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Nonlinear model predictive control from data: a set membership approach

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SUMMARY

A new approach to design a Nonlinear Model Predictive Control law that employs an approximate model, derived directly from data, is introduced. The main advantage of using such models lies in the possibility to obtain a finite computable bound on the worst-case model error. Such a bound can be exploited to analyze the robust convergence of the system trajectories to a neighborhood of the origin. The effectiveness of the proposed approach, named Set Membership Predictive Control, is shown in a vehicle lateral stability control problem, through numerical simulations of harsh maneuvers. Copyright © 2012 John Wiley & Sons, Ltd.

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KEY WORDS: predictive control; robust stability; nonlinear control

1. INTRODUCTION

In Nonlinear Model Predictive Control (NMPC), the current control action is computed by solving online a constrained Finite Horizon Optimal Control Problem (FHOCP) according to the Receding Horizon (RH) strategy, see for example [1]. A nonlinear model of the system to be controlled is employed in the FHOCP to predict the future state behavior, starting from the state value x_t at time step t. In most cases, such a model is based on first-principle laws, yielding a *physical model*; in other cases, it is a black-box model in the form of a nonlinear parametric function (e.g., a neural network), whose parameters are identified from measured process data. However, since such models only approximate the dynamic behavior of the system under control, to guarantee stability of the controlled system, robustness issues should be taken into account in the design procedure. Robust stability analysis and robust design of predictive controllers in the presence of modeling uncertainty are a very active field of research, and different approaches have been considered in this context. A first contribution has been given in [2], where the contraction principle has been used to derive necessary and sufficient conditions for robust stability. In [3], a sufficient condition has been introduced by means of the computation of suitable weights in the cost functions, whereas in [4], robust stability has been achieved by enforcing a robust state contraction constraint. A min–max approach has been employed for the case of linear time varying systems in the presence of polytopic and parametric uncertainties in [5] and [6], respectively. A robust design that adopts constraint tightening techniques has been introduced in [7]. All the aforementioned contributions were developed for linear systems. More recently, the attention moved to nonlinear systems. In this context, several interesting results have been obtained within the framework of Input-to-State Stability (ISS). In particular, in [8], an ISS approach has been employed to analyze the robustness of NMPC schemes;

in [9], a suboptimal NMPC law with ISS guarantees has been derived, whereas in [10], the problem of robust NMPC design has been solved in the presence of state-dependent uncertainties and additive bounded perturbations. Finally, in [11], a combined MPC and integral sliding mode strategy has been introduced whose robust stability properties are proved through an ISS regional analysis. Besides, the theory of invariant sets has been employed in [12] and [13] to obtain robust stability through the use of interval arithmetic methodologies and a tube-based approach, respectively. As witnessed by the aforementioned review, several contributions have been introduced for the robust analysis and design of nonlinear model-based predictive controllers. A common feature of the such approaches is that the analysis/design procedures of the robust MPC controller have been performed by using a nominal nonlinear state space model and describing the uncertainty through additive bounded disturbances whose characteristics, in general, do not depend on the nominal model, and they do not guarantee that the real plant belongs to such a description. This issue stems from the difficulty to derive a useful description of the model uncertainty in nonlinear parametric models, whose parameters are usually obtained via physical modeling or identified from input/output data, and in such modeling procedures, it is not trivial to derive also some uncertainty estimate. In order to deal with the previously described issue, this paper investigates the robustness of a new NMPC scheme in which a nonparametric model, identified directly from measured input–output data, is used in the FHOCP. Such a model is derived via a Nonlinear Set Membership (NSM) methodology (see [14] for details); thus, the proposed approach will be referred to as Set Membership Predictive Control (SMPC). As a matter of fact, SM identification methods were employed in [6] and [15] to derive parametric and nonparametric uncertain linear models, respectively. Indeed, in this paper, a different SM setting is considered because a nonparametric approach and a nonlinear model are considered. In the NSM approach, an additive uncertainty model is derived together with the model of the plant dynamics, and an upper bound on such uncertainty can be computed. Then, by using the NSM model in the design of a SMPC control law, many techniques for robustness analysis and robust design can be suitably used. In particular, in this paper, a nominal (i.e., assuming zero uncertainty) controller design is carried out using the identified model. Then, by exploiting the upper bound of the modeling error provided by the NSM approach, a theoretical result is introduced, which allows us to perform an a posteriori analysis of the robust stability properties of the controlled system. In particular, such a result defines a relationship between the bound on the modeling error and the asymptotic regulation accuracy of the closed-loop system. The effectiveness of the approach is tested on a vehicle stability control case study. The paper is organized as follows. A summary of NMPC and of the NSM modeling approach is given in Sections 2.1 and 2.2, respectively. The design of a SMPC law using a NSM model and the related robust stability analysis are presented in Section 2.3. An automotive case study is treated in Section 3, and conclusions are included in Section 4.

2. NONLINEAR MODEL PREDICTIVE CONTROL USING NONLINEAR APPROXIMATE MODELS

2.1. Nonlinear Model Predictive Control

Consider the following nonlinear discrete-time state space model:

$$x_{t+1} = f^{M}(x_t, u_t),\ t \in \mathbb{Z} \tag{1}$$

where $f^{M} : \mathbb{R}^{m+1} \to \mathbb{R}^{m}$ is a nonlinear function and $u_t \in \mathbb{R}$ and $x_t \in \mathbb{R}^{m}$ are the system input and state, respectively.

Assume that the problem is to regulate the system state to the origin under some input and state constraints. By defining the prediction horizon N_p and the control horizon $N_c \leqslant N_p$, it is possible to define a cost function $J_t(U, x_{t|t})$ of the form

$$J_t(U, x_{t|t}) = \sum_{j=0}^{N_p-1} L\left(x_{t+j|t}, u_{t+j|t}\right) + \Psi\left(x_{t+Np|t}\right) \tag{2}$$

The per-stage cost function $L(\cdot)$ and the terminal state cost $\Psi(\cdot)$ are suitably chosen and tuned according to the desired control performance. $L(\cdot)$ is typically continuous and convex in its arguments. The cost function $J_t(\cdot)$ is evaluated on the basis of the predicted state values $x_{t+j|t}$, $j \in [1, N_p]$ obtained using the model (1), the input sequence $U_t = [u_{t|t} \dots u_{t+N_c-1|t}]$, and the initial state $x_{t|t} = x_t$. The sequence U_t is a decision variable in the problem, whereas the remaining input values $[u_{t+N_c|t}, \dots, u_{t+N_p-1|t}]$ can be fixed according to different strategies [1, 16]. The NMPC control is computed according to the RH strategy:

(i) At time instant t, obtain the state $x_{t|t} \doteq x_t$.
(ii) Solve the optimization problem

$$\begin{aligned} &\min_{U} \quad J_t(U, x_{t|t}) \\ &\text{subject to} \end{aligned} \tag{3a}$$

$$x_{t+j+1|t} = f^M(x_{t+j|t}, u_{t+j|t}),\ j \in [0, Np-1] \tag{3b}$$

$$x_{t+j|t} \in \mathbb{X},\ j \in [1, N_p] \tag{3c}$$

$$u_{t+k|t} \in \mathbb{U}, k \in [0, N_p - 1] \tag{3d}$$

$$\textit{stabilizing} \quad \textit{constraints} \tag{3e}$$

where the input and state constraints are represented by a set $\mathbb{X} \subseteq \mathbb{R}^m$ and a compact set $\mathbb{U} \subseteq \mathbb{R}$, both containing the origin in their interiors; denote with $U_t^* = \left[u_{t|t}^* \dots u_{t+N_c-1|t}^*\right]$ the minimizer of (3).
(iii) Apply the first element of U_t^* as the actual control action $u_t = u_{t|t}^*$.
(iv) Repeat the whole procedure at the next sampling time $t + 1$.

Possible additional *stabilizing constraints* (e.g., state contraction and terminal set) can be included in (3e) to ensure stability of the controlled system.

A model derived from physical laws is usually employed as model of plant (1) to be controlled. In this work, instead, an approximate model derived from data by means of the NSM methodology is used. In the sequel, a brief overview of the NSM technique is presented.

2.2. Nonlinear Set Membership identification

Suppose that the plant to be controlled **P** is a nonlinear discrete-time dynamic system described by the following regression:

$$\begin{aligned} y_{t+1} &= P(\mathbf{y}_t, \mathbf{u}_t),\ t \in \mathbb{Z} \\ \mathbf{y}_t &= [y_t, \dots, y_{t-n_y}] \\ \mathbf{u}_t &= [u_t, \dots, u_{t-n_u}] \end{aligned} \tag{4}$$

where $u_t, y_t \in \mathbb{R}$, $P : \mathbb{R}^n \to \mathbb{R}$, and $n = n_y + n_u + 2$.

Suppose that **P** is not known, but a set of noise-corrupted measurements is available

$$(\widetilde{y}_t, \widetilde{u}_t) \quad t \in \mathcal{T} \doteq \{-\nu + 1, -\nu + 2, \dots, 0\} \tag{5}$$

Measurements (5) can be collected in an initial experiment on the plant to be controlled. Note that such measurements are needed also in the case of physical models, to tune the model parameters. Let $\widetilde{\varphi}_t \doteq [\widetilde{\mathbf{y}}_t; \widetilde{\mathbf{u}}_t]$, where $\widetilde{\mathbf{y}}_t = [\widetilde{y}_t, \dots, \widetilde{y}_{t-n_y}]$ and $\widetilde{\mathbf{u}}_t = [\widetilde{u}_t, \dots, \widetilde{u}_{t-n_u}]$.

Then, (4) can be rewritten as

$$\widetilde{y}_{t+1} = P(\widetilde{\varphi}_t) + d_t,\ t \in \mathcal{T} \tag{6}$$

where the term d_t accounts for the fact that y_t and $\varphi_t \doteq [\mathbf{y}_t; \mathbf{u}_t]$ are not exactly know

The aim is to derive a model M of P from the available measurements $(\widetilde{y}_t, \widetilde{u}_t)$. The estimate M should be chosen to give small (possibly minimal) L_p error $||P - M||_p$, where the p-norm of a given function $F(\varphi)$ is defined as $\|F\|_p \doteq \left[\int_\phi |F(\varphi)|^p \, \mathrm{d}\varphi\right]^{\frac{1}{p}}$, $p \in [1, \infty]$, where $\|F\|_\infty \doteq \operatorname{ess\,sup}_{\varphi \in \Phi} |F(\varphi)|$, $|\cdot|$ denotes the Euclidean norm, and Φ is a bounded set in $\mathbb{R}^n$.

Whatever estimate is chosen, no information on the identification error can be derived, unless some assumptions are made on the function P and on the noise d.

Assumption 1

- $P \in \mathcal{F}(\gamma)$

$$\mathcal{F}(\gamma) \doteq \{F \in C^0 : |F(\varphi) - F(\bar{\varphi})| \leqslant \gamma |\varphi - \bar{\varphi}| \quad \forall \varphi, \bar{\varphi} \in \Phi \subset \mathbb{R}^n\}$$

- $|d_t| \leqslant \varepsilon < \infty$, $t \in \mathcal{T}$.

Thus, $\mathcal{F}(\gamma)$ is the set of Lipschitz continuous functions on Φ with Lipschitz constant γ. It is assumed that Φ is a compact set. Assumption 1 represents the only restriction imposed on the model *structure* in this approach.

A key role in the SM framework is played by the Feasible System Set (FSS), often called *unfalsified systems set*, that is, the set of all systems consistent with prior assumptions and measured data.

Definition 1
Feasible System Set:

$$FSS \doteq \{F \in \mathcal{F}(\gamma) : |\widetilde{y}_t - F(\widetilde{\varphi}_t)| \leqslant \varepsilon, t \in \mathcal{T}\} \tag{7}$$

If Assumption 1 is *true* , then $P \in FSS$.

For a given estimate $M \simeq P$, the related L_p error $\|P - M\|_p$ cannot be exactly computed, but its tightest bound is given by

$$\|P - M\|_p \leqslant \sup_{F \in FSS} \|F - M\|_p$$

This motivates the following definition of worst-case identification error.

Definition 2
The worst-case identification error of the estimate M is

$$E(M) \doteq \sup_{F \in FSS} \|F - M\|_\infty$$

Looking for estimates that minimize the worst-case identification error leads to the following optimality concept.

Definition 3
An estimate F^* is optimal if

$$E\left(F^*\right) = \inf_M E(M) = \mathcal{R}_I$$

The quantity $\mathcal{R}_I$, called *radius of information*, gives the minimal worst-case identification error that can be guaranteed by any estimate on the basis of the available information.

Let us define the following functions:

$$\begin{aligned}\overline{F}(\varphi) &\doteq \min_{t\in\mathcal{T}}\left(\overline{h}_t + \gamma|\varphi - \widetilde{\varphi}_t|\right)\\ \underline{F}(\varphi) &\doteq \max_{t\in\mathcal{T}}\left(\underline{h}_t - \gamma|\varphi - \widetilde{\varphi}_t|\right)\\ \overline{h}_t &\doteq \widetilde{y}_t + \varepsilon\\ \underline{h}_t &\doteq \widetilde{y}_t - \varepsilon\end{aligned} \tag{8}$$

The next theorem shows that the estimate

$$M_c \doteq \frac{1}{2}\left(\underline{F} + \overline{F}\right) \tag{9}$$

is optimal for any L_p norm.

Theorem 1 (Theorem 7 in [14]*.)*
For any L_p norm, with $p \in [1, \infty]$:

- $M_c \in FSS$
- The estimate $M_c = \frac{1}{2}\left(\underline{F} + \overline{F}\right)$ is optimal
- The worst-case identification error $E(M_c)$ is given by

$$E(M_c) = \frac{1}{2}\left\|\overline{F} - \underline{F}\right\|_\infty \tag{10}$$

Model M_c (9) can be written as a nonlinear regression:

$$y_{t+1} = M_c(y_t, \ldots, y_{t-n_y}, u_t, \ldots, u_{t-n_u}),\ t \in \mathbb{Z} \tag{11}$$

where M_c is a Lipschitz continuous function with Lipschitz constant γ (see [14] for more details).

The next section focuses on the proposed SMPC. In such a methodology, a nonlinear predictive controller is designed using model (11) to predict the state behavior in the FHOCP, whereas the worst-case error $E(M_c)$ (10) is employed to perform an a posteriori robust stability analysis of the SMPC closed-loop system.

2.3. Set Membership Predictive Control

In order to employ an NSM model within the context of NMPC, the plant P (4) and the model M_c (11) are rewritten here in state space form, by using u_t as input variable and x_t as pseudo-state,

$$\begin{aligned}x_t &\doteq [y_t \ldots y_{t-n_y} u_{t-1} \ldots u_{t-n_u}]\\ &= \left[x_t^{(1)} \cdots x_t^{(n_y+1)} x_t^{(n_y+2)} \cdots x_t^{(n_y+n_u+1)}\right] \in \mathbb{R}^{n-1}\end{aligned} \tag{12}$$

The state space equation of plant (4) results to be

$$x_{t+1} = f^P(x_t, u_t) \tag{13}$$

where $f^P : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is defined as

$$f^P(x_t, u_t) \doteq \begin{bmatrix} P\left(x_t^{(1)}, \ldots, x_t^{(n_y+1)}, u_t, x_t^{(n_y+2)}, \ldots, x_t^{(n_y+n_u+1)}\right)\\ x_t^{(1)}\\ \vdots\\ x_t^{(n_y)}\\ u_t\\ \vdots\\ x_t^{(n_y+n_u)}\end{bmatrix} \tag{14}$$

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Note that, since $P(\cdot)$ is assumed to be Lipschitz continuous with constant γ, function $f^P(\cdot)$ in (14) results to be Lipschitz continuous too, with constant $L_P = \sqrt{1+\gamma^2}$:

$$
\begin{aligned}
&||f^P(x_1,u_1) - f^P(x_2,u_2)||_2^2 = \\
&= ||P(\cdot) - P(\cdot)||_2^2 + ||x_1^{(1)} - x_2^{(1)}||_2^2 + \cdots + ||u_1 - u_2||_2^2 + \cdots + ||x_1^{(n_u+n_y)} - x_2^{(n_u+n_y)}||_2^2 = \\
&\leqslant (1+\gamma^2)\left(||x_1^{(1)} - x_2^{(1)}||_2^2 + \cdots + ||u_1 - u_2||_2^2 + \cdots + ||x_1^{(n_u+n_y)} - x_2^{(n_u+n_y)}||_2^2\right) = \\
&= (1+\gamma^2)\left(||x_1 - x_2||_2^2 + ||u_1 - u_2||_2^2\right) = L_P^2\left(||x_1 - x_2||_2^2 + ||u_1 - u_2||_2^2\right)
\end{aligned} \tag{15}
$$

Applying the same procedure to the model M_c (11) leads to the state space description

$$x_{t+1} = f^{M_c}(x_t, u_t) \tag{16}$$

where

$$
f^{M_c}(x_t,u_t) \doteq \begin{bmatrix} M_c\left(x_t^{(1)},\ldots,x_t^{(n_y+1)},u_t,x_t^{(n_y+2)},\ldots,x_t^{(n_y+n_u+1)}\right) \\ x_t^{(1)} \\ \vdots \\ x_t^{(n_y)} \\ u_t \\ \vdots \\ x_t^{(n_y+n_u)} \end{bmatrix} \tag{17}
$$

By construction, taking into account Theorem 1, function $f^{M_c}(\cdot)$ in (17) is also Lipschitz continuous with constant $L_M = \sqrt{1+\gamma^2}$ (i.e., $L_M = L_P$).

Moreover, it can be shown that the estimation error is upper bounded by $\mathcal{R}_I$:

$$\|f^P - f^{M_c}\|_\infty \leqslant \mathcal{R}_I \tag{18}$$

The SMPC law is derived according to the RH strategy described in Section 2.1, by using model (11). The resulting controller is a static function of the pseudo-state x_t, that is, $u_t = \kappa(x_t)$, defined on a compact set $\mathcal{X}$ of values of x where the optimization problem is feasible. Note that $\mathcal{X} \subseteq \mathbb{R}^{n-1}$ is a subset of $\Phi \subseteq \mathbb{R}^n$. Now, from (18), the following bound on the model uncertainty can be obtained:

$$\sup_{x\in\mathcal{X}} \|f^P - f^{M_c}\|_2 \leqslant \mathcal{R}_I \tag{19}$$

Note that bound (19) can be derived as only the first components of the model equations (14) and (17) are different, so that the 2-norm of the difference $f^P(x) - f^{M_c}(x)$ is equivalent to the absolute value of the difference $P(\phi) - M_c(\phi)$.

When the predictive controller $u_t = \kappa(x_t)$ is applied to systems (13) and (16), the following autonomous systems are obtained:

$$x_{t+1} = f^P(x_t, \kappa(x_t)) = F^P(x_t) \tag{20}$$

$$x_{t+1} = f^{M_c}(x_t, \kappa(x_t)) = F^{M_c}(x_t) \tag{21}$$

Now, notations $\phi^P(t, x_0) = \underbrace{F^P(F^P(\ldots F^P}_{t \text{ times}}(x_0)\ldots))$ and $\phi^{M_c}(t, x_0) = \underbrace{F^{M_c}(F^{M_c}(\ldots F^{M_c}}_{t \text{ times}}(x_0)\ldots))$ denote the state trajectories of systems (20) and (21), respectively.

Moreover, the following assumptions are made:

Assumption 2
The control law $u_t = \kappa(x_t)$ is continuous over $\mathcal{X}$.

Assumption 3
The autonomous system (21) is uniformly asymptotically stable at the origin for any initial condition $x_0 \in \mathcal{X}$, that is, it is stable and

$$\forall \epsilon > 0, \forall \xi > 0 \quad \exists \tau \in \mathbb{N} \text{ subject to } ||\phi^M(t+\tau, x_0)||_2 < \epsilon, \quad \forall t \geqslant 0, \forall x_0 \in \mathcal{X} : ||x_0||_2 \leqslant \xi \tag{22}$$

Assumption 2 is related to the structure of the optimization problem (i.e., the regularity of model (18) and employed cost function and constraint sets, see e.g., [17–19]). According to the optimal control theory (see e.g., [20]), the higher is the weight on the control input, the smoother is the derived control law. On the contrary, if such a weight is not introduced, the control law could be discontinuous. An example of this phenomenon is given in [21], where a numerical example is reported, in which the absence of weights on the input leads to discontinuities in the controller, whereas an even small quadratic cost on the input makes the control law continuous again. Another well-known condition for the continuity of the control law is the joint convexity of cost and constraints as a function of both the state and the input [22], yet this condition is also hard to check, except for cases such as linear quadratic MPC. Assumption 3 can be satisfied with a suitable choice of the cost function J_t (2) and of the stabilizing constraints (3e) (see e.g., [1–16]).

Define the closed-loop one step prediction error as

$$\begin{aligned} e(x_t) &\doteq x^P_{t+1} - x^{M_c}_{t+1} = \\ &= f^P(x_t, \kappa(x_t)) - f^{M_c}(x_t, \kappa(x_t)) \end{aligned} \tag{23}$$

As a consequence of (19), prediction error (23) results to be bounded:

$$\|e(x_t)\|_2 \leqslant \mathcal{R}_I = \mu \tag{24}$$

According to Assumption 2, compactness of $\mathcal{X}$ implies that function κ is uniformly continuous over $\mathcal{X}$, so that there exists a class-$\mathcal{K}_\infty$ function α_κ such that (see e.g., [23])

$$|\kappa(x^1) - \kappa(x^2)| \leqslant \alpha_\kappa(\|x^1 - x^2\|), \ \forall x^1, \ x^2 \in \mathcal{X} \tag{25}$$

Recall that a class-$\mathcal{K}_\infty$ function $\alpha : [0, +\infty) \to [0, +\infty)$ is a monotonically increasing function such that $\alpha(0) = 0$ and $\lim_{d \to \infty} \alpha(d) = +\infty$.

Moreover, the function $F^{M_c}(x)$ also results to be uniformly continuous over the compact set $\mathcal{X}$, being the composition of a Lipschitz continuous function with a continuous one. Therefore, there exists a corresponding class-$\mathcal{K}_\infty$ function $\alpha_{F^{M_c}}$ with properties analogous to (25):

$$\|F^{M_c}(x^1) - F^{M_c}(x^2)\|_2 \leqslant \alpha_{F^{M_c}}\left(\|x^1 - x^2\|_2\right), \ \forall x^1, \ x^2 \in \mathcal{X} \tag{26}$$

2.4. Set Membership Predictive Control robustness analysis

In this section, a theorem showing that the trajectory ϕ^P of system (20) converges to a neighborhood of the origin, whose size depends on the accuracy of the model f^{M_c} in (21), will be introduced.

Before stating the theorem, the following candidate Lyapunov function $V : \mathcal{X} \to \mathbb{R}^+$ for system (21) is defined:

$$V(x) \doteq \sum_{j=0}^{\widehat{T}-1} \|\phi^{M_c}(j, x)\|_2 \tag{27}$$

where $\widehat{T} \geqslant T$ and $T = \inf_{x \in \mathcal{X}} \left(T \in \mathbb{N} : \|\phi^{M_c}(t+T, x)\|_2 < \|x\|_2,\ \forall t \geqslant 0\right)$.

$V(x)$ is given by the sum of a finite number of compositions of function F^{M_c}. Because the latter is uniformly continuous over the compact $\mathcal{X}$, $V(x)$ is also uniformly continuous over $\mathcal{X}$; hence, there exists a class-$\mathcal{K}_\infty$ function α_V such that

$$|V(x^1) - V(x^2)| \leqslant \alpha_V \left(\|x^1 - x^2\|\right),\ \forall x^1,\ x^2 \in \mathcal{X} \tag{28}$$

Moreover, because of bound (24), the following inequality holds:

$$\forall x \in \mathcal{X},\ \forall e : (F^{M_c}(x) + e) \in \mathcal{X}$$

$$V\left(F^{M_c}(x) + e\right) \leqslant V\left(F^{M_c}(x)\right) + \alpha_V(e) \leqslant V\left(F^{M_c}(x)\right) + \alpha_V(\mu) \tag{29}$$

Furthermore,

$$\|x\|_2 \leqslant V(x) \leqslant \sup_{x \in \mathcal{X}} \frac{V(x)}{\|x\|_2} \|x\|_2 = b\,\|x\|_2,\ \forall x \in \mathcal{X} \tag{30}$$

$$V(F^{M_c}(x)) - V(x) = \Delta V(x) = -\frac{\|x\|_2 - \|\phi^{M_c}(\widehat{T}, x)\|_2}{\|x\|_2} \|x\|_2 \leqslant -K\|x\|_2,\ \forall x \in \mathcal{X} \tag{31}$$

where

$$b = \sup_{x \in \mathcal{X}} \frac{V(x)}{\|x\|_2} \tag{32}$$

and

$$K = \inf_{x \in \mathcal{X}} \frac{\|x\|_2 - \|\phi^{M_c}(\widehat{T}, x)\|_2}{\|x\|_2},\ 0 < K < 1 \tag{33}$$

Note that b in (32) exists finite by the definition of $V(x)$.

Therefore, $V(x)$ is a Lyapunov function for system (21) over $\mathcal{X}$.

Theorem 2

Suppose that Assumptions 2 and 3 hold, then $\forall x_0 \in \mathcal{X}$ such that $\phi^P(t, x_0) \in \mathcal{X}, \forall t \geqslant 0$:

(i) the trajectory distance $d(t, x_0) = \|\phi^P(t, x_0) - \phi^{M_c}(t, x_0)\|_2$ is bounded by Δ which increases monotonically with the bound μ introduced in (24), i.e.

$$\|\phi^P(t, x_0) - \phi^{M_c}(t, x_0)\|_2 \leqslant \Delta = \Delta(\mu)$$

(ii) the trajectory ϕ^P asymptotically converges to a neighborhood of the origin whose size grows monotonically with the value of μ introduced in (24) (i.e. the worst-case accuracy of the model f^{M_c})

$$\lim_{t \to \infty} \|\phi^P(t, x_0)\|_2 \leqslant q(\mu);,\ q(0) = 0;\ \mu^1 > \mu^2 \iff q(\mu^1) > q(\mu^2)$$

Proof

(i) Choose any $x_0 \in \mathcal{X}$ as initial condition for system (20) and model (21). On the basis of (24) and of the uniform continuity of f^{M_c}, it can be noted that

$$\begin{aligned}
d(1,x_0) &= \|\phi^P(1,x_0) - \phi^{M_c}(1,x_0)\|_2 \leqslant \mu \\
d(2,x_0) &= \|\phi^P(2,x_0) - \phi^{M_c}(2,x_0)\|_2 = \\
&= \|F^P(F^P(x_0,\kappa(x_0))) - F^{M_c}(F^{M_c}(x_0,\kappa(x_0)))\|_2 \leqslant \\
&\leqslant \|F^{M_c}(F^P(x_0)) + e - F^{M_c}(F^{M_c}(x_0))\|_2 \leqslant \\
&\leqslant \|F^{M_c}(F^P(x_0)) - F^{M_c}(F^{M_c}(x_0))\|_2 + \|e\|_2 \leqslant \\
&\leqslant \alpha_{F^{M_c}}(\|F^P(x_0) - F^{M_c}(x_0)\|) + \mu \leqslant \\
&\leqslant \alpha_{F^{M_c}}(\mu) + \mu \\
d(3,x_0) &= \|\phi^P(3,x_0) - \phi^{M_c}(3,x_0)\|_2 = \\
&= \|F^P(F^P(F^P(x_0))) - F^{M_c}(F^{M_c}(F^{M_c}(x_0)))\|_2 \leqslant \\
&\leqslant \cdots \leqslant \alpha^2_{F^{M_c}}(\mu) + \alpha_{F^{M_c}}(\mu) + \mu \\
d(t,x_0) &= \|\phi^P(t,x_0) - \phi^{M_c}(t,x_0)\|_2 \leqslant \sum_{k=0}^{t-1} \alpha^k_{F^{M_c}}(\mu)
\end{aligned}$$

with the convention that $\alpha^0_{F^{M_c}}(\mu) = \mu$ and $\alpha^k_{F^{M_c}}(\mu) = \underbrace{\alpha_{F^{M_c}}(\alpha_{F^{M_c}}(\ldots(\mu)\ldots))}_{k\text{times}}$.

Thus, the following upper bound of the distance between trajectories $\phi^P(t,x_0)$ and $\phi^{M_c}(t,x_0)$ is obtained:

$$d(t,x_0) \leqslant \sum_{k=0}^{t-1} \alpha^k_{F^{M_c}}(\mu) = \Delta_1(t,\mu)\,, \quad \forall x_0 \quad \forall t \geqslant 1 \tag{34}$$

The quantity $\sum_{k=0}^{t-1} \alpha^k_{F^{M_c}}(\mu) = \Delta_1(t,\mu)$ is generally not bounded as $t \to \infty$, so it cannot be proved, on the basis of inequality (34) alone, that the trajectory distance $d(t,x_0)$ is bounded. However, by means of the properties of Lyapunov function (27), a second upper bound $\Delta_2(t,\mu)$ of $d(t,x_0)$ can be computed. In fact, through (29) and (31), the following inequality holds:

$$V(F^{M_c}(x) + e) \leqslant V(x) - K\|x\|_2 + \alpha_V(\mu) \tag{35}$$

then, on the basis of (30) and (35), the state trajectory $\phi^P(t,x_0)$ is such that

$$\begin{aligned}
\|\phi^P(t,x_0)\|_2 &\leqslant V\left(\phi^P(t,x_0)\right) = V\left(F^P(\phi^P(t-1,x_0))\right) = \\
&= V(F^{M_c}(\phi^P(t-1,x_0)) + e(\phi^P(t-1,x_0))) \leqslant \\
&\leqslant V(\phi^P(t-1,x_0)) - K\|\phi^P(t-1,x_0)\|_2 + \alpha_V(\mu) \leqslant \\
&\leqslant V(\phi^P(t-1,x_0)) - \frac{K}{b}V(\phi^P(t-1,x_0)) + \alpha_V(\mu) = \\
&= \left(1 - \frac{K}{b}\right)V(\phi^P(t-1,x_0)) + \alpha_V(\mu) = \\
&= \eta V(\phi^P(t-1,x_0)) + \alpha_V(\mu) \leqslant \\
&\leqslant \eta^t V(x_0) + \sum_{j=0}^{t-1} \eta^j \alpha_V(\mu) \leqslant \\
&\leqslant \eta^t V(x_0) + \frac{\alpha_V(\mu)}{1-\eta}
\end{aligned} \tag{36}$$

with $\eta = \left(1 - \frac{K}{b}\right) < 1$. Thus, the following result is obtained:

$$\|\phi^P(t, x_0)\|_2 \leqslant \eta^t V(x_0) + \frac{b}{K}\alpha_V(\mu) \tag{37}$$

$$\|\phi^{M_c}(t, x_0)\|_2 \leqslant \eta^t V(x_0) \tag{38}$$

By means of inequalities (37) and (38), an upper bound $\Delta_2(t, \mu)$ can be computed as

$$\begin{aligned} d(t, x_0) &= \|\phi^P(t, x_0) - \phi^{M_c}(t, x_0)\|_2 \leqslant \\ &\leqslant \|\phi^P(t, x_0)\|_2 + \|\phi^{M_c}(t, x_0)\|_2 \leqslant \\ &\leqslant 2\,\eta^t V(x_0) + \frac{b}{K}\alpha_V(\mu) \leqslant \\ &\leqslant 2\,\eta^t \sup_{x_0 \in \mathcal{X}} V(x_0) + \frac{b}{K}\alpha_V(\mu) = \\ &= \Delta_2(t, \mu)\,, \ \forall x_0 \quad \forall t \geqslant 0 \end{aligned} \tag{39}$$

Note that, since $\mu < \infty$ and $\mathcal{X}$ is compact,

$$\Delta_2(t, \mu) < \infty, \ \forall t \geqslant 0$$

$$\lim_{t \to \infty} \Delta_2(t, \mu) = \frac{b}{K}\alpha_V(\mu) = q(\mu)$$

$$q < \Delta_2(t, \mu) < \infty, \ \forall t \geqslant 0$$

Thus, as t increases towards ∞, the bound $\Delta_2(t, \mu)$ (39) decreases monotonically from a finite positive value equal to $2 \sup_{x_0 \in \mathcal{X}} V(x_0) + \frac{b}{K}\alpha_V(\mu)$ towards a finite positive value $q(\mu)$, whereas the bound $\Delta_1(t, \mu)$ (34) generally increases monotonically from 0 to ∞. Therefore, for a fixed value of μ, there exists a finite discrete-time instant $\widehat{t} > 0$ such that $\Delta_1(\widehat{t}, \mu) > \Delta_2(\widehat{t}, \mu)$. As a consequence, by considering the lowest bound between $\Delta_1(t, \mu)$ and $\Delta_2(t, \mu)$ for any $t \geqslant 0$, the following bound $\Delta(\mu)$ of $d(t, x)$, which depends only on μ, is obtained:

$$\begin{aligned} &\Delta(\mu) = \sup_{t \geqslant 0} \min(\Delta_1(t, \mu), \Delta_2(t, \mu)) \\ &q(\mu) \leqslant \Delta(\mu) < \infty \\ &\|\phi^P(t, x_0) - \phi^{M_c}(t, x_0)\|_2 \leqslant \Delta(\mu), \ \forall x_0 \in \mathcal{X} \quad \forall t \geqslant 0 \end{aligned}$$

As both $\Delta_1(t, \mu)$ and $\Delta_2(t, \mu)$ increase monotonically with μ, also their pointwise minimum with respect to t does.

(ii) On the basis of (37), it can be noted that

$$\begin{aligned} \lim_{t \to \infty} \|\phi^P(t, x_0)\|_2 &\leqslant \lim_{t \to \infty} \eta^t b \|x_0\|_2 + \frac{b}{K}\alpha_V(\mu) \\ &= \frac{b}{K}\alpha_V(\mu) = q(\mu), \ \forall x_0 \in \mathcal{X}. \end{aligned} \tag{40}$$

The claim follows by the properties of the class-$\mathcal{K}_\infty$ function α_V.

□

Remark 1

In the proof of Theorem 2, the case of diverging bound $\Delta_1(\mu)$ has been considered. Depending on the properties of function α_{FM_c} and on the value of μ, in principle, the quantity $\sum_{k=0}^{t-1} \alpha_{FM_c}^k(\mu) =$

$\Delta_1(t,\mu)$ can also converge, as $t \to \infty$, to some finite quantity $q_1(\mu)$. The latter would still be monotonically strictly increasing with μ and such that $q_1(0) = 0$. In this case, all the results of the theorem would still hold true; eventually, the bound $q(\mu) = \frac{b}{K}\alpha_V(\mu)$ would be replaced by $q_1(\mu) = \lim_{t\to\infty} \sum_{k=0}^{t-1} \alpha^k_{F^{M_c}}(\mu)$, whichever is smaller.

In the sequel, the following notation will be used:

$$\mathbb{B}(\mathcal{A}, r) = \bigcup_{x\in\mathcal{A}} \mathbb{B}(x, r) \quad \mathcal{A} \subset \mathbb{R}^n$$

where $\mathbb{B}(x,r) \doteq \{\xi \in \mathbb{R}^n : \|x - \xi\|_2 \leqslant r\}$.

Proposition 1
Suppose there exists a positively invariant set $\mathcal{G} \subset \mathcal{X}$ such that

(i) $\phi^{M_c}(t, x_0) \in \mathcal{G}, \quad \forall x_0 \in \mathcal{G}, \forall t \geqslant 0$
(ii) $\mathbb{B}(\mathcal{G}, \Delta) \subset \mathcal{X}$

then points (i) and (ii) of Theorem 2 hold $\forall x_0 \in \mathcal{G}$.

The main consequence of Proposition 1 is that for any initial condition $x_0 \in \mathcal{G}$, it is guaranteed that the state trajectory is kept inside the set $\mathcal{X}$ and converges to the set $\mathbb{B}(0, q)$, whose size depends on the accuracy of the model f^{M_c}. Theorem 2 is to be intended mainly as a qualitative result that establishes local robust attractivity of the origin of the closed-loop system. The difficulty of using this result also for quantitative analysis lies in the practical computation of the involved quantities and in the related conservativeness. On the other hand, to find nonconservative and practically useful results for nonlinear systems is quite a hard task, unless some more restrictive assumptions on the structure of the system and of the problem are made.

Remark 2
If the control law κ is Lipschitz continuous with constant L_κ (i.e., if stronger regularity properties than those of Assumption 2 are assumed), the results of Theorem 2 can be refined into linear relationships between the trajectory bounds and the worst-case modeling error μ. More specifically, in this case, function F^{M_c} is also Lipschitz continuous, with constant $L_{M_{CL}} = \sqrt{(1 + L_\kappa^2)(1 + L_M^2)}$ so is the Lyapunov function V, with constant $L_V = \sum_{j=0}^{\widehat{T}-1} (L_{M_{CL}})^j$. Then, the bounds involved in Theorem 2 become

$$\Delta_1(t,\mu) = \left(\sum_{k=0}^{t-1} (L_{M_{CL}})^k\right)\mu$$

$$\Delta_2(t,\mu) = 2\,\eta^t \sup_{x_0\in\mathcal{X}} V(x_0) + \frac{b}{K} L_V\,\mu$$

$$q(\mu) = \frac{b}{K} L_V\,\mu$$

Lipschitz continuity is indeed a strong assumption in the case of general nonlinear systems. In some cases, numerical inspection procedures can be used to check whether such condition holds.

3. APPLICATION TO VEHICLE YAW CONTROL

In order to show the effectiveness of the proposed SMPC methodology, an application to a vehicle yaw stability control system is presented here.

3.1. Control requirements

Yaw stability control systems have been introduced to significantly enhance safety and handling properties of vehicles (see e.g., [24] and [25]) by modifying their passive dynamic behavior using suitable control structures and actuation devices. In particular, in this paper, a vehicle equipped with a front steer-by-wire actuator, based on a classical rack and pinion steering system (see e.g., [26]), is considered.

The control objective is the tracking of a reference yaw rate value $\dot{\psi}_{\text{ref}}(t)$, whose course is designed to improve the vehicle maneuverability and to assist the driver in keeping directional stability under different driving conditions. In the considered situation, the vehicle front steering angle δ represents the control input, whereas the yaw rate $\dot{\psi}$ is the controlled output.

A feedback control law receives as input the reference yaw rate value, together with the measured yaw rate $\dot{\psi}$, and computes a suitable command current for the steer-by-wire device that imposes accordingly the pinion angle and, consequently, the steering angle δ of the front wheels. The desired vehicle behavior is taken into account in the control design by a suitable choice of the reference signal $\dot{\psi}_{\text{ref}}$. Details on the computation of the $\dot{\psi}_{\text{ref}}$ can be found in [25]. The tracking of $\dot{\psi}_{\text{ref}}$ can be taken into account by minimizing the amount of the tracking error e, defined as

$$e = \dot{\psi}_{\text{ref}} - \dot{\psi}$$

The value of the front steering angle δ, generated by the employed active device, is subject to its physical limits. In particular, the range of allowed front steering angles that can be mechanically generated is $\pm$ 35°; thus, saturation of the control input (i.e., the angle δ) has to be taken into account in the control design.

3.2. Vehicle model identification and Set Membership Predictive Control design

A set of measured values of $\tilde{\dot{\psi}}_t$ and $\tilde{\delta}_t$ are collected to identify NSM vehicle model M_c (11).

The number of output and input regressors, n_y and n_u, have to be chosen to achieve a suitable tradeoff between model complexity and accuracy, whereas the values of the Lipschitz constant γ and of the noise bound ε are estimated from the data to achieve a nonempty *FSS* (for more details on the regressor choice and on the computation of γ and ε, the interested reader is referred to [14]).

According to (12), the pseudo-state results to be

$$x_t = \left[\begin{array}{cccc} \dot{\psi}_t & \dots & \dot{\psi}_{t-n_y} & \delta_{t-1} \dots \delta_{t-n_u} \end{array} \right] \in \mathbb{R}^{n-1} \tag{41}$$

The control move is obtained, according to the RH strategy (Section 2.1), by optimizing the following cost function:

$$\min_{U} \quad \sum_{j=1}^{N_p} Q\, e_{t+j+1|t}^2 + R\, \delta_{t+j|t}^2 \tag{42}$$

subject to

$$x_{t+j+1|t} = f_c^M \left(x_{t+j|t}, \delta_{t+j|t} \right),\ j \in [0, Np-1] \tag{43}$$

$$U \in \mathbb{U} = \left\{ \delta_{t+j|t} \,:\, |\delta_{t+j|t}| \leqslant \bar{\delta},\ j \in [1, N_c] \right\} \tag{44}$$

$$e_{t+Np|t} = 0 \tag{45}$$

$$u_{t+N_c|t}, \dots, u_{t+N_p-1|t} = u_{t+N_c-1|t} \tag{46}$$

where $Q,\ R \in \mathbb{R}^+$ are suitable weights, $e_{t+j|k}$ is the jth step ahead prediction of the tracking error obtained as

$$e_{t+j|t} \doteq \dot{\psi}_{\text{ref},t} - \dot{\psi}_{t+j|t}$$

Terminal state (45) equality constraint induces asymptotic closed-loop stability of the nominal model as required by Assumption 3, since in the presence of this constraint, the optimal cost function turns out to be a Lyapunov function for the tracking error's dynamics. Please refer to [27] for more details.

The saturation value $\bar{\delta}$ is the maximum steering angle that can be mechanically generated, that is, $\bar{\delta} = \pm 35°$. The values of Q, R, N_p, N_c are design parameters suitably chosen to achieve a good compromise between closed-loop stability and performance (see e.g., [1]).

3.3. Simulation results

In order to evaluate the effectiveness of the proposed approach in a realistic way, a detailed 14-degree of freedom (dof) Simulink® vehicle model is employed. Such a model gives an accurate description of the vehicle dynamics as compared with the actual measurements and includes nonlinear suspension, steer, and tire characteristics, obtained on the basis of measurements on the real vehicle (see [25] for a detailed description of such a model). The 14-dof model has been employed at first stance to generate $\tilde{\dot{\psi}}_t$ and $\tilde{\delta}_t$ data with sampling time $T_s = 0.01$ s, by simulating a series of standard maneuvers. Then, such data have been divided into two subsets: that is, the identification and the validation data. The identification data have been employed to derive NSM vehicle model (11), whereas the validation ones to evaluate its accuracy and to tune the values of n_u, n_y, γ, and ε. In particular, after a series of trial-and-error iterations, the values $n_y = 1$, $n_u = 3$, $\gamma = 3$ and $\varepsilon = 0.02$ rad/s have been chosen. The SMPC law has then been designed using the parameters $N_p = 30$, $N_c = 3$, $Q = 10$, and $R = 5$.

The performance of the proposed SMPC approach have been compared with the one of an NMPC controller designed using the nonlinear single-track vehicle model described by state equation (47) (see [24] and Figure 1 for more details):

$$\begin{aligned} mv(t)\dot{\beta}(t) + mv(t)\dot{\psi}(t) &= F_{yf}(t) + F_{yr}(t) \\ J_z\ddot{\psi}(t) &= aF_{yf}(t) - bF_{yr}(t) \end{aligned} \tag{47}$$

where m is the vehicle mass, J_z is the moment of inertia around the vertical axis, β is the sideslip angle, ψ is the yaw angle, v is the vehicle speed, and a and b are the distances between the center of gravity and the front and rear axles, respectively.

The nominal parameter values used are $m = 1715\,\text{kg}$, $J_z = 2700\,\text{kgm}^2$, $a = 1.07$ m, and $b = 1.47$ m. The sampling time $T_s = 0.01$ s is used to discretize the model by means the

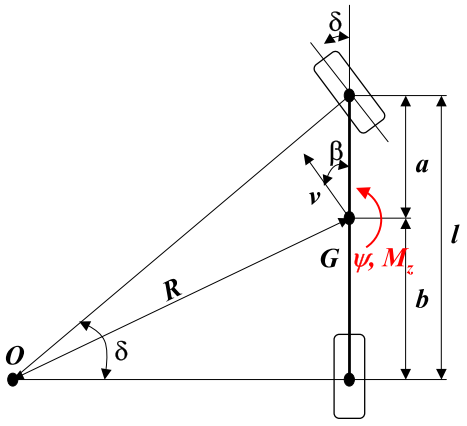

Figure 1. Single track model schematic.

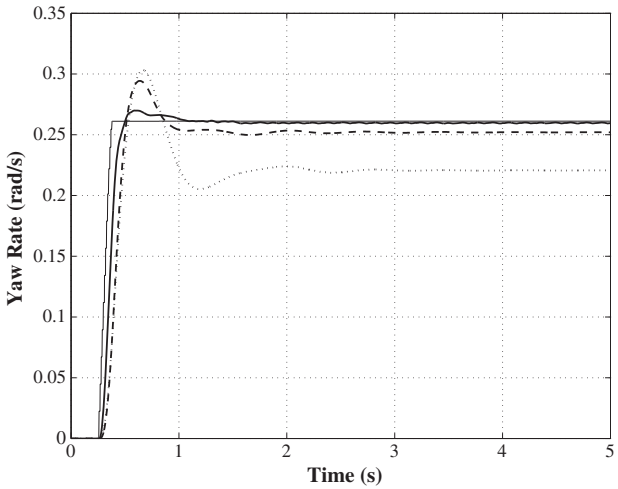

Figure 2. Results of the 50° handwheel step test with nominal parameters. Uncontrolled vehicle yaw rate (dotted), reference yaw rate (thin solid line), and yaw rate are obtained with the Set Membership Predictive Control (solid) and Nonlinear Model Predictive Control based on a physical model (dashed) control laws.

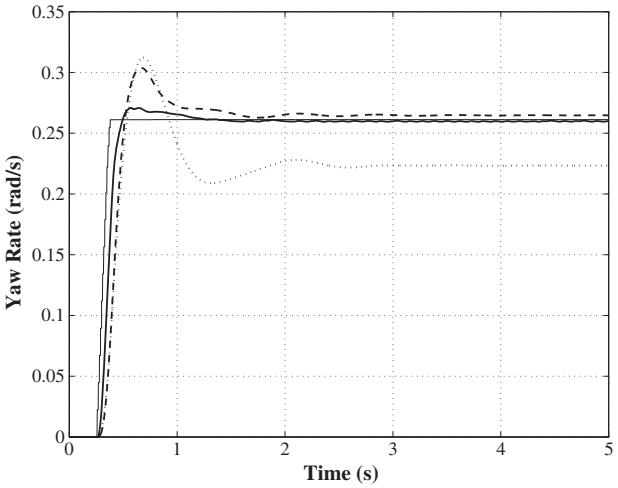

Figure 3. Results of the 50° handwheel step test with increased mass. Uncontrolled vehicle yaw rate (dotted), reference yaw rate (thin solid line), and yaw rate are obtained with the Set Membership Predictive Control (solid) and Nonlinear Model Predictive Control based on a physical model (dashed) control laws.

forward difference approximation. F_{yf} and F_{yr} are the front and rear tire lateral forces, which can be expressed as nonlinear functions of the state, the input and of the vehicle speed (see [25] and [28] for more details):

$$\begin{aligned} F_{yf} &= F_{yf}(\beta, \dot{\psi}, v, \delta) \\ F_{yr} &= F_{yr}(\beta, \dot{\psi}, v, \delta) \end{aligned} \tag{48}$$

At first, the NMPC parameters Q, R, N_p, and N_c were set to be the same of the SMPC ones. In such a way, the performance of the NMPC controller was significantly worse than that of the SMPC. After that, to make a fair comparison, the NMPC parameters have been tuned via a trial-and-error procedure to improve the performance. The final choice was $Q = 2$, $R = 10$, $N_p = 80$, and $N_c = 2$.

In both cases, the control move computation has been performed using the MatLab® optimization function `fmincon`.

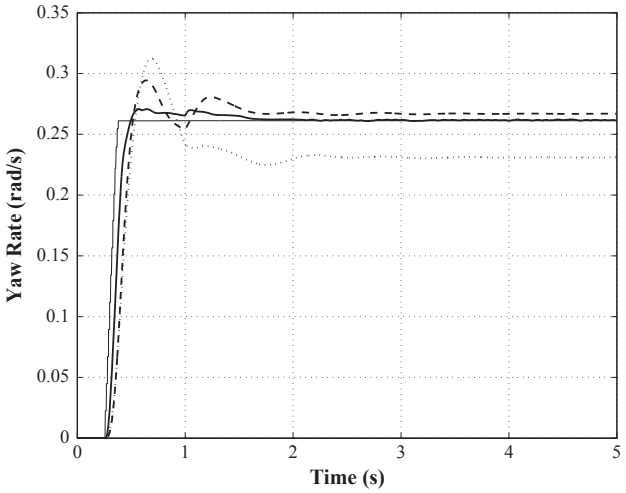

Figure 4. Results of the 50° handwheel step test when a blust of wind occurs at 1 s. Uncontrolled vehicle yaw rate (dotted), reference yaw rate (thin solid line), and yaw rate obtained with the Set Membership Predictive Control (solid) and Nonlinear Model Predictive Control based on a physical model (dashed) control laws.

Assumptions 1–3 have been checked through numerical inspection by using an approach similar to the estimation of the Lipschitz constant proposed in [14] and [29].

An open loop (i.e., without driver's feedback) maneuver has been chosen to test the control effectiveness and to compare the two approaches. In particular, a 50° handwheel step at 100 km/h, with a handwheel speed of 400°/s, has been performed in different conditions:

- With nominal vehicle parameters
- With increased vehicle mass, +100 kg, with consequent variations of the other involved inertial and geometrical characteristics
- With a lateral wind gust, which exerts on the vehicle a lateral force and a moment of 800 N and 500 Nm, respectively, for a period of 1 s

Such tests aimed at evaluating both the transient and steady state performance of the controlled vehicle.

The obtained results in nominal conditions (i.e., when the parameters of the 14-dof model match with those of physical model (47)) are reported in Figure 2. It can be noted that the NMPC law based on the physical model achieves a steady-state regulation error of about 3%. This is due to the neglected dynamics and undermodeling of the physical model. On the other hand, the SMPC approach, by employing a model identified directly from data, achieves better regulation precision with 0.9% steady-state tracking error. The advantages of the SMPC technique are also evident in an handwheel step test with increased vehicle mass. The result of this test is shown in Figure 3: whereas the SMPC law is able to keep a nearly zero tracking error (0.9%), the NMPC law based on the physical model achieves a slightly higher steady-state tracking error (1.9%) with respect to the SMPC law. Further, Figure 4 shows that the SMPC controller is more robust with respect to the NMPC controller when an external disturbance occurs (0.9% versus 3.8%).

Thus, the presented simulation results highlight that the proposed SMPC methodology improves both robustness and regulation precision of the closed-loop system with respect to the NMPC one, on the basis of physical modeling of the system.

4. CONCLUSION

The paper presented a novel approach, denoted as SMPC, to design a predictive control law. The proposed SMPC technique relies on a model derived using a NSM identification method and

input/output data collected during preliminary tests. The NSM approach is able to provide a model with a bounded model uncertainty; by exploiting this feature, a theoretical result that allows an a posteriori analysis of the robust stability of the closed-loop system has been introduced. The effectiveness of the proposed approach has been shown through a vehicle lateral stability control problem.

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