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Congruence curves of the Goldstein-Petrich flows

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Abstract. We study the existence of contours which evolve retaining their shapes under the second Goldstein-Petrich flow. We present a proof of the existence, for each integer $n \geq 2$, of a 1-parameter family of non-congruent Goldstein-Petrich contours of $\mathbb{R}^2$ with symmetry group of order n. Explicit algorithms to compute and visualize the contours and their evolution are given.

1. Introduction

In ref. [**GP1**], R.E. Goldstein and D.M. Petrich showed that the mKdV equation

$$\kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0 \tag{1}$$

is associated to the flow on the space of unit-speed plane curves $\mathbf{z} : \mathbb{R} \to \mathbb{C}$ defined by

$$\mathbf{z}_t = -(\frac{\kappa^2}{2} + i\kappa_s)\mathbf{z}_s, \quad |\mathbf{z}_s| = 1, \quad \kappa = -i\mathbf{z}_{ss}\overline{\mathbf{z}}_s \tag{2}$$

A simple closed curve which evolves retaining its shape under (2) is said to be a GP contour. The existence of GP contours was considered in [**GP2, NSW**] and examples of closed, non-simple congruence curves of the flow (2) have been examined by Chou and Qu in ref. [**CQ**]. In [**Mu**], we exhibited explicit numerical examples of GP contours. Based on these results we wish to prove the following theorem :

Theorem 1. *For every integer $n \geq 2$ there exist $q_n \in (0,1)$ and a 1-parameter family $\{\gamma_{[q,n]}\}_{q\in[0,q_n)}$ of non-congruent GP contours with symmetry group of order*

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n. The evolution of $\gamma_{[q,n]}$ under the second Goldstein-Petrich flow is given by

$$\mathbf{z}_{[q,n]} : (s,t) \in \mathbb{R} \times \mathbb{R} \to \mathrm{Exp}(t\mu_{[q,n]}) \cdot \gamma_{[q,n]}(s - v_{[q,n]}t) \in \mathbb{R}^2, \tag{3}$$

where $\mu_{[q,n]} \in \mathfrak{e}(2)$ and $v_{[q,n]} \in \mathbb{R}$ are the momentum and the wave velocity of $\gamma_{[q,n]}$. Moreover, there exist a countable set $\mathcal{T}_n \subset [0, q_n)$ such that $\mathbf{z}_{[q,n]}$ is periodic in time, for each $q \in \mathcal{T}_n$.

The material is organized as follows. Section 2 recalls the basic definitions and collects the preliminary results from the existing literature. Section 3 analyzes the explicit integration of GP contours and proves the Theorem. Section 4 develops the numerical algorithms for the construction and the visualization of the 1-parameter families of GP contours with assigned symmetry group.

2. Preliminaries

2.1. Local motions. Denote by $J(\mathbb{R},\mathbb{R})$ the *total jet space* of smooth $\mathbb{R}$-valued functions of one independent variable, endowed with its standard coordinates

$$(s, u_{(0)}, u_{(1)}, \dots, u_{(h)}, \dots).$$

If $u : \mathbb{R} \to \mathbb{R}$ is a smooth function, its *prolongation* is defined by

$$j(u) : s \in \mathbb{R} \mapsto \left(s, u|_s, \frac{du}{ds}|_s, \dots, \frac{d^h u}{ds^h}|_s, \dots \right) \in J(\mathbb{R},\mathbb{R}).$$

A map $\mathfrak{w} : J(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ is said a *polynomial differential function* if there exists $w \in \mathbb{R}[x_0, \dots, x_h]$ such that

$$\mathfrak{w}(\mathbf{u}) = w(u_{(0)}, u_{(1)}, \dots, u_{(h)}),$$

for each $\mathbf{u} = (s, u_{(0)}, u_{(1)}, \dots, u_{(h)}, \dots) \in J(\mathbb{R},\mathbb{R})$. The algebra of polynomial differential functions, $J[\mathbf{u}]$, is endowed with the *total derivative*, defined by

$$D\mathfrak{w} = \sum_{p=0}^{\infty} \frac{\partial w}{\partial u_{(p)}} u_{(p+1)}.$$

A differential function $\mathfrak{w} \in J[\mathbf{u}]$ is a *total divergence* if there exists $\mathfrak{p} \in J[\mathbf{u}]$ such that $\mathfrak{w} = D(\mathfrak{p})$. The primitive $\mathfrak{p}$ is unique up to an additive constant. By $D^{-1}(\mathfrak{w})$ we denote the unique primitive of $\mathfrak{w}$ which vanishes at $\mathbf{u} = \mathbf{0}$. There is another natural differential operator, known as the *Euler operator*, defined by

$$\delta(\mathfrak{w}) = \sum_{\ell=0}^{\infty} (-1)^{\ell} D^{\ell} \left(\frac{\partial \mathfrak{w}}{\partial u_{(\ell)}} \right).$$

We now recall three elementary properties :

- $\mathfrak{w} \in J[\mathbf{u}]$ is a total divergence if and only if $\delta(\mathfrak{w}) = 0$;
- for each $\mathfrak{w} \in J[\mathbf{u}]$, $u_{(1)}\delta(\mathfrak{w})$ is a total divergence;
- for each $\mathfrak{w} \in J[\mathbf{u}]$, $u_{(0)}D(\delta(\mathfrak{w}))$ is a total divergence.

We let $\mathcal{M}$ be the space of unit-speed curves $\gamma : \mathbb{R} \to \mathbb{R}^2 \cong \mathbb{C}$. The arc-length parameter and the curvature are denoted by s and k respectively. Tangent vectors to $\mathcal{M}$ at γ are vector fields $V = (v_1 + iv_2)\gamma'$ along γ satisfying $v_1' = kv_2$. For each $\mathfrak{v} \in J[\mathbf{u}]$ such that $\delta(u_{(0)}\mathfrak{v}) = 0$, we define a cross section of $T(\mathcal{M})$ by

$$\mathcal{V} : \gamma \in \mathcal{M} \to \left(D^{-1}(u_{(0)}\mathfrak{v}) + i\mathfrak{v} \right)|_{j(k)}\gamma' \in T_{\gamma}(\mathcal{M}).$$

Definition 1. We call $\mathcal{V}$ the *local vector field* associated to $\mathfrak{v} \in J[\mathbf{u}]$. If $\mathfrak{v} = D(\delta(\mathfrak{w}))$, then $u_{(0)}\mathfrak{v}$ is a total divergence and the corresponding local vector field is said to be the *Hamiltonian vector field* with energy $\mathfrak{w}$. By a *local motion of plane curves* is meant an integral curve of a local vector field.

In other words, a local motion associated to $\mathfrak{v}$ is a smooth map

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} : (s,t) \in \mathbb{R} \times (a,b) \to \mathbb{C} \cong \mathbb{R}^2$$

such that

$$\mathbf{z}_t = (D^{-1}(u\mathfrak{v})|_{j_s(\kappa)} + i\mathfrak{v}|_{j_s(\kappa)})\mathbf{z}_s, \quad |\mathbf{z}_s| = 1, \tag{4}$$

where

$$\kappa = -i\mathbf{z}_{ss}\bar{\mathbf{z}}_s \tag{5}$$

is the *curvature function.* The *Frenet frame* along $\mathbf{z}$ is the map $\mathcal{A} : \mathbb{R} \times (a,b) \to \mathbb{E}(2)$ defined by

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{x} & \mathbf{x}' & -\mathbf{y}' \\ \mathbf{y} & \mathbf{y}' & \mathbf{x}' \end{pmatrix}.$$

If we set

$$\mathfrak{u} = D^{-1}(u_{(0)}\mathfrak{v}), \quad \mathfrak{p} = D\mathfrak{v} + u_{(0)}\mathfrak{u}, \tag{6}$$

then

$$\Theta := \mathcal{A}^{-1}d\mathcal{A} = \mathcal{K}|_{j_s(\kappa)}ds + \mathcal{P}(\mathfrak{v})|_{j_s(\kappa)}ds, \tag{7}$$

where $\mathcal{K}$ and $\mathcal{P}(\mathfrak{v})$ are the $\mathfrak{e}(2)$-valued differential functions

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -u_{(0)} \\ 0 & u_{(0)} & 0 \end{pmatrix}, \quad \mathcal{P}(\mathfrak{v}) = \begin{pmatrix} 0 & 0 & 0 \\ \mathfrak{u} & 0 & -\mathfrak{p} \\ \mathfrak{v} & \mathfrak{p} & 0 \end{pmatrix}. \tag{8}$$

The Maurer-Cartan equation $d\Theta + \Theta \wedge \Theta = 0$ yields

$$\kappa_t = D\left(D\mathfrak{v} + u_{(0)}D^{-1}(u_{(0)}\mathfrak{v})\right)|_{j_s(\kappa)}. \tag{9}$$

If the local vector field is Hamiltonian with energy $\mathfrak{w}$, then (9) takes the form

$$\kappa_t = \mathcal{E}(\delta(\mathfrak{w}))|_{j_s(\kappa)} \tag{10}$$

where

$$\mathcal{E} = (D^3 + D \cdot u_{(0)}D^{-1} \cdot u_{(0)}D)$$

is the canonical Hamiltonian structure of the mKdV hierarchy (cf. chapter 7 of ref. [**Ol**]).

2.2. The Goldstein-Petrich flows and the mKdV hierarchy.

According to [**GP1**] we consider the sequence $\{\mathfrak{v}_n\}_{n\in\mathbb{N}}$ of polynomial differential functions

$$\mathfrak{v}_1 = -u_{(1)} \quad \mathfrak{v}_n = D(D(\mathfrak{v}_{n-1}) + u_{(0)}D^{-1}(u_{(0)}\mathfrak{v}_{n-1})), \quad n \geq 2. \tag{11}$$

Then, the *mKdV hierarchy* is given by

$$u_t = \mathfrak{v}_n|_{j_s(u)}, \quad n \geq 1. \tag{12}$$

Setting

$$\mathfrak{w}_n = \int_0^1 D^{-1}(\mathfrak{v}_{n+1})|_{\epsilon u}ud\epsilon, \quad n \geq 0,$$

we obtain another sequence $\{\mathfrak{w}_n\}_{n\in\mathbb{N}}$ of polynomial differential functions such that

$$\mathfrak{v}_1 = D(\delta(\mathfrak{w}_0)), \quad \mathfrak{v}_n = D(\delta(\mathfrak{w}_{n-1})) = \mathcal{E}(\delta(\mathfrak{w}_{n-2})), \quad n \geq 2. \tag{13}$$

This leads to the bi-Hamiltonian representations of the mKdV hierarchy, namely

$$u_t = D(\delta(\mathfrak{w}_{n-1}))|_{j_s(u)} = \mathcal{E}(\delta(\mathfrak{w}_{n-2}))|_{j_s(u)}, \quad n \geq 2. \tag{14}$$

The first three equations of the mKdV hierarchy are

$$\begin{cases} u_t + u_s = 0 \\ u_t + \frac{3}{2}u^2 u_s + u_{sss} = 0, \\ u_t + u_{sssss} + \frac{5}{2}u^2 u_{sss} + 10u^2 u_s u_{ss} + \frac{5}{2}u_s^3 + \frac{15}{8}u^4 u_s = 0. \end{cases}$$

Definition 2. The local vector field $\mathcal{V}_n$ associated to $\mathfrak{v}_n$ is called the *n-th flow of Goldstein-Petrich.*

Remark 1. The Goldstein-Petrich flow $\mathcal{V}_n$ is Hamiltonian with energy $\mathfrak{w}_{n-2}$, for each $n \geq 2$. Moreover, the curvature function of a local motion of $\mathcal{V}_n$ evolves accordingly to the n-th member of the mKdV hierarchy.

3. Goldstein-Petrich contours

3.1. Congruence curves. A unit-speed curve γ which moves without changing its shape under the Goldstein-Petrich flow $\mathcal{V}_n$ is said to be a *congruence curve* of class n. From now we consider curves with non-constant curvature. Then, γ is a congruence curve of order n if and only if there exist $B : (a, b) \to \mathbb{E}(2)$ and $\upsilon : (a, b) \to \mathbb{R}$ such that

$$\mathbf{z} : (s, t) \in \mathbb{R} \times (a, b) \to B(t)\gamma(s - \upsilon(t)) \tag{15}$$

is a local motion of $\mathcal{V}_n$.

Lemma 2. *The function υ is linear.*

Proof. Equation (15) implies that the curvature function of $\mathbf{z}$ is given by

$$\kappa(s, t) = k(s + \upsilon(t)),$$

where k is the curvature of γ. From $\kappa_t = \mathfrak{v}_n|_{j_s(\kappa)}$ we find

$$k'|_{s+\upsilon(t)} \frac{d\upsilon}{dt}|_t = (\mathfrak{v}_n|_{j_s(k)})|_{s+\upsilon(t)}. \tag{16}$$

Taking $s_0 \in \mathbb{R}$ such that $k'|_{s_0} \neq 0$ and setting $s = -\upsilon(t) + s_0$ in (16) we obtain

$$\frac{d\upsilon}{dt}|_t = \frac{\upsilon_n|_{j_s(k)})|_{s_0}}{k'|_{s_0}} = \text{constant}.$$

$\square$

As a consequence, we assume that the evolution of a congruence curve is

$$\mathbf{z}(s, t) = B(t) \cdot \gamma(s - vt), \tag{17}$$

where the constant $v \in \mathbb{R}$ is the *wave velocity.* The curvature of a congruence curve of class n and wave velocity v is a solution of the *stationary mKdV equation*

$$\mathfrak{v}_n|_{j(k)} + vk' = 0. \tag{18}$$

In analogy with (6) we put

$$\mathfrak{u}_n = D^{-1}(u_{(0)}\mathfrak{v}_n), \quad \mathfrak{p}_n = D(\mathfrak{v}_n) + u_{(0)}\mathfrak{u}_n$$

and we consider the $\mathfrak{e}(2)$-valued polynomial differential function

$$\mathcal{H}(\mathfrak{v}_n) = \mathcal{P}(\mathfrak{v}_n) + v\mathcal{K}, \tag{19}$$

where $\mathcal{K}$ and $\mathcal{P}(\mathfrak{v}_n)$ are defined as in (8). An easy inspection shows that k satisfies (18) if and only if

$$(\mathcal{H}(\mathfrak{v}_n)|_{j(k)})' = [\mathcal{H}(\mathfrak{v}_n), \mathcal{K}]|_{j(k)}. \tag{20}$$

This implies that there exists $\mathbf{m} \in \mathfrak{e}(2)$ such that

$$A \cdot \mathcal{H}(\mathfrak{v}_n)|_{j(k)} \cdot A^{-1} = \mathbf{m}, \tag{21}$$

where $A : \mathbb{R} \to \mathbb{E}(2)$ is the Frenet frame along γ. We call $\mathbf{m}$ the *momentum* of γ.

Proposition 3. *Let γ be a congruence curve of class n, with wave velocity $v \in \mathbb{R}$ and momentum $\mathbf{m}$, then its evolution under $\mathcal{V}_n$ is given by*

$$\mathbf{z}(s,t) = \mathrm{Exp}(t\mathbf{m}) \cdot \gamma(s - vt). \tag{22}$$

Proof. Let $\mathbf{z}(s,t) = B(t)\gamma(s-vt)$ be the evolution of γ under $\mathcal{V}_n$. The Frenet frame of $\mathbf{z}$ is

$$\mathcal{A}(s,t) = B(t)A_\gamma(s - vt), \tag{23}$$

where A is the Frenet frame along the curve γ. From (7) we have

$$\mathcal{A}^{-1}d\mathcal{A} = \mathcal{K}|_{j_s(\kappa)}ds + \mathcal{P}(\mathfrak{v}_n)|_{j_s(\kappa)}dt. \tag{24}$$

Then, (21), (23) and (24) imply

$$B^{-1}|_t \frac{dB}{dt}|_t = A_\gamma(s - vt) \cdot (\mathcal{H}(\mathfrak{v}_n)|_{j(k)})|_{s-vt} \cdot A_\gamma(s + v_\gamma t)^{-1} = \mathbf{m}.$$

This yields the required result. □

3.2. Congruence curves of class 2. The curvature of a congruence curve of class two satisfies

$$k''' + (\frac{3}{2}k^2 - v)k' = 0,$$

where v is the wave velocity. From this we get

$$(k')^2 = -\frac{1}{4}\left(k^4 + c_2k^2 + c_1k - c_0\right),$$

where $c_2 = -4v$ and c_1, c_0 are constants of integration. Solutions with $c_1 = 0$ are plane elastic curves. Since closed planar elasticae are not simple [**BG**], we suppose $c_1 \neq 0$. Eventually scaling γ by a similarity factor, we normalize the curve by $c_1 = 1$ and we assume that the curvature is a periodic solution of

$$(k')^2 = -\frac{1}{4}\left(k^4 + c_2k^2 + k + c_0\right). \tag{25}$$

In addition, we require that the polynomial

$$P(t|c_2, c_0) = t^4 + c_2t^2 + t + c_0$$

has two distinct real roots $r_1 > r_2$ and two complex conjugate roots r_3 and r_4, with $\mathrm{Im}(r_3) > 0$. The coefficients c_2 and c_0 can be written in terms of the parameters $p < 0$ and $q \in (-1, 1)$ by

$$c_{0,p,q} = \frac{(1 + 4p^3q^2)(1 + 4p^3(q^2 - 1))}{16p^4}, \quad c_{2,p,q} = -\frac{1}{2p^2} + p(2q^2 - 1). \tag{26}$$

We set
(27)
$$g_{p,q}=-\frac{1}{2p}\left(1+p^6+p^3(4q^2-2)\right)^{1/4},\quad m_{p,q}=\frac{1}{2}+\frac{-1+p^3(1-2q^2)}{2\left(1+p^6+p^3(4q^2-2)\right)^{1/2}},$$
and we define
$$\begin{cases}A_{1,p,q}=\frac{1}{2p^2}\sqrt{1-p^3+2q(-p)^{3/2}}(1-2q(-p)^{3/2}),\\ A_{2,p,q}=\frac{1}{2p^2}\sqrt{1-p^3-2q(-p)^{3/2}}(1+2q(-p)^{3/2}),\\ B_{1,p,q}=\frac{1}{p}\sqrt{1-p^3+2q(-p)^{3/2}},\\ B_{2,p,q}=\frac{1}{p}\sqrt{1-p^3-2q(-p)^{3/2}}.\end{cases}$$
We denote by $\mathrm{cn}(-|m)$ the Jacobi elliptic cn-function with parameter $m\in(0,1)$ and we put
$$\begin{cases}\alpha_{1,p,q}=A_{1,p,q}-A_{2,p,q}, & \alpha_{2,p,q}=-(A_{1,p,q}+A_{2,p,q}),\\ \beta_{1,p,q}=B_{1,p,q}-B_{2,p,q}, & \beta_{2,p,q}=-(B_{1,p,q}+B_{2,p,q}).\end{cases}\tag{28}$$
Then,
$$k_{p,q}(s)=\frac{\alpha_{1,p,q}\mathrm{cn}(g_{p,q}s|m_{p,q})+\alpha_{2,p,q}}{\beta_{1,p,q}\mathrm{cn}(g_{p,q}s|m_{p,q})+\beta_{2,p,q}}\tag{29}$$
is a periodic solution[1] of (25), with coefficients $c_{0,p,q}$ and $c_{2,p,q}$ and period
$$\omega_{p,q}=\frac{4}{g_{p,q}}\int_0^{\pi/2}\frac{d\vartheta}{\sqrt{1-m_{p,q}\sin^2(\vartheta)}}.\tag{30}$$
For each $p<0$ and $q\in(-1,1)$ we let $\gamma_{p,q}:\mathbb{R}\to\mathbb{R}^2$ be the unit-speed curve with curvature $k_{p,q}$ such that
$$\gamma_{p,q}(0)=(-2p+4q(-p)^{-1/2},0),\quad \gamma'_{p,q}(0)=(0,-1)^t.$$
Since $k_{p,-q}(s)=k_{p,q}(s+\omega_{p,q})$, the curves $\gamma_{p,q}$ and $\gamma_{p,-q}$ are congruent each to the other. If $q=0$, the curvature is constant and $\gamma_{p,0}$ is a circle with signed radius $2p$. The angular function
$$\theta_{p,q}(s):=\int_0^s k_{p,q}(u)du$$
can be computed in terms of elliptic integrals of the third kind[2]. As a result we obtain
$$\theta_{p,q}(s)=h_{1,p,q}s+h_{2,p,q}\Phi_{2,p,q}(s)+h_{3,p,q}\Phi_{3,p,q}(s),\tag{31}$$
the coefficients $h_{j,p,q}$ and the functions $\Phi_{i,p,q}$ are defined by
$$\begin{cases}h_{1,p,q}=\frac{\alpha_{1,p,q}}{\beta_{1,p,q}},\\ h_{2,p,q}=\frac{\alpha_{2,p,q}\beta_{1,p,q}-\alpha_{1,p,q}\beta_{2,p,q}}{g_{p,q}\sqrt{(\beta_{2,p,1}-\beta_{1,p,q})(\beta_{1,p,q}+\beta_{2,p,q})(\beta_{1,p,q}^2(1-m_{p,q})-\beta_{2,p,q}^2m_{p,q})}},\\ h_{3,p,q}=-\frac{\alpha_{2,p,q}\beta_{1,p,q}-\alpha_{1,p,q}\beta_{2,p,q}}{g_{p,q}\beta_{1,p,q}\beta_{2,p,q}\sqrt{1-m_{p,q}}}\end{cases}\tag{32}$$

[1]See ref. [**BF**], pg. 133
[2]See ref. [**La**], pg. 67-69.

and by
(33)
$$\begin{cases}\Phi_{2,p,q}(s) = \operatorname{arctanh}\left(\sqrt{\frac{(1-m_{p,q})\beta_{1,p,q}^2+m_{p,q}\beta_{2,p,q}}{(\beta_{1,p,q}^2-\beta_{2,p,1})(\beta_{1,p,q}+\beta_{2,p,1})}}\operatorname{sd}(g_{p,q}s|m_{p,q})\right),\\ \Phi_{3,p,q}(s) = \Pi\left(\frac{\beta_{1,p,q}^2}{\beta_{2,p,q}^2}, \frac{1}{2}(\pi - 2\operatorname{am}(g_{p,q}s|m_{p,q}), \frac{m_{p,q}}{1-m_{p,q}})\right) - \Pi\left(\frac{\beta_{1,p,q}^2}{\beta_{2,p,q}^2}, \frac{\pi}{2}, \frac{m_{p,q}}{1-m_{p,q}})\right),\end{cases}$$

where

$$\begin{cases}\Pi(n,\phi,m) = \int_0^\phi \frac{d\theta}{(1-n\sin^2(\theta))\sqrt{1-m\sin^2(\theta)}},\\ \operatorname{am}(s,m) = \int_0^s \operatorname{dn}(u|m)du\end{cases}$$

are the integral of the third kind and the Jacobi amplitude respectively.

Proposition 4. *The curve $\gamma_{p,q}$ is given by*

$$\gamma_{p,q} = 2e^{i\theta_{p,q}}\left((2k_{p,q}^2 + c_{2,p,q}) + 4i\kappa'_{p,q}\right). \tag{34}$$

Proof. We set

$$\eta_{1,p,q} = -\frac{1}{2}k_{p,q} - \frac{1}{4}c_{2,p,q}, \quad \eta_{2,p,q} = -k'_{p,q}. \tag{35}$$

Then,

$$\mathcal{H}(\mathfrak{v}_2)|_{j(k_{p,q})} = \begin{pmatrix} 0 & 0 & 0\\ \eta_{1,p,q} & 0 & -1/8\\ \eta_{2,p,q} & 1/8 & 0\end{pmatrix}. \tag{36}$$

The Frenet frame field of a unit-speed curve γ with curvature $k_{p,q}$ and initial condition $\gamma'(0) = (1,0)^t$ is

$$A = \begin{pmatrix} 1 & 0 & 0\\ \gamma_1 & \cos(\theta_{p,q}) & -\sin(\theta_{p,q})\\ \gamma_2 & \sin(\theta_{p,q}) & \cos(\theta_{p,q})\end{pmatrix}. \tag{37}$$

Denote by

$$\mathbf{m} = \begin{pmatrix} 0 & 0 & 0\\ m_1 & 0 & -m_3\\ m_2 & m_3 & 0\end{pmatrix}$$

the momentum of γ. From (21) we have

$$A^{-1}\cdot\mathcal{H}(\mathfrak{v}_2)|_{j(k_{p,q})}\cdot A = \mathbf{m}. \tag{38}$$

Combining (36), (37) and (38) we obtain

$$\gamma = 8i(e^{i\theta_{p,q}}(\eta_{1,p,q} + i\eta_{2,p,q}) + (m_1 + im_2)).$$

Then,

$$\tilde{\gamma} = 8e^{i\theta_{p,q}}(\eta_{1,p,q} + i\eta_{2,p,q})$$

is a unit-speed curve with curvature $k_{p,q}$ and initial conditions

$$\tilde{\gamma}(0) = (-2p + 4q(-p)^{-1/2}, 0)^t, \quad \tilde{\gamma}(0) = (0,-1)^t.$$

This implies the required result. □

Since $\eta_{1,p,q} + i\eta_{1,p,q}$ is periodic, with period $\omega_{p,q}$, we deduce :

Corollary 5. *The curve $\gamma_{p,q}$ is closed if and only if*

$$\Lambda_{p,q} = \frac{1}{2\pi}\int_0^{\omega_{p,q}} k_{p,q}(u)du = \frac{\ell}{n} \in \mathbb{Q}, \tag{39}$$

where $\ell, n \in Z$ are relatively prime integers, with $\ell \geq 0$.

Remark 6. The integer ℓ is the turning number, $|n|$ is the order of the symmetry group. In particular, for a simple curve the integer ℓ is 1. If $q \neq 0$, the elliptic curve parameterized by $k_{p,q}$ and $k'_{p,q}$ intersects the Ox-axis in two points. Then, the four vertex theorem implies $|n| > 1$.

3.3. Proof of Theorem 1. We fix a positive integer $n > 1$. We define the *characteristic curve*

$$\Sigma_n = \{(p,q) \in \mathbb{R}^{-1} \times (-1,1) : \Lambda_{p,q} = -1/n\},$$

and we let Σ_n^+ be the set of all $(p,q) \in \Sigma_n$ such that $q \geq 0$. Since the function

$$\Lambda : (p,q) \in \mathbb{R}^- \times (-1,1) \to \Lambda_{p,q} \in \mathbb{R}$$

satisfies

$$\Lambda_{p,q} = \Lambda_{p,-q}, \quad \Lambda_{p,0} = -\frac{1}{\sqrt{1-p^3}}, \quad \partial_p\Lambda|_{p,0} = -\frac{3p^2}{2(1-p^3)^{3/2}} < 0, \tag{40}$$

then there exist a maximal $\epsilon_n \in (0,1]$ and a unique real-analytic even function

$$\phi_n : (-\epsilon_n, \epsilon_n) \to \mathbb{R}^- \tag{41}$$

such that

$$\phi_n(0) = (1-n^2)^{1/3}, \quad (\phi_n(q), q) \in \Sigma_n, \quad \forall q \in (-\epsilon_n, \epsilon_n).$$

We define

$$\gamma_{[q,n]} := \gamma_{\phi_n(q),q}, \tag{42}$$

and we consider the one-parameter family $\{\gamma_{[q,n]}\}_{q\in(-\epsilon_n,\epsilon_n)}$ of closed curves with curvature functions $k_{[q,n]} = k_{\phi_n(q),q}$. We let $\omega[q,n]$ be the period of $k_{[q,n]}$. Then,

$$\mathfrak{K}_{[n]} : (s,q) \in \mathbb{R} \times (-\epsilon_n, \epsilon_n) \to k_{[q,n]}(s) \in \mathbb{R}$$

is a real-analytic function, periodic in s, satisfying

$$\mathfrak{K}_{[n]}(s,0) = (2\phi_n(q))^{-1} < 0.$$

It follows that there exists $\epsilon'_n \in (0, \epsilon_n]$ such that $\gamma_{[q,n]}$ is strictly convex and satisfies

$$\frac{1}{2\pi}\int_0^{\omega[q,n]} k_{[q,n]}(u)du = -\frac{1}{n},$$

for each $q \in (-\epsilon'_n, \epsilon'_n)$. This implies (cf. [**MN**]) that $\gamma_{[q,n]}$ is a simple curve, for every $q \in (-\epsilon'_n, \epsilon'_n)$. We set

$$q_n = \mathrm{Sup}\{q \in (0,\epsilon_n) : \gamma_{[\widetilde{q},n]} \text{ is a simple curve}, \forall \widetilde{q} \in (0,q)\}.$$

Then, $\{\gamma_{[q,n]}\}_{q\in[0,q_n)}$ is a one-parameter family of simple congruence curves of class 2, with symmetry group of order n. Since the curves of the family have different lengths, they are not congruent each to the other. The momentum $\mathbf{m}_{[q,n]}$ and the wave velocity $v_{[q,n]}$ of $\gamma_{[q,n]}$ are given by

$$\mathbf{m}_{[q,n]} = \frac{1}{8}\begin{pmatrix} 0 & 0 & 0 \\ \mu_{[q,n]} & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mu_{[q,n]} = -2\phi_n(q) + \frac{4q}{\sqrt{-\phi_n(q)}}. \tag{43}$$

and by

$$v_{[q,n]} = \frac{1}{4}\left(\frac{1}{2\phi_n(q)^2} - \phi_n(q)(-1+2q^2)\right). \tag{44}$$

From (22), (43) and (44) we see that the evolution of $\gamma_{[q,n]}$ is

$$\mathbf{z}_{[q,n]}(s,t) = e^{it/8}\gamma_{[q,n]}(s - v_{[q,n]}t) + \mu_{[q,n]}\rho(t), \tag{45}$$

where

$$\rho(t) = \sin(t/8) + i(1-\cos(t/8)).$$

Therefore, $\mathbf{z}_{[q,n]}(s,t)$ is periodic in time if and only if

$$\frac{4\pi v_{[q,n]}}{n\omega_{[q,n]}} \in \mathbb{Q}.$$

Since the function

$$T_n : q \in [0, q_n) \to \frac{4\pi v_{[q,n]}}{n\omega_{[q,n]}} \tag{46}$$

is non-constant and real-analytic, then there exists a countable set $\mathcal{T}_n \subset [0, q_n)$ such that the evolution of $\gamma_{[q,n]}$ is periodic, for all $q \in \mathcal{T}_n$.

3.4. Example. Consider the family $\{\gamma_{[q,7]}\}_{q\in[0,1)}$. The function ϕ_7 is defined for all $q \in (-1,1)$ and the upper part Σ_7^+ of the characteristic curve is parameterized $q \in [0,1) \to (\phi_7(q), q) \in \Sigma_7^+$. The approximate value of q_7 is 0.8013658294677735. The behavior of the family is illustrated in the Figures 1, 2, 3 and 4.

Remark 7. Numerical experiments show that the characteristic curve Σ_n is always the graph of the function $\phi_n : (-1,1) \to \mathbb{R}$. Furthermore, there exists a well defined *separating value* $q_n \in (0,1)$ such that $\gamma_{[q,n]}$ is simple if and only if $|q| < q_n$. The experimental evidence also suggest that each GP-contour is equivalent, up to a similarity of $\mathbb{R}^2$, to a curve of the form $\gamma_{[q,n]}$, with $n \geq 2$ and $q \in (0, q_n)$.

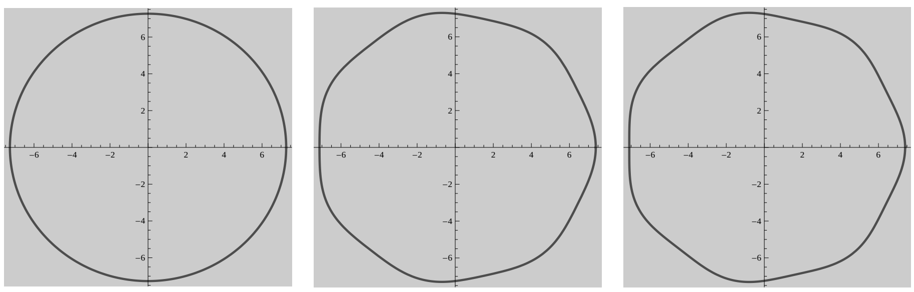

Figure 1. The curves $\gamma_{[0,7]}$, $\gamma_{[0.06,7]}$ and $\gamma_{[q_1,7]}$.

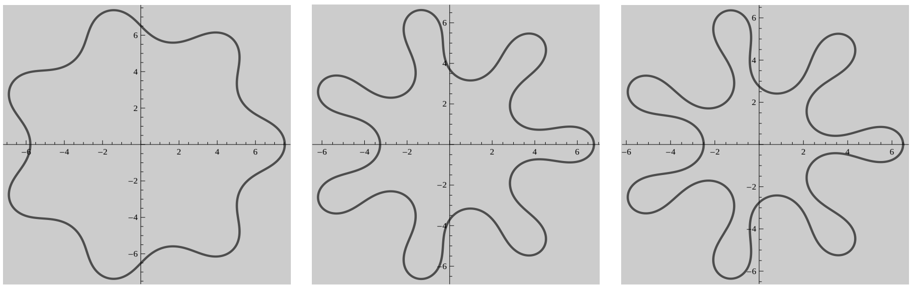

FIGURE 2. The curves $\gamma_{[0.4,7]}$, $\gamma_{[q_2,7]}$ and $\gamma_{[0.75,7]}$.

The numerical value of q_n can be found by the mean of the following procedure (see also Step 6 of Section 4) :

- compute q_n' such that $c_2(\phi_n(q_n'), q_n') = 0$;
- compute $q_n'' \in (q_1', 1)$ such that

$$\eta_{1,\phi_n(q_n''),q_n''}\left(\frac{\omega_{[q_n'',n]}}{2}\right) = 0,$$

where $\eta_{1,p,q}$ is defined as in (35).

- for each $q \in (q_1', q_1'']$ compute $s_{[q,n]} \in [0, \frac{\omega_{[q_n'',n]}}{2}]$ such that

$$g_{\phi_n(q),q}\mathrm{cn}(s_{[q,n]}|m_{\phi_n(q),q}) = -\frac{\alpha_{2,\phi_n(q),q} - \beta_{2,\phi_n(q),q}\sqrt{-c_{2,\phi_n(q),q}}}{\alpha_{1,\phi_n(q),q} - \beta_{1,\phi_n(q),q}\sqrt{-c_{2,\phi_n(q),q}}};$$

- q_n is the unique zero of the function

$$q \in (q_1', q_1''] \to |\gamma_{[q,n]}(s_{[q,n]}) - \gamma_{[q,n]}(-s_{[q,n]})|.$$

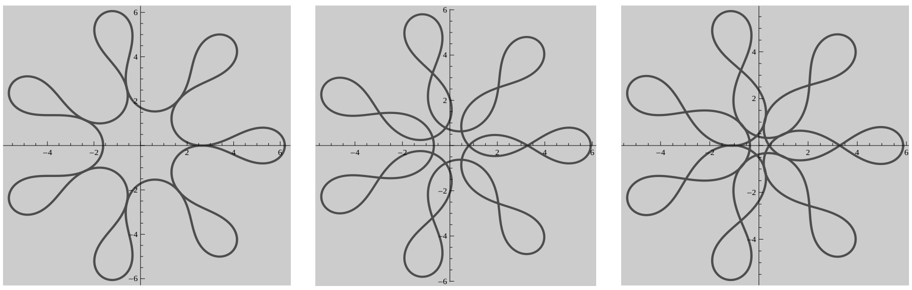

FIGURE 3. The curves $\gamma_{[q^*,7]}$, $\gamma_{[0.845,7]}$ and $\gamma_{[q_3,7]}$.

REMARK 8. Congruence curves which evolve periodically in time can be computed as follows: consider the function T_n and set $I_n = \mathrm{Im}(T_n)$. For each $\ell/h \in I_n \cap \mathbb{Q}$ there exists a unique $\hat{q}_n(\ell, h) \in [0, q_n)$ such that $T_n(\hat{q}_n(\ell, h)) = \ell/h$. The explicit evaluation of $\hat{q}_n(\ell, h)$ can be made via numerical routines (see Steps 9 and 10 of Section 4). Thus, the family of simple, closed congruence curves with symmetry group of order n and periodic evolution in time is $\{\gamma_{[\hat{q}_n(u),n]}\}_{u \in I_n \cap \mathbb{Q}}$.

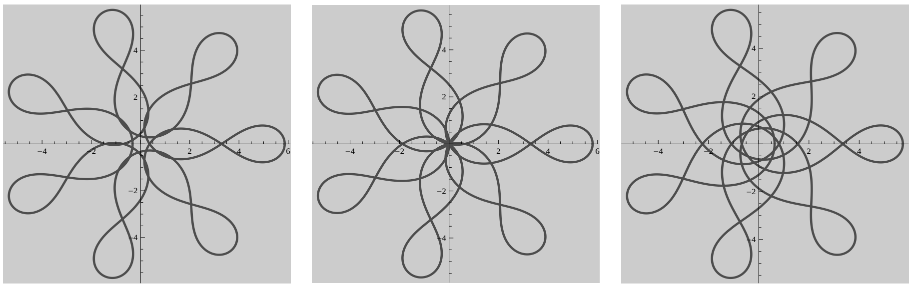

FIGURE 4. The curves $\gamma_{[0.86,7]}$, $\gamma_{[q_4,7]}$ and $\gamma_{[0.9,7]}$.

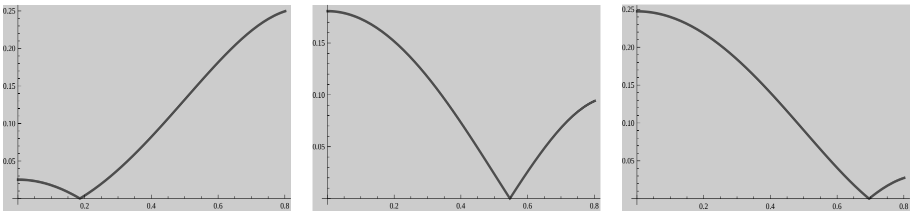

FIGURE 5. The functions $T_7(q) - \ell/h$, with $\ell/h = -2/9, -1/15, 0$.

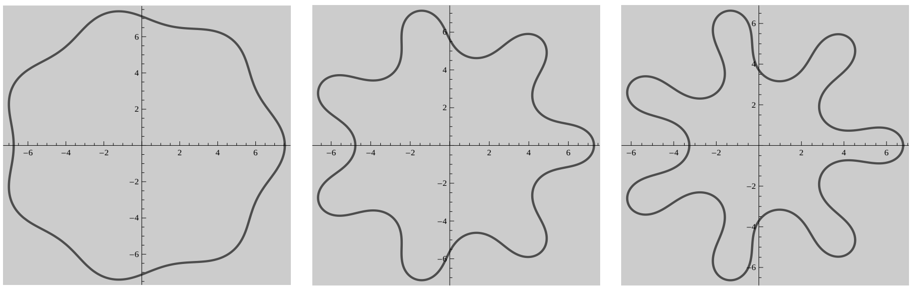

FIGURE 6. The curves $\gamma_{[\hat{q}_7(-2/9),7]}$, $\gamma_{[\hat{q}_7(-1/15),7]}$ and $\gamma_{[\hat{q}_7(0),7]}$.

4. Numerical computations and visualization

In this section we show how to translate the results and the computations of Section 2 into numerical and graphical routines implemented the software *Mathematica 7.0.*

• **Step 1.** Define the coefficients $c_{0,p,q}$, $c_{2,p,q}$, $\alpha_{j,p,q}$, $\beta_{j,p,q}$, $j = 1, 2$, $g_{p,q}$ and $m_{p,q}$ as in (26), (28) and (27) :

$$\mathbf{C0[p\_, q\_]:=}\frac{\left(1+4p^3q^2\right)\left(1+4p^3\left(-1+q^2\right)\right)}{16p^4};$$

$$\mathbf{C2[p\_, q\_]:=}-\frac{1}{2p^2}+p\left(-1+2q^2\right);$$

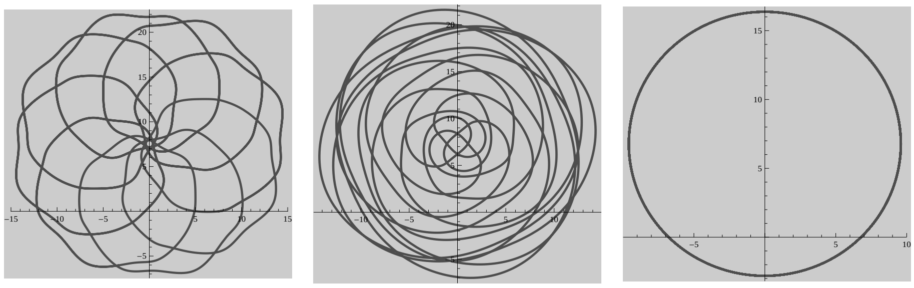

FIGURE 7. Trajectories of the points $\gamma_{[\hat{q}_7(-\frac{2}{9}),7]}(0)$, $\gamma_{[\hat{q}_7(-\frac{2}{15}),7]}(0)$ and $\gamma_{[\hat{q}_7(0),7]}(0)$.

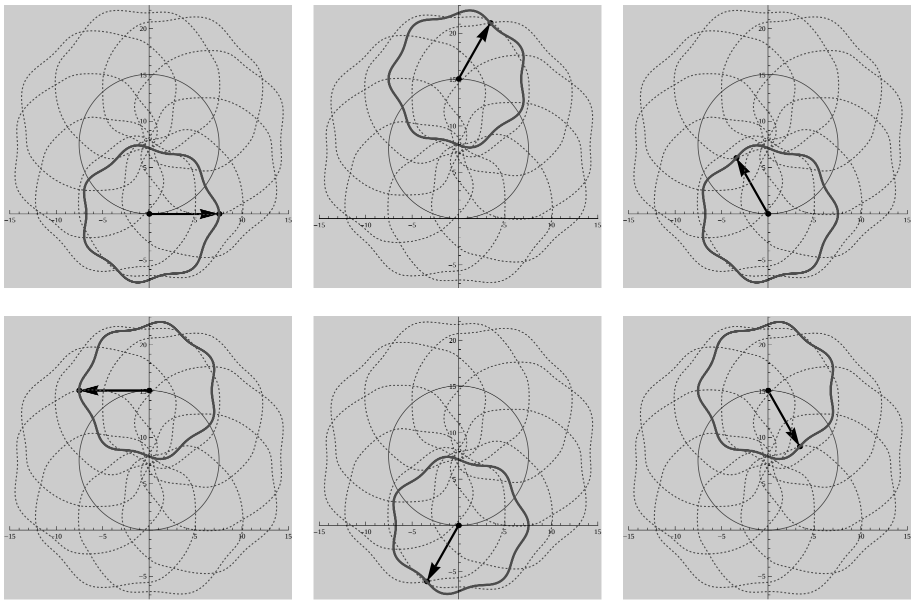

FIGURE 8. Evolution of the curve $\gamma_{[\hat{q}_7(-\frac{2}{9}),7]}$.

$$\boldsymbol{\alpha 1[\mathrm{p\_},\mathrm{q\_}]:=\frac{\sqrt{\frac{1}{p^2}-p-\frac{2q}{\sqrt{-p}}}\left(1+2(-p)^{3/2}q\right)}{2p}+\frac{\sqrt{1-p^3+2(-p)^{3/2}q}\left(1+2\sqrt{-p}pq\right)}{2p^2};}$$

$$\boldsymbol{\alpha 2[\mathrm{p\_},\mathrm{q\_}]:=\frac{\sqrt{\frac{1}{p^2}-p-\frac{2q}{\sqrt{-p}}}\left(1+2(-p)^{3/2}q\right)}{2p}-\frac{\sqrt{1-p^3+2(-p)^{3/2}q}\left(1+2\sqrt{-p}pq\right)}{2p^2};}$$

$$\boldsymbol{\beta 1[\mathrm{p\_},\mathrm{q\_}]:=\sqrt{\frac{1}{p^2}-p-\frac{2q}{\sqrt{-p}}}+\frac{\sqrt{1-p^3+2(-p)^{3/2}q}}{p};}$$

$$\boldsymbol{\beta 2[\mathrm{p\_},\mathrm{q\_}]:=\sqrt{\frac{1}{p^2}-p-\frac{2q}{\sqrt{-p}}}-\frac{\sqrt{1-p^3+2(-p)^{3/2}q}}{p};}$$

$$\boldsymbol{m[\mathrm{p\_},\mathrm{q\_}]:=\frac{-1+p^3\left(1-2q^2\right)+\sqrt{1+p^6+p^3\left(-2+4q^2\right)}}{2\sqrt{1+p^6+p^3\left(-2+4q^2\right)}};}$$

$$\boldsymbol{g[\mathrm{p\_},\mathrm{q\_}]:=-\frac{\left(1+p^6+p^3\left(-2+4q^2\right)\right)^{1/4}}{2*p};}$$

• **Step 2.** Define the curvature $k_{p,q}$ and its period $\omega_{p,q}$ (cf. (29) and (30)) :

$$k[\mathrm{s\_, p\_, q\_}] := \frac{\alpha 1[p,q] * \mathrm{JacobiCN}[g[p,q]*s, m[p,q]] + \alpha 2[p,q]}{\beta 1[p,q] * \mathrm{JacobiCN}[g[p,q]*s, m[p,q]] + \beta 2[p,q]};$$
$$\mathrm{Dk}[\mathrm{s\_, p\_, q\_}] := \mathrm{Evaluate}[D[k[s,p,q], s]];$$
$$\omega[\mathrm{p\_, q\_}] := \frac{4}{g[p,q]} \mathrm{EllipticK}[m[p,q]];$$

• **Step 3.** Compute the angular function $\theta_{p,q}$ (cf. (31),(32)) and (33)) :

$$\mathrm{h1}[\mathrm{p\_, q\_}] := \frac{\alpha 1[p,q]}{\beta 1[p,q]};$$
$$\mathrm{h2}[\mathrm{p\_, q\_}] := \frac{(\alpha 2[p,q]\beta 1[p,q] - \alpha 1[p,q]\beta 2[p,q])}{g[p,q] * \sqrt{(-\beta 1[p,q] + \beta 2[p,q])(\beta 1[p,q] + \beta 2[p,q])((-1 + m[p,q])\beta 1[p,q]^2 - m[p,q]\beta 2[p,q]^2)}};$$
$$\mathrm{h3}[\mathrm{p\_, q\_}] := -\sqrt{\frac{1}{1 - m[p,q]}} * \frac{(\alpha 2[p,q]\beta 1[p,q] - \alpha 1[p,q]\beta 2[p,q])}{g[p,q]\beta 1[p,q]\beta 2[p,q]};$$
$$\Phi 2[\mathrm{s\_, p\_, q\_}] := \mathrm{ArcTanh}\left[\frac{\mathrm{JacobiSD}[sg[p,q], m[p,q]]}{\sqrt{\frac{(\beta 1[p,q] - \beta 2[p,q])(\beta 1[p,q] + \beta 2[p,q])}{-(-1 + m[p,q])\beta 1[p,q]^2 + m[p,q]\beta 2[p,q]^2}}}\right];$$
$$\Phi 3[\mathrm{s\_, p\_, q\_}] := -\,\mathrm{EllipticPi}\left[\frac{\beta 1[p,q]^2}{\beta 2[p,q]^2}, \frac{m[p,q]}{-1 + m[p,q]}\right] +$$
$$\mathrm{EllipticPi}\left[\frac{\beta 1[p,q]^2}{\beta 2[p,q]^2}, \frac{1}{2}(\pi - 2\mathrm{JacobiAmplitude}[sg[p,q], m[p,q]]), \frac{m[p,q]}{-1 + m[p,q]}\right];$$

$$\theta[\mathrm{s\_, p\_, q\_}] := \mathrm{h1}[p,q] * s + \mathrm{h2}[p,q] * \Phi 2[s,p,q] + \mathrm{h3}[p,q] * \Phi 3[s,p,q];$$

• **Step 4.** Compute the function ϕ_n and the one-parameter family of closed curves $\{\gamma_{[q,n]}\}_{q\in[0,1)}$ (cf. (35),(34),(41) and (42)) :

$$\Lambda[\mathrm{p\_, q\_}] := \frac{1}{2\mathrm{Pi}}\theta[\omega[p,q], p, q];$$
$$\eta[\mathrm{s\_, p\_, q\_}] := \left\{\frac{\mathrm{C2}[p,q]}{4} + \frac{k[s,p,q]\hat{}2}{2}, +\mathrm{Dk}[s,p,q]\right\};$$
$$R[\mathrm{s\_, p\_, q\_}] := \{\{\mathrm{Cos}[\theta[s,p,q]], -\mathrm{Sin}[\theta[s,p,q]]\}, \{\mathrm{Sin}[\theta[s,p,q]], \mathrm{Cos}[\theta[s,p,q]]\}\};$$
$$\varphi[\mathrm{q\_, n\_}] := \mathrm{Evaluate}\left[\mathrm{FindRoot}\left[\Lambda[p,q] + \frac{1}{n} == 0, \left\{p, -\sqrt[3]{-1 + n\hat{}2}, -0.1\right\}, \mathrm{Method} \to \text{"Brent"}\right]\right][[1]][[2]];$$
$$\gamma[\mathrm{s\_, q\_, n\_}] := 8 * R[s, \varphi[q,n], q].\eta[s, \varphi[q,n], q];$$

• **Step 5.** Visualize Σ_n and the curve $\gamma_{[q,n]}$:

$$\Sigma[\mathrm{n\_}] := \mathrm{ContourPlot}\left[\Lambda[p,q] == -1/n, \left\{p, -\sqrt[3]{-1 + n\hat{}2}, -0.01\right\}, \{q, 0.00001, 0.99\},\right.$$
$$\mathrm{ContourStyle} \to \{\mathrm{GrayLevel}[0.3], \mathrm{Thickness}[0.008]\}, \mathrm{Background} \to \mathrm{GrayLevel}[0.8],$$
$$\left.\mathrm{PlotPoints} \to 50\right];$$
$$\mathrm{CURVE}[\mathrm{q\_, n\_}] := \mathrm{ParametricPlot}[\mathrm{Evaluate}[\gamma[s,q,n]], \{s, 0, n * \omega[\varphi[q,n], q]\},$$
$$\mathrm{PlotStyle} \to \{\mathrm{GrayLevel}[0.3], \mathrm{Thickness}[0.008]\}, \mathrm{Background} \to \mathrm{GrayLevel}[0.8],$$
$$\mathrm{PlotRange} \to \mathrm{All}];$$

• **Step 6.** Specify the order of the symmetry group and compute q_n :

$$\mathrm{nn} := 5$$
$$\mathrm{f1}[\mathrm{q\_}] := \mathrm{Abs}[\mathrm{C2}[\varphi[q, \mathrm{nn}], q]];$$
$$\mathrm{F1} := \mathrm{Plot}[\mathrm{f1}[q], \{q, 0, 0.99\}];$$
$$\mathrm{f2}[\mathrm{q\_}] := \mathrm{Abs}\left[\frac{\mathrm{C2}[\varphi[q,\mathrm{nn}], q]}{4} + \frac{k\left[\frac{1}{2}*\omega[\varphi[q,\mathrm{nn}], q], \varphi[q,\mathrm{nn}], q\right]\hat{}2}{2}\right];$$
$$\mathrm{F2} := \mathrm{Plot}[\mathrm{f2}[q], \{q, 0, 0.99\}];$$
$$\mathrm{Q1} := \mathrm{Evaluate}[\mathrm{First}[\mathrm{Sort}[\mathrm{InputForm}[\mathrm{F1}][[1,1]][[1,3]][[2]][[1]], \#1[[2]] < \#2[[2]]\&]]];$$
$$\mathrm{Q2} := \mathrm{Evaluate}[\mathrm{First}[\mathrm{Sort}[\mathrm{InputForm}[\mathrm{F2}][[1,1]][[1,3]][[2]][[1]], \#1[[2]] < \#2[[2]]\&]]];$$
$$\mathrm{s1}[\mathrm{q\_}] :=$$

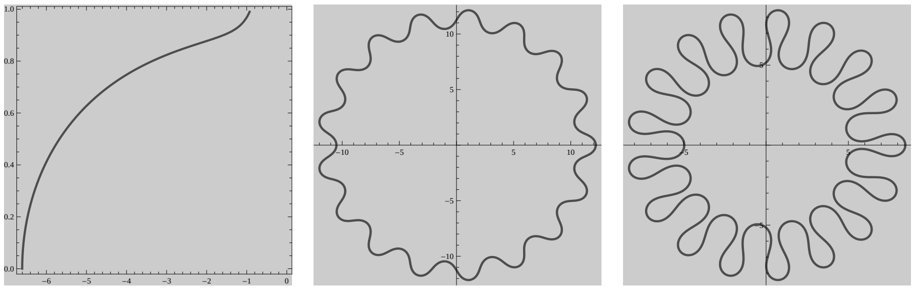

FIGURE 9. Σ_{17} and the curves $\gamma_{[0.5,17]}$, $\gamma_{[0.8,17]}$.

$\frac{1}{g[\varphi[q,\mathrm{nn}],q]}$**InverseJacobiCN** $\left[-\frac{\alpha 2[\varphi[q,\mathrm{nn}],q]-\beta 2[\varphi[q,\mathrm{nn}],q]\sqrt{-\mathrm{C2}[\varphi[q,\mathrm{nn}],q]/2}}{\alpha 1[\varphi[q,\mathrm{nn}],q]-\beta 1[\varphi[q,\mathrm{nn}],q]\sqrt{-\mathrm{C2}[\varphi[q,\mathrm{nn}],q]/2}},\right.$
$m[\varphi[q,\mathrm{nn}],q]]$;
f3[q_]:=Norm$[\gamma[\mathrm{s1}[q],q,\mathrm{nn}]-\gamma[-\mathrm{s1}[q],q,\mathrm{nn}]]$;
steps:=7; initialpoint:=$\frac{1}{2}(\mathrm{Q1}[[1]]+\mathrm{Q2}[[1]])$; **internalparameter[1]:=1/30;**
internalparameter[2]:=20;
QQ$[\mathrm{y\_},\delta\_,\mathrm{k\_}]$**:=First[Sort[Table**$[\{\mathrm{f3}[q],q\},\{q,y-\delta,y+\delta,1/k\}]]]$;
$S[1,\mathrm{y\_},\delta\_,\mathrm{k\_}]$:=QQ$[y,\delta,k]$;
$S[\mathrm{m\_},\mathrm{y\_},\delta\_,\mathrm{k\_}]:=S\left[m-1,S[m-1,y,\delta,k][[2]],\delta/\left(2^{m-1}\right),k*\left(2^{m-1}\right)\right]$;
Qn:=Evaluate$[S$**[steps, initialpoint, internalparameter[1], internalparameter[2]]];**

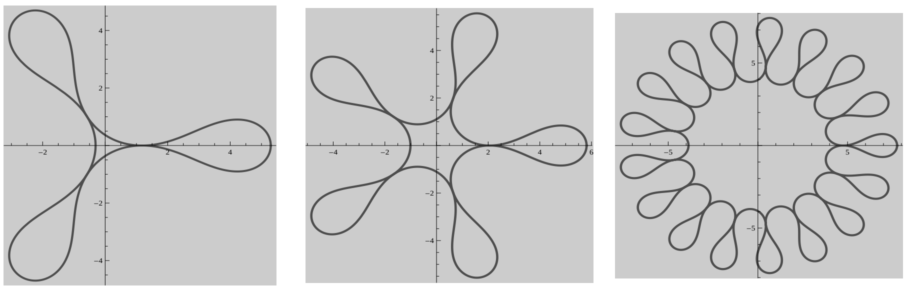

FIGURE 10. The "separating" curves $\gamma_{[q_3,3]}$, $\gamma_{[q_5,5]}$ and $\gamma_{[q_{17},17]}$.

• **Step 7.** Compute the evolution of the congruence curves :
$\mu 1[\mathrm{q\_},\mathrm{n\_}]:=-2*\varphi[q,n]+\frac{4*q}{\sqrt{-\varphi[q,n]}}$;
$\mu[\mathrm{q\_},\mathrm{n\_}]:=\frac{1}{8}\{\{0,0,0\},\{\mu 1[q,n],0,-1\},\{0,1,0\}\}$;
$v[\mathrm{q\_},\mathrm{n\_}]:=\frac{1}{4}\left(\frac{1}{2*\varphi[q,n]\wedge 2}-\varphi[q,n](2q\wedge 2-1)\right)$;
$z[\mathrm{s\_},\mathrm{t\_},\mathrm{q\_},\mathrm{n\_}]:=\left\{\left\{\mathrm{Cos}\left[\frac{t}{8}\right],-\mathrm{Sin}\left[\frac{t}{8}\right]\right\},\left\{\mathrm{Sin}\left[\frac{t}{8}\right],\mathrm{Cos}\left[\frac{t}{8}\right]\right\}\right\}.\gamma[s-v[q,n]*t,q,n]+$
$\mu 1[q,n]\{\mathrm{Sin}[t/8],-\mathrm{Cos}[t/8]+1\}$;
• **Step 9.** Compute the function T_n :
$T[\mathrm{q\_},\mathrm{n\_}]:=4\frac{\mathrm{Pi}*v[q,n]}{n*\omega[\varphi[q,n],q]}$;,

• **Step 10.** Specify the order of the symmetry group, take $u/w \in I_7 \cap \mathbb{Q}$ and compute $\hat{q}_n(u,w)$:

$$\mathbf{nnn{:=}7; u{:=}-2; w{:=}9;}$$
$$\mathbf{T1{:=}Plot\left[Abs\left[T[q,nnn]-\frac{u}{w}\right],\{q,0,0.99\}\right];}$$
$$\mathbf{QPA{:=}Evaluate[First[Sort[InputForm[T1][[1,1]][[1,3]][[2]][[1]],\#1[[2]]<\#2[[2]]\&]]];}$$
$$\mathbf{steps{:=}6}$$
$$\mathbf{initialpoint{:=}QPA[[1]];}$$
$$\mathbf{internalparameter[1]{:=}1/30;}$$
$$\mathbf{internalparameter[2]{:=}20;}$$
$$\mathbf{QP1[y\_,\delta\_,k\_]{:=}First\left[Sort\left[Table\left[\left\{Abs\left[T[q,nnn]-\frac{u}{w}\right],q\right\},\{q,y-\delta,y+\delta,1/k\}\right]\right]\right];}$$
$$\mathbf{SQP1[1,y\_,\delta\_,k\_]{:=}QP1[y,\delta,k];}$$
$$\mathbf{SQP1[m\_,y\_,\delta\_,k\_]{:=}SQP1\left[m-1,SQP1[m-1,y,\delta,k][[2]],\delta\big/\left(2^{m-1}\right),k*\left(2^{m-1}\right)\right];}$$
$$\mathbf{QPA2{:=}Evaluate[SQP1[steps,initialpoint,internalparameter[1],internalparameter[2]]];}$$
$$\mathbf{QP{:=}QPA2[[2]];}$$

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