strains must be interpolated in a consistent manner. If the element has edges which are aligned with the coordinate system (x, y), the terms in shear strains $(\gamma_{xz}, \gamma_{yz})$ are approximated by [41] by the following substitute shape functions:

$$\begin{aligned}
\tilde{N}_1(\eta) &= \frac{1}{4}\left[\begin{array}{cccc} 1-\eta & 1-\eta & 1+\eta & 1+\eta \end{array}\right] \\
\tilde{N}_2(\xi) &= \frac{1}{4}\left[\begin{array}{cccc} 1-\xi & 1+\xi & 1+\xi & 1-\xi \end{array}\right].
\end{aligned} \tag{21}$$

Note that, no special integration rule is required for evaluating the shear terms. A numerical integration based on the 2×2 Gaussian rule is used to evaluate all the terms.

Remark 2.1. In this study, curvature based smoothing technique proposed by Chen *et al.,* [9] for meshfree methods and applied for plates by Nguyen *et al.,* [33] is employed to approximate the extension and the bending strains.

Remark 2.2. Field consistent shape functions are employed to approximate the shear strains.

3 Numerical examples

In this section, we present the static response and the natural frequencies of laminated composite plates using four noded quadrilateral element. In this study, the in-plane displacements u, v and the transverse displacement w are expressed by sinus shear deformation theory (SINUS-ZZ) as:

$$\begin{aligned}
u &= u_o + zu_1 + \sin\left(\frac{\pi z}{h}\right) u_2 \\
v &= v_o + zv_1 + \sin\left(\frac{\pi z}{h}\right) v_2 \\
w &= w_o + zw_1 + z^2 w_2
\end{aligned} \tag{22}$$

where u_o, v_o and w_o are translations of a point at the middle-surface of the plate, w_2 is higher order translations, and u_1, v_1, u_3 and v_3 denote rotations [42] and considers a quadratic variation of the transverse displacement w allowing for through-the-thickness deformations. The effect of the plate aspect ratio, the ply angle and the ratio of Young's modulus E_1/E_2 is numerically studied. Before proceeding with a detailed study on the effect of different

parameters on the response of cross-ply laminated plates, the formulation developed here is compared against available results pertaining to static bending and free vibration of cross-ply laminated plates. Based on a progressive mesh-refinement, a 20×20 structured quadrilateral mesh is found to be adequate to model the full laminated plate for the present analysis. For sake of brevity, the mesh convergence is not shown here. In this study, the results from the smoothing technique is denoted by CS-FEMQ4. The results with standard Q4 element with field consistent approach is also presented, denoted by FEM-Q4.

3.1 Static bending

The static analysis is conducted for cross-ply laminated plates with three and four layers under following sinusoidal load:

$$p_z = P_o \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right) \tag{23}$$

where P_o is the amplitude of the mechanical load. The origin of the coordinate system is located at the lower-left corner on the midplane. The physical quantities are non-dimensionalized by following relations, unless otherwise mentioned:

$$\begin{aligned} \overline{w} = w(a/2, a/2, 0)\frac{100h^3E_2}{Pa^4}; \quad \overline{\sigma}_{xx} = \sigma_{xx}(a/2, a/2, h/2)\frac{h^2}{Pa^2}; \\ \overline{\sigma}_{yy} = \sigma_{yy}(a/2, a/2, h/4)\frac{h^2}{Pa^2}; \quad \overline{\tau}_{xx} = \tau_{xx}(0, a/2, h)\frac{h}{Pa}; \\ \overline{\tau}_{xy} = \tau_{xy}(a, a, h/2)\frac{h^2}{Pa^2}. \end{aligned} \tag{24}$$

3.1.1 Four layer $(0^\circ/90^\circ)_\mathrm{s}$ square cross-ply laminated plate under sinusoidal load

A square simply supported laminate of side a and thickness h, composed of four equally thick layers oriented at $(0^\circ/90^\circ)_\mathrm{s}$ is considered. The plate is subjected to a sinusoidal vertical pressure given by Equation (23). The material properties are as follows: $E_1 = 25E_2$; $G_{12} = G_{13} = 0.5E_2$; $G_{23} = 0.2E_2$; $\nu_{12} = 0.25$. For this example, there is a three-dimensional exact solution by Pagano [35]. In Table 1, we present results for the SINUS-ZZ theory with strain smoothing and field consistent approach. Also presented are the solutions from the standard 4-noded element with field consistent approach, denoted by FEM-Q4. We compare the results with higher order plate theories [36, 15], first order theory [37] and an exact solution [35]. The effect of plate thickness is also shown in Table 1. It is clear

that the FSDT cannot be used for thick laminates. It can be seen that the results from the present formulation show very good agreement with those in the literature and very precise transverse displacements and stresses are obtained. The main features of the present formulation are: (1) theories from ESL to higher order layer descriptions can be implemented within a single code (since it is based on CUF); (2) the strain smoothing technique reduces the computational effort and also improves the accuracy of the field variables and (3) the present formulation is insensitive to shear locking.

3.1.2 Three layer (0°/90°/0°) square cross ply laminated plate under sinusoidal load

In this case, a square laminate of side a and thickness h, composed of three equally thick layers oriented at (0°/90°/0°) is considered. It is simply supported on all edges and subjected to a sinusoidal vertical pressure of the form, given by Equation (23). The material properties for this example are: E_1 =132.38 GPa, $E_2 = E_3$ =10.756 GPa, G_{12} =3.606 GPa, $G_{13} = G_{23} = 5.6537$ GPa, $\nu_{12} = \nu_{13} = 0.24$, $\nu_{23} = 0.49$. In Table 2, we present results for the SINUS-ZZ theory with strain smoothing and field consistent approach. The results from the present formulation are compared with the analytical solution [4, 5] and MITC4 formulation [7]. It can be seen that the numerical results from the present formulation are found to be in good agreement with the existing solutions. Moreover, the present formulation shows an improvement in the accuracy of the field variables when compared to other formulations and is insensitive to shear locking.

3.2 Free vibration - cross-ply laminated plates

In this example, all layers of the laminate are assumed to be of the same thickness, density and made up of the same linear elastic material. The following material parameters are considered for each layer

$$\frac{E_1}{E_2} = 10,20,30, \text{ or } \ 40; \quad G_{12} = G_{13} = 0.6E_2; \\ G_3 = 0.5E_2; \nu_{12} = 0.25.$$

The subscripts 1 and 2 denote the directions normal and the transverse to the fiber direction in a lamina, which may be oriented at an angle to the plate axes. The ply angle of each layer is measure from the global $x-$axis to the fiber direction. The example considered is a simply supported square plate of the cross-ply lamination $(0°/90°)_s$. The thickness and the length of the plate are denoted by h and a, respectively. The thickness-to-span ratio $h/a =$

Table 1: The normalized central deflection $\overline{w} = w(a/2, a/2, 0)\frac{100E_2h^3}{Pa^4}$, stresses, $\overline{\sigma}_{xx} = \sigma_{xx}(a/2, a/2, h/2)\frac{h^2}{Pa^2}, \overline{\sigma}_{yy} = \sigma_{yy}(a/2, a/2, h/4)\frac{h^2}{Pa^2}$ and $\overline{\tau}_{xz} = \tau_{xz}(0, /a2, 0)\frac{h}{Pa}$ of a simply supported cross-ply laminated square plate $[0°/90°/90°/0°]$, with $E_1 = 25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, ν_{12} =0.25.

a/h	Method	w	σ_{xx}	σ_{yy}	τ_{xz}
	HSDT [36]	1.8937	0.6651	0.6322	0.2064
	FSDT [37]	1.7100	0.4059	0.5765	0.1398
4	Elasticity [35]	1.9540	0.7200	0.666	0.2700
	RBF [15]	1.9783	0.6765	0.5872	0.2332
	Present (FEM Q4)	1.8949	0.6617	0.5615	0.2913
	CS-FEM Q4 (4 subcells)	1.9089	0.7067	0.6273	0.2201
	HSDT [36]	0.7147	0.5456	0.3888	0.2640
	FSDT [37]	0.6628	0.4989	0.3615	0.1667
10	Elasticity [35]	0.7430	0.5590	0.4030	0.3010
	RBF [15]	0.7325	0.5627	0.3908	0.3321
	Present (FEM Q4)	0.7135	0.5670	0.3682	0.3517
	CS-FEM Q4 (4 subcells)	0.7195	0.5597	0.3905	0.2952
	HSDT [36]	0.4343	0.5387	0.2708	0.2897
	FSDT [37]	04337	0.5382	0.2705	0.1780
100	Elasticity [35]	0.4347	0.5390	0.2710	0.3390
	RBF [15]	0.4307	0.5431	0.2730	0.3768
	Present (FEM Q4)	0.4302	0.5559	-	0.3766
	CS-FEM Q4 (4 subcells)	0.4304	0.5368	-	0.3285

Table 2: Transverse displacement $\overline{w} = w(a/2, a/2, h/2)$ at the center of a multilayered plate $[0°/90°/0°]$ with $E_1 = 132.38$ GPa, $E_2 = E_3 = 10.756$ GPa, $G_{12} = 3.606$ GPa, $G_{13} = G_{23} = 5.6537$ GPa, $\nu_{12} = \nu_{13} = 0.24$, $\nu_{23} = 0.49$. A structured quadrilateral mesh with 20×20 elements

$\overline{w}$	a/h				
	10	50	100	500	1000
Analytical (ESL-2) [4, 5]	0.9249	0.7767	0.7720	0.7705	0.7704
MITC4 [7]	0.9195	0.7713	0.7666	0.7650	0.7650
Present (FEM Q4)	0.9152	0.7700	0.7651	0.7636	0.7635
CS-FEM Q4 (4 subcells)	0.9235	0.7703	0.7655	0.7639	0.7639

0.2 is employed in the computations. In this study, we present the non dimensionalized free flexural frequencies as, unless specified otherwise:

$$\Omega = \omega \frac{a^2}{h} \sqrt{\frac{\rho}{E_2}}$$

Table 3 lists the fundamental frequency for a simply supported cross-ply laminated square plate with $h/a = 0.2$ and for different ratio of Young's modulus, E_1/E_2. It can be seen that the results from the present formulation are in very close agreement with the values of [21] based on higher order theory, the meshfree results of Liew *et al.,* [22] and Ferreira *et al.,* based on FSDT and higher order theories with radial basis functions [16]. The effect of plate thickness on the fundamental frequency is shown in Table 4. It can be seen that the results agree with the results available in the literature. The present formulation is insensitive to shear locking.

4 Conclusion

In this article, the cell based smoothing technique for implementation of the Carrera Unified Formulation was detailed and discussed. The efficiency and accuracy of the present approach is demonstrated with few numerical examples. The shear locking is suppressed by employing a field consistent approach. This improved finite element technique shows insensitivity to shear locking and produce excellent results in static bending and free vibration of cross-ply laminated plates.

Table 3: The normalized fundamental frequency $\Omega = \omega a^2/h\sqrt{\rho/E_2}$ of a simply supported cross-ply laminated square plate $(0°/90°)_s$ with $h/a = 0.2$, $\frac{E_1}{E_2} = 10$, 20, 30 or 40, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = 0.25$.

Method	Mesh	subcell(s)	E_1/E_2			
			10	20	30	40
Liew [22]			8.2924	9.5613	10.3200	10.8490
Reddy, Khdeir [21]			8.2982	9.5671	10.3260	10.8540
FSDT [16]	21×21		8.2982	9.5671	10.3258	10.8540
HSDT [16] ($\nu_{23} = 0.18$)	21×21		8.2999	9.5411	10.2687	10.7652
Present FEM Q4	20×20		8.3651	9.5801	10.2980	10.7894
CS-FEM Q4	20×20	1	8.3604	9.5755	10.2937	10.7853
		2	8.3615	9.5767	10.2950	10.7866
		4	8.3639	9.5790	10.2970	10.7883
		8	8.3642	9.5793	10.2973	10.7887

Table 4: Variation of fundamental frequencies, $\Omega = \omega a^2/h\sqrt{\rho/E_2}$ with a/h for a simply supported square laminated plate $[0°/90°/90°/0°]$, $\Omega = \omega a^2/h\sqrt{\rho/E_2}$, with $E_1/E_2 = 40$, $G_{12} = G_{13} = 0.6E_2$, $G_{23} = 0.5E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$.

Method	a/h					
	2	4	10	20	50	100
FSDT [43]	5.4998	9.3949	15.1426	17.6596	18.6742	18.8362
Model-1 (12dofs) [19]	5.4033	9.2870	15.1048	17.6470	18.6720	18.8357
Model-2 (9dofs) [19]	5.3929	9.2710	15.0949	17.6434	18.6713	18.8355
HSDT [36]	5.5065	9.3235	15.1073	17.6457	18.6718	18.8356
HSDT [39]	6.0017	10.2032	15.9405	17.9938	18.7381	18.8526
Present(FEM Q4)	5.4029	9.3005	15.1790	17.7578	18.7993	18.9657
CS-FEM Q4 (4 subcells)	5.4026	9.2998	15.1766	17.7540	18.7947	18.9611

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