On Mixed-Integer Random Convex Programs

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Abstract— We consider a class of mixed-integer optimization problems subject to N randomly drawn convex constraints. We provide explicit bounds on the tails of the probability that the optimal solution found under these N constraints will become infeasible for the next random constraint. First, we study constraint sets in general mixed-integer optimization problems, whose continuous counterpart is convex. We prove that the number of support constraints (i.e., constraints whose removal strictly improve the optimal objective) is bounded by a number depending geometrically on the dimension of the decision vector. Next, we use these results to show that the tails of the violation probability are bounded by a binomial distribution. Finally, we apply these bounds to an example of robust truss topology design. The findings in this paper are a first step towards an extension of previous results on continuous random convex programs to the case of problems with mixed-integer decision variables that naturally occur in many real-world applications.

I. INTRODUCTION

Whenever optimization techniques are used to find solutions to real-world problems, e.g., in machine learning or optimal control, uncertainty in the input data is very likely to occur. In order to make the solutions robust against uncertainties and to assure they do not violate unanticipated constraints, the uncertainties have to be accounted for *ex-ante* in the optimization process. There are several approaches to counteract these difficulties and make optimization robust to uncertainty. The *robust convex optimization* approach [1], [2] finds a solution to a convex problem that is robust to all uncertainty realizations bound to lie in a given bounded uncertainty set. *Chance-constrained* approaches assume that there is a probability measure over the uncertainty set and try to find an optimal solution that satisfies uncertain constraints with a guaranteed high probability, see [3] and the references therein. *Random convex programs* find an optimal solution subject to a finite number of randomly drawn constraints, which can be done fast and efficiently with modern convex solvers. This solution will then remain optimal with a high probability for the next "unseen" constraint realization. More precisely, a random convex program (RCP) is a convex optimization problem with linear cost objective, subject to N randomly drawn convex constraints. Since the constraints are randomly drawn, the optimal solution of an RCP is a random variable. For RCPs with continuous decision vectors, [4], [5] initially provided bounds on the tails of the

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probability that an optimal solution found under N random constraints will become infeasible for the next randomly drawn constraint. These bounds were refined in [6] under the restriction that the RCP is feasible with probability one for all random constraints. The work [7] lifted this restriction and proved bounds without any assumptions on feasibility. Further it extends previous results to the case that some of the random constraints may be violated with the goal of further improving the optimal objective.

In this paper, we extend the existing work on continuous RCPs to the case of random programs with mixed-integer decision vectors lying in the intersection of $\mathbb{R}^n \times \mathbb{Z}^d$ and random convex constraints, and we prove bounds on the tails of the violation probability in this more general case. This class of optimization problems is clearly not convex in the usual sense, since the domain has discrete components, however it seems appropriate to name them *mixed-integer random convex programs* (MI-RCP) to highlight the fact that the feasible set is the intersection of $\mathbb{R}^n \times \mathbb{Z}^d$ with a convex set. It is worth to stress that the focus of this paper is *not* on the numerical or computational issues related to the solution of an instance of a MI-RCP. Rather, the contribution is on the probabilistic properties of such a solution, no matter how the solution itself is obtained in practice.

To begin with, we study properties of constraints sets of general deterministic mixed-integer convex programs. More specifically, we prove that the number of *support constraints*, i.e., constraints whose removal in the convex program leads to an improvement of the optimal objective, is bounded by a number depending on the space in which the decision vector lies (the so-called Helly dimension, named after the classical result due to Helly, see [8], [9], [10]). Unfortunately, unlike the continuous case where this number is essentially equal to the dimension of the decision variable, in the mixed-integer case the Helly dimension grows geometrically with the number of discrete decision variables. These results on mixed-integer convex problems then allow us to prove novel bounds on the tails of the violation probability for MI-RCPs. Again, the important property of RCPs is that the optimal solution, although it was found on only a finite subset of all possible realizations of the random constraints, will with high probability *generalize* and be feasible for the next, yet "unseen", constraint. In this paper we show that this property carries over from continuous to mixed-integer problems and, hence, extend the theory of RCPs even further (and with that also the range of possible applications).

The paper is structured as follows. In Section II we first study properties of constraint sets of mixed-integer deterministic convex problems. In Section III we give a formal definition of mixed-integer RCPs and of their violation probability, and then we derive the bounds on the tails of the violation probability. In Section IV we apply these results to an example of robust truss topology design. Section V concludes the paper and points to future work.

II. Mixed-Integer Convex Constraint Sets

In this section we consider mixed-integer "convex" optimization problems of the form

$$\mathrm{P}[K]: \min_{x\in\Omega} c^\top x \qquad (1)$$
$$\text{s.t. } f_j(x) \le 0, \; j \in K$$

with a linear objective $c \neq 0$ and decision variable x, confined to lie within a compact domain Ω that will either be a subset of $\mathbb{Z}^d$, $\mathbb{R}^n \times \mathbb{Z}^d$ or $\mathbb{R}^n$. The constraints are given by convex and lower-semicontinuous functions $f_j \colon \mathbb{R}^m \to \mathbb{R}$ (where either $m = d$, $n + d$, or n) that are indexed over the finite set K. We denote the optimal solution of $\mathrm{P}[K]$ by $x^*(K)$ and the optimal objective by $J^*(K)$.

Definition 1 (Support Constraints): A constraint $j \in K$ is a *support constraint* of $\mathrm{P}[K]$ if $J^*(K \setminus j) < J^*(K)$, i.e., if the optimal objective strictly improves when constraint j is removed from $\mathrm{P}[K]$. Denote the set of support constraints of $\mathrm{P}[K]$ by $\mathrm{Sc}(K) \subset K$.

The following definition of convex sets in mixed-integer spaces will conclude this section.

Definition 2: Let $M \subset \mathbb{R}^m$ be a closed set. We say that a subset $C \subset M$ is *M-convex* if there is a convex subset $\tilde{C}$ of $\mathbb{R}^m$ such that $C = \tilde{C} \cap M$.
For example, on the integer lattice $\mathbb{Z}^d$, M-convex sets are simply the intersection of standard convex sets in $\mathbb{R}^d$ with the integer lattice.

A. Support Constraints on the Integer Lattice

In this section we will prove the crucial result that the number of support constraints of a problem $\mathrm{P}[K]$ is less than or equal to 2^d if $\Omega \subset \mathbb{Z}^d$, that is, if the decision variables are confined to lie on the *integer* lattice. The proof uses the following fact from [9, Proof of Proposition 4.2, p.83]:

Fact 1: Let $S \subset \mathbb{Z}^d$ be a set with $2^d + 1$ points and let $\mathrm{ci}(S)$ denote the smallest convex set in $\mathbb{Z}^d$ that includes S. Then it holds that

$$\bigcap_{x \in S} \mathrm{ci}(S \setminus \{x\}) \neq \emptyset . \qquad (2)$$

This means that, if the number of points in a set $S \subset \mathbb{Z}^d$ is large enough, then the intersection of $\mathbb{Z}^d$ and the convex hulls of all subsets $S \setminus \{x\}$, consisting of S with the point x removed, is nonempty.

Theorem 1: For a feasible convex optimization problem $\mathrm{P}[K]$ with $\Omega \subset \mathbb{Z}^d$, the number of support constraints is less than or equal to $2^d - 1$.

Proof: The proof is by contradiction. Assume that there are more than $2^d - 1$ support constraints. Without loss of generality we consider the case with 2^d support constraints and also assume that the support constraints are the first 2^d constraints $k = 1, \ldots, 2^d$.

Let $x_0^* := x^*(K) \in \mathbb{Z}^d$ denote the optimal solution of the optimization problem and let $x_k^* := x^*(K \setminus k) \in \mathbb{Z}^d$ be the optimal solution when support constraint k is removed. Let $J^* := c^\top x_0^*$ and $J_k^* := c^\top x_k^*$, for $k = 1, \ldots, 2^d$, denote the respective optimal objective values. Define the set $\mathcal{X}$ to be

$$\mathcal{X} := \{x_0^*, x_1^*, \ldots, x_q^*\} \subset \mathbb{Z}^d, \; q = 2^d . \qquad (3)$$

Notice that all points in $\mathcal{X}$ are distinct. To prove this, assume for contradiction that two of them coincide e.g. $x_{k_1}^* = x_{k_2}^*$. The point $x_{k_1}^*$ satisfies all constraints except for k_1 and the same holds for the point $x_{k_2}^*$ with constraint k_2. Since they are equal, they satisfy all constraints. Since k_1 and k_2 are support constraints, the points have a better objective value than x_0^*: $J_{k_1}^* = J_{k_2}^* < J_0^*$ by definition. So the point $x_{k_1}^* = x_{k_2}^*$ satisfies all constraints in K and has a better objective value than x_0^*, which is a contradiction to x_0^* being optimal for $\mathrm{P}[K]$.

Define η_{min} as the smallest objective improvement when discarding a support constraint

$$\eta_{min} := \min_{k=1,\ldots,2^d} (J^* - J_k^*) \qquad (4)$$

and let η be such that $0 < \eta < \eta_{min}$. Consider the halfspace

$$\mathcal{H} := \{x \colon c^\top x < J^* - \eta\} . \qquad (5)$$

By construction, all points x_k^*, $k = 1, \ldots, 2^d$, lie in $\mathcal{H}$ while x_0^* does not.

Since there are 2^d support constraints and also $x_0^* \in \mathcal{X}$ we have that $|\mathcal{X}| = 2^d + 1$. We now apply Fact 1 and obtain that

$$\bigcap_{x_k^* \in \mathcal{X}} \mathrm{ci}(\mathcal{X} \setminus \{x_k^*\}) \neq \emptyset \qquad (6)$$

and hence there exists a $z \in \mathbb{Z}^d$ that is in the intersection.

Since z is in the intersection (6) we have $z \in \mathrm{ci}(\mathcal{X} \setminus \{x_0^*\})$ and hence it is in the convex hull of all the points x_k^*, $k = 1, \ldots, 2^d$. It follows that $z \in \mathcal{H}$ since all the x_k^* for $k \neq 0$ are in $\mathcal{H}$.

For all support constraints $k \in \mathrm{Sc}(K)$ we know that $z \in \mathrm{ci}(\mathcal{X} \setminus \{x_k^*\})$, and since all other points $x_j^*, j \neq k$, satisfy constraint k (and all other non-support constraints), z does too. So z is feasible for $\mathrm{P}[K]$. Hence, z is an integer point that is feasible for $\mathrm{P}[K]$ and also $z \in \mathcal{H}$, which means that $c^\top z < J^* - \eta < J^* = c^\top x_0^*$. We have thus found a feasible integer solution with better objective than x_0^*, which is a contradiction to x_0^* being optimal. ∎

B. Support Constraints for Mixed-Integer Problems

In this section, we prove an upper bound on the number of support constraints of an *mixed-integer* convex optimization problem, i.e., one where the decision variables are confined to lie in $\Omega \subset \mathbb{R}^n \times \mathbb{Z}^d$. In fact, we will prove an upper bound on the number of support constraints for any $\Omega \subset M$ where M is a general closed subset of $\mathbb{R}^m$. This can be an integer or a mixed-integer or a some other space.

Definition 3: We define the *Helly's dimension* $h(M)$ of M to be the smallest integer h such that for every finite collection of M-convex sets $C_1, \ldots, C_m$ with $m \geq h$ for which every subcollection of h of the sets has nonempty intersection

$$\bigcap_{i \in I} C_i \neq \emptyset , \qquad (7)$$

with $|I| = h$, it follows that the intersection $C_1 \cap C_2 \cap \cdots \cap C_m$ of all sets is also nonempty.

Theorem 2: For a feasible convex optimization problem $\mathrm{P}[K]$ with $\Omega \subset M$, the number of support constraints is less than or equal to $h - 1$.

Proof: The proof is again by contradiction. We assume that we have q support constraints and that $q \geq h$. Let the points x_0^* and x_k^*, $k = 1, \ldots, q$ and the halfspace $\mathcal{H}$ be defined as in the proof of Theorem 1. With the same

argumentation as above all the points x_k^* for $k = 0, \ldots, q$, are distinct. Define the polytopes

$$P_k := \operatorname{conv}_M \{x_i^* : i \in \{0, \ldots, q\} \setminus \{k\}\}, \qquad (8)$$

for $k = 0, \ldots, q$ to be the convex hull in M of the points $\{x_0^*, x_1^*, \ldots, x_q^*\}$ except for the point x_k^*. We have $q + 1$ polytopes P_k, since there are $q+1$ points x_k^*. Since the points in $\{x_0^*, x_1^*, \ldots, x_q^*\} \setminus \{x_k^*\}$ satisfy constraint k, by convexity of the constraints it follows that all points in the convex hull P_k satisfy constraint k.

Let $I \subset \{0, \ldots, q\}$ be an arbitrary index set of cardinality $|I| = h$. Since $h < q + 1$ there is an index $j \in \{0, \ldots, q\}$ that is not in the index set I. So there is a point $x_j^* \in M$ that lies in all the sets P_i, $i \in I$, by construction of the polytopes P_i, and hence x_j^* lies in the intersection

$$x_j^* \in \bigcap_{i \in I} P_i \ . \qquad (9)$$

It follows that for arbitrary index sets $I \subset \{0, \ldots, q\}$ with cardinality $|I| = h$ the intersection $\bigcap_{i \in I} P_i \neq \emptyset$ is nonempty. Since h is the Helly dimension of M, we can conclude from Definition 3 of the Helly dimension, that the intersection of *all* P_k is not empty

$$\bigcap_{k=0,\ldots,q} P_k \neq \emptyset \qquad (10)$$

and there exists a $z \in M$ that lies in the intersection of all the P_k. This point z is in $P_0 = \operatorname{conv}_M \{x_1^*, \ldots, x_q^*\}$, which does not include x_0^*, hence z is in $\mathcal{H}$ by construction. The point z also satisfies all q support constraints (and all other constraints) because it is in each of the P_k. So, $c^\top z < J^* - \eta < J^*$, and z is feasible for problem (1) and has better objective than x_0^* which is a contradiction to x_0^* being optimal. ∎

Corollary 1: For a feasible convex optimization problem P[K] with $\Omega \subset \mathbb{R}^n \times \mathbb{Z}^d$, the number of support constraints is less than or equal to $(n+1)2^d - 1$.

Proof: Averkov and Weismantel proved in [10] that the Helly's dimension of $\mathbb{R}^n \times \mathbb{Z}^d$ is $h(\mathbb{R}^n \times \mathbb{Z}^d) = (n+1)2^d$. The claim then follows from Theorem 2. ∎

We have thus established that a feasible mixed-integer convex problem P[K] has at most $(n+1)2^d - 1$ support constraints.

Remark 1: For an infeasible convex optimization problem as in (1) with $\Omega \subset M$ the number of support constraints is less than or equal to h. The proof is a straightforward extension of the proof of Lemma 2.3 in [7] to spaces with general Helly dimension h. It is omitted here due to space restrictions.

We conclude the section by introducing the concept of fully supported problems, i.e., problems in which the number of support constraints is exactly $h - 1$.

Definition 4 (Fully Supported Problem): A feasible problem P[K] is called *fully supported* if the number of it has exactly $h - 1$ support constraints.

C. Examples

Previous results in [5], [6], [7] show that for continuous spaces the number of support constraints depends linearly on the dimension of the decision variable. In the case of integer or mixed-integer decision variables our results show that there is an exponential dependence of the number of support constraints on the dimension of the integer decision variables. In this section, we show that these bounds on the number of support constraints can actually be attained, i.e., we construct two examples of integer linear problems (ILP) with exactly $2^d - 1$ support constraints.
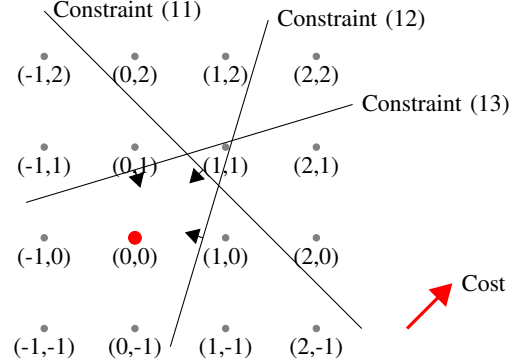

Fig. 1. Illustration of the feasible region of a two-dimensional ILP. The gray dots depict the integer lattice and the black lines the linear constraints. The bold arrows are the normal directions pointing into the feasible region. The red arrow on the bottom right depicts the direction of cost improvement and the red point is the optimal integer solution.

A two-dimensional Example Consider the ILP

$$\begin{aligned}
\min. \ &-x - y \\
\text{s.t. } &x + y \leq 1.5 &(11) \\
&x - 0.3y \leq 0.75 &(12) \\
&-0.3x + y \leq 0.75 &(13) \\
&(x, y) \in \mathbb{Z}^2 \cap [-10.10]^2 \ .
\end{aligned}$$

The optimal solution is $(x^*, y^*) = (0, 0)$, as it can be easily checked. If we remove Constraint (11), the integer point $(x_1, y_1) = (1, 1)$ has a better objective than (x^*, y^*). If we remove Constraint (12), the integer point $(x_2, y_2) = (1, 0)$ has a better objective than (x^*, y^*). If we remove Constraint (13), the integer point $(x_3, y_3) = (0, 1)$ has a better objective than (x^*, y^*). Hence, Constraints 11-13 are support constraints and the ILP has $2^d - 1 = 3$ support constraints.

A three-dimensional Example Consider the ILP

$$\begin{aligned}
\min. \ &-x - y - z \\
\text{s.t. } &Ax \leq b \\
&(x, y, z) \in \mathbb{Z}^2 \cap [-10.10]^3 \ .
\end{aligned}$$

with

$$A^\top = \begin{bmatrix} 1 & -0.35 & 1 & 1 & 1 & -0.35 & -0.35 \\ 1 & 1 & -0.35 & 1 & -0.35 & 1 & -0.35 \\ 1 & 1 & 1 & -0.35 & -0.35 & -0.35 & 1 \end{bmatrix}$$

and $b^\top = [2.5 \ \ 1.7 \ \ 1.7 \ \ 1.7 \ \ 0.85 \ \ 0.85 \ \ 0.85]$. It can be checked that the optimal integer solution of this problem is $(x^*, y^*, z^*) = (0, 0, 0)$. If the first row of A is removed, the integer point $(x_1, y_1, z_1) = (1, 1, 1)$ has a better objective than (x^*, y^*, z^*) and, hence, this row is a support constraint. The same can be checked for the removal of each of the other rows of A and we obtain an ILP that has $2^3 - 1 = 7$ support constraints.

III. Mixed-Integer Random Convex Programs

In this section, we consider random convex programs in which the decision variable can be integer or continuous or a mixture of both. Consider a function $f(x, \delta): (\mathbb{R}^n \times \mathbb{Z}^d) \times \mathbb{R}^l \to \mathbb{R}$ that is convex and lower-semicontinuous in x for any fixed δ. Let $\delta \in \Delta \subset \mathbb{R}^l$ denote a random vector and let $\mathbb{P}$ be a probability measure over Δ. Denote by $\omega := (\delta^{(1)}, \ldots, \delta^{(N)})$ N independent extractions drawn from Δ.

Definition 5 (Mixed-Integer Random Convex Program): A MI-RCP is an optimization problem of the form

$$P[\omega]: \min_{x \in \Omega} c^\top x \qquad (14)$$
$$\text{s.t. } f(x, \delta^{(j)}) \le 0, \; j = 1, \ldots, N ,$$

where the decision variables x lie in the compact domain $\Omega \subset \mathbb{R}^n \times \mathbb{Z}^d$ ($n = 0$ for pure integer programs).

Definition 6 (Violation Probability): For a feasible MI-RCP $P[\omega]$ the violation probability is defined as

$$V^*(\omega) := \mathbb{P}\{\delta \in \Delta: f(x^*(\omega), \delta) > 0\} , \qquad (15)$$

that is, the probability that the optimal solution $x^*(\omega)$ of $P[\omega]$ found under extraction ω will become infeasible under the next realization $\delta \in \Delta$.

$V^*(\omega)$ is itself a random variable with values in $[0, 1]$, depending on the random extraction ω. We make the following assumptions regarding $P[\omega]$:

Assumption 1:

1) Problem $P[\omega]$ is feasible with probability one.
2) The optimal solution of $P[\omega]$ is unique.
3) $P[\omega]$ is fully supported with probability one.

The second assumption is no severe restriction, since the uniqueness of the optimal solution can always be achieved by introducing suitable tie-breaking rules (see e.g. [5]). Also, the assumption that $P[\omega]$ is fully supported can be lifted if refinement techniques as in [7] are applied. We state it solely to streamline the proof of the main result.

A. Main Result

We are now in a position to state our main result on the tails of the random variable $V^*(\omega)$ for random convex programs with mixed-integer decision variables.

Theorem 3: Consider a MI-RCP $P[\omega]$ as in Definition 5 and let $N \ge h$, where $h = (n+1)2^d$ is the Helly dimension of $\mathbb{R}^n \times \mathbb{Z}^d$ and let Assumption 1 hold. Then for $\epsilon \in (0, 1]$

$$\mathbb{P}^N\{\omega \in \Delta^N: V^*(\omega) > \epsilon\} = \Phi(\epsilon; h - 2, N) , \qquad (16)$$

where $\Phi(\epsilon; h - 2, N)$ denotes the cumulative distribution of the binomial random variable

$$\Phi(\epsilon; q, N) := \sum_{j=0}^{q} \binom{N}{j} \epsilon^j (1 - \epsilon)^{N-j} . \qquad (17)$$

Proof: The proof follows the lines of Theorem 3.3 in [7]. Since $P[\omega]$ is fully supported with probability one, the support constraint set has cardinality exactly $h - 1$. Define $\zeta := h - 1$.

Let $I_\zeta^i(\omega), i = 1, \ldots, C_{N,\zeta}$ with $C_{N,\zeta} = \binom{N}{\zeta}$ denote the subsets of ζ elements of $\omega = (\delta^{(1)}, \ldots, \delta^{(N)})$. Without loss of generality let $I_\zeta^1(\omega)$ be the first ζ elements of ω.

Denote by $x_i^* := x^*(I_\zeta^i(\omega))$ the optimal solution found on the constraint set $I_\zeta^i(\omega)$. Define the events S_i as

$$S_i := \{\omega \in \Delta^N: x_i^* = x^*(\omega)\}$$

for all $i = 1, \ldots, C_{N,\zeta}$, i.e., S_i is the event that constraints $I_\zeta^i(\omega)$ are the support constraints of problem $P[\omega]$. It holds that

$$S_i = \{\omega: f(x_i^*, \delta^{(j)}) \le 0, \; j \notin I_\zeta^i(\omega)\} \qquad (18)$$

where the equality is understood as equal except on a set of zero probability mass. The right hand side of (18) is the set of ω for which the optimal solution found on the subset $I_\zeta^i(\omega)$ does not violate any other of the constraints in ω.

This can be seen with an argumentation similar to [6, Proof of $S_{\tilde{\mathcal{I}}} = \tilde{S}_{\tilde{\mathcal{I}}}$ a.s.] and is omitted here due to space restrictions.

The events S_i are mutually exclusive $S_i \cap S_j = \emptyset$ for $i \ne j$ as there is exactly one support constraint set of cardinality ζ and

$$\Delta^N = \bigcup_{i=1}^{C_{N,\zeta}} S_i . \qquad (19)$$

It follows that

$$1 = \mathbb{P}^N\{\Delta^N\} = \sum_{i=1}^{C_{N,\zeta}} \mathbb{P}^N\{S_i\} = C_{N,\zeta} \mathbb{P}^N\{S_1\} \qquad (20)$$

and, hence, $\mathbb{P}^N\{S_1\} = \frac{1}{C_{N,\zeta}}$. Define now

$$V_i(\omega) := \mathbb{P}\{\delta \in \Delta: f(x_i^*, \delta) > 0\} , \qquad (21)$$

where x_i^* is again the optimal solution found with constraint set $I_\zeta^i(\omega)$. We will now, without loss of generality, consider V_1, as all V_i have the same distribution, because no constraint set $I_\zeta^i(\omega)$ has a higher probability of occuring than the others. Define for $\alpha \in [0, 1]$ the probability distribution of V_1 as

$$F_1(\alpha) := \mathbb{P}^\zeta\{\omega^\zeta: V_1(\omega^\zeta) \le \alpha\} , \qquad (22)$$

where $\omega^\zeta = (\delta^{(1)}, \ldots, \delta^{(\zeta)})$. Assume for now that $V_1 = v$ is given. Because of equality (18), the probability of S_1 equals the probability that none of the constraints from the extractions $\delta^{(\zeta+j)}, \; j = 1, \ldots, N - \zeta$ are violated by x_1^*. Since all extractions are drawn independently the probability of S_1 equals the probability that x_1^* does not violate any of $N - \zeta$ independent realizations drawn from Δ. We obtain that

$$\mathbb{P}^N\{S_1 | V_1 = v\} = (1 - v)^{N-\zeta} . \qquad (23)$$

If we now decondition V_1 and recall (20), we obtain

$$\frac{1}{C_{N,\zeta}} = \mathbb{P}^N\{S_1\} = \int_0^1 (1 - v)^{N-\zeta} dF_1(v) \qquad (24)$$

which is a Hausdorff moment problem and with the same argumentation as in [7], [6] it follows that $F_1(\alpha) = \alpha^\zeta$. It follows for the set $B := \{\omega: V^*(\omega) > \epsilon\}$ that

$$B = \bigcup_{i=1}^{C_{N,\zeta}} B \cap S_i , \qquad (25)$$

where $B \cap S_i = \{\omega: V_i(\omega) > \epsilon\}$. All intersections $B \cap S_i$ are disjoint and have the same probability because all constraint

sets have the same probability of being a support constraint set. From

$$B \cap S_i = \bigcup_{\alpha \in (\epsilon, 1]} \{\omega : V_i(\omega) = \alpha\} \tag{26}$$

we conclude that

$$\mathbb{P}^N \{B \cap S_1\} = \int_\epsilon^1 \mathbb{P}^N \{S_1, V_1 = \alpha\} \, d\alpha \tag{27}$$

$$= \int_\epsilon^1 \mathbb{P}^N \{S_1 | V_1 = \alpha\} \, dF_1(\alpha) \tag{28}$$

$$\stackrel{(a)}{=} \zeta \int_\epsilon^1 (1 - v)^{N - \zeta} \alpha^{\zeta - 1} d\alpha , \tag{29}$$

where equality (a) follows from Eq. (23) and $F_1(\alpha) = \alpha^\zeta$. The integral (29) is the incomplete beta function $\zeta B(1 - \epsilon; N - \zeta + 1, \zeta)$ and it follows that

$$\mathbb{P}^N \{B \cap S_1\} = \zeta B(1 - \epsilon; N - \zeta + 1, \zeta) \tag{30}$$

$$= \binom{N}{\zeta}^{-1} \Phi(\epsilon; \zeta - 1, N) \tag{31}$$

(for more details on this derivation please see [7, Line 3.16]). Finally observe that

$$\mathbb{P}^N \{B\} = C_{N,\zeta} \mathbb{P}^N \{B \cap S_1\} = \Phi(\epsilon; \zeta - 1, N) \tag{32}$$

which concludes the proof. ∎

Corollary 2: Let $P[\omega]$, N, and h be as in Theorem 3 and let Assumptions 1.1 and 1.2 hold (i.e. we remove the assumption that $P[\omega]$ is fully-supported with probability one), then the inequality

$$\mathbb{P}^N \{\omega \in \Delta^N : V^*(\omega) > \epsilon\} \le \Phi(\epsilon; h - 2, N) \tag{33}$$

holds.

Proof: As mentioned above, the use of a regularized refinement analogously as in [7] in combination with Theorem 3 can be used to prove this corollary. ∎

Remark 2: For a true mixed-integer problem the bound is

$$\mathbb{P}^N \{\omega \in \Delta^N : V^*(\omega) > \epsilon\} \le \Phi(\epsilon; (n + 1)2^d - 2, N) .$$

IV. A NUMERICAL EXAMPLE

We apply the presented results to a problem of truss structure design (see, e.g., [11] and the references therein). In particular, we consider the 2-dimensional structure in Fig. 2, composed by at most 7 bars connected in 5 nodes, of which 2 are constrained in all directions, and 3 are free. The free nodes are ordered as shown in Fig. 2. A cartesian coordinate system is considered, with axes (a, b) as in Fig. 2, so that the horizontal and vertical displacements of the free nodes are given by (a_i, b_i), $i \in \{1, 2, 3\}$. The displacements are collected in a vector $\xi \in \mathbb{R}^6$, $\xi \doteq [a_1, b_1, \ldots, a_3, b_3]^\top$. The external loads applied on the free nodes are represented by a vector of random variables $F(\delta) \in \mathbb{R}^6$, $F(\delta) \doteq [F_{a_1}, F_{b_1}, \ldots, F_{a_3}, F_{b_3}]^\top$, containing the vertical and lateral loads applied at each node. Vector $F(\delta)$ is computed as:

$$F(\delta) = \overline{F} + \overline{\Phi}(\delta) \tag{34}$$

with

$$\overline{F}^\top = [0 \quad 0 \quad 0 \quad -1500 \quad 0 \quad 0] \tag{35}$$
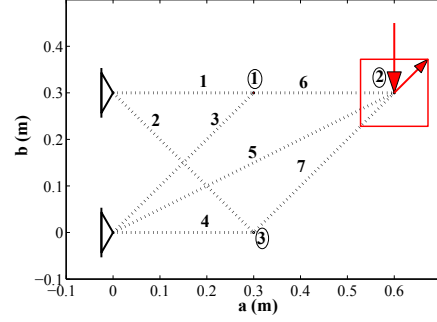

Fig. 2. Numerical example. Topology of the considered 2-dimensional truss structure, with numbered free nodes and bars. Circled numbers indicate the free nodes. The larger arrow indicates the nominal load configuration, while the smaller arrow and rectangle around node 2 illustrate the load uncertainty.

is the vector of nominal loads in Newton, and

$$\overline{\Phi}(\delta)^\top = [0 \quad 0 \quad \delta_{a_2} \quad \delta_{b_2} \quad 0 \quad 0] \tag{36}$$

is a vector of random perturbations on the nominal loads.

The random variables δ_{a_2}, δ_{b_2} are computed as follows:

$$\delta_{a_2} = \max \left(\min \left(720, \eta_{a_2} \right), -720 \right);$$
$$\delta_{b_2} = \max \left(\min \left(720, \eta_{b_2} \right), -720 \right);$$

where η_{a_2}, η_{b_2} are independent random variables distributed according to $\mathcal{N}(0, 180)$ (a Gaussian distribution with zero mean and variance 180). The nominal load and its maximal variation are shown in Fig. 2. The cross-sections S_i and the Young's moduli θ_i, $i \in [1, 7]$, of the bars are also uncertain:

$$\begin{aligned} S_i &= \overline{S}(1 + 0.005 \, \delta_{S,i}), \ i = 1, \ldots, 7 \\ \theta_i &= \overline{\theta}(1 + 0.05 \, \delta_{\theta,i}), \ i = 1, \ldots, 7, \end{aligned} \tag{37}$$

where $\overline{S} = 10^{-4} \, \text{m}^2$ and $\overline{\theta} = 10^{10} \, \text{N/m}^2$ are the nominal cross section and Young's modulus, respectively, and $\delta_S \in \mathbb{R}^7$, $\delta_\theta \in \mathbb{R}^7$ are vectors of independent random variables, each one with uniform distribution in $[-0.5, 0.5]$. The uncertainty in this problem can be collected in a 16-dimensional random vector $\delta = [\delta_{a_2}, \delta_{b_2}, \delta_S^\top, \delta_\theta^\top]^\top$. The lengths of the 7 bars are indicated as l_i, $i = 1, \ldots, 7$; their numerical values in meters can be easily derived from Fig. 2. Under the assumption of small displacements, the following linear relationship holds, between the loads F and the displacements ξ:

$$\xi = K^{-1} F, \tag{38}$$

where the stiffness matrix $K \in \mathbb{R}^{6 \times 6}$ is computed as:

$$K = A \begin{bmatrix} \frac{\theta_1 S_1}{l_1} x_1 & & \\ & \ddots & \\ & & \frac{\theta_7 S_7}{l_7} x_7 \end{bmatrix} A^\top \tag{39}$$

and $A \in \mathbb{R}^{6 \times 7}$ is a matrix describing the topology of the structure. In particular, the i-th column of a_i of A is a vector of direction cosines, such that $a_i^\top \xi$ measures the elongation of the i-th bar. In (39), the decision variables x_i are binary, i.e. $x_i \in \{0, 1\} \forall i \in [1, 7]$, so that $x_i = 0$ means that the i-th bar is not present, while $x_i = 1$ means that the i-th bar is included in the structure. Finally, the compliance of the structure, i.e. the total stored elastic energy, is given by:

$$g(x, \delta) = \frac{1}{2} F^\top \xi = \frac{1}{2} F^\top K^{-1} F. \tag{40}$$
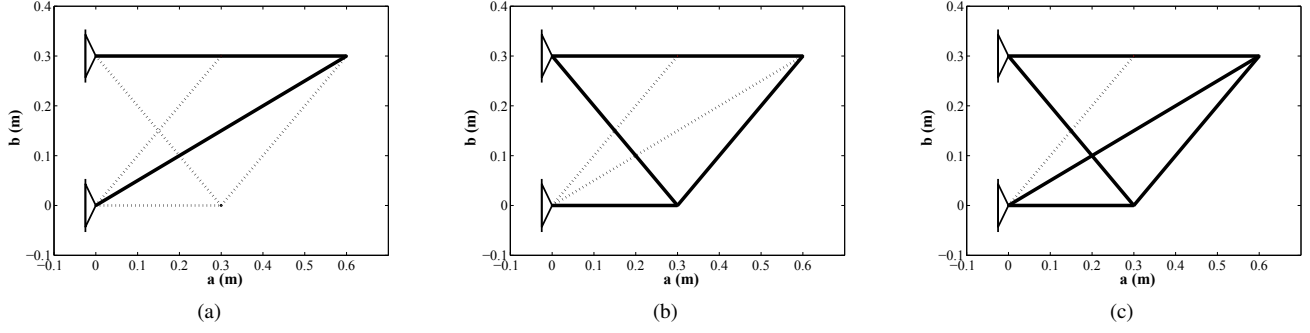
Fig. 3. Two-dimensional truss structure example. Solutions obtained with (a) the nominal load configuration, (b) $\epsilon = 0.9$ and (c) $\epsilon = 0.05$. Employed value of β: $1\,10^{-9}$.

The aim is to minimize the number of bars $n_b(x) = c^\top x$, $c = [1, \ldots, 1]^\top$, to be used in the structure, subject to the constraint that the compliance is lower than a prescribed maximal allowed value $\overline{f} = 8\,\text{J}$. By using the Schur complement rule [12], the nonlinear (and non-convex) constraint $g \leq \overline{g}$ can be converted into the LMI:

$$G(x, \delta) \succeq 0, \qquad (41)$$

where

$$G(x, \delta) \doteq$$

$$\begin{bmatrix} A \begin{bmatrix} \frac{\theta_1(\delta)\,S_1(\delta)}{l_1}x_1 & & \\ & \ddots & \\ & & \frac{\theta_7(\delta)S_7(\delta)}{l_7}x_7 \end{bmatrix} A^\top & F(\delta) \\ F(\delta)^\top & 2\overline{g} \end{bmatrix}. \qquad (42)$$

Therefore, by considering a multi-sample $\omega = (\delta^{(1)}, \ldots, \delta^{(N)}) \in \Delta^N$, the problem can be formalized as the following integer optimization program with affine cost and random convex constraints:

$$P[\omega] : \min_{x \in \Omega} . \qquad c^\top x \qquad (43)$$

$$\text{subject to:} \quad G(x, \delta_j) \succeq 0, \quad j = 1, \ldots, N, \quad (44)$$

with $\Omega = \{x \in \mathbb{R}^7 : x_i \in \{0, 1\}, i = 1, \ldots, 7\}$. We first solve a nominal problem, i.e with $\delta = 0$. The obtained optimal solution, $x^{*,\text{nom}}$, is shown in Fig. 3(a), and the related optimal cost is $n_b(x^{*,\text{nom}}) = 3$ bars. The related (nominal) compliance is $g(x^{*,\text{nom}}, 0) = 6.47\,\text{J}$.
We now choose $\beta = 10^{-9}$ and $\epsilon = 0.9$ and, according to Theorem 3, we extract $N = 171$ samples of δ. The obtained scenario solution, $x^{*,0.9}$, is shown in Fig. 3(b), the optimal number of bars is $n_b(x^{*,0.9}) = 5$ and the related (nominal) compliance is $g(x^{*,0.9}, 0) = 3.93\,\text{J}$. The worst-case compliance, among all of the extracted samples, is $\max_{i \in [1,N]} g(x^{*,0.9}, \delta_i) = 7.53\,\text{J}$.
Finally, we set again $\beta = 10^{-9}$ and decrease ϵ to $5\,10^{-2}$. Using Theorem 3 we have $N = 4100$ samples of δ. The obtained scenario solution, $x^{*,0.05}$, is shown in Fig. 3(c): the optimal number of bars is $n_b(x^{*,0.05}) = 6$, and the related (nominal) compliance is $g(x^{*,0.05}, 0) = 3.06\,\text{J}$. The worst-case compliance, among all of the extracted samples, is $\max_{i \in [1,N]} g(x^{*,0.05}, \delta_i) = 6.44\,\text{J}$.

V. CONCLUSIONS

We extended existing work on continuous RCPs to the case of random programs with mixed-integer decision variables, and we provided bounds on the tails of the violation probability. We showed that the important generalization property of continuous RCPs carries over to mixed-integer problems. However, different from the continuous case, where the upper bound on the number of support constraints increases linearly with the number of variables, in the mixed-integer setting we obtained a geometric dependence of this upper bound with respect to the number of discrete variables (also, we showed examples in dimension two and three where the upper bound is attained). This suggests that mixed-integer convex problems, besides being harder computationally with respect to their continuous counterparts, may also be more difficult to immunize against uncertainty via the RCP approach. It is not excluded, however, that a different approach than the one presented in this work may lead to improved bounds on the violation probability tail, and this is indeed the subject of ongoing study.

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