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Abstract

We consider a quasilinear equation, involving the p-Laplace operator, with a p-superlinear nonlinearity. We prove the existence of a nontrivial solution, also when there is no mountain pass geometry, without imposing a global sign condition. Techniques of Morse theory are employed.

Keywords: p-Laplace equations; nontrivial solutions; Morse theory.

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1 Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p\, u = \lambda\, V\, |u|^{p-2}\, u + g(x,u) & \text{in } \Omega\,, \\[2ex] \qquad u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 1$, $\Delta_p\, u = \mathrm{div}\left(|\nabla u|^{p-2}\, \nabla u\right)$ is the p-Laplacian of u, $p \in]1,\infty[$, $\lambda \in \mathbb{R}$ is a parameter, $V \in L^\infty(\Omega)$ and g is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the following conditions:

$(g1)$ there exist $C > 0$ and

$$p < q < \begin{cases} p^* := \dfrac{np}{n-p} & \text{if } p < n \\[2ex] \infty & \text{if } p \geq n \end{cases}$$

such that

$$|g(x,s)| \leq C\left(|s|^{q-1} + 1\right)\,;$$

$(g2)$ we have

$$g(x,s) = \mathrm{o}(|s|^{p-1}) \;\text{ as }\; s \to 0, \;\text{ uniformly in }\; x\,;$$

$(g3)$ there exist $\mu > p$ and $R > 0$ such that

$$0 < \mu\, G(x,s) := \int_0^s g(x,t)\,dt \le s\, g(x,s), \quad \text{whenever } |s| \ge R\,.$$

In particular, $g(x,0) \equiv 0$ and hence we have the trivial solution $u = 0$, and we seek another.

In the case $p = 2$, the existence of a nontrivial solution u for (1.1) can be obtained via the Linking Theorem (see e.g. Rabinowitz [21, Theorem 5.16]). More precisely, let us assume, without loss of generality, that $\lambda \ge 0$. If the set

$$\mathcal{M} = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega V\,|u|^p\,dx = 1 \right\}$$

is empty or if $\mathcal{M} \ne \emptyset$ and

$$\lambda < \lambda_1 := \min\left\{ \int_\Omega |\nabla u|^p\,dx : \quad u \in \mathcal{M} \right\}\,,$$

then the existence of a nontrivial solution can be proved, without any further assumption, by the Mountain Pass Theorem for any $p > 1$ (see Ambrosetti and Rabinowitz [1] for the case $p = 2$ and Dinca, Jebelean and Mawhin [10] for the case $p \ne 2$). On the contrary, if $\mathcal{M} \ne \emptyset$ and $\lambda \ge \lambda_1$, the classical proof is based on the fact that each eigenvalue λ_k of $-\Delta_2$ induces a suitable direct sum decomposition of $W_0^{1,2}(\Omega)$. On the other hand, if $p \ne 2$, such decompositions are not available. Nevertheless, a linking argument over cones, rather than over linear subspaces, has been developed for $p \ne 2$, when λ is close to λ_1 by Fan and Z. Li [12] and for any λ by Degiovanni and Lancelotti [9]. In such a way, the mentioned result of Rabinowitz has been completely extended to the case $p \ne 2$.

When $\lambda \ge \lambda_1$, in all these results a global sign condition like $G(x,s) \ge 0$ needs to be imposed, in order to recognize the linking geometry. However, such an assumption can be relaxed by means of Morse theory or nonstandard linking constructions.

When $p = 2$, in Benci [2, Theorem 7.14] it is show that, if a nonresonance condition at the origin is satisfied, the existence of a nontrivial solution can be obtained without any further assumption. On the other hand, S.J. Li and Willem [15, Theorem 4] are able to treat the resonant case under a local sign condition on G. Related results are also contained in J.Q. Liu and S.J. Li [14].

The approach based on Morse theory has been extended to the case $p \neq 2$ by S. Liu [16] when λ is close to λ_1 and by Perera [19] when λ does not belong to the spectrum of the p-Laplace operator.

Our purpose is to develop this approach, in order to remove any condition on λ and require only a local sign condition on G. Our result is the following

Theorem 1.1. *Let us suppose that assumptions* $(g1) - (g3)$ *hold and let* $V \in L^\infty(\Omega)$*. Then, for every* $\lambda \in \mathbb{R}$*, problem* (1.1) *has a nontrivial solution* $u \in W_0^{1,p}(\Omega)$ *in each of the following cases:*

(a) *there exists* $\delta > 0$ *such that* $G(x,s) \geq 0$ *for a.e.* $x \in \Omega$ *and every* $s \in \mathbb{R}$ *with* $|s| \leq \delta$*;*

(b) *there exists* $\delta > 0$ *such that* $G(x,s) \leq 0$ *for a.e.* $x \in \Omega$ *and every* $s \in \mathbb{R}$ *with* $|s| \leq \delta$*.*

This is a natural extension to the case $p \neq 2$ of the mentioned result of S.J. Li and Willem, although the argument is based there on a nonstandard linking construction and here on Morse theory.

In the next section we recall and prove some preliminary facts, while in section 3 we prove the main result in a more general setting. In the last section we recover Theorem 1.1 as a particular case.

2 Preliminaries

Let Φ be a C^1-functional defined on a real Banach space W. We denote by B_ρ and S_ρ the closed ball and sphere of center 0 and radius ρ. We also denote by H the Alexander-Spanier cohomology with $\mathbb{Z}_2$-coefficients (see Spanier [22]). For a symmetric subset X of $W \setminus \{0\}$, $i(X)$ denotes its $\mathbb{Z}_2$-cohomological index (see Fadell and Rabinowitz [11]). The following notion has been introduced, in a slightly different form, by Perera, Agarwal, and O'Regan [20] and is in turn a variant of the homological local linking of Perera [18]. It should also be compared with the local linking of S.J. Li and Willem [15].

Definition 2.1. We say that Φ has a cohomological local splitting near 0 in dimension $k < \infty$, if there are two symmetric cones W_-, W_+ in W and $\rho > 0$ such that

$$W_- \cap W_+ = \{0\} \,, \qquad i(W_- \setminus \{0\}) = i(W \setminus W_+) = k \tag{2.1}$$

and

$$\begin{cases} \Phi(u) \leq \Phi(0) & \text{for every } u \in B_\rho \cap W_- \,, \\ \\ \Phi(u) \geq \Phi(0) & \text{for every } u \in B_\rho \cap W_+ \,. \end{cases} \tag{2.2}$$

As we will see, in such a case 0 must be a critical point of Φ.

Recall that the cohomological critical groups of Φ at a point $u \in W$ are defined by

$$C^q(\Phi, u) = H^q(\Phi^c, \Phi^c \setminus \{u\}) \,, \quad q \geq 0 \,,$$

where $c = \Phi(u)$ is the corresponding value and Φ^c is the closed sublevel set $\big\{w \in W : \Phi(w) \leq c\big\}$ (see, e.g., Chang [3] or Mawhin and Willem [17]). By the excision property, we have

$$C^q(\Phi, u) \approx H^q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\})$$

for every neighborhood U of u. Therefore, the concept has local nature. Moreover, it is well known that all critical groups are trivial, if u is not a critical point of Φ (see e.g. Corvellec [5, Proposition 3.4]). Finally, the next result shows a stability property and is a particular case of Corvellec and Hantoute [7, Theorem 5.2] (see also Benci [2, Theorem 5.16]). The case in which W is a Hilbert space and Φ is of class C^2 can be also found in Chang [3, Theorem I.5.6] and in Mawhin and Willem [17, Theorem 8.8].

Theorem 2.2. *Let $\Phi_t : W \longrightarrow \mathbb{R}$, $t \in [0, 1]$, be a family of functionals of class C^1. Assume that there exists $\rho > 0$ such that each Φ_t satisfies the Palais-Smale condition over B_ρ and has no critical point in B_ρ other than 0. Suppose also that the map $\{t \mapsto \Phi_t\}$ is continuous from $[0, 1]$ into $C^1(B_\rho)$.*

Then $C^q(\Phi_t, 0)$ is independent of t.

The cohomological local splitting allows to give an estimate of the critical groups, also in the absence of a direct sum decomposition.

Proposition 2.3. *If Φ has a cohomological local splitting near 0 in dimension k, then 0 is a critical point of Φ. Moreover, if 0 is an isolated critical point of Φ, then we have $C^k(\Phi, 0) \neq 0$.*

This proposition is a variant of a result of Perera, Agarwal, and O'Regan [20]. We need the following lemma from Degiovanni and Lancelotti (see [9, Theorem 2.7] and also Cingolani and Degiovanni [4, Theorem 3.6]), which establishes a connection between equivariant index and nonequivariant cohomology.

Lemma 2.4. *If X is a symmetric subset of $W \setminus \{0\}$ with $k = i(X) < \infty$ and A is a symmetric subset of X with $i(A) = k$, then the homomorphism $i^* : H^k(W, X) \to H^k(W, A)$, induced by the inclusion $i : (W, A) \subseteq (W, X)$, is nontrivial.*

Proof of Proposition 2.3. It is enough to prove that, if 0 does not accumulate critical points of Φ, then $C^k(\Phi, 0) \neq 0$. Therefore assume, without loss of generality, that Φ has no critical point u with $0 < \|u\| \leq \rho$. Let $c = \Phi(0)$.

There exists a deformation $\eta : W \times [0,1] \longrightarrow W$ such that

$$\begin{array}{ll} \Phi(\eta(u,t)) < \Phi(u) & \text{if } \Phi'(u) \neq 0 \text{ and } t > 0\,, \\ \eta(u,t) = u & \text{otherwise}\,, \end{array}$$

(see e.g. Benci [2, Theorem 5.5] or Corvellec [6]). Let $0 < r \leq \rho$ be such that $\eta(B_r \times [0, r]) \subseteq B_\rho$. Since $B_r \cap W_-$ is contractible and $S_r \cap W_-$ is a deformation retract of $W_- \setminus \{0\}$, from Lemma 2.4 and (2.1) we deduce that the homomorphism

$$i^* : H^k(W, B_\rho \setminus W_+) \longrightarrow H^k(B_r \cap W_-, S_r \cap W_-)\,,$$

induced by the inclusion $i : (B_r \cap W_-, S_r \cap W_-) \subseteq (W, B_\rho \setminus W_+)$, is nontrivial. On the other hand, since (2.2) implies

$$B_r \cap W_- \subseteq \Phi^c \cap B_r\,,\ S_r \cap W_- \subseteq \Phi^c \cap B_r \setminus \{0\}\,,\ \eta\left(\Phi^c \cap B_r \setminus \{0\}, r\right) \subseteq B_\rho \setminus W_+\,,$$

we may also consider the composition

$$H^k(W, B_\rho \setminus W_+) \xrightarrow{\eta(\cdot,r)^*} H^k(\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\}) \xrightarrow{j^*} H^k(B_r \cap W_-, S_r \cap W_-)$$

where $j : (B_r \cap W_-, S_r \cap W_-) \subseteq (\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\})$ is the inclusion. Again (2.2) yields

$$\eta\left((S_r \cap W_-) \times [0, r]\right) \subseteq B_\rho \setminus W_+\,,$$

so that $\eta(\cdot, r) \circ j$ is homotopic to i. Therefore $j^* \circ \eta(\cdot, r)^* = i^*$ is nontrivial, which in turn implies that $H^k(\Phi^c \cap B_r, \Phi^c \cap B_r \setminus \{0\}) \neq 0$. $\square$

Now, let us recall a situation in which one can build two symmetric cones satisfying (2.1). Let Ω be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$ and let

$$\mathcal{V}(\Omega) := \begin{cases} \displaystyle\bigcup_{r>n/p} L^r(\Omega) & \text{if } p \leq n, \\ L^1(\Omega) & \text{if } p > n. \end{cases}$$

Take $V \in \mathcal{V}(\Omega)$ and consider the eigenvalue problem

$$\begin{cases} -\Delta_p \, u = \lambda \, V \, |u|^{p-2} \, u & \text{in } \Omega \, , \\ \qquad u = 0 & \text{on } \partial\Omega \, . \end{cases} \tag{2.3}$$

We refer the reader to Cuesta [8] and Szulkin and Willem [23] for general properties concerning (2.3).

Now, assume that $\big\{x \in \Omega : \, V(x) > 0\big\}$ has positive measure, denote by $\mathcal{F}$ the class of symmetric subsets of

$$\mathcal{M} = \left\{ u \in W^{1,p}_0(\Omega) : \, \int_\Omega V \, |u|^p \, dx = 1 \right\}$$

and set

$$\lambda_k = \inf_{\substack{M \in \mathcal{F} \\ i(M) \geq k}} \sup_{u \in M} \int_\Omega |\nabla u|^p \, dx, \quad k \geq 1 \, . \tag{2.4}$$

Then $\lambda_k \nearrow +\infty$ are eigenvalues of (2.3) and the following result holds (see Degiovanni and Lancelotti [9, Theorem 3.2]).

Proposition 2.5. *Let $k \geq 1$ be such that $\lambda_k < \lambda_{k+1}$ and let*

$$W_- = \left\{ u \in W^{1,p}_0(\Omega) : \, \int_\Omega |\nabla u|^p \, dx \leq \lambda_k \int_\Omega V \, |u|^p \, dx \right\} ,$$
$$W_+ = \left\{ u \in W^{1,p}_0(\Omega) : \, \int_\Omega |\nabla u|^p \, dx \geq \lambda_{k+1} \int_\Omega V \, |u|^p \, dx \right\} .$$

Then W_-, W_+ are two symmetric cones in $W^{1,p}_0(\Omega)$ satisfying (2.1).

3 The main result

Let Ω be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$, let $V \in \mathcal{V}(\Omega)$ and let $g : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying the following assumptions:

$(g1')$ we have that

for every $\varepsilon > 0$ there exists $a_\varepsilon \in \mathcal{V}(\Omega)$ such that

$$|g(x,s)| \leq a_\varepsilon(x)\,|s|^{p-1} + \varepsilon\,|s|^{p^*-1}, \qquad \text{if } p < n;$$

there exist $a \in \mathcal{V}(\Omega)$, $C > 0$ and $q > p$ such that

$$|g(x,s)| \leq a(x)|s|^{p-1} + C|s|^{q-1}, \qquad \text{if } p = n;$$

for every $S > 0$ there exists $a_S \in \mathcal{V}(\Omega)$ such that

$$|g(x,s)| \leq a_S(x)|s|^{p-1} \text{ whenever } |s| \leq S, \qquad \text{if } p > n;$$

$(g2')$ for a.e. $x \in \Omega$, we have $\displaystyle\lim_{s\to 0} \frac{G(x,s)}{|s|^p} = 0$ and $\displaystyle\lim_{|s|\to\infty} \frac{G(x,s)}{|s|^p} = +\infty$, where $\displaystyle G(x,s) = \int_0^s g(x,t)\,dt$;

$(g3')$ there exist $\mu > p$, $\gamma_0 \in L^1(\Omega)$ and $\gamma_1 \in \mathcal{V}(\Omega)$ such that

$$-\gamma_0(x) - \gamma_1(x)|s|^p \leq \mu G(x,s) \leq s g(x,s) + \gamma_0(x) + \gamma_1(x)|s|^p$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

In order to study the quasilinear problem

$$\begin{cases} -\Delta_p u = \lambda V |u|^{p-2} u + g(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.1}$$

let us define a functional $\Phi : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ of class C^1 by

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega V|u|^p\,dx - \int_\Omega G(x,u)\,dx$$

and set $\|u\| = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$ for every $u \in W_0^{1,p}(\Omega)$. Recall also that, for every $\gamma \in \mathcal{V}(\Omega)$, the map $\{u \mapsto \gamma|u|^p\}$ is weak-to-strong sequentially continuous from $W_0^{1,p}(\Omega)$ into $L^1(\Omega)$.

Lemma 3.1. *The following facts hold:*

(*a*) *for every $c \in \mathbb{R}$, we have*

$$\limsup_{\substack{\|u\|\to\infty \\ \Phi(u)\leq c}} \frac{\Phi'(u)u - p\,\Phi(u)}{\|u\|^p} < 0\,;$$

(*b*) *for every $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, we have*

$$\lim_{|t|\to\infty} \frac{\Phi(tu)}{|t|^p} = -\infty\,.$$

Proof. (*a*) Let $c \in \mathbb{R}$. By contradiction, let $d_k \to 0$ and let (u_k) be a sequence in Φ^c such that $\|u_k\| \to \infty$ and

$$\Phi'(u_k)u_k - p\,\Phi(u_k) \geq -d_k\|u_k\|^p \qquad \text{for every } k \in \mathbb{N}\,.$$

If we set $v_k = u_k/\|u_k\|$, up to a subsequence (v_k) is convergent to some $v \in W_0^{1,p}(\Omega)$ weakly and a.e. in Ω.

From $(g3')$ it follows that

$$\begin{aligned}
&-d_k\|u_k\|^p \leq \Phi'(u_k)u_k - p\,\Phi(u_k) = \int_\Omega \left(pG(x,u_k) - u_k g(x,u_k)\right)\,dx \\
&= \int_\Omega \left(\mu G(x,u_k) - u_k g(x,u_k)\right)\,dx - (\mu - p)\int_\Omega G(x,u_k)\,dx \\
&\qquad\qquad\qquad \leq \int_\Omega \left(\gamma_0 + \gamma_1|u_k|^p\right)\,dx - (\mu - p)\int_\Omega G(x,u_k)\,dx\,,
\end{aligned}$$

whence

$$(\mu - p)\int_\Omega G(x,u_k)\,dx \leq d_k\|u_k\|^p + \int_\Omega \left(\gamma_0 + \gamma_1|u_k|^p\right)\,dx\,. \tag{3.2}$$

Therefore we have

$$\limsup_k \frac{\int_\Omega G(x,u_k)\,dx}{\|u_k\|^p} < +\infty\,. \tag{3.3}$$

On the other hand, $(g3')$ also yields

$$\frac{G(x,u_k)}{\|u_k\|^p} + \frac{\gamma_0}{\|u_k\|^p} \geq -\gamma_1|v_k|^p\,,$$

hence, by the (generalized) Fatou lemma and (3.3),

$$\int_\Omega \left(\liminf_k \frac{G(x,u_k)}{\|u_k\|^p}\right) dx \le \liminf_k \frac{\int_\Omega G(x,u_k)\,dx}{\|u_k\|^p} < +\infty\,.$$

Since by $(g2')$ we have

$$\lim_k \frac{G(x,u_k)}{\|u_k\|^p} = \lim_k \left(\frac{G(x,u_k)}{|u_k|^p}\,|v_k|^p\right) = +\infty \qquad \text{where } v \neq 0\,,$$

we deduce that $v = 0$ a.e. in Ω.

Formula (3.2) can also be rewritten as

$$\left(\frac{\mu}{p}-1\right)\int_\Omega |\nabla u_k|^p\,dx \le (\mu-p)\Phi(u_k) + d_k\|u_k\|^p + \int_\Omega \gamma_0\,dx + \int_\Omega \left[\left(\frac{\mu}{p}-1\right)V + \gamma_1\right]|u_k|^p\,dx\,,$$

namely

$$\left(\frac{\mu}{p}-1\right) \le d_k + (\mu-p)\frac{\Phi(u_k)}{\|u_k\|^p} + \frac{\int_\Omega \gamma_0\,dx}{\|u_k\|^p} + \int_\Omega \left[\left(\frac{\mu}{p}-1\right)V + \gamma_1\right]|v_k|^p\,dx\,.$$

Going to the limit as $k \to \infty$, we get $\frac{\mu}{p} - 1 \le 0$ and a contradiction follows.

(b) Since by $(g3')$

$$\frac{G(x,tu)}{|t|^p} + \frac{\gamma_0}{|t|^p} \ge -\gamma_1|u|^p\,,$$

applying as before Fatou's lemma, the assertion follows. □

Lemma 3.2. *There exists $a < 0$ such that Φ^a is contractible in itself.*

Proof. By (a) of Lemma 3.1, there exists $b \in \mathbb{R}$ such that

$$\Phi'(u)u - p\,\Phi(u) \le b \qquad \text{for every } u \in \Phi^0\,.$$

In particular, there exists $a < 0$ such that

$$\Phi'(u)u < 0 \qquad \text{for every } u \in \Phi^a\,. \tag{3.4}$$

If we set, taking into account (b) of Lemma 3.1,

$$t(u) = \min\{t \geq 1 : \ \Phi(tu) \leq a\} \ ,$$

from (3.4) we deduce that the function $\{u \mapsto t(u)\}$ is continuous. Then $r(u) = t(u)u$ is a retraction of $W^{1,p}_0(\Omega) \setminus \{0\}$ onto Φ^a. Since $W^{1,p}_0(\Omega) \setminus \{0\}$ is contractible in itself, the same is true for Φ^a. □

Lemma 3.3. *Assume that* 0 *is an isolated critical point of* Φ. *Then the following facts hold:*

(a) *if there exists* $\delta > 0$ *such that* $G(x,s) \geq 0$ *for a.e.* $x \in \Omega$ *and every* $s \in \mathbb{R}$ *with* $|s| \leq \delta$, *we have* $C^q(\Phi, 0) \neq 0$ *for*

$$q = i\left(\left\{u \in W^{1,p}_0(\Omega) \setminus \{0\} : \ \int_\Omega |\nabla u|^p\,dx \leq \lambda \int_\Omega V\,|u|^p\,dx\right\}\right) \ ;$$

(b) *if there exists* $\delta > 0$ *such that* $G(x,s) \leq 0$ *for a.e.* $x \in \Omega$ *and every* $s \in \mathbb{R}$ *with* $|s| \leq \delta$, *we have* $C^q(\Phi, 0) \neq 0$ *for*

$$q = i\left(\left\{u \in W^{1,p}_0(\Omega) : \ \int_\Omega |\nabla u|^p\,dx < \lambda \int_\Omega V\,|u|^p\,dx\right\}\right) \ .$$

Proof. By replacing (λ, V) with $(-\lambda, -V)$ if necessary, we may assume that $\lambda \geq 0$. Let $\vartheta : \mathbb{R} \longrightarrow [0,1]$ be a C^∞-function such that $\vartheta(s) = 0$ for $|s| \leq \delta/2$ and $\vartheta(s) = 1$ for $|s| \geq \delta$. For every $t \in [0,1]$, define

$$G_t(x,s) = G(x, (1 - t\vartheta(s))s)$$

and $\Phi_t : W^{1,p}_0(\Omega) \longrightarrow \mathbb{R}$ by

$$\Phi_t(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx - \frac{\lambda}{p}\int_\Omega V|u|^p\,dx - \int_\Omega G_t(x,u)\,dx \,.$$

From $(g1')$ it follows that each Φ_t satisfies the Palais-Smale condition over every bounded subset of $W^{1,p}_0(\Omega)$ (see also [9, Proposition 4.3]). Moreover the map $\{t \mapsto \Phi_t\}$ is continuous from $[0,1]$ into $C^1(B)$ for every bounded subset B of $W^{1,p}_0(\Omega)$. We claim that there exists $\rho > 0$ such that each Φ_t has no critical point in B_ρ other than 0. By contradiction, let (t_j) be a sequence in $[0,1]$ and (u_j) a sequence convergent to 0 with $\Phi'_{t_j}(u_j) = 0$

and $u_j \neq 0$. Then the same regularity argument of Guedda and Veron [13, Propositions 1.2 and 1.3] shows that (u_j) is convergent to 0 also in $L^\infty(\Omega)$. Therefore we have $\Phi'(u_j) = 0$ eventually as $j \to \infty$. Since 0 is an isolated critical point of Φ, a contradiction follows. From Theorem 2.2 we deduce that $C^q(\Phi, 0) \approx C^q(\Phi_1, 0)$ for any $q \geq 0$.

Observe also that

$$\int_\Omega G_1(x,u)\,dx = \mathrm{o}(\|u\|^p) \;\text{ as }\; \|u\| \to 0 \tag{3.5}$$

(see, e.g., [9, Proposition 4.3]).

In case (a), we have $G_1(x,s) \geq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. If the set $\big\{x \in \Omega : V(x) > 0\big\}$ has positive measure and $\lambda \geq \lambda_1$, where (λ_k) is the sequence defined in (2.4), take $k \geq 1$ such that $\lambda_k \leq \lambda < \lambda_{k+1}$ and define W_-, W_+ as in Proposition 2.5. Otherwise, let $W_- = \{0\}$ and $W_+ = W_0^{1,p}(\Omega)$. In any case, W_-, W_+ are two symmetric cones in $W_0^{1,p}(\Omega)$ satisfying (2.1) and

$$\begin{aligned} &i(W_- \setminus \{0\}) \\ &\quad = i\left(\left\{u \in W_0^{1,p}(\Omega) \setminus \{0\} :\ \int_\Omega |\nabla u|^p\,dx \leq \lambda \int_\Omega V\,|u|^p\,dx\right\}\right)\,, \end{aligned} \tag{3.6}$$

$$\int_\Omega |\nabla u|^p\,dx \leq \lambda \int_\Omega V|u|^p\,dx \qquad \forall u \in W_-\,, \tag{3.7}$$

$$\exists \sigma \in]0,1[:\ (1-\sigma)\int_\Omega |\nabla u|^p\,dx \geq \lambda \int_\Omega V|u|^p\,dx \qquad \forall u \in W_+\,. \tag{3.8}$$

From (3.7) and the sign information on G_1, it follows

$$\Phi_1(u) \leq 0 \qquad \text{for every } u \in W_- \text{ with } \|u\| \leq \rho \tag{3.9}$$

for any $\rho > 0$. On the other hand, combining (3.5) and (3.8), we get

$$\Phi_1(u) \geq 0 \qquad \text{for every } u \in W_+ \text{ with } \|u\| \leq \rho \tag{3.10}$$

provided that ρ is sufficiently small. Therefore also (2.2) is satisfied and the assertion follows from Proposition 2.3 and (3.6).

In case (b), we have $G_1(x,s) \leq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. If the set $\big\{x \in \Omega : V(x) > 0\big\}$ has positive measure and $\lambda > \lambda_1$, take now

$k \geq 1$ such that $\lambda_k < \lambda \leq \lambda_{k+1}$ and define W_-, W_+ as in Proposition 2.5. Otherwise, let $W_- = \{0\}$ and $W_+ = W_0^{1,p}(\Omega)$. In any case, W_-, W_+ are two symmetric cones in $W_0^{1,p}(\Omega)$ satisfying (2.1) and

$$\begin{aligned} & i(W_0^{1,p}(\Omega) \setminus W_+) \\ & \qquad = i\left(\left\{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p \, dx < \lambda \int_\Omega V\, |u|^p \, dx\right\}\right) , \end{aligned} \tag{3.11}$$

$$\exists \sigma \in]0,1[: \ (1+\sigma) \int_\Omega |\nabla u|^p \, dx \leq \lambda \int_\Omega V |u|^p \, dx \qquad \forall u \in W_- \,, \tag{3.12}$$

$$\int_\Omega |\nabla u|^p \, dx \geq \lambda \int_\Omega V |u|^p \, dx \qquad \forall u \in W_+ \,. \tag{3.13}$$

Combining (3.5) with (3.12), we get again (3.9) if ρ is sufficiently small. On the other hand, from (3.13) and the sign information on G_1 we deduce (3.10) for any $\rho > 0$. Then (2.2) is satisfied and the assertion follows from Proposition 2.3 and (3.11). □

Now we can prove the main result of the section.

Theorem 3.4. *Let us suppose that assumptions $(g1') - (g3')$ hold and let $V \in \mathcal{V}(\Omega)$. Then, for every $\lambda \in \mathbb{R}$, problem* (3.1) *has a nontrivial solution $u \in W_0^{1,p}(\Omega)$ in each of the following cases:*

(a) *there exists $\delta > 0$ such that $G(x,s) \geq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq \delta$,*

(b) *there exists $\delta > 0$ such that $G(x,s) \leq 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq \delta$.*

Proof. A standard argument shows that Φ satisfies the Palais-Smale compactness condition (see, e.g., [9, Proposition 4.3]).

Suppose, for a contradiction, that the origin is the only critical point of Φ. By Lemma 3.2 there exists $a < 0$ such that Φ^a is contractible in itself. On the other hand, by the second deformation lemma (see e.g. Chang [3] or Mawhin and Willem [17]), Φ^0 is a deformation retract of W and Φ^a is a deformation retract of $\Phi^0 \setminus \{0\}$, so

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}) \approx H^q(W, \Phi^a) = 0 \qquad \text{for every } q \geq 0 \,.$$

By Lemma 3.3 a contradiction follows. □

For the sake of completeness, let us state a simple extension of a result of Perera [19], which can be proved by the same argument.

Theorem 3.5. *Let us suppose that assumptions* $(g1') - (g3')$ *hold, let* $V \in \mathcal{V}(\Omega)$ *and let* $\lambda \geq 0$. *If the set* $\{x \in \Omega : V(x) > 0\}$ *has positive measure, assume also that* $\lambda \notin \{\lambda_k : k \geq 1\}$, *where* (λ_k) *is the sequence defined in* (2.4).

Then problem (3.1) *has a nontrivial solution* $u \in W_0^{1,p}(\Omega)$.

Thus, the extra assumption on λ is compensated by the fact that there is no sign condition on G. Observe that the union of Theorems 3.4 and 3.5 provides a complete extension to the case $p \neq 2$ of S.J. Li and Willem [15, Theorem 4].

4 Proof of Theorem 1.1

Since $V \in L^\infty(\Omega)$, we have $V \in \mathcal{V}(\Omega)$. It is also standard that assumptions $(g1) - (g3)$ imply $(g1') - (g3')$ (see e.g. Degiovanni and Lancelotti [9]). Then the assertion follows from Theorem 3.4.

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