Cumulative distribution of the stretching of vortical structures in isotropic turbulence.

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Abstract. By using a Navier-Stokes isotropic turbulent field numerically simulated in the box with a discretization of 1024^3 [Biferale *et al.* (2005)], we show that the probability of having a stretching-tilting larger than twice the local enstrophy is very small. This probability decreases if we try to filter out the large scales, while it increases filtering out the small scales. This is basically due to the suppression of the compact structures (blobs).

1. Introduction

Many aspects of the behaviour of turbulent fields, as the onset of instability, vorticity intensification or damping, the three-dimensionalization of the flow field, have been associated to the formation of spatial and temporal internal scales in part associated to the stretching and tilting of vortical structures [Monin & Yaglom (1971); Tennekes & Lumley (1972); Pope (2000)]. The energy cascade to smaller scales in the standard picture of turbulence is interpreted in terms of the stretching of vortices due to the interaction with similar eddy sizes [Frisch (1995)]. In this study, we consider statistics related to the intensity of the stretching-tilting of vortical filaments, sheets and blobs with reference to their vorticity magnitude in steady forced isotropic turbulence at $Re_\lambda = 280$ using data from [Biferale *et al.* (2005)].

2. Local measure of three-dimensional inner scale formation process

With reference to the phenomena described by the inertial nonlinear nonconvective part of the vorticity transport equation, let us introduce a local measure of the process of three-dimensional inner scales formation

$$f(\mathbf{x}) = \frac{|\boldsymbol{\omega} \cdot \nabla \mathbf{u}|}{|\boldsymbol{\omega}|^2}(\mathbf{x}) \tag{1}$$

where $\mathbf{u}$ is the fluctuating velocity field and $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ is the vorticity vector.

The numerator, $|\boldsymbol{\omega} \cdot \nabla \mathbf{u}|$, is the so called stretching-tilting term of the vorticity equation, and is zero in two-dimensional flows. In three-dimensional fields, it is responsible for the transfer of the kinetic energy from larger to smaller scales (positive or extensional stretching) and vice-versa (negative or compressional stretching) and for the three-dimensionality of the vorticity field. In equation (1) it is normalized by the magnitude of the vorticity, which, leaving aside a factor

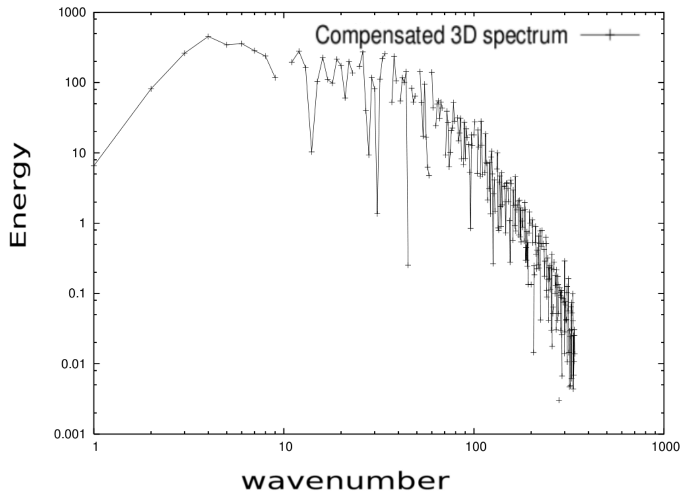

Figure 1. Compensated three-dimensional energy spectrum. The three-dimensional spectrum has been obtained from the computed one-dimensional spectrum by using the isotropy of the flow, see [Monin & Yaglom (1971), vol.2.]

1/2, is usually referred to as enstrophy, the only invariant of the rate-of-rotation tensor different from zero.

In order to look for the typical range of values of $f(x)$ and to relate them to the behaviour of the various turbulence scales present in an isotropic field, we have evaluated the function f over a fully resolved homogeneous and isotropic incompressible turbulence . The database consists of 1024^3 resolution grid point Direct Numerical Simulation (DNS) of an isotropic Navier-Stokes forced field, at Reynolds $Re_\lambda = 280$ [Biferale *et al.* (2005)]. All instants in the simulation are statistically equivalent and provide a statistical set of a slightly more than 10^9 elements. We considered the statistics that were obtained averaging over the full domain in one instant. The instantaneous effects of the low wavenumber forcing introduces a turbulent kinetic energy inhomogeneity of about 20% when we average in parallel planes. The field has been slightly modified in order to filter out this effects. As this bias was generated by the energy supply at the large scale range, the two lowest wavenumbers have been damped out. The resolved part of the energy spectrum extends up to $\kappa \sim 330$. The inertial range extends from $\kappa \sim 10$ to $\kappa \sim 70$, see the compensated version of the three-dimensional spectrum in figure 1. The higher wave-numbers, which are affected by the aliasing error, are not shown.

The range of values attained by $f(x)$ is wide but only, at a few spatial points, values as high as a few hundreds were observed. In order to read the typical values of $f(x)$, we study its *survival function.* By denoting $F(s) = P(f(x) \leq s)$ the cumulative distribution function (referred as cdf in the following) of $f(x)$, the survival function is defined as the complement to 1 of the cdf, that is

$$S(s) = P(f(x) > s) = 1 - F(s). \tag{2}$$

For each value of the threshold s, $S(s)$ describes the *probability that $f(x)$ takes values larger than s. It has been found that, when $f(x)$ is evaluated on the reference homogeneous and isotropic turbulent field, the probability that $f(x) > 2$ is almost zero (see figure 2). Thus, $f(x) = 2$ can be considered the maximum statistical value that $f(x)$ can reach when the turbulence is simulated with a fine grain.*

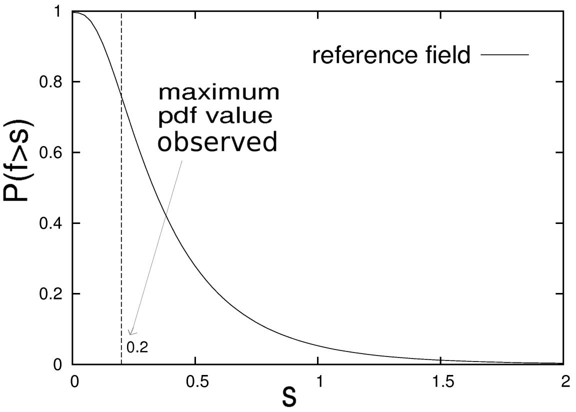

Figure 2. Survival probability of the normalized stretching-tilting function in a fully resolved isotropic 3D turbulent field ($P(f(x)) \geq s$). Unfiltered velocity field.

3. Properties of the survival function of the normalized stretching-tilting: analysis on filtered fields

The application of filters to the velocity field, carried out in the wavenumber space by means of suitable convolutions, allows to analyse the behaviour of the function $f(x)$ in the different scale ranges of the turbulence. This analysis is mainly performed by using two kinds of filters, a high pass filter and a band-cut filter. The first filter is essentially a cut-off filter, which we refer to as *"cross" filter* and which allows the contribution of the structures that are characterized by at least one *large dimension* to be removed. In the Fourier space, this means that we are filtering out all structures whose wavenumber vector has at least one small component. A graphical scheme of the filtering in the wave number plane k_1, k_2 is provided in figure 3 (i), where the blue coloured bands represent the regions of the wavenumber space which are filtered out.We are thus using a sort of high-pass filter, which affects all wavenumbers that, along any possible direction, have at least one component under a certain threshold. Given the threshold k_{MAX}, the filter reduces the contribution of the modes with wave number components

$$k_1 < k_{MAX} \;\text{ or }\; k_2 < k_{MAX} \;\text{ or } k_3 < k_{MAX}.$$

The representation of this high-pass filter, g_{hp}, in Fourier space can be given by the following function [Tordella & Iovieno (2006)]:

$$g_{hp}(\underline{k}) \;=\; \prod_i \phi(k_i),\; \phi(k_i) = \frac{1}{1 + e^{-(k_i - k_{MAX})}}. \tag{3}$$

Since function g_{hp} filters any wavenumber that has at least one component lower than the threshold k_{MAX}, it reduces the kinetic energy of the filamentous (one component lower than k_{MAX}), layered (two components lower than k_{MAX}) and blobby (three components lower than k_{MAX}) structures. This filter is efficient in reducing the integral scale of the turbulence [Tordella & Iovieno (2006)].

The second filter can be obtained by reducing the contribution of a variable band (see figure 3 the part in red (ii))

$$k_{MIN} < k_1 < k_{MAX} \quad \text{or} \quad k_{MIN} < k_2 < k_{MAX} \;\text{or}$$
$$k_{MIN} < k_3 < k_{MAX}.$$

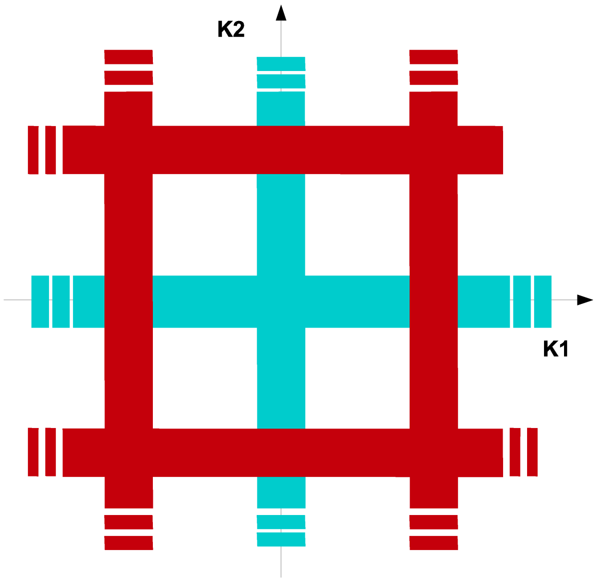

Figure 3. Scheme of the filter *cross* that can be used as i) a high-pass filter: the wave-numbers under a certain threshold are partially removed, see equation (3) (region in blue in the k_1, k_2 wavenumber plane), ii) band-cut filter : the wave-numbers inside a range, or above a certain threshold, are cut (region in red).

The visualizations in figure 4 show the effect of the $0-20$ high pass filter and $30-150$ band cut filter on $f(x)$. If the high pass filter is used, the value of the function f is reduced (bottom left panel), while, if the band-cut filter is used, the function grows up (bottom right panel). In fact, it is possible to observe that if the high pass filter is applied the values inside $[0.8, 2]$ are less probable than in the unfiltered field (the frequency of the red spots is lower in the entire domain). In contrast, if a 30-150 filter is used, the values $[0.8, 2]$ are more dense. So, the visualization allows to verify that the high-pass filter has the effect of *decreasing* the values attained by $f(x)$ in the whole domain. From the figure we can see that the distribution of values of the two filtered fields is uniform over the domain. This suggests that the cross filter here used does not spoil the self-similarity of the field.

The reduction of the survival function S grows up as the threshold K_{MAX} of the high pass filter increases (see figure 5). In fact by varying the value of the threshold, K_{MAX}, it is possible to consider the removal of different scale ranges. The effects of the filtering out of the low wavenumbers, in the ranges $0-10, 0-20, 0-40$, are compared in figure 5. The first filtering affects the energy-containing range, while the other two also include a part of the inertial range.

A different behaviour can be expected when we try to filter smaller scales, by applying the band-cut filter to the inertial range of scales. The $10-40$, $40-70$, $70-100$, $100-130$, $30-150$ (intermediate-inertial/small scale filtering), $150-330$ (dissipative) ranges considered are compared in figure 6. All the filtered ranges qualitatively induce the same effect, a slight *increase* in the survival probability. The most effective result is obtained by filtering over the whole inertial range, $30 < \kappa < 150$ for small values, i.e. $0.5 < s < 2$. An increasing of about 60% is observed for $s = 1$ and of more than 100% for $s = 1.5$ This highlights the fact that the structures of the inertial range contribute more to the intensity of the vorticity field than to stretching and tilting. The general trend is almost inverted with respect to the case of the high pass filtered turbulence (compare the 0-40 and 10-40 results in figures 4 and 5, respectively) and this can be confirmed, with slight differences, as long as we enlarge the amplitude of the filtering band to

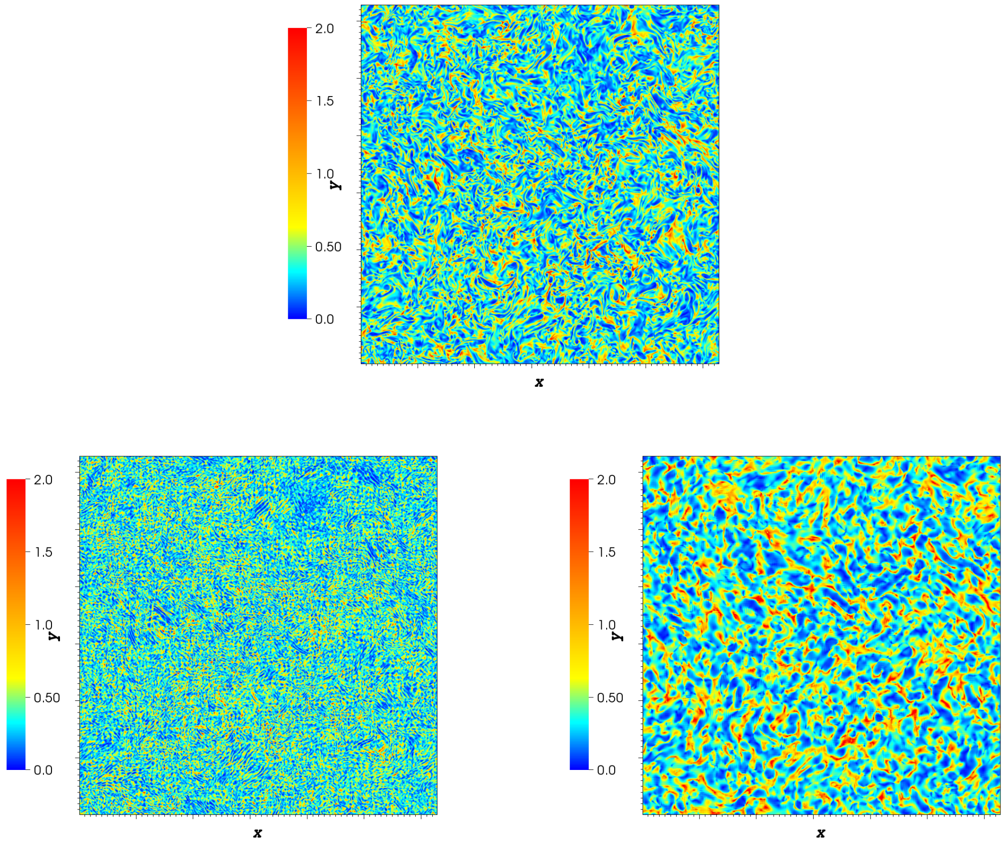

Figure 4. Visualization of the values of function f, see (1), in a plane (a two-dimensional section of the cubic domain, 1024^3 grid points). Top: reference field. Bottom left: The wave number range 0-20 is filtered out. Bottom right: The wave number range 30-150 is filtered out.

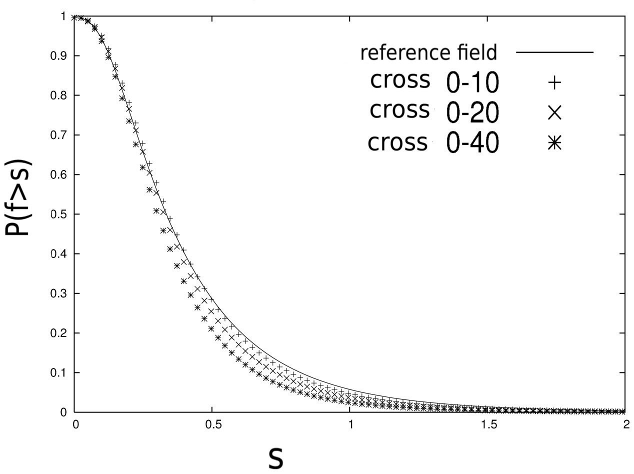

Figure 5. Survival probability of the normalized stretching-tilting function in a high pass filtered isotropic turbulent field.

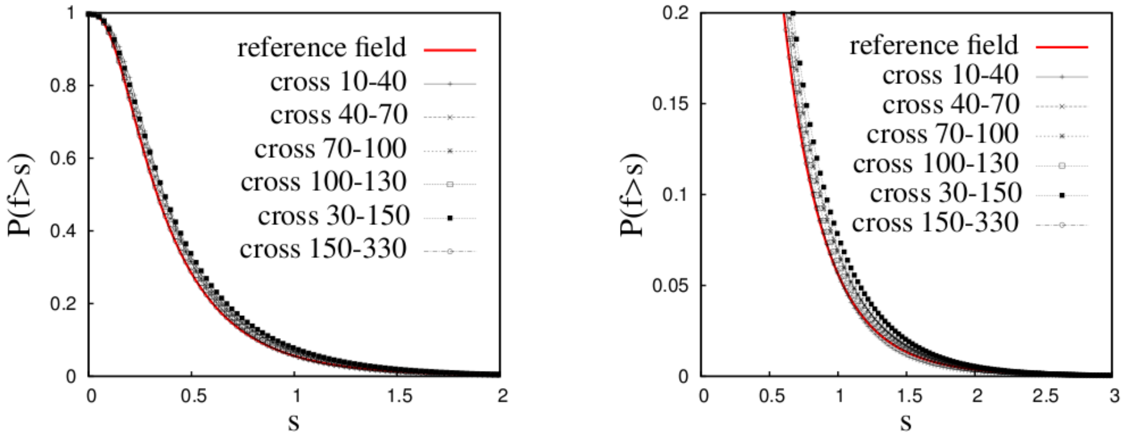

Figure 6. Probability that the normalized stretching-tilting function in a band cut filtered (inertial range) isotropic turbulent field is higher than a given threshold s. Right: enlarged view.

get closer to the dissipative range. Moving toward the dissipative range ($150 < \kappa < 330$), the band-cut filter becomes a sort of low-pass filter. By filtering these wave numbers, the obtained effect is minimum, although we have removed the contribution of the highest 200 wave-numbers (see figure 6).

4. Conclusions

From the statistical information collected in homogeneous and isotropic turbulence on the function $f(x)$, the stretching term normalized over the enstrophy, we draw two observations. First, there is an almost zero probability of having a stretching/tilting of intensity larger than twice the square of the vorticity magnitude. Second, when compact structures (blobs) in the inertial range are filtered out, the probability of having f higher than a given threshold, s, increases by 20% at $s = 0.5$, and by 60-70% at $s = 1.0$. If larger blobs are instead filtered, an opposite situation occurs. The present observations must be associated to the non discriminating effect of filtering on filaments and sheets, which is due to their specific nature that cannot be reconciled inside either a category of small or large scales.

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