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# ASYMPTOTIC CONTROLLABILITY BY MEANS OF EVENTUALLY PERIODIC SWITCHING RULES*

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**Abstract.** In this paper we introduce the notion of eventually periodic switching signal. We prove that if a family of linear vector fields satisfies a mild finite time controllability condition, and for each initial state there exists a time-dependent switching signal which asymptotically drives the system to the origin (possibly allowing different signals for different initial states), then the same goal can be achieved by means of an eventually periodic switching signal. This enables us to considerably reduce the dependence of the control law on the initial state. In this sense, the problem addressed in this paper can be reviewed as a switched system theory version of the classical problem of investigating whether, or to what extent, a nonlinear asymptotically controllable system admits stabilizing feedback laws.

**Key words.** switched systems, asymptotic controllability, time-dependent switching rules, state-dependent switching rules, stabilization

**AMS subject classifications.** 93C10, 93C65, 93B27, 93D20, 34D05

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## 1. Introduction.

This paper deals with the so-called stabilization problem for switched systems. In engineering literature, the term "switched system" denotes a special type of hybrid system whose time evolution is described by a continuous but not necessarily differentiable curve in the state space. Examples of switched systems can be frequently found in applications and technology. Switched systems are often defined by assigning a family of vector fields $\{f_n(x)\}_{n\in\mathcal{N}}$ on $\mathbb{R}^d$ (where $\mathcal{N}$ is some set of indices, and $d$ is a positive integer) and a piecewise constant function $\sigma$ with values in the index set $\mathcal{N}$. The function $\sigma$ is called the *switching rule*: It specifies the vector field which determines the system evolution at each instant. The switching rule may be either time-dependent or state-dependent; although a unified treatment has been attempted by some authors, we prefer to keep the two options distinct, at least at the beginning. Indeed, the mathematical treatment of systems with a state-dependent switching rule encounters difficulties which do not arise when the switching rule is time-dependent.

A *time-dependent switching rule* consists of a piecewise constant map $\sigma(t) : [0, +\infty) \to \mathcal{N}$. It may, or may not, be dependent on the initial state. Provided that all the vector fields $f_n(x)$ are forward complete, the trajectories of a switched system with a time-dependent switching rule exist for each initial state and are continuable on the whole interval $[0, +\infty)$. Moreover, they cannot exhibit chattering or Fuller's phenomenon (sometimes called Zeno phenomenon; see [13]). In particular, if the switching rule is the same for each initial state, the trajectories can be determined by solving a time-varying differential system

$$\dot{x} = f(t, x),$$

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where $f(t,x) = f_n(x)$ on each interval on which $\sigma(t) = n$. However, this simplification is not possible if the switching rule is different for each initial state.

A *state-dependent switching rule*, in its simpler version, corresponds to a discontinuous feedback $k(x) : \mathbb{R}^d \to \mathcal{N}$. To compute the trajectories, one needs to solve the differential system

$$\dot{x} = f_{k(x)}(x),$$

whose right-hand side is, in general, discontinuous. Now, it is well known that in this case solutions (in classical or Carathéodory sense) do not necessarily exist or are not continuable. Moreover, chattering and Fuller's phenomenon may arise. It turns out that, in general, state-dependent switching rules and time-dependent switching rules cannot be immediately converted into each other (see [7]).

In this paper we address the open-loop version of the stabilization problem, better known in the classical control theory literature as the *asymptotic controllability* problem. We focus on a special class of time-dependent switching rules, called *eventually periodic switching rules*, already introduced in [6]. We prove that if the family of vector fields is linear and satisfies a mild finite time controllability condition, then asymptotic controllability by means of generic time-dependent switching rules implies asymptotic controllability by means of eventually periodic switching rules. The construction exploits the existence of a stable manifold of a discrete-time dynamical system associated to the given family of vector fields.

The main feature of eventually periodic switching rules is that they exhibit a strongly reduced dependence on the initial state. Moreover, they can be reinterpreted in terms of state dependence. Thus, our results contribute to partially bridging the gap between the notions of time-dependent and state-dependent switching rule. In fact, we can recognize an analogy between the problem addressed in this paper and the problem of investigating whether, for a nonlinear system, asymptotic controllability implies feedback stabilization. For sake of conciseness, we do not report here the details of this classical problem. The reader can find the precise statement in [18] and some remarkable contributions in [8, 1, 17, 9]. We limit ourselves to point out that the notion of sampled feedback law exploited in [8] combines both time-dependence and state-dependence.

This paper is organized as follows. In section 2 we present the basic definitions and some preliminary facts to be used later. In particular, we introduce the discrete-time dynamical system associated to a periodic switching law. In section 3 we expose the main results. The notion of finite time controllability we need in this paper is introduced and discussed in section 4. The proof of the main results is given in section 5. In section 6 we suggest the state-dependent interpretation of eventually periodic switching signals. In section 7 we propose some simple results (independent of the controllability assumption) ensuring the existence of a nontrivial stable manifold for the associated discrete-time system. Finally, section 8 contains some illustrative examples and section 9 the final comments. The appendix is devoted to families of nonlinear vector fields on compact manifolds; in the spirit of classical geometric control theory, we prove some facts which are essential for the proof of the main results of this paper.

**2. Preliminary definitions.** In this paper we will be mainly concerned with families of linear vector fields of $\mathbb{R}^d$. Let $\mathcal{N} = \{1, \dots, N\}$, where $N \geq 2$ is a fixed integer, and let us denote by $\mathcal{F} = \{f_n(x)\}_{n \in \mathcal{N}}$ a family of vector fields of $\mathbb{R}^d$. The vector field $f_n(x)$ is called the $n$th component of $\mathcal{F}$.

The family $\mathcal{F}$ is said to be *linear* if for each $n \in \mathcal{N}$, $f_n(x) = A_n x$, where $A_n$ is a $d \times d$ real matrix. For each linear family $\mathcal{F}$ and each $n \in \mathcal{N}$, the curve $\varphi_n(t, x_0) : \mathbb{R} \to \mathbb{R}^d$ uniquely defined by the conditions $\dot{x} = A_n x$ and $\varphi_n(0, x_0) = x_0$ is called the *trajectory* of the $n$th component, issued from the initial state $x_0 \in \mathbb{R}^d$. It is represented, as usual, by $\varphi_n(t, x_0) = e^{tA_n} x_0$.

**2.1. Switching signals and switched trajectories.** Let $\mathcal{N}$ be equipped with the discrete topology. By *switching signal* we mean any right continuous, piecewise constant[1] function $\sigma : [0, +\infty) \to \mathcal{N}$. The discontinuity points of a switching signal $\sigma$ form a finite or infinite (possibly empty) subset of the open half line $(0, +\infty)$. They are called the *switching times* of $\sigma$. We denote by $I_\sigma$ the set whose elements are $t_0 = 0$ and all the switching times of $\sigma$, indexed in such a way that $0 = t_0 < t_1 < t_2 < \dots$. If the set $I_\sigma$ is infinite, then clearly $\lim_{i \to +\infty} t_i = +\infty$. The positive numbers $\theta_i = t_{i+1} - t_i$ are called *durations*. The number of switching times of $\sigma$ in the interval $(0, T)$ $(T > 0)$ is denoted $\mathbf{s}_\sigma(T)$.

The set of all the switching signals is denoted by $\mathcal{U}_\mathcal{N}$. It possesses the so-called *concatenation property*: If $\sigma_1, \sigma_2 \in \mathcal{U}_\mathcal{N}$ and $T > 0$, then $\sigma \in \mathcal{U}_\mathcal{N}$, where

$$\sigma(t) = \begin{cases} \sigma_1(t) & \text{for } t \in [0, T), \\ \sigma_2(t - T) & \text{for } t \geq T. \end{cases}$$

Let $\sigma \in \mathcal{U}_\mathcal{N}$ and let a linear family $\mathcal{F}$ of $\mathbb{R}^d$ be given. For each $x_0 \in \mathbb{R}^d$, there is a unique continuous curve $t \mapsto \varphi_\mathcal{F}(t, x_0, \sigma) : [0, +\infty) \to \mathbb{R}^d$ satisfying the conditions $\varphi_\mathcal{F}(0, x_0, \sigma) = x_0$ and

$$\varphi_\mathcal{F}(t, x_0, \sigma) = \varphi_{\sigma(t_i)}(t - t_i, \varphi_\mathcal{F}(t_i, x_0, \sigma)) \qquad \forall t \in [t_i, t_{i+1}), \quad \forall t_i \in I_\sigma.$$

We say that $\varphi_\mathcal{F}(t, x_0, \sigma)$ is the *switched trajectory* of $\mathcal{F}$, issued from the initial state $x_0$, and corresponding to the switching signal $\sigma$. It can be represented[2] as

$$\begin{aligned} \varphi_\mathcal{F}(t, x_0, \sigma) &= e^{(t - t_i) A_{\sigma(t_i)}} \varphi_\mathcal{F}(t_i, x_0, \sigma) \\ &= e^{(t - t_i) A_{\sigma(t_i)}} e^{(t_i - t_{i-1}) A_{\sigma(t_{i-1})}} \dots e^{t_1 A_{\sigma(0)}} x_0 \end{aligned}$$

for each $t \in [t_i, t_{i+1})$, and $t_i \in I_\sigma$. We emphasize that the operator

$$x \mapsto \Phi(t, \sigma) x \underset{def}{=} \varphi_\mathcal{F}(t, x, \sigma)$$

is linear and nonsingular for each $t \geq 0$ and each $\sigma$.

*Remark* 1. A switched trajectory of a linear family $\mathcal{F}$ can be reviewed as a trajectory of a bilinear control system of the form $\dot{x} = \sum_{n=1}^{N} u_n A_n x$, where the input $u$ is piecewise constant and takes value on the set $\{(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\} \subset \mathbb{R}^d$.

**2.2. Linear switched systems.** A *linear switched system* is defined by a linear family $\mathcal{F}$ of $\mathbb{R}^d$, together with a map $\Sigma : \mathbb{R}^d \to \mathcal{U}_\mathcal{N}$ which assigns a switching signal $\sigma(t) = \Sigma_{x_0}(t)$ to each point $x_0 \in \mathbb{R}^d$, regarded as the initial state. A linear switched

[1] This means that there are at most finitely many jumps in each compact interval.

[2] Of course, a switched trajectory of a linear family can be represented in a much simpler form if the matrices $A_n$ commute; however, in this paper we do not make this assumption.

system is denoted by $(\mathcal{F}, \Sigma)$. The map $\Sigma$ is referred to as a (time-dependent) *switching map*.

The switched trajectory of $\mathcal{F}$ issued from any point $x_0 \in \mathbb{R}^d$ and corresponding to the switching signal $\Sigma_{x_0}$ will also be called a trajectory of $(\mathcal{F}, \Sigma)$.

A switched system for which $\Sigma$ is constant, i.e., the same switching signal $\sigma(t)$ is applied for each initial state $x_0$, will be simply written as $(\mathcal{F}, \sigma)$.

In what follows, we denote by $||\cdot||$ the Euclidean norm of a vector or the Frobenius norm of a matrix. Moreover, we write $\mathcal{S}_r = \{x \in \mathbb{R}^d : ||x|| = r\}$ $(r > 0)$. The following lemma will be used later.

LEMMA 1. *Let $\mathcal{F}$ be a linear family of $\mathbb{R}^d$. Then there exist $\alpha > 0$ and $\gamma > 1$ with the following property. For each switching signal $\sigma$, each $T > 0$, each $t \in [0, T]$, and each $x_0 \in \mathbb{R}^d$*

$$||\varphi_{\mathcal{F}}(t, x_0, \sigma)|| \leq \gamma^{k+1} e^{\alpha T} ||x_0||, \tag{1}$$

*where $k = \mathbf{s}_\sigma(T)$.*

*Proof.* As is well known, for each $n \in \mathcal{N}$ there exist $\alpha_n \in \mathbb{R}$ and $\gamma_n > 0$ such that

$$||e^{tA_n}\xi|| \leq \gamma_n e^{\alpha_n t} ||\xi||$$

for each $t \geq 0$ and $\xi \in \mathbb{R}^d$.

Let $\gamma > \max\{1, \gamma_1, \ldots, \gamma_n\}$ and $\alpha > \max\{0, \alpha_1, \ldots, \alpha_n\}$. Let $t_0 < t_1 < t_2 < \ldots$ be the sequence of switching times of $\sigma$. If $T \in (t_0, t_1]$, then (1) is obvious. Assuming that (1) is true for $T \in (t_k, t_{k+1}]$. Then, it is not difficult to prove that it is true also for $T \in (t_{k+1}, t_{k+2}]$. The result follows by induction. □

**2.3. Asymptotic controllability.** Given a linear family of vector fields $\mathcal{F}$, it is interesting to characterize those switching maps $\Sigma$, if any, such that for each initial state $x_0$

$$\lim_{t \to +\infty} \varphi_{\mathcal{F}}(t, x_0, \Sigma_{x_0}) = 0. \tag{P1}$$

As is well known, this problem is not trivial, since even if all the matrices $A_n$ are Hurwitz (i.e., all their eigenvalues lie in the open left complex plane), it may happen that the trajectory corresponding to some switching signals and some initial states diverges [13, 20].

DEFINITION 1. *The linear family of vector fields $\mathcal{F}$ is said to be* asymptotically controllable *if there exists a switching map $\Sigma$ such that property* (P1) *holds for each $x_0 \in \mathbb{R}^d$. In this case, we also say that $\Sigma$ is an AC-switching map for $\mathcal{F}$.*

DEFINITION 2. *The linear family of vector fields $\mathcal{F}$ is said to be* uniformly asymptotically controllable *if there exists a switched signal $\sigma(t)$ such that property* (P1) *holds for each $x_0 \in \mathbb{R}^d$, with $\Sigma_{x_0} \equiv \sigma$. In this case, we also say that $\sigma$ is a UAC-switching signal for $\mathcal{F}$.*

The notion of asymptotic controllability (sometimes shortened to asycontrollability; see [18]) is classical: It means that all the initial states can be eventually driven toward the origin, but different switching signals might be required for different initial states. On the contrary, uniform asymptotic controllability means that the same switching signal works for all the initial states.[3]

[3] In [20, 6] asymptotic controllability and uniform asymptotic controllability are, respectively, termed *pointwise stabilizability* and *consistent stabilizability*.

Clearly, if $\mathcal{F}$ is uniformly asymptotically controllable, then it is asymptotically controllable, but the converse is false in general: This is shown by an example in [20, p. 58], and also by examples presented later in this paper.

*Remark* 2. Asymptotic controllability has been merely defined here in terms of the attraction property (P1). However, the vector fields of $\mathcal{F}$ being linear, (P1) automatically implies stability, as is specified in the following proposition.

PROPOSITION 1. *If a linear family of vector fields $\mathcal{F}$ is asymptotically controllable, then it is possible to find an AC-switching map $\Sigma$ such that besides* (P1)*, the following additional property holds:*

(P2) $\forall \varepsilon > 0\ \exists \delta > 0$ *such that* $||x_0|| < \delta$ *implies* $||\varphi_{\mathcal{F}}(t, x_0, \Sigma_{x_0})|| < \varepsilon\ \forall t \geq 0$.

We report the proof of Proposition 1 for the reader's convenience, although similar arguments can be found in [19] and in [21].

*Proof.* For each $p \in \mathcal{S}_1$, the asymptotic controllability assumption yields a switching signal $\sigma_p$ and a time $T_p$ such that

$$||\varphi_{\mathcal{F}}(T_p, p, \sigma_p)|| < \frac{1}{3}.$$

Since $x \mapsto \varphi_{\mathcal{F}}(T_p, x, \sigma_p)$ is a nonsingular linear map, there exists a neighborhood $U_p$ of $p$ such that

$$||\varphi_{\mathcal{F}}(T_p, x, \sigma_p)|| < \frac{1}{2}$$

for each $x \in U_p$. The sets $V_p = U_p \cap \mathcal{S}_1$ form a (relatively) open covering of $\mathcal{S}_1$. By the compactness argument, we can extract a finite covering $V_{p_1}, \ldots, V_{p_L}$. For each $\ell = 1, \ldots, L$, rename $\sigma_\ell = \sigma_{p_\ell}$, $T_\ell = T_{p_\ell}$ and let $\mathcal{O}_\ell = \{x \neq 0 : x/||x|| \in V_{p_\ell}\}$. It is clear that $\cup_{1 \leq \ell \leq L} \mathcal{O}_\ell = \mathbb{R}^d \setminus \{0\}$. If $x \in \mathcal{O}_\ell$, then

$$||\varphi_{\mathcal{F}}(T_\ell, x, \sigma_\ell)|| = ||x|| \cdot ||\varphi_{\mathcal{F}}(T_\ell, x/||x||, \sigma_\ell)|| < \frac{||x||}{2}. \tag{2}$$

Moreover, as long as $t \in [0, T_\ell]$, by Lemma 1 we have

$$||\varphi_{\mathcal{F}}(t, x, \sigma_\ell)|| \leq \gamma^{k+1} e^{\alpha T} ||x||, \tag{3}$$

where $T = \max\{T_1, \ldots, T_L\}$ and $k = \max\{\mathbf{s}_{\sigma_1}(T_1), \ldots, \mathbf{s}_{\sigma_L}(T_L)\}$.

A new switching signal $\bar{\sigma}_{x_0}$ can now be defined for each initial state $x_0 \neq 0$ by concatenation, according to the following procedure:

1. if $x_0 \in \mathcal{O}_{\ell_0}$ then apply the input $\sigma_{\ell_0}$ restricted on the interval $[0, T_{\ell_0}]$;
2. set $x_1 = \varphi_{\mathcal{F}}(T_{\ell_0}, x_0, \sigma_{\ell_0})$;
3. replace $x_0$ by $x_1$ and repeat the previous steps.

Taking into account both (2) and (3), and using induction, it is not difficult to see that

$$||\varphi_{\mathcal{F}}(t, x_0, \bar{\sigma}_{x_0})|| \leq \gamma^{k+1} e^{\alpha T} ||x_0||$$

for each $t \geq 0$. The conclusion easily follows. ∎

**2.4. Periodic switching signals.** In this paper, we are interested in switching rules whose structure is required to satisfy particular restrictions.

A *periodic switched system* is a pair $(\mathcal{F}, \sigma)$ such that $\sigma$ is periodic for $\mathcal{F}$. More precisely, a switching signal $\sigma(t)$ is said to be *periodic* (of period $T$) for $\mathcal{F}$ if there exist a string of real numbers $\tau_0, \ldots, \tau_H$ (where $H$ is an integer, $H \geq 1$) and a string of indices $n_1, \ldots, n_H \in \mathcal{N}$ such that

- $0 = \tau_0 < \tau_1 < \ldots < \tau_H = T$;
- $\sigma(t) = n_h$ for $t \in [\tau_{h-1}, \tau_h)$ for each $h = 1, \ldots, H$;
- $\sigma(t) = \sigma(t - T)$ for $t \geq T$.

The points $\tau_h + mT$, with $h = 1, \ldots, H$ and $m = 0, 1, \ldots$, coincide with the switching times, provided that $n_1 \neq n_2, n_2 \neq n_3, \ldots, n_H \neq n_0$. Note that $\sigma$ is constant when $H = 1$.

DEFINITION 3. *The linear family $\mathcal{F}$ is said to be* periodically asymptotically controllable *if it is uniformly asymptotically controllable by means of a periodic switching signal.*

*Remark* 3. In [20] it is proven that periodic asymptotic controllability and uniform asymptotically controllability are actually equivalent.

Asymptotic controllability can be achieved by means of high frequency periodic switching signals (see [22, 20]) under the assumption that for some integer $H > 1$ and some indices $n_1, \ldots, n_H \in \mathcal{N}$ there exists a Hurwitz convex combination of the matrices $A_{n_1}, \ldots, A_{n_H}$. This assumption was originally introduced in [16] to prove the existence of state dependent stabilizing switching rules (see [5] for an extension to nonlinear systems).

Asymptotic controllability can also be achieved by means of switching signals with large dwell time (see Lemma 2 of [15]) provided that all the matrices $A_n$ are Hurwitz. The case of a pair of planar oscillators has been studied in [3].

**2.5. Eventually periodic switching signals.** As already mentioned, there exist families of linear vector fields not uniformly asymptotically controllable (and hence, not periodically asymptotically controllable) but which can be asymptotically driven toward zero by applying a different switching signal for each initial state. We are especially interested in switching rules of the following type.

DEFINITION 4. *A switching map $\Sigma : \mathbb{R}^d \to \mathcal{U}_{\mathcal{N}}$ is said to be* eventually periodic *(of period $T$) if there exist a periodic switching signal $\sigma(t)$ (of period $T$) and a map $T_0 : \mathbb{R}^d \to [0, +\infty)$ such that for each $x_0 \in \mathbb{R}^d$,*

$$\Sigma_{x_0}(t) = \sigma(t - T_0(x_0)) \qquad \forall t \geq T_0(x_0) \ .$$

*The switched system $(\mathcal{F}, \Sigma)$ is said to be* eventually periodic *if the map $\Sigma$ is eventually periodic.*[4]

In other words, a switched system is eventually periodic if it behaves as a periodic one after an initial transient interval (depending on the initial state).

DEFINITION 5. *We say that a linear family $\mathcal{F}$ is* eventually periodically asymptotically controllable *if there exists an eventually periodic map $\Sigma$ such that property* (P1) *holds for each $x_0 \in \mathbb{R}^d$.*

In the remaining part of this section we illustrate a systematic way to construct eventually periodic AC-switching maps (compare with the first example of section 8).

[4]Eventually periodic switching maps have been introduced in [6], with the name of near-periodic switching rules.

Let $\mathcal{F}$ be a linear family of $\mathbb{R}^d$. Let $\sigma(t)$ be any periodic switching signal of period $T$. With the notation introduced above, we associate to $(\mathcal{F}, \sigma)$ the linear discrete dynamical system

$$x_{k+1} = \Phi(T, \sigma)x_k = \varphi_{\mathcal{F}}(T, x_k, \sigma) \qquad k \in \mathbb{N} \,. \tag{4}$$

Of course, if the origin is an asymptotically stable fixed point for the associated discrete dynamical system (4), then $\sigma$ is a UAC-switching signal for $\mathcal{F}$ [20, 22]. As is well known, the origin is asymptotically stable for (4) if and only if all the eigenvalues of the linear operator $\Phi(T, \sigma)$ lie in the unit disc of the complex plane. If $\nu < d$ (counting multiplicity) among the eigenvalues of $\Phi(T, \sigma)$ meet this condition, then the initial states which are asymptotically driven to the origin along the solutions of (4) form a $\nu$-dimensional invariant subspace of $\mathbb{R}^d$, called the *maximal stable subspace* of (4) and denoted $W_s$. A subspace $W \subset \mathbb{R}^d$ will be called a *stable subspace* of (4) if it is invariant and contained in $W_s$.

Now, the construction of eventually periodic AC-switching maps for $\mathcal{F}$ can be easily achieved if the following conditions are met:

(C1) there exists a periodic switching signal whose associated discrete dynamical system (4) has a nontrivial stable subspace $W$;

(C2) for each $x_0 \in \mathbb{R}^d$ $(x_0 \neq 0)$, there exists a switching signal (depending on $x_0$) such that the corresponding switched trajectory of $\mathcal{F}$ issued from $x_0$ intersects $W$ in finite time.

**3. The main results.** In order to state the main results of this paper, we need to formulate a special controllability assumption about the family $\mathcal{F}$.

Let $x, y \in \mathbb{R}^d$. We say that $y$ is *reachable* from $x$ in time $t > 0$ if there exists a switching signal $\sigma$ such that $\varphi_{\mathcal{F}}(t, x, \sigma) = y$. We denote by $\mathcal{R}_{\mathcal{F}}(T, x)$ the set of all points $y$ of $\mathbb{R}^d$ which are reachable from $x$ in some time $t \leq T$.

If $y \in \mathcal{R}_{\mathcal{F}}(T_1, x)$ and $z \in \mathcal{R}_{\mathcal{F}}(T_2, y)$, then $z \in \mathcal{R}_{\mathcal{F}}(T_1 + T_2, x)$ (this follows immediately from the concatenation properties).

DEFINITION 6. *The family $\mathcal{F}$ of linear vector fields is said to be* radially controllable *if for each pair of points $x, y \in \mathbb{R}^d$ $(x \neq 0, y \neq 0)$ there exist $\lambda > 0$ and $T > 0$ such that $\lambda y \in \mathcal{R}_{\mathcal{F}}(T, x)$.*

The notion of radial controllability will be discussed in section 4, where we also explain how it can be checked in practice.

THEOREM 1. *Let $\mathcal{F}$ be a linear family of $\mathbb{R}^d$. Assume that it is radially controllable and asymptotically controllable. Then, for each $x_0 \in \mathbb{R}^d$ $(x_0 \neq 0)$ there exists a periodic switching signal $\sigma$ such that the discrete dynamical system* (4) *associated to $(\mathcal{F}, \sigma)$ has a nontrivial stable subspace containing the point $x_0$.*

THEOREM 2. *Let $\mathcal{F}$ be a linear family of $\mathbb{R}^d$. Assume that it is radially controllable and asymptotically controllable. Then, $\mathcal{F}$ is eventually periodically asymptotically controllable. Moreover, an eventually periodic AC-switching map $\Sigma$ can be found in such a way that property* (P2) *is fulfilled, as well.*

The proof of Theorems 1 and 2 will be given in section 5; it relies crucially on the results of nonlinear nature presented in the appendix.

*Remark* 4. Theorems 1 and 2 can be easily extended to families of linear systems $\mathcal{F} = \{A_n x\}_{n \in \mathcal{N}}$ for which the index set $\mathcal{N}$ is not necessarily finite, provided that for some $\gamma > 1$ and some $\alpha > 0$, the inequality $||e^{tA_n}|| \leq \gamma e^{\alpha t}$ holds for each $t \in \mathbb{R}$ and each $n \in \mathcal{N}$. In particular Theorem 2 remains true when the index set $\mathcal{N}$ is a compact topological space.

*Remark* 5. Let us remark that, with respect to asymptotic controllability, eventually periodic asymptotic controllability considerably reduces the dependence of the control policy on the initial state. Actually, after the initial transient we need to drive the system to reach the stable subspace of the associated discrete dynamical system, the control becomes periodic and can be implemented in an automatic way.

**4. Radial controllability.** In this section we discuss the notion of radial controllability. In particular, we show that the radial controllability of a family of linear vector fields on $\mathbb{R}^d$ can be checked by looking at the global controllability of a family of (in general, nonlinear) vector fields on a $(d-1)$-dimensional compact manifold.

To each family $\mathcal{F} = \{f_n(x) = A_n x\}_{n\in\mathcal{N}}$ of linear vector fields of $\mathbb{R}^d$, we associate a family $\mathcal{F}^\sharp$ formed by the (in general, nonlinear) vector fields

$$f_n^\sharp(x) = A_n x - \left(\frac{x^{\mathbf{t}} A_n x}{x^{\mathbf{t}} x}\right) x \ , \qquad n = 1, \dots, N, \tag{5}$$

where $x \in \mathbb{R}^d$ $(x \neq 0)$, and $^{\mathbf{t}}$ denotes transposition. For each $n \in \mathcal{N}$, $f_n^\sharp(x)$ is homogeneous of degree one[5] and analytic on $\mathbb{R}^d \setminus \{0\}$. It is immediate to verify that $f_n^\sharp(x)$ is tangent to any sphere $\mathcal{S}_r$ $(r > 0)$. In fact, $f_n^\sharp(\bar{x})$ is the projection of $A_n\bar{x}$ on the tangent space of the sphere $\mathcal{S}_{\bar{r}}$ (with $\bar{r} = ||\bar{x}||$) at the point $\bar{x}$.

By virtue of homogeneity, we can limit ourselves to $r = 1$. Let the vector fields $f_n^\circ(p)$ be the restriction of the vector fields $f_n^\sharp(x)$ to the $(d-1)$-dimensional sphere $\mathcal{S}_1$ and let $\mathcal{F}^\circ = \{f_n^\circ(p)\}_{n\in\mathcal{N}}$.

PROPOSITION 2. *$\mathcal{F}$ is radially controllable if and only if $\mathcal{F}^\circ$ is globally controllable on $\mathcal{S}_1$.*

*Proof.* For any $n \in \mathcal{N}$, let $\varphi(t)$ be a solution of $\dot{x} = A_n x$ and let $\varphi^o(t)$ be a solution of $\dot{p} = f_n^o(p)$. By direct computation, we see that if $\varphi^o(0) = \varphi(0)/||\varphi(0)||$, then $\varphi^o(t) = \varphi(t)/||\varphi(t)||$ for each $t \in \mathbb{R}$. Now, assume that $\mathcal{F}$ is radially controllable, and let $p, q \in \mathcal{S}_1$. There exists a switching signal $\sigma$ which steers $p$ to $\lambda q$ for some $\lambda > 0$. Clearly, the same switching law applied to $\mathcal{F}^o$ steers $p$ to $q$. Vice versa, if $\mathcal{F}^o$ is globally controllable, for each pair $z, y \in \mathbb{R}^d$ we can find a switched signal steering $p = z/||z||$ to $q = y/||y||$. Of course, the same switching law, applied to $\mathcal{F}$, steers $z$ to $\lambda y$ for some $\lambda > 0$. □

Thus, the problem is reduced to test controllability of a family of vector fields on a $(d-1)$-dimensional manifold. Combining Proposition 2 with Propositions 6 and 7 (see the appendix), we get the following corollary.

COROLLARY 1. *Let $\mathcal{F}$ be a linear family of $\mathbb{R}^d$. If $\mathcal{F}$ is radially controllable, then there exist $\tilde{T} > 0$ and $\tilde{S}$ such that for each pair of points $z, y \in \mathbb{R}^d$ $(z \neq 0, y \neq 0)$ there exist $\lambda > 0$ such that $\lambda y \in \mathcal{R}_{\mathcal{F}}(T, z)$ for some $T \leq \tilde{T}$. In addition, a switching signal $\sigma$ steering $z$ to $\lambda y$ in time $T$ can be found with $\mathbf{s}_\sigma(T) \leq \tilde{S}$.*

In fact, the following proposition provides an important, additional information.

PROPOSITION 3. *Let $\mathcal{F}$ be a radially controllable, linear family of $\mathbb{R}^d$. Let $\tilde{T} > 0$ and $\tilde{S}$ be as in the statement of Corollary* 1. *Then, there exists a real number $\tilde{\lambda} > 0$ enjoying the following property: for each pair of points $z, y \in \mathbb{R}^d$ $(z \neq 0, y \neq 0)$ there exist a positive number $\lambda$ and a switching signal $\sigma$ such that*

$$||\lambda y|| \leq \tilde{\lambda}||z|| \ , \tag{6}$$

[5]This means that for each $a \in \mathbb{R} \setminus \{0\}$ one has $f_n^\sharp(ax) = a f_n^\sharp(x)$.

*and* $\lambda y = \varphi_{\mathcal{F}}(T, z, \sigma)$ *with* $T \leq \tilde{T}$, $\mathbf{s}_\sigma(T) \leq \tilde{S}$.

*Proof.* By Corollary 1, there exist a positive number $\lambda$ and a switched signal $\sigma$ which connects $z$ to $\lambda y$ in time $T \leq \tilde{T}$, such that $\mathbf{s}_\sigma(T) \leq \tilde{S}$. In other words, we can write

$$\lambda y = e^{\theta_K A_{n_K}} \dots e^{\theta_1 A_{n_1}} z, \tag{7}$$

for some integer $K \leq \tilde{S}$, some indices $n_1, \dots, n_K \in \mathcal{N}$ and some positive durations $\theta_1, \dots, \theta_K$ with $\theta_1 + \dots + \theta_K = T$. Let us define $\tilde{\lambda} = \gamma^{\tilde{S}+1} e^{\alpha \tilde{T}}$. $\tilde{\lambda}$ is independent of $z, y$, and it remains only to prove that (6) holds. With the same notation as in Proposition 1, from (7) we have

$$\begin{aligned} ||\lambda y|| &\leq \gamma_{n_K} \cdot \dots \cdot \gamma_{n_1} e^{\alpha_{n_K} \theta_K} \dots e^{\alpha_{n_1} \theta_1} ||z|| \\ &\leq \gamma^K e^{\alpha(\theta_1 + \dots + \theta_K)} ||z|| \leq \gamma^{\tilde{S}+1} e^{\alpha \tilde{T}} ||z|| \end{aligned}$$

as required. □

## 5. Proof of the main results.

**5.1. Proof of Theorem 1.** We prove that for each $w \neq 0$ there exists a periodic switching signal $\sigma$ of some period $T > 0$ such that for the associated discrete dynamical system one has

$$\Phi(T, \sigma) w = \lambda w \tag{8}$$

with $\lambda \in (0, 1/2]$. In other words, we prove that $\sigma$ can be found in such a way that $w$ is a real eigenvector of the linear operator $\Phi(T, \sigma)$, corresponding to a real eigenvalue $\lambda$ lying inside the open unit disc. It follows immediately that the one-dimensional subspace generated by $w$ is a nontrivial stable subspace of the linear map $\Phi(T, \sigma)$.

Let $w \in \mathbb{R}^d$, $w \neq 0$. Since $\mathcal{F}$ is asymptotically controllable, there exists a switching signal $\sigma_1$ (depending on $w$) such that $\lim_{t \to +\infty} \varphi_{\mathcal{F}}(t, w, \sigma_1) = 0$. In particular, there exists a time $T_1 > 0$ such that

$$||z|| \leq \frac{1}{2\tilde{\lambda}} ||w||, \tag{9}$$

where $z = \varphi_{\mathcal{F}}(T_1, w, \sigma_1)$ and $\tilde{\lambda} > 0$ is the number in the statement of Proposition 3. Recall that $\tilde{\lambda}$ depends only on $\mathcal{F}$.

Since the system is radially controllable, we can use again Propositions 1 and 3 to find a switched signal $\sigma_2$, a time $T_2 > 0$, and a positive number $\lambda$ such that $\lambda w = \varphi_{\mathcal{F}}(T_2, z, \sigma_2)$, with $T_2 \leq \tilde{T}$ and

$$||\lambda w|| \leq \tilde{\lambda} ||z||. \tag{10}$$

Let us now define a periodic switching law $\sigma$, whose period is $T = T_1 + T_2$, in such a way that

$$\sigma(t) = \begin{cases} \sigma_1(t) & \text{if } t \in [0, T_1), \\ \sigma_2(t - T_1) & \text{if } t \in [T_1, T). \end{cases}$$

The switching law $\sigma$ is well defined because of the concatenation property. We clearly have

$$\lambda w = \varphi_{\mathcal{F}}(T_2, z, \sigma_2) = \varphi_{\mathcal{F}}(T_2, \varphi_{\mathcal{F}}(T_1, w, \sigma_1), \sigma_2) = \varphi_{\mathcal{F}}(T, w, \sigma) = \Phi(T, \sigma) w.$$

Moreover, by virtue of (9) and (10),

$$\lambda||w|| = ||\lambda w|| \leq \tilde{\lambda}||z|| \leq \frac{\tilde{\lambda}}{2\tilde{\lambda}}||w|| = \frac{||w||}{2}$$

from which we easily get $\lambda \leq 1/2$ as desired.

**5.2. Proof of Theorem 2.** According to Theorem 1, condition (C1) is fulfilled by $\mathcal{F}$. Let $\sigma$ be a switched signal, periodic for $\mathcal{F}$, and let $w$ be such that (8) holds with some $\lambda \leq 1/2$. By virtue of the radial controllability assumption, for each $x_0 \neq 0$ there exist a switching signal $\sigma_{x_0}$ and a positive time $T_0(x_0)$ such that

$$\varphi_{\mathcal{F}}(T_0(x_0), x_0, \sigma_{x_0}) = w_0,$$

where $w_0$ is parallel to $w$. We get an eventually periodic AC-switching map by setting

$$\Sigma_{x_0}(t) = \begin{cases} \sigma_{x_0}(t), & t \in [0, T_0(x_0)], \\ \sigma(t - T_0(x_0)), & t > T_0(x_0). \end{cases} \tag{11}$$

All the switched trajectories of $(\mathcal{F}, \Sigma)$ satisfy property (P1).

Finally, we prove that the eventually periodic switching signal (11) satisfies also property (P2). Recall that by Proposition 6 we can choose $\sigma_{x_0}$ in such a way that $\mathbf{s}_{\sigma_{x_0}}(T(x_0)) \leq \tilde{S}$ and $T_0(x_0) \leq \tilde{T}$. By repeating similar arguments as in the proof of Proposition 1, we have

$$||\varphi_{\mathcal{F}}(t, x_0, \Sigma_{x_0})|| = ||\varphi_{\mathcal{F}}(t, x_0, \sigma_{x_0})|| \leq \gamma^{\tilde{S}+1} e^{\alpha \tilde{T}} ||x_0|| \tag{12}$$

for $t \in [0, T_0(x_0)]$. Hence, in particular, $||w_0|| \leq \gamma^{\tilde{S}+1} e^{\alpha \tilde{T}} ||x_0||$. Analogously, for $t \in [T_0(x_0), T_0(x_0) + T]$,

$$\begin{aligned} ||\varphi_{\mathcal{F}}(t, x_0, \Sigma_{x_0})|| &= ||\varphi_{\mathcal{F}}(t - T_0(x_0), \varphi_{\mathcal{F}}(T_0(x_0), x_0, \sigma_{x_0}), \sigma)|| \\ &= ||\varphi_{\mathcal{F}}(t - T_0(x_0), w_0, \sigma)|| \\ &\leq \gamma^{H+1} e^{\alpha T} ||w_0||, \end{aligned}$$

where $T$ is the period of $\sigma$ and $H = \mathbf{s}_{\sigma}(T)$.

By construction, if we set $w_1 = \varphi_{\mathcal{F}}(T, w_0, \sigma)$, then $w_1$ is parallel to $w_0$ and $||w_1|| \leq \frac{1}{2}||w_0||$.

Now let $t \in [T_0(x_0) + T, T_0(x_0) + 2T]$. As before, we have

$$||\varphi_{\mathcal{F}}(t, x_0, \Sigma_{x_0})|| = ||\varphi_{\mathcal{F}}(t - T_0(x_0) - T, w_1, \sigma)|| \leq \gamma^{H+1} e^{\alpha T} ||w_1|| \leq \gamma^{H+1} e^{\alpha T} ||w_0|| \ .$$

The reasoning can actually be iterated on each interval of the form $[T_0(x_0) + mT, T_0(x_0) + (m+1)T]$. Taking into account (12), we finally get

$$||\varphi_{\mathcal{F}}(t, x_0, \Sigma_{x_0})|| \leq \gamma^{\tilde{S}+H+2} e^{\alpha(\tilde{T}+T)} ||x_0|| \ .$$

Property (P2) is easily achieved with $\delta = \varepsilon / (\gamma^{\tilde{S}+H+2} e^{\alpha(\tilde{T}+T)})$.

**6. State-dependent switching rules.** Let the linear family $\mathcal{F}$ of $\mathbb{R}^d$ be given. The construction of state-dependent switching rules is frequently achieved in the literature on the base of the following procedure (see, for instance, [4]).

1. Find a finite family of open, pairwise disjoint subsets of $\mathbb{R}^d$, $\Omega_1, \ldots, \Omega_L$, such that

$$\cup_{1\leq \ell \leq L} \overline{\Omega}_\ell = \mathbb{R}^d \setminus \{0\}.$$

2. Associate an index $n_\ell \in \mathcal{N}$ to each region $\Omega_\ell$.
3. Define $\sigma(x) = n_\ell$ whenever $x \in \Omega_\ell$.

Typically, this can been accomplished by the aid of a Lyapunov-like function if $\mathcal{F}$ is quadratically stabilizable (see [13]), and hence, in particular, if there exists a Hurwitz convex combination of the matrices $A_1, \ldots, A_N$. In the latter case the regions $\Omega_1, \ldots, \Omega_L$ are conic.

As already mentioned in the introduction, this procedure leads to a system of equations with discontinuous right-hand side, for which the existence of Carathéodory solutions (and a fortiori, switched solutions) is not sure. To overcome the difficulty, one can try to introduce hysteresis in the systems, as in [16] and [13], or to resort to generalized (Filippov or Krasowski) solutions as in [2]. However, sometimes an approach based on a eventually periodic switching law might be preferable.

Recall that, although not reproducible in general by means of a purely static memoryless feedback law, an eventually periodic switching map has a reduced dependence on the initial state (Remark 5). Under certain circumstances, this allows us to identify appropriate "switching loci" with associated appropriate indices.

Let $\sigma$ be a periodic switching signal. With the notation of section 2, let

$$\Phi_1 = \Phi(T, \sigma) = e^{\theta_H A_{n_H}} \ldots e^{\theta_1 A_{n_1}},$$

where $\theta_h = \tau_h - \tau_{h-1}$ $(h = 1, \ldots, H)$ and $\theta_1 + \ldots + \theta_H = T$. Assume that the discrete dynamical system associated to $\Phi_1$ has a nontrivial stable subspace $W_1$.

Then it is possible to prove that for each $h = 2, \ldots, H$,

$$W_h = e^{\theta_{h-1} A_{n_{h-1}}} \ldots e^{\theta_1 A_{n_1}} W_1$$

is a stable subspace for the discrete dynamical system associated to the operator

$$\Phi_h = e^{\theta_{h-1} A_{n_{h-1}}} \ldots e^{\theta_1 A_{n_1}} e^{\theta_H A_{n_H}} \ldots e^{\theta_h A_{n_h}}.$$

Note that $\Phi_h$ corresponds to the same periodic signal $\sigma$ as $\Phi_1$, translated of a quantity $\theta_1 + \ldots + \theta_h$. Assume further that the subspaces $W_h$ are pairwise transversal, and associate the index $n_h$ with the subspace $W_h$. An eventually periodic AC-switching map whose periodic part coincides with $\sigma$ can be redescribed according to the following steps:

1. (transient initial interval) starting from any initial state $x_0 \neq 0$, drive the system to hit one of the subspaces $W_h$;
2. (steady state behavior) when the system trajectory hits $W_h$, switch on the $n_h$-component.

Note that during the transient interval, the system is operated in open-loop. However, according to the results of section 4 the length of the transient interval can be predicted. During the steady state, the control procedure can be implemented automatically. To compare our approach with the more traditional one sketched above, the reader may find useful to look at the examples of section 8.

**7. Existence of stable subspaces.** Note that in the statement of Theorems 1 and 2 we do not impose any assumption about the asymptotic behavior of the single components of $\mathcal{F}$. Constructing UAC-switching signals is trivial if there exists at least one $n \in \mathcal{N}$ such that $A_n$ is Hurwitz. In this case indeed, one can take $\sigma(t) \equiv n$. Hence, the natural motivation of Theorems 1 and 2 apparently relies on the case where none of the components of $\mathcal{F}$ is asymptotically stable. However, other reasons of interest might come from certain practical applications. Indeed, it may happen that switching among two or more components is compulsory. The admissible switching rules might have a partially fixed structure, in the sense that the activation of the various components must obey to a preassigned sequence, while other details, such as durations (i.e., the times elapsed between two consecutive switches) are available for design. In such a situation, we know that a bad choice of the durations can lead to instability for certain initial states, even if some or all the matrices $A_n$ are Hurwitz. To avoid similar drawbacks, the ideas developed in the proof of Theorems 1 and 2 can be fruitfully applied.

In this section we limit ourselves, for simplicity, to consider the case where all the components of $\mathcal{F}$ must be cyclically activated following a prescribed order, each one for a nonvanishing interval of time. More precisely, we present some simple conditions which allows us to predict the existence of a nontrivial stable subspace for the discrete dynamical system defined by an operator of the form

$$\Phi_{\theta_1,\dots,\theta_N} = e^{\theta_N A_N} \dots e^{\theta_1 A_1} \tag{13}$$

(we simply write $\Phi$ instead of $\Phi_{\theta_1,\dots,\theta_N}$ when the string $\theta_1, \dots, \theta_N$ is clear from the context).

We emphasize that the criteria of this section are independent of the radial controllability assumption.

PROPOSITION 4. *Let $\mathcal{F}$ be a linear family of $\mathbb{R}^d$, with index set $\mathcal{N} = \{1, \dots, N\}$. Assume that for some $\alpha_1, \dots, \alpha_N$ ($\alpha_n > 0$, $\sum_{n=1}^N \alpha_n = 1$) the matrix $\bar{A} = \sum_n \alpha_n A_n$ has at least one eigenvalue with negative real part. Then, there exists a sequence of positive durations $\theta_1, \dots, \theta_N$ such that the discrete dynamical system defined by the operator* (13) *has a nontrivial stable subspace.*

*Proof.* Let $\theta_n = \alpha_n T$ for each $n \in \mathcal{N}$ and some $T > 0$. For sufficiently small $T$, there exists a matrix $C(T)$ such that

$$\Phi_{\theta_1,\dots,\theta_N} = e^{C(T)} .$$

Such a matrix $C(T)$ can be represented by the Baker–Campbell–Hausdorff expansion [23]

$$C(T) = \left( \sum_n \alpha_n A_n \right) T + G(T) T^2,$$

where $G(T)$ is bounded. Recall that the eigenvalues depend continuously on the elements of a matrix. By taking a small enough $T$ and using the assumption that $\sum_n \alpha_n A_n$ has at least one eigenvalue with negative real part, we arrive at the conclusion that $C(T)$ has at least one eigenvalue with negative real part as well. The statement easily follows. □

*Remark* 6. As already recalled, the much stronger assumption that for some choice of $\alpha_1, \dots, \alpha_N$ the matrix $\bar{A}$ is Hurwitz, has been used in [16] in order to construct

state-dependent switching rules and in [22] (see also [20]) in order to construct high frequency periodic UAC-switching signals.

The assumption of Proposition 4 is fulfilled, in particular, if for some index $\bar{n}$ the matrix $A_{\bar{n}}$ has at least one eigenvalue with negative real part. Indeed, we can take a convex combination with $\alpha_n \ll 1$ for each $n \neq \bar{n}$, so that $\sum_n \alpha_n A_n$ can be viewed as a small perturbation of $A_{\bar{n}}$. On the other hand, the discrete dynamical system associated to (13) may have a nontrivial stable subspace even if all the matrices $A_n$ and their convex combinations have all their eigenvalues in the open right half-plane: this happens, for instance, in the reversed time version of Example 1.

It can be proven that if the matrices $A_1, \ldots, A_N$ are symmetric, then the existence of an index $\bar{n}$ such that the matrix $A_{\bar{n}}$ has at least one eigenvalue with negative real part is a necessary condition for asymptotic controllability [20]. Combining these observations, we obtain the following result.

COROLLARY 2. *Let $\mathcal{F}$ be a linear family of $\mathbb{R}^d$, such that each matrix $A_n$ is symmetric. If $\mathcal{F}$ is asymptotically controllable, then there exists a sequence of positive durations $\theta_1, \ldots, \theta_N$ such that the discrete dynamical system defined by the operator* (13) *has a nontrivial stable subspace.*

The assumption of Corollary 2 can be slightly relaxed by asking that for each $A \in \mathcal{F}$, $A^{\mathbf{t}} \in \mathcal{F}$.

PROPOSITION 5. *Assume that all the matrices $A_1, \ldots, A_N$ are Hurwitz. Then, for each sequence of positive durations $\theta_1, \ldots, \theta_N$, the discrete dynamical system defined by the operator* (13) *has a nontrivial stable subspace.*

In fact, the conclusion of Proposition 5 can be proved to be valid under the following slightly weaker assumption: For each $n = 1, \ldots, N$, $\operatorname{tr} A_n < 0$ (where tr denotes the trace of the matrix $A_n$).

*Proof.* Recall that for each $t \in \mathbb{R}$ and each square matrix $A$, $\det e^{tA} = e^{\operatorname{tr}(tA)} = e^{t(\operatorname{tr} A)}$. If $\operatorname{tr} A_n < 0$, then

$$0 < \det e^{tA_n} < 1$$

for each $t$ and each $n \in \mathcal{N}$. Hence,

$$\det \Phi = \det e^{\theta_N A_N} \ldots e^{\theta_1 A_1} = \det e^{\theta_N A_N} \cdot \ldots \cdot \det e^{\theta_1 A_1} < 1.$$

On the other hand, $\det \Phi = \lambda_1 \cdot \ldots \cdot \lambda_d$, where $\lambda_1, \ldots, \lambda_d$ are the (nonnecessarily distinct) eigenvalues of $\Phi$. It follows that at least one eigenvalue of $\Phi$ lies in the interior of the unit disc of the complex plane. □

Proposition 5 implies that a family $\mathcal{F}$ whose components are all asymptotically stable cannot be completely destabilized by applying a periodic switching signal.

**8. Examples.** To illustrate the problems investigated in this paper, the related notions and the approach based on the associated discrete dynamical system (4), we revisit below the example of a pair of linear systems, both with spiral configuration. Although the same example can be found elsewhere, we point out many new aspects and details.

*Example* 1. Consider in $\mathbb{R}^2$ the family $\mathcal{F}$ formed by the pair of vector fields $f_1(x) = A_1 x$, $f_2(x) = A_2 x$, where[6]

$$A_1 = \begin{pmatrix} \frac{1}{4} & \alpha \\ -\frac{1}{\alpha} & \frac{1}{4} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \frac{1}{4} & \frac{1}{\alpha} \\ -\alpha & \frac{1}{4} \end{pmatrix}, \quad \alpha > 1.$$

[6]The diagonal entries of $A_1$ and $A_2$ have been taken equal to 1/4 since this value is convenient for numerical simulations, but in fact any $\beta > 0$ works.

$A_1$ and $A_2$ are both completely unstable. It is easy to check that there exist no Hurwitz convex combinations of $A_1$ and $A_2$. In fact, the system is not quadratically stabilizable. Nevertheless, the switching rule

$$\sigma(x) = \begin{cases} 1 & \text{if } xy < 0, \\ 2 & \text{if } xy > 0 \end{cases}$$

is of the type considered at the beginning of section 6. It gives rise to an asymptotically stable system of ordinary differential equations with discontinuous right-hand side. Note that in spite of the discontinuities, in this case switched trajectories exist for each initial state and for every $t \in \mathbb{R}$, and are unique.

It is also possible to verify that $\mathcal{F}$ is not uniformly asymptotically controllable (the necessary condition in [20, p. 59] is not met). Now, we show how an eventually periodic AC-switching map can be constructed for $\mathcal{F}$.

Let $\tau > 0$ be fixed. We first consider the periodic switching law of period $2\tau$ such that $\sigma(t) = 1$ for $t \in [0, \tau)$, and $\sigma(t) = 2$ for $t \in [\tau, 2\tau)$. The fundamental matrices for $A_1$ and $A_2$ are, respectively,

$$e^{\tau A_1} = \begin{pmatrix} \cos\tau & \alpha \sin\tau \\ -\frac{1}{\alpha}\sin\tau & \cos\tau \end{pmatrix} e^{\frac{1}{4}\tau}, \qquad e^{\tau A_2} = \begin{pmatrix} \cos\tau & \frac{1}{\alpha}\sin\tau \\ -\alpha\sin\tau & \cos\tau \end{pmatrix} e^{\frac{1}{4}\tau},$$

and their product is

$$e^{\tau A_2} e^{\tau A_1} = \begin{pmatrix} 1 - \frac{\alpha^2+1}{\alpha^2}\sin^2\tau & \frac{\alpha^2+1}{\alpha}\sin\tau\cos\tau \\ -\frac{\alpha^2+1}{\alpha}\sin\tau\cos\tau & 1 - (\alpha^2+1)\sin^2\tau \end{pmatrix} e^{\frac{1}{2}\tau}. \tag{14}$$

In order to analyze the stability of the periodic switched system of period $2\tau$ defined by $\mathcal{F}$ and $\sigma$, we study the stability of the associated discrete dynamical system defined by (14). Therefore, let us consider the characteristic equation of (14):

$$\lambda^2 - e^{\frac{1}{2}\tau}\left(2 - \frac{(\alpha^2+1)^2}{\alpha^2}\sin^2\tau\right)\lambda + e^{\tau} = 0. \tag{15}$$

Since $e^{\tau} > 1$ for any $\tau > 0$, at least one eigenvalue $\lambda_1$ lies out of the unit circle. To check whether the eigenvalue $\lambda_2$ lies out of the unit circle too, we may check the behavior of the inverse matrix $(e^{\tau A_2} e^{\tau A_1})^{-1}$, whose eigenvalues are $\lambda_1^{-1}$ and $\lambda_2^{-1}$. The characteristic equation of the inverse matrix is

$$\lambda^2 - e^{-\frac{1}{2}\tau}\left(2 - \frac{(\alpha^2+1)^2}{\alpha^2}\sin^2\tau\right)\lambda + e^{-\tau} = 0. \tag{16}$$

Since $e^{-\tau} < 1$, by the Schur–Cohn lemma [12], both of the eigenvalues of the inverse matrix lie in the unit circle if and only if

$$\left| e^{-\frac{1}{2}\tau}\left(2 - \frac{(\alpha^2+1)^2}{\alpha^2}\sin^2\tau\right)\right| < 1 + e^{-\tau}. \tag{17}$$

Therefore, $\lambda_2$ lies out of the unit circle if and only if (17) is verified, while it belongs to the unit circle if the inequality is reversed.

Inequality (17) is verified if either

$$\text{(a)} \begin{cases} 2 - \frac{(\alpha^2+1)^2}{\alpha^2}\sin^2\tau > 0 \\ 2 - \frac{(\alpha^2+1)^2}{\alpha^2}\sin^2\tau < e^{\frac{1}{2}\tau} + e^{-\frac{1}{2}\tau} \end{cases} \text{ or } \text{(b)} \begin{cases} 2 - \frac{(\alpha^2+1)^2}{\alpha^2}\sin^2\tau < 0 \\ \frac{(\alpha^2+1)^2}{\alpha^2}\sin^2\tau - 2 < e^{\frac{1}{2}\tau} + e^{-\frac{1}{2}\tau}. \end{cases}$$

The second inequality of system (a) is verified for all $\alpha > 0$ and for all $\tau > 0$, since $e^{\frac{1}{2}\tau} + e^{-\frac{1}{2}\tau} - 2 > 0$ for all $\tau > 0$.

In system (b), the second inequality is equivalent to

$$g_\alpha(\tau) = \frac{(\alpha^2+1)^2}{\alpha^2}\sin^2\tau < 2 + e^{\frac{1}{2}\tau} + e^{-\frac{1}{2}\tau} = f(\tau).$$

We remark that $\min g_\alpha(\tau) = 0 < f(\tau)$ for all $\tau > 0, \alpha > 0$. Since $f(\tau)$ is strictly increasing for $\tau > 0$ if, for some $k \in \mathbb{N}$,

$$\max g_\alpha(\tau) = g_\alpha\left(\frac{2k+1}{2}\pi\right) = \frac{(\alpha^2+1)^2}{\alpha^2} > f\left(\frac{2k+1}{2}\pi\right), \tag{18}$$

there exists a neighborhood of $\frac{2k+1}{2}\pi$ where $g_\alpha(\tau) > f(\tau)$.

Since $\frac{(\alpha^2+1)^2}{\alpha^2} \to +\infty$ as $\alpha \to +\infty$, it is to be expected that there exists $\bar{\alpha}$ such that, for all $\alpha > \bar{\alpha}$, inequality (18) is verified for at least $k = 0$. The simulations illustrated in Figures 1, 2, and 3 show that this actually happens, with $1.3 < \bar{\alpha} < 1.5$. For $\alpha = 2$, $g_2\left(\frac{\pi}{2}\right) > f\left(\frac{\pi}{2}\right)$ but $g_2\left(\frac{3\pi}{2}\right) < f\left(\frac{3\pi}{2}\right)$, while $g_{10}(\tau) > f(\tau)$ in three intervals. It is clear that, as $\alpha$ grows, the number of intervals grows as well.

For those values of $\tau$ for which (17) holds with reversed sign, the discrete dynamical system associated to (14) has a nontrivial stable subspace. On the other hand, it is immediately seen that the system is radially controllable. Hence, conditions (C1) and (C2) are fulfilled, and the existence of eventually periodic AC-switching maps is therefore guaranteed. Moreover, it can be checked that the stable subspace of (14) coincides with the $x$-axis, while the stable subspace of the dynamical system defined by $e^{\tau A_1}e^{\tau A_2}$ coincides with the $y$-axis. This illustrates the state-space interpretation of an eventually periodic AC-switching signal suggested in section 6.

*Remark* 7. The reversed time version of Example 1 is interesting as well. It is formed by a pair of asymptotically stable linear vector fields. By applying a periodic switching law, we see that the stability properties may actually depend on the value of the period $T$. The fact that stability is preserved for periodic switching signals of both small and large values of the period agrees with the results of [20, 22, 5, 15].

Note that as predicted at the end of section 7, it is not possible to completely destabilize this pair of vector fields by means of a periodic switching signal.

*Example* 2. Consider now the family $\mathcal{F}$ defined by the matrices

$$A_1 = \begin{pmatrix} \frac{1}{4} & \alpha \\ -\frac{1}{\alpha} & \frac{1}{4} \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \frac{1}{4} & -\frac{1}{\alpha} \\ \alpha & \frac{1}{4} \end{pmatrix}, \quad \alpha > 1.$$

We still have two unstable sink configurations, but rotating in opposite directions. The system is not quadratically stabilizable. A state-dependent switching rule can be defined by assigning

$$\sigma(x,y) = \begin{cases} 1 & \text{if } x + y > 0, \\ 2 & \text{if } x + y < 0. \end{cases}$$

In this case, there are initial states (all the points of the line $y + x = 0$ with $x > 0$) for which Carathéodory solutions of the resulting system of ordinary differential equations do not exist. However, the closed-loop system is asymptotically stable with respect to generalized Krasowski solutions (but Krasowski solutions are not switched solutions, in general).

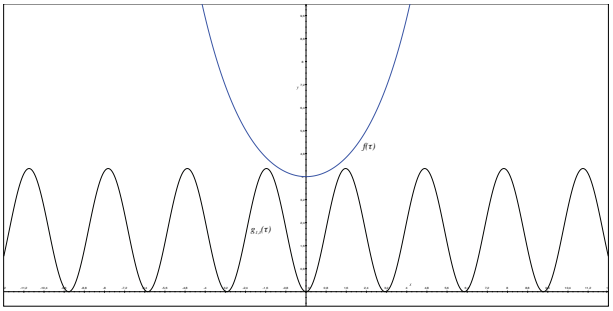


FIG. 1. *For* $\alpha = 1.3$, $g_\alpha(\tau) < f(\tau)\ \forall \tau > 0$.

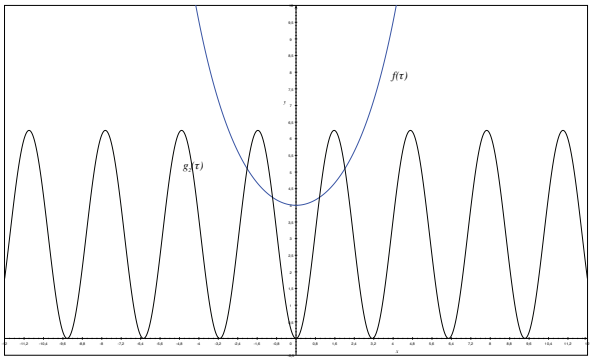


FIG. 2. *For* $\alpha = 2$ *and for* $\tau > 0$, $g_\alpha(\tau) < f(\tau)$ *out of one interval.*

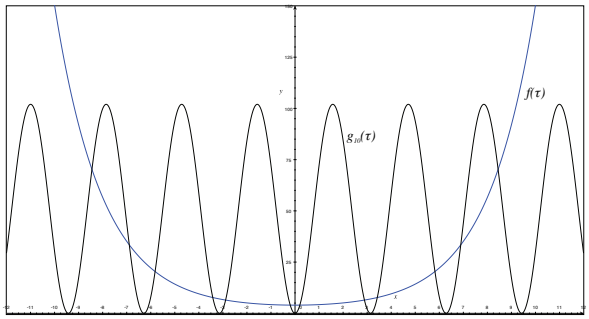


FIG. 3. *For* $\alpha = 10$ *and for* $\tau > 0$, $g_\alpha(\tau) < f(\tau)$ *out of three intervals.*

Proposition 4 can be applied to $\mathcal{F}$. Moreover, $\mathcal{F}$ is radially controllable. Hence, $\mathcal{F}$ is eventually periodically asymptotically controllable. More precisely, it can be checked that using periodically the switching sequence

$$e^{\frac{\pi}{2}A_2} e^{\frac{\pi}{2}A_1}$$

and starting from a point on the $x$-axis, one obtains a trajectory convergent to the origin.

*Example* 3. The family $\mathcal{F}$ defined by the matrices

$$A_1 = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$$

is quadratically stabilizable. Using the Lyapunov function $V = x^2 + y^2$, one can define a state-dependent switching rule, for instance,

$$\sigma(x, y) = \begin{cases} 1 & \text{if } (y - \frac{x}{2})(y + 2x) < 0, \\ 2 & \text{otherwise.} \end{cases}$$

Even in this case, the closed-loop system is asymptotically stable with respect to Krasowski solutions, but switched trajectories do not exist for some initial states.

It is not difficult to check that $\mathcal{F}$ is radially controllable. Using again Proposition 4, we see that $\mathcal{F}$ is eventually periodically asymptotically controllable.

The next example indicates that, at least for pairs of linear vector fields of $\mathbb{R}^2$, the radial controllability assumption is not necessary for the existence of eventually periodic, or even periodic, AC-switching signals.

*Example* 4. Consider the pair of planar vector fields defined by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The first component has a saddle configuration, while the second component is characterized by a line of stable equilibria. It is clear that the system is not radially controllable, since no trajectory crosses the $y$-axis. The associated discrete-time linear operator is

$$\Phi = e^{\tau_2 A_2} e^{\tau_1 A_1} = \begin{pmatrix} e^{\tau_1 - \tau_2} & 0 \\ 0 & e^{-\tau_2} \end{pmatrix}.$$

Thus, we see that $\Phi$ has one eigenvalue in the interior of the unit disc if $\tau_2 > 0$, and two eigenvalues in the interior of the unit disc if $\tau_2 > \tau_1 > 0$. As a consequence, we can construct both eventually periodic AC-switching maps and periodic AC-switching signals.

**9. Concluding remarks.** In this paper we propose a method which can be used to design in a systematic way switching signals such that the corresponding trajectories converge to the origin. The method applies to systems whose components are linear, but the underlying idea can be extended, in principle, to systems with nonlinear components. Although our method basically generates time-dependent switching signals, a state-dependent interpretation is plausible.

Since the method exploits the stable subspace of a discrete dynamical system associated to a suitable periodic switching signal, it is natural to expect that in numerical simulations the generated trajectories are well behaved only within a finite number of iterations. For large time, because of the accumulation of round-off errors, the trajectories might diverge. Hence, in order to achieve the desired goal, the process needs to be monitored and restarted from time to time, after updating the initial state. On the other hand, the interpretation in terms of state-dependence suggests that at least when only practical semi-global stability is required, some modification can be successfully introduced in our construction, in order to enhance the robustness. Some results in this direction will be the object of a forthcoming paper.

**Appendix. Some facts about geometric control theory.** The proofs of Theorems 1 and 2 rely on certain facts concerning the controllability of general families of nonlinear vector fields on compact manifolds. The required material is recalled and developed here, following the classical approach of geometric control theory [10].

Let $M$ be an analytic differentiable manifold, $\dim M = c$. For the purposes of this appendix, and in order to avoid notational ambiguity, we denote by $\mathcal{G} = \{g_n(p)\}_{n\in\mathcal{N}}$ a family of analytic, forward complete vector fields on $M$. Moreover, we denote by $\psi_n(t,p)$ the flow map[7] generated by $g_n(p)$ and by $\psi(t,p,\sigma)$ the switched trajectory corresponding to a switching signal $\sigma$ (to simplify the notation, when we want to focus on the trajectory rather than the switching signal, we may also write $\psi(t,p)$). The definition of reachable set extends easily to this nonlinear context. The following definition is classical.

DEFINITION 7. *A family of vector fields $\mathcal{G} = \{g_n(p)\}_{n\in\mathcal{N}}$ on $M$ is said to be* globally controllable *when for each pair of points $p, q \in M$, there exists some $T \geq 0$ such that $q \in \mathcal{R}_{\mathcal{G}}(T,p)$.*

To check global controllability, one usually needs to compute Lie brackets and to look for some additional recurrence condition: Results of this type can be found for instance in [10, p. 114], [11], and [14].

The following propositions state that if $M$ is compact, then there are a uniform transferring time and a uniform number of switches.

PROPOSITION 6. *Let $M$ be a compact, analytic manifold of dimension $c$. Let $\mathcal{G}$ be a globally controllable family of analytic vector fields. Then, there exists $\hat{T} > 0$ such that, for all $p, q \in M$, $q \in \mathcal{R}_{\mathcal{G}}(\hat{T}, p)$.*

PROPOSITION 7. *Under the same hypotheses of Proposition 6, there exist $\tilde{T} > 0$ and a positive integer $\tilde{S}$ satisfying the following property: For all $p, q \in M$, there exist a switching signal and a time $T \leq \tilde{T}$ such that $q = \psi(T,p,\sigma)$ and $\mathbf{s}_\sigma(T) \leq \tilde{S}$.*

The proofs of Propositions 6 and 7 require the notion of normal reachability. Given the set

$$\Theta(T) = \{(\theta_1, \ldots, \theta_c) \in \mathbb{R}^c \ : \ \theta_1 + \ldots + \theta_c < T, \ \theta_1 > 0, \ldots, \theta_c > 0\},$$

given $n_1, \ldots, n_c \in \mathcal{N}$, and given $q \in M$, we define a map

$$\begin{array}{rccl} \Psi^q_{n_1\ldots n_c}: & \Theta(T) & \longrightarrow & M \\ & (\theta_1, \ldots, \theta_c) & \mapsto & \psi_{n_c}(\theta_c, \psi_{n_c-1}(\theta_{c-1}, \ldots, \psi_{n_1}(\theta_1, q)\ldots)). \end{array}$$

Clearly $\Psi^q_{n_1\cdots n_c} = \psi(\theta_1 + \ldots + \theta_c, q, \sigma)$, where $\sigma$ is defined in the obvious way. A point $q'$ is said to be *normally reachable* from $q$ in time $T$ if there exist $n_1, \ldots, n_c \in \mathcal{N}$ such that $\Psi^q_{n_1\ldots n_c}(\Theta(T))$ contains a neighborhood $U_{q'}$ of $q'$.

By the Hermann–Nagano theorem [10, p. 66]), the Lie algebra generated by a globally controllable family $\mathcal{G}$ has full rank at each point of $M$. Therefore, we may apply the following theorem.

THEOREM 3 (see [10, Thm. 6, p. 48]). *Under the hypotheses of Proposition 6, for any $T > 0$, for any $\bar{q} \in M$, there exists $\bar{q}'$ normally reachable from $\bar{q}$ in time $T$.*

Observe that this theorem guarantees that there exists a neighborhood of $\bar{q}'$ whose points are reachable from $\bar{q}$ after as many switches as the dimension of $M$.

*Proof of Proposition* 6. For each $p, q \in M$, let us denote by

$$T(p,q) = \inf\{T : q \in \mathcal{R}_{\mathcal{G}}(T,p)\} \ . \tag{19}$$

[7]The forward completeness assumption guarantees that $\psi_n$ is defined for all $t \in \mathbb{R}$.

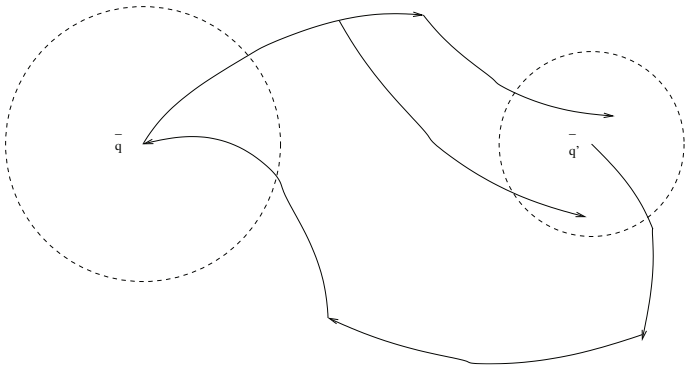


FIG. 4. *Construction of the neighborhood $U_{\bar q}$ of $\bar q$.*

Assume that the statement is false. Then, for each integer $j$ there exist $p_j$, $q_j$ such that

$$T(q_j, p_j) \geq j \ . \tag{20}$$

Since $M$ is compact, we can extract convergent subsequences $\{p_{j_k}\}$, $\{q_{j_k}\}$. Let

$$\lim_{k\to+\infty} p_{j_k} = \bar p \ , \qquad \lim_{k\to+\infty} q_{j_k} = \bar q \ .$$

*Step* 1 (see Figure 4). There exist a time $T_{\bar q} > 0$ and a neighborhood $U_{\bar q}$ of $\bar q$ such that $U_{\bar q} \subset \mathcal{R}_{\mathcal{G}}(T_{\bar q}, \bar q)$.

By Theorem 3, we may take $\bar q'$ normally reachable from $\bar q$ in a time $T_1$; let $U_{\bar q'}$ be the neighborhood introduced in the definition of normal reachability. By the global controllability assumption, there exist $T_2 > 0$ and a switched trajectory of $\mathcal{G}$ such that $\psi(T_2, \bar q') = \bar q$. Since the map $q \mapsto \psi(T_2, q)$ is a homomorphism on $M$, $U_{\bar q} = \psi(T_2, U_{\bar q'})$ is a neighborhood of $\bar q$ so that for each $q \in U_{\bar q}$ there exists a switched trajectory $\bar\psi$ such that $\psi(T_2, \bar\psi(T, \bar q)) = q$, with $T \leq T_1$. If we set $T_{\bar q} = T_1 + T_2$, by the concatenation property we get

$$U_{\bar q} \subseteq \mathcal{R}_{\mathcal{G}}(T_{\bar q}, \bar q).$$

*Step* 2. There exist $T_{\bar p} > 0$ and a neighborhood $U_{\bar p}$ of $\bar p$ such that $T(p, \bar p) \leq T_{\bar p}$ for all $p \in U_{\bar p}$.

To get this result, we apply Step 1 to the family $-\mathcal{G} = \{-g_n(p)\}$. For all $p \in U_{\bar p}$, there exist $T_1'$ and $T_2'$ and switched trajectories $\psi'$ and $\bar\psi'$ of $-\mathcal{G}$ such that $\psi'(T_2', \bar\psi'(T', \bar p)) = p$, with $T' \leq T_1'$.

The reversed time trajectory is a switched trajectory connecting $p$ to $\bar p$ in a time not greater than $T_{\bar p} = T_1' + T_2'$.

*Step* 3 (see Figure 5). We are now ready to prove the statement.

Let us consider a switched trajectory of $\mathcal{G}$ connecting $\bar p$ to $\bar q$ in a time $\bar T > 0$. We choose an integer $k$ such that

$$p_{j_k} \in U_{\bar p}, \quad q_{j_k} \in U_{\bar q}, \quad j_k > T_{\bar p} + T_{\bar q} + \bar T.$$

By Step 2, there exists a switched trajectory connecting $p_{j_k}$ to $\bar p$ in a time not greater than $T_{\bar p}$. By Step 1, there exists a switched trajectory connecting $\bar q$ to $q_{j_k}$

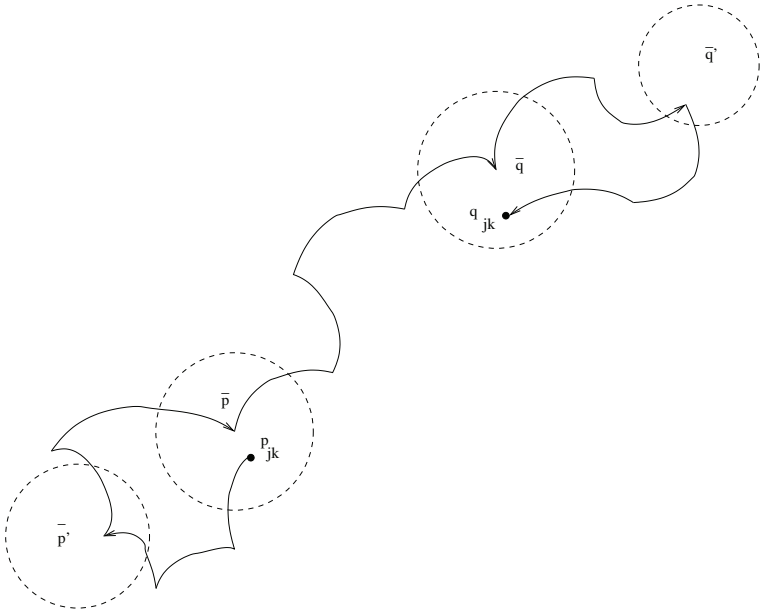


FIG. 5. *Construction of the trajectory joining $p_{j_k}$ to $q_{j_k}$.*

in a time not greater than $T_{\bar{q}}$. Thus, we are able to connect $p_{j_k}$ to $q_{j_k}$ by using the trajectories constructed above, so that

$$T(p_{j_k}, q_{j_k}) \leq T_{\bar{p}} + T_{\bar{q}} + \bar{T} < j_k,$$

which is a contradiction to (20).

*Proof of Proposition* 7. Let $\tilde{T} = 5\hat{T}$, where $\hat{T}$ is determined as in Proposition 6. Let $\mathbf{s}(p, q)$ be the minimum number of switching times necessary to steer $p$ to $q$ in a time $T \leq \tilde{T}$. We want to prove that there exists an upper bound for $\mathbf{s}(p, q)$. By contradiction, let us suppose that, for any integer $j \geq 0$, there exist $p_j,\ q_j$ such that

$$\mathbf{s}(p_j, q_j) > j. \tag{21}$$

As in Proposition 6, we may extract convergent subsequences $p_{j_k} \to \bar{p}$ and $q_{j_k} \to \bar{q}$. Steps 1 and 2 of Proposition 6 permit us to connect $p_{j_k}$ to $q_{j_k}$ via five switched trajectories, with times and number of switches depending only on $\bar{p}$ and $\bar{q}$. More precisely, we have the following steps.

*Step* 1. By Theorem 3, there exist $\bar{q}'$ normally reachable from $\bar{q}$ in a time $T > 0$ (which we may choose smaller than $\hat{T}$) and a neighborhood $U_{\bar{q}'}$ such that any point $q' \in U_{\bar{q}'}$ is reachable from $\bar{q}$ in a time not greater than $T$ and with exactly $c$ switches. As observed in Step 1 of the proof of Proposition 6, the trajectory connecting $\bar{q}'$ to $\bar{q}$ in time $\hat{T}$ (which exists by Proposition 6) defines a homomorphism of $U_{\bar{q}'}$ over a neighborhood $U_{\bar{q}}$ of $\bar{q}$. If $j_k$ is large enough so that $q_{j_k} \in U_{\bar{q}}$, we may connect $\bar{q}$ to $q_{j_k}$ via two switched trajectories

depending only on $\bar{q}$ and $\bar{q}'$. To be more precise,

$$\begin{cases} q_{j_k} = \psi(T_1, q'_{jk}, \sigma_1), \quad \text{with } q'_{j_k} \in U_{\bar{q}'}, \\ \mathbf{s}_{\sigma_1}(T_1) = \mathbf{s}_1, \\ T_1 \leq \hat{T}, \end{cases}$$

$$\begin{cases} q'_{j_k} = \psi(T_2, \bar{q}, \sigma_2), \\ \mathbf{s}_{\sigma_2}(T_2) = c - 1, \\ T_2 \leq \hat{T}. \end{cases}$$

*Step* 2. By Proposition 6, there exist a switching law $\sigma_3$ and a time $T_3 \leq \hat{T}$ such that

$$\begin{cases} \bar{q} = \psi(T_2, \bar{p}, \sigma_3), \\ \mathbf{s}_{\sigma_3}(T_1) = \mathbf{s}_3. \end{cases}$$

*Step* 3. By applying Step 1 to the family $-\mathcal{G}$, we obtain neighborhoods of $\bar{p}'$ (chosen as in Step 2 of Proposition 6) and $\bar{p}$. If we choose $j_k$ large enough so that $p_{j_k} \in U_{\bar{p}}$, we may find a point $p'_{j_k} \in U_{\bar{p}'}$ such that

$$\begin{cases} \bar{p} = \psi(T_4, p'_{j_k}, \sigma_4), \\ \mathbf{s}_{\sigma_4}(T_4) = c - 1, \\ T_4 \leq \hat{T}, \end{cases}$$

$$\begin{cases} p'_{j_k} = \psi(T_5, \bar{p}_{j_k}, \sigma_5), \\ \mathbf{s}_{\sigma_5}(T_5) = \mathbf{s}_5, \\ T_5 \leq \hat{T}. \end{cases}$$

By the concatenation property, we obtain a switching law $\sigma$ and a time $T$ such that

$$\begin{cases} q_{j_k} = \psi(T, p_{j_k}, \sigma), \\ \mathbf{s}_\sigma(T) = \mathbf{s}_1 + \mathbf{s}_3 + \mathbf{s}_5 + 2c - 2, \\ T \leq 5\hat{T} = \tilde{T} \ . \end{cases}$$

Therefore, $s(p_{j_k}, q_{j_k}) \leq \mathbf{s}_1 + \mathbf{s}_3 + \mathbf{s}_5 + 2c - 2$. Since we may choose $j_k 1 \geq \mathbf{s}_1 + \mathbf{s}_3 + \mathbf{s}_5 + 2c - 2$, we get a contradiction to (21).

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