Monomials as sums of powers: the Real binary case

MONOMIALS AS SUMS OF POWERS: THE REAL BINARY CASE

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ABSTRACT. We generalize an example, due to Sylvester, and prove that any monomial of degree d in $\mathbb{R}[x_0, x_1]$, which is not a power of a variable, cannot be written as a linear combination of fewer than d powers of linear forms.

1. INTRODUCTION

It is well-known, and easy to prove, that if k is a field of characteristic zero and $R = k[x_0, \ldots, x_n] = \bigoplus_{i=0}^{\infty} R_i$ is the standard graded polynomial algebra, then the k-vector space R_d (for any d) has a basis consisting of polynomials $\{L_1^d, \ldots, L_s^d\}$ where $s = \binom{d+n}{n} = \dim_k R_d$ and the L_i are pairwise linearly independent forms in R_1. It follows that every form in R_d is a k-linear combination of at most s d^{th} powers of linear forms and, if k is algebraically closed, simply a sum of at most s d^{th} powers of linear forms. We will call such a way of writing $F \in R_d$ a *Waring expansion of* F because of the echo of Waring's problem from number theory. We will further refer to such an expression as a *minimal Waring expansion for* F if the number of summands in such an expression for F is minimal among all such representations.

If $n > 0$ and $d = 2$ it is a classical fact that although $s = \binom{n+2}{2}$ every quadratic form has a Waring expansion involving $\leq n + 1 < s$ summands and that, in general, i.e. for $[F]$ belonging to a non-empty Zariski open subset of $\mathbb{P}(R_2)$ a minimal Waring expansion for F has exactly $n + 1$ summands.

These observations have led to a series of problems, usually called **Waring Problems**, which ask for information on minimal Waring expansions for forms of degree d in R.

The long outstanding problem of finding the number of summands in a minimal Waring expansion of the generic form of degree d was solved, after being open for almost 100 years, by J. Alexander and A. Hirschowitz (see [AH95]), when k is an algebraically closed field.

Of course, solving this problem for the generic form of degree d does not always give information about any specific form of degree d and the problem of finding the length of the minimal Waring expansion for specific forms has also been a continuing source of interesting speculations and lovely results. E.g. it was Sylvester ([Har92]) who first observed that although for $R = \mathbb{C}[x_0, x_1]$, the generic form of degree d has a Waring Expansion with $s = \lceil \frac{d+1}{2} \rceil$ summands, the monomial $x_0 x_1^{d-1}$ has d summands in its minimal Waring expansion (the maximum possible).

The Waring problem for specific forms has been considered in depth by B. Reznick in his monograph (see [Rez92]) and by Comas and Seiguer who, to our knowledge, were the first to resolve the problem completely and algorithmically in $\mathbb{C}[x_0, x_1]$ in their unpublished work ([CS01]).

It is interesting to note that although the Waring problem is a very interesting and stimulating problem in purely algebraic terms, it has a surprising number of intimate connections with problems in areas as seemingly disparate as algebraic geometry and communication theory (see for example [RS00],[CC03] and [CM96])

Indeed, if $k = \mathbb{R}$, the field of real numbers, the connection with real world problems is very direct. This has prompted a re-examination of the Waring problem for $R = \mathbb{R}[x_0, x_1]$, and a recent very suggestive paper of Comon and Ottaviani (see [CO09]) considered this very problem for degrees $d \leq 5$.

Our main result in this paper follows the line of Sylvester's examples and concerns the minimal Waring expansion for monomials in $\mathbb{R}[x_0, x_1]$. We first give a new proof of the fact that the minimal Waring expansion of the monomial $x_0^a x_1^b$ in $\mathbb{C}[x_0, x_1]$ with $0 < a \leq b$ has $b + 1$ summands. In sharp contrast to this we show that in $\mathbb{R}[x_0, x_1]$ every monomial of degree d (except x_0^d and x_1^d) has d summands in its minimal Waring expansion.

2. Basic results

Let $S = k[x_0, x_1]$ and $T = k[y_0, y_1]$. We make S into a T-module using differentiation, i.e. we think of $y_0 = \partial/\partial x_0$ and $y_1 = \partial/\partial x_1$. We refer to a polynomial in T as ∂ instead of using capital letters. In particular, for any form F in S_d we define the ideal $F^\perp \subseteq T$ as follows:

$$F^\perp = \{\partial \in T : \partial F = 0\}.$$

The following *Apolarity Lemma* is due to Iliev and Ranestad [IR01].

Lemma 2.1. *A homogeneous form $F \in S$ can be written as*

$$F(x_0, x_1) = \sum_{i=1}^{r} \alpha_i (L_i)^d, \ L_i \textit{ pairwise linearly independent, } \alpha_i \in k$$

i.e. has a Waring expansion with r summands, if and only if the ideal $F^\perp$ contains the product of r distinct linear forms.

3. Binary monomials: the complex case

The complex case is straightforward for monomials.

Proposition 3.1. *Let $M = x_0^a x_1^b$ be a monomial in $\mathbb{C}[x_0, x_1]$. If $0 < a \leq b$, then M has a minimal Waring expansion with $b+1$ summands, i.e. is a sum of $b+1$ powers of linear forms and no fewer.*

Proof. Let $I = M^\perp = (y_0^{a+1}, y_1^{b+1})$ and notice that the linear system defined by I_{b+1} is base point free on $\mathbb{P}^1 = \mathbb{P}S_1$. Applying Bertini's Theorem, we get that the generic element of I_{b+1} defines a set of $b+1$ distinct points and hence it is the product of $b+1$ distinct linear forms. Thus the apolarity lemma yields that M is the sum of $b+1$ powers of linear forms. If $r < b+1$, then r powers do not suffice as no element in $I_r = (y_0^{a+1})_r$ is a product of r distinct linear forms. □

4. Binary monomials: the real case

We can also ask for a real Waring expansion of a monomial M. More precisely, we want to write

$$M(x_0, x_1) = \sum_{i=1}^{r} \alpha_i (L_i)^d, \quad \alpha_i \in \{1, -1\}$$

where the linear forms L_i are in $\mathbb{R}[x_0, x_1]$. In order to do this, we have to increase the number of summands in Proposition 3.1.

The following elementary facts will be extremely useful.

Lemma 4.1. *Consider the degree d polynomial*

$$F(x) = c_d x^d + \ldots c_1 x + c_0 \in \mathbb{R}[x].$$

If $c_i = c_{i-1} = 0$ for some $1 \leq i \leq d$, then $F(x)$ does not have d real roots.

Proof. The proof is obvious if $i = 1$ or $i = d$, so we may as well assume that $1 < i < d$.

Consider all the pairs (c_r, c_s) of non-zero coefficients such that $r > s$ and $c_j = 0$ if $r > j > s$. Let α be number of pairs such that $r - s$ is odd and β the number of pairs such that $r - s$ is even. Notice that, by hypothesis, $\alpha + 2\beta < d - 1$

Now we apply Descartes' rule of signs. For a pair (c_r, c_s) such that $r - s$ is odd we get a real root of $F(x)$. For a pair (c_r, c_s) such that $r - s$ is even we get either two real roots of $F(x)$ or none.

In conclusion, the number of real roots of $F(x)$ is at most $\alpha + 2\beta$ and we are done. □

Lemma 4.2. *For each $i < d$ there exists a degree d polynomial $F(x) = c_d x^d + \dots c_1 x + c_0 \in \mathbb{R}[x]$ having d real roots and such that $c_i = 0$.*

Proof. Choose $a_1, \dots, a_d \in \mathbb{R}$ and consider the polynomial $F(x) = (x - a_1) \cdot \dots \cdot (x - a_d)$. This polynomial can also be written as

$$F(x) = \sum_{i=0}^{d} E_i(a_1, \dots, a_d) x^i,$$

where E_i is the degree i elementary symmetric function in its arguments. The vanishing of the i-th coefficient of $F(x)$ can be written as

$$E_i(a_1, \dots, a_{d-1}) + a_d E_{i-1}(a_1, \dots, a_{d-1}) = 0.$$

Hence, if we choose the $a_1, \dots, a_{d-1} > 0$ and distinct there exists a unique, negative value of a_d such that the coefficient of x^i in $F(x)$ is zero. As the roots of $F(x)$ are $a_1, \dots, a_d$ the polynomial has d real, distinct roots.

□

Using the previous results we immediately get a lower bound on the number of summands in the minimal Waring expansion of a monomial in $\mathbb{R}[x_0, x_1]$.

Lemma 4.3. *Let $M = x_0^a x_1^b$ be a monomial in $\mathbb{R}[x_0, x_1]$. If $0 < a \leq b$, then M does not have a Waring expansion with $r \leq a + b - 1$ real summands.*

Proof. Let $I = M^\perp = (y_0^{a+1}, y_1^{b+1})$. The general degree r element in I has the form $F(y_0, y_1) =$

$$c_r y_0^r + c_{r-1} y_0^{r-1} y_1 + \dots + c_{a+1} y_0^{a+1} y_1^{r-a-1} + c_{r-b-1} y_0^{r-b-1} y_1^{b+1} + \dots + c_0 y_1^r.$$

If $a+1 \geq r-b+2$, then by Lemma 4.1 $F(y_0, y_1)$ is not the product of r real linear forms. The conclusion follows by the apolarity lemma. □

Proposition 4.4. *Let $M = x_0^a x_1^b$ be a monomial in $\mathbb{R}[x_0, x_1]$. If $0 < a \leq b$, then M has a minimal Waring expansion with $a + b$ summands which are powers of real linear forms.*

Proof. We have that $M^\perp = I = (y_0^{a+1}, y_1^{b+1})$. Notice that I_{a+b} is the subspace of T_{a+b} of polynomials which are missing all the monomials having factor y_0^a or y_1^b. Thus, Lemma 4.2 and the apolarity lemma yield the result. □

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