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Superstrings on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ from supergravity

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We derive from a general formulation of pure spinor string theory on type IIA backgrounds the specific form of the action for the $\mathrm{AdS}_4 \times \mathbb{C}P^3$ background. We provide a complete geometrical characterization of the structure of the superfields involved in the action.

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I. INTRODUCTION

The recent developments on the duality between $\mathcal{N} = 6$ superconformal Chern-Simons theory in three dimensions and superstrings moving on $\mathrm{AdS}_4 \times \mathbb{P}^3$ [1–9] have prompted the study of superstrings on $\mathrm{Osp}(\mathcal{N}|4)$ backgrounds [10–13]. The main issue is of course the integrability of the system and this has been already studied in a series of papers [14–25]. On the other side, one would like also to consider the string theory in a framework where all symmetries are manifest and which takes the RR fields of the background properly into account. In [13], the limit for large RR fields is analyzed and it has been shown the relation with a topological model on the Grassmannian $\mathrm{Osp}(6|4)/\mathrm{SO}(6) \times \mathrm{Sp}(4)$. The exactness of the background is also discussed in [13].

The pure spinor formalism is well suited to the present situation and in a previous paper [12] two of the present authors provided the pure spinor version of the $\mathrm{AdS}_4 \times \mathbb{P}^3$ sigma model, described as the coset space $\mathrm{Osp}(6|4)/\mathrm{SO}(1,3) \times \mathrm{U}(3)$. Furthermore, the four authors published another paper [26] where a systematic study of pure spinor superstring on type IIA backgrounds has been completely performed. This analysis has been based on the previous studies by Berkovits and Howe [27], by Oda and Tonin [28] and on the geometric (a.k.a. rheonomic) formulation of supergravity [29]. There it has been shown how to derive from the geometrical formulation of supergravity (in type IIA case) the pure spinor sigma model and the relative pure spinor constraints [30,31]. It has been proved that the action is BRST invariant and, only in the case of type IIA, has a peculiar structure since it can be written in terms of four pieces which are the Green-Schwarz action, a Q-exact piece, a $\bar{Q}$-exact piece and a $Q\bar{Q}$-exact piece. This allows us to derive the complete expression of the sigma model where all superfields are made explicit. One of the advantages of the geometrical formulation of supergravity is that it provides a superspace framework where all bosonic fields are extended to be superfields and the rheonomic conditions ensure the integrability of the extension, leading to the correct field content. The advantage stays in the fact that one can very easily read off the sigma model action in terms of the background solution. As an example, here we derive of the pure spinor sigma model for the $\mathrm{AdS}_4 \times \mathbb{P}^3$ background.

In this case we have to take into account the RR field strengths $\mathbf{G}^{[2]}$ and $\mathbf{G}^{[4]}$ which are, respectively, proportional to the Kähler form on $\mathbb{P}^3$ and to the Levi-Civita invariant tensor in AdS_4. This background has 24 Killing spinors parametrized by the combinations $\chi_x \otimes \eta^A$ where χ_x are the Killing spinors of AdS_4 and η^A are the 6 Killing spinors of $\mathbb{P}^3$. Therefore, it is convenient to use a superspace with 24 fermionic coordinates. Now, the problem is whether this superspace is sufficient to provide a complete description of the supergravity states and, whether the vertex operators constructed in terms of this superspace describe on-shell $\mathrm{AdS}_4 \times \mathbb{P}^3$-supergravity fluctuations. It is established that all supergravity models with more than 16 supercharges are described by an on-shell superspace, since an auxiliary-field formulation does not exist, and therefore we expect that the 24-extended superspace is sufficient for the present formulation. There is also another aspect to be noticed: the formulation of GS superstrings on the same coset has been studied extensively in [10] and it has been argued that 24 fermions are indeed sufficient to formulate the model. Indeed, κ-symmetry removes exactly 8 fermions leading to a supersymmetric model. In our case, κ-symmetry is replaced by BRST symmetry plus pure spinor constraints, so that we have to check whether the pure spinors satisfying the new constraints [30] cancel the central charge. In fact, we will see that by reducing the spinor space from 32 dimensions to the 24 dimensions adapted to the present background, there exists a solution of the pure spinor constraints with only 14 degrees of freedom, matching the bosonic and fermionic degrees of freedom.

In addition, by means of the formalism constructed in [26], we provide and explicit expression for the sigma model where all couplings are exhibited. We devote a particular attention on the quartic part of the action for the ghosts.

The paper is organized as follows. In Sec. II we review the description of Type IIA supergravity in terms of its Free Differential Algebra (FDA) in the string frame and the corresponding rheonomic parametrization. In Sec. III we describe the compactification of type IIA on $\mathrm{AdS}_4 \times \mathbb{P}^3$. In Sec. IV we introduce the pure spinors of OSp(6|4). Finally in Sec. V we give the complete pure spinor superstring action on $\mathrm{AdS}_4 \times \mathbb{P}^3$. The reader is referred to the appendices for a definition of the $D = 4$ and $D = 6$ spinor conventions and for some useful formulas.

II. SUMMARY OF TYPE IIA SUPERGRAVITY AND OF ITS FDA

In order to pursue our program we have to consider the structure of the Free Differential Algebra of type IIA supergravity, the rheonomic parametrization of its curvatures and the corresponding field equations that are the integrability conditions of such rheonomic parametrizations. All these necessary ingredients were recently determined in [26]. In this section, we summarize those results collecting all the items for our subsequent discussion.

A. Definition of the curvatures

The p-forms entering the FDA of the type IIA theory are listed below:

Form	degree p	f(ermion)/b(oson)	Name	String Sector	Curvature
$\omega^{\underline{ab}}$	1	b	spin connection	NS-NS	$R^{\underline{ab}}$
$V^{\underline{a}}$	1	b	Vielbein	NS-NS	$T^{\underline{a}}$
$\psi_{L/R}$	1	f	gravitino	NS-R	$\rho_{L/R}$
$\mathbf{C}^{[1]}$	1	b	RR 1-form	R-R	$\mathbf{G}^{[2]}$
φ	0	b	dilaton	NS-NS	$\mathbf{f}^{[1]}$
$\chi_{L/R}$	0	f	dilatino	NS-RS	$\nabla_{\chi_{L/R}}$
$\mathbf{B}^{[2]}$	2	b	Kalb-Ramond field	NS-NS	$\mathbf{H}^{[3]}$
$\mathbf{C}^{[3]}$	3	b	RR 3-form	R-R	$\mathbf{G}^{[4]}$

The explicit definition of the FDA curvatures, constructed with the above fields is displayed below:

$$R^{\underline{ab}} \equiv d\omega^{\underline{ab}} - \omega^{\underline{ac}} \wedge \omega^{\underline{cb}} \tag{2.1}$$

$$T^{\underline{a}} \equiv \mathcal{D}V^{\underline{a}} - \mathrm{i}\frac{1}{2}(\bar{\psi}_L \wedge \Gamma^{\underline{a}}\psi_L + \bar{\psi}_R \wedge \Gamma^{\underline{a}}\psi_R) \tag{2.2}$$

$$\rho_{L,R} \equiv \mathcal{D}\psi_{L,R} \equiv d\psi_{L,R} - \frac{1}{4}\omega^{\underline{ab}} \wedge \Gamma_{\underline{ab}}\psi_{L,R} \tag{2.3}$$

$$\mathbf{G}^{[2]} \equiv d\mathbf{C}^{[1]} + \exp[-\varphi]\bar{\psi}_R \wedge \psi_L \tag{2.4}$$

$$\mathbf{f}^{[1]} \equiv d\varphi \tag{2.5}$$

$$\nabla\chi_{L/R} \equiv d\chi_{L,R} - \frac{1}{4}\omega^{\underline{ab}} \wedge \Gamma_{\underline{ab}}\chi_{L,R} \tag{2.6}$$

$$\mathbf{H}^{[3]} = d\mathbf{B}^{[2]} + \mathrm{i}(\bar{\psi}_L \wedge \Gamma_{\underline{a}}\psi_L - \bar{\psi}_R \wedge \Gamma_{\underline{a}}\psi_R) \wedge V^{\underline{a}} \tag{2.7}$$

$$\mathbf{G}^{[4]} = d\mathbf{C}^{[3]} + \mathbf{B}^{[2]} \wedge d\mathbf{C}^{[1]} - \frac{1}{2}\exp[-\varphi](\bar{\psi}_L \wedge \Gamma_{\underline{ab}}\psi_R + \bar{\psi}_R \wedge \Gamma_{\underline{ab}}\psi_L) \wedge V^{\underline{a}} \wedge V^{\underline{b}}. \tag{2.8}$$

The 0-form dilaton φ appearing in Eq. (2.4) introduces a dynamic coupling constant. Furthermore, as mentioned in the table, $V^{\underline{a}}$, and $\omega^{\underline{ab}}$ respectively denote the vielbein and the spin connection, which together with the gravitino $\psi_{L/R}$ complete the multiplet of 1-forms gauging the type IIA super Poincaré algebra in $D = 10$. The two fermionic 1-forms $\psi_{L/R}$ are Majorana-Weyl spinors of opposite chirality:

$$\Gamma_{11}\psi_{L/R} = \pm\psi_{L/R}. \tag{2.9}$$

The flat metric $\eta_{\underline{ab}} = \mathrm{diag}(+, -, \ldots, -)$ is the mostly minus one and Γ_{11} is Hermitian and squares to the identity $\Gamma_{11}^2 = \mathbf{1}$.

B. Rheonomic parametrizations of the curvatures in the string frame

As explained in [26] the form of the rheonomic parametrization required in order to construct the pure spinor action of superstrings is that corresponding to the string frame and not that corresponding to the Einstein frame. This parametrization was derived in [26] and it is formulated in terms of a certain set of tensors, which involve both the supercovariant field strengths $\mathcal{G}_{\underline{ab}}$, $\mathcal{G}_{\underline{abcd}}$ of the Ramond-Ramond p-forms and also bilinear currents in the dilatino field $\chi_{L/R}$. The needed tensors are those listed below:

$$\begin{aligned}
\mathcal{M}_{\underline{ab}} &= \left(\frac{1}{8}\exp[\varphi]\mathcal{G}_{\underline{ab}} + \frac{9}{64}\bar{\chi}_R\Gamma_{\underline{ab}}\chi_L\right)\\
\mathcal{M}_{\underline{abcd}} &= -\frac{1}{16}\exp[\varphi]\mathcal{G}_{\underline{abcd}} - \frac{3}{256}\bar{\chi}_L\Gamma_{\underline{abcd}}\chi_R\\
\mathcal{N}_0 &= \frac{3}{4}\bar{\chi}_L\chi_R\\
\mathcal{N}_{\underline{ab}} &= \frac{1}{4}\exp[\varphi]\mathcal{G}_{\underline{ab}} + \frac{9}{32}\bar{\chi}_R\Gamma_{\underline{ab}}\chi_L = 2\mathcal{M}_{\underline{ab}}\\
\mathcal{N}_{\underline{abcd}} &= \frac{1}{24}\exp[\varphi]\mathcal{G}_{\underline{abcd}} + \frac{1}{128}\bar{\chi}_R\Gamma_{\underline{abcd}}\chi_L\\
&= -\frac{2}{3}\mathcal{M}_{\underline{abcd}}.
\end{aligned} \tag{2.10}$$

The above tensors are conveniently assembled into the following spinor matrices

$$\mathcal{M}_{\pm} = \mathrm{i}(\mp\mathcal{M}_{\underline{ab}}\Gamma^{\underline{ab}} + \mathcal{M}_{\underline{abcd}}\Gamma^{\underline{abcd}}) \tag{2.11}$$

$$\mathcal{N}_{\pm}^{(\mathrm{even})} = \mp\mathcal{N}_0\mathbf{1} + \mathcal{N}_{\underline{ab}}\Gamma^{\underline{ab}} \mp \mathcal{N}_{\underline{abcd}}\Gamma^{\underline{abcd}} \tag{2.12}$$

$$\begin{aligned}
\mathcal{N}_{\pm}^{(\mathrm{odd})} &= \pm\frac{i}{3}f_{\underline{a}}\Gamma^{\underline{a}} \pm \frac{1}{64}\bar{\chi}_{R/L}\Gamma_{\underline{abc}}\chi_{R/L}\Gamma^{\underline{abc}}\\
&\quad - \frac{i}{12}\mathcal{H}_{\underline{abc}}\Gamma^{\underline{abc}}
\end{aligned} \tag{2.13}$$

$$\mathcal{L}_{a\pm}^{(\mathrm{odd})} = \mathcal{M}_{\mp}\Gamma_{\underline{a}}; \qquad \mathcal{L}_{a\pm}^{(\mathrm{even})} = \mp\frac{3}{8}\mathcal{H}_{\underline{abc}}\Gamma^{\underline{bc}}. \tag{2.14}$$

In terms of these objects the rheonomic parametrizations of the curvatures, solving the Bianchi identities can be written as follows:

1. Bosonic curvatures

$$T^{\underline{a}} = 0 \tag{2.15}$$

$$\begin{aligned}
R^{\underline{ab}} &= R^{\underline{ab}}{}_{\underline{mn}}V^{\underline{m}} \wedge V^{\underline{n}} + \bar{\psi}_R\Theta^{\underline{ab}}_{\underline{m}|L} \wedge V^{\underline{m}} + \bar{\psi}_L\Theta^{\underline{ab}}_{\underline{m}|R} \wedge V^{\underline{m}}\\
&\quad + \mathrm{i}\frac{3}{4}(\bar{\psi}_L \wedge \Gamma_{\underline{c}}\psi_L - \bar{\psi}_R \wedge \Gamma_{\underline{c}}\psi_R)\mathcal{H}^{\underline{abc}}\\
&\quad + 2i\bar{\psi}_L \wedge \Gamma^{[\underline{a}}\mathcal{M}_{+}\Gamma^{\underline{b}]}\psi_R
\end{aligned} \tag{2.16}$$

$$\mathbf{H}^{[3]} = \mathcal{H}_{\underline{abc}}V^{\underline{a}} \wedge V^{\underline{b}} \wedge V^{\underline{c}} \tag{2.17}$$

$$\begin{aligned}
\mathbf{G}^{[2]} &= \mathcal{G}_{\underline{ab}}V^{\underline{a}} \wedge V^{\underline{b}} + \mathrm{i}\frac{3}{2}\exp[-\varphi]\\
&\quad \times(\bar{\chi}_L\Gamma_{\underline{a}}\psi_L + \bar{\chi}_R\Gamma_{\underline{a}}\psi_R) \wedge V^{\underline{a}}
\end{aligned} \tag{2.18}$$

$$\mathbf{f}^{[1]} = f_{\underline{a}}V^{\underline{a}} + \frac{3}{2}(\bar{\chi}_R\psi_L - \bar{\chi}_L\psi_R) \tag{2.19}$$

$$\begin{aligned}
\mathbf{G}^{[4]} &= \mathcal{G}_{\underline{abcd}}V^{\underline{a}} \wedge V^{\underline{b}} \wedge V^{\underline{c}} \wedge V^{\underline{d}} - \mathrm{i}\frac{1}{2}\exp[-\varphi]\\
&\quad \times(\bar{\chi}_L\Gamma_{\underline{abc}}\psi_L - \bar{\chi}_R\Gamma_{\underline{abc}}\psi_R) \wedge V^{\underline{a}} \wedge V^{\underline{b}} \wedge V^{\underline{c}}.
\end{aligned} \tag{2.20}$$

2. Fermionic curvatures

$$\begin{aligned}
\rho_{L/R} &= \rho^{L/R}_{\underline{ab}}V^{\underline{a}} \wedge V^{\underline{b}} + \mathcal{L}^{(\mathrm{even})}_{\underline{a}\pm}\psi_{L/R} \wedge V^{\underline{a}}\\
&\quad + \mathcal{L}^{(\mathrm{odd})}_{\underline{a}\mp}\psi_{R/L} \wedge V^{\underline{a}} + \rho^{(0,2)}_{L/R}
\end{aligned} \tag{2.21}$$

$$\nabla\chi_{L/R} = \mathcal{D}_{\underline{a}}\chi_{L/R}V^{\underline{a}} + \mathcal{N}^{(\mathrm{even})}_{\pm}\psi_{L/R} + \mathcal{N}^{(\mathrm{odd})}_{\mp}\psi_{R/L}. \tag{2.22}$$

Note that the components of the generalized curvatures along the bosonic vielbeins do not coincide with their spacetime components, but rather with their supercovariant extension. Indeed expanding, for example, the four-form along the spacetime differentials one finds that

$$\begin{aligned}
\tilde{G}_{\mu\nu\rho\sigma} &\equiv \mathcal{G}_{\underline{abcd}}V^{\underline{a}}_{\mu} \wedge V^{\underline{b}}_{\nu} \wedge V^{\underline{c}}_{\rho} \wedge V^{\underline{d}}_{\sigma}\\
&= \partial_{[\mu}C^{[4]}_{\nu\rho\sigma]} + B^{[2]}_{[\mu\nu}\partial_\rho C^{[1]}_{\sigma]} - \frac{1}{2}e^{-\varphi}(\bar{\psi}_{L[\mu}\Gamma_{\nu\rho}\psi_{R\sigma]}\\
&\quad + \bar{\psi}_{R/\mu}\Gamma_{\nu\rho}\psi_{L\sigma]}) + \mathrm{i}\frac{1}{2}\exp[-\varphi]\\
&\quad \times(\bar{\chi}_L\Gamma_{[\mu\nu\rho}\psi_{L\sigma]} - \bar{\chi}_R\Gamma_{[\mu\nu\rho}\psi_{R\sigma]}),
\end{aligned}$$

where $\tilde{G}$ is the supercovariant field strength.

In the parametrization (2.16) of the Riemann tensor we have used the following definition:

$$\Theta_{\underline{ab|c}L/R} = -i(\Gamma_{\underline{a}}\rho_{\underline{bc}R/L} + \Gamma_{\underline{b}}\rho_{\underline{ca}R/L} - \Gamma_{\underline{c}}\rho_{\underline{ab}R/L}). \tag{2.23}$$

Finally by $\rho^{(0,2)}_{L/R}$ we have denoted the fermion-fermion part of the gravitino curvature whose explicit expression can be written in two different forms, equivalent by Fierz re-arrangement:

$$\begin{aligned}
\rho^{(0,2)}_{L/R} &= \pm\frac{21}{32}\Gamma_{\underline{a}}\chi_{R/L}\bar{\psi}_{L/R} \wedge \Gamma^{\underline{a}}\psi_{L/R}\\
&\quad \mp\frac{1}{2560}\Gamma_{\underline{a_1a_2a_3a_4a_5}}\chi_{R/L}(\bar{\psi}_{L/R}\Gamma^{\underline{a_1a_2a_3a_4a_5}}\psi_{L/R})
\end{aligned} \tag{2.24}$$

or

$$\begin{aligned}
\rho^{(0,2)}_{L/R} &= \pm\frac{3}{8}\,\mathrm{i}\psi_{L/R} \wedge \bar{\chi}_{R/L}\psi_{L/R} \pm \frac{3}{16}\,\mathrm{i}\Gamma_{\underline{ab}}\psi_{L/R}\\
&\quad \wedge \bar{\chi}_{R/L}\Gamma^{\underline{ab}}\psi_{L/R}.
\end{aligned} \tag{2.25}$$

C. Field equations of type IIA supergravity in the string frame

The rheonomic parametrizations of the supercurvatures displayed above imply, via Bianchi identities, a certain number of constraints on the inner components of the same curvatures which can be recognized as the field equations of type IIA supergravity in the string frame. These are the equations that have to be solved in constructing any specific supergravity background and read as follows.

We have an Einstein equation of the following form:

$$\mathcal{R}_{\underline{ab}} = \hat{T}_{\underline{ab}}(f) + \hat{T}_{\underline{ab}}(\mathcal{G}_2) + \hat{T}_{\underline{ab}}(\mathcal{H}) + \hat{T}_{\underline{ab}}(\mathcal{G}_4) \tag{2.26}$$

where the stress-energy tensor on the right hand side are defined as

$$\hat{T}_{\underline{ab}}(f) = -\mathcal{D}_{\underline{a}}\mathcal{D}_{\underline{b}}\varphi + \frac{8}{9}\mathcal{D}_{\underline{a}}\varphi\mathcal{D}_{\underline{b}}\varphi - \eta_{\underline{ab}}\left(\frac{1}{6}\varphi + \frac{5}{9}\mathcal{D}^{\underline{m}}\varphi\mathcal{D}_{\underline{m}}\varphi\right) \tag{2.27}$$

$$\hat{T}_{\underline{ab}}(\mathcal{G}_2) = \exp[2\varphi]\mathcal{G}_{\underline{ax}}\mathcal{G}_{\underline{by}}\eta^{\underline{ab}} \tag{2.28}$$

$$\hat{T}_{\underline{ab}}(\mathcal{H}) = -\exp\left[\frac{1}{3}\varphi\right]\left(\frac{9}{8}\mathcal{H}_{\underline{axy}}\mathcal{H}_{\underline{bwt}}\eta^{\underline{xw}}\eta^{\underline{yt}} - \frac{1}{8}\eta_{\underline{ab}}\mathcal{H}_{\underline{xyz}}\mathcal{H}^{\underline{xyz}}\right) \tag{2.29}$$

$$\hat{T}_{\underline{ab}}(\mathcal{G}_4) = \exp[2\varphi]\left(6\mathcal{G}_{\underline{ax_1x_2x_3}}\mathcal{G}_{\underline{by_1y_2y_3}}\eta^{\underline{x_1y_1}}\eta^{\underline{x_2y_2}}\eta^{\underline{x_3y_3}} - \frac{1}{2}\eta_{\underline{ab}}\mathcal{G}_{\underline{x_1\ldots x_4}}\mathcal{G}^{\underline{x_1\ldots x_4}}\right). \tag{2.30}$$

Next we have the equations for the dilaton and the Ramond 1-form:

$$0 = \Box\varphi - 2f_{\underline{a}}f^{\underline{a}} + \frac{3}{2}\exp[2\varphi]\mathcal{G}^{\underline{x_1x_2}}\mathcal{G}_{\underline{x_1x_2}} + \frac{3}{2}\exp[2\varphi]\mathcal{G}^{\underline{x_1x_2x_3x_4}}\mathcal{G}_{\underline{x_1x_2x_3x_4}} + \frac{3}{4}\exp\left[\frac{4}{3}\varphi\right]\mathcal{H}^{\underline{x_1x_2x_3}}\mathcal{H}_{\underline{x_1x_2x_3}} \tag{2.31}$$

$$0 = \mathcal{D}_{\underline{m}}\mathcal{G}^{\underline{ma}} - \frac{5}{3}f^{\underline{m}}\mathcal{G}_{\underline{ma}} + 3\mathcal{G}^{\underline{ax_1x_2x_3}}\mathcal{H}_{\underline{x_1x_2x_3}} \tag{2.32}$$

and the equations for the NS 2-form and for the RR 3-form:

$$0 = \mathcal{D}_{\underline{m}}\mathcal{H}^{\underline{mab}} - \frac{2}{3}f^{\underline{m}}\mathcal{H}_{\underline{mab}} - \exp\left[\frac{4}{3}\varphi\right] \times\left(4\mathcal{G}^{\underline{x_1x_2ab}}\mathcal{G}_{\underline{x_1x_2}} - \frac{1}{24}\epsilon^{\underline{abx_1\ldots x_8}}\mathcal{G}_{\underline{x_1x_2x_3x_4}}\mathcal{G}_{\underline{x_5x_6x_7x_8}}\right) \tag{2.33}$$

$$0 = \mathcal{D}_{\underline{m}}\mathcal{G}^{\underline{ma_1a_2a_3}} + \frac{1}{3}f_m\mathcal{G}^{\underline{ma_1a_2a_3}} + \exp\left[\frac{2}{3}\varphi\right] \times\left(\frac{3}{2}\mathcal{G}^{\underline{m[a_1}}H^{\underline{a_2a_3]n}}\eta_{\underline{mn}} + \frac{1}{48}\epsilon^{\underline{a_1a_2a_3x_1\ldots x_7}}\mathcal{G}_{\underline{x_1x_2x_3x_4}}H_{\underline{x_5x_6x_7}}\right). \tag{2.34}$$

Any solution of these bosonic set of equations can be uniquely extended to a full superspace solution involving 32 theta variables by means of the rheonomic conditions. The implementation of such a fermionic integration is the *supergauge completion*.

III. COMPACTIFICATIONS OF TYPE IIA ON $\mathbf{AdS_4} \times \mathbb{P}^3$

In this section we construct a compactification of type IIA supergravity on the following direct product manifold:

$$\mathcal{M}_{10} = \mathrm{AdS}_4 \times \mathbb{P}^3. \tag{3.1}$$

The local symmetries of the effective theory on this background is encoded in the supergroup $\mathrm{OSp}(6|4)$. The supergauge completion of the $\mathrm{AdS}_4 \times \mathbb{P}^3$ space consists in expressing the ten-dimensional superfields, satisfying the rheonomic parametrizations in terms of the coordinates of the *mini-superspace* associated with this background, namely, of the 10 space-time coordinates $x^{\underline{\mu}}$ and the 24 fermionic ones θ, parametrizing the preserved supersymmetries only. This procedure relies on the representation of the *mini-superspace* in terms of the following supercoset manifold

$$\mathcal{M}^{10|24} = \frac{\mathrm{OSp}(6|4)}{\mathrm{SO}(1,3)\times\mathrm{U}(3)}. \tag{3.2}$$

The bosonic subgroup of $\mathrm{OSp}(6|4)$ is $\mathrm{Sp}(4,\mathbb{R})\times\mathrm{SO}(6)$. The Maurer-Cartan 1-forms of $\mathfrak{sp}(4,\mathbb{R})$ are denoted by Δ^{xy} $(x, y = 1,\ldots,4)$, the $\mathfrak{so}(6)$ 1-forms are denoted by $\mathcal{A}_{AB}$ $(A, B = 1,\ldots,6)$ while the (real) fermionic 1-forms are denoted by Φ_A^x and transform in the fundamental representation of $\mathrm{Sp}(4,\mathbb{R})$ and in the fundamental representation of $\mathrm{SO}(6)$. These forms satisfy the $\mathrm{OSp}(6|4)$ Maurer-Cartan equations:

$$\begin{aligned} &d\Delta^{xy} + \Delta^{xz}\wedge\Delta^{ty}\epsilon_{zt} = -4\mathrm{i}e\Phi_A^x\wedge\Phi_A^y,\\ &d\mathcal{A}_{AB} - e\mathcal{A}_{AC}\wedge\mathcal{A}_{CB} = 4\mathrm{i}\Phi_A^x\wedge\Phi_B^y\epsilon_{xy}\\ &d\Phi_A^x + \Delta^{xy}\wedge\epsilon_{yz}\Phi_A^z - e\mathcal{A}_{AB}\wedge\Phi_B^x = 0, \end{aligned} \tag{3.3}$$

where

$$\epsilon_{xy} = -\epsilon_{yx} = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.4}$$

The Maurer-Cartan equations are solved in terms of the supercoset representative of (3.2). We rely for this analysis on the general discussion in [12]. It is convenient to express this solution in terms of the 1-forms describing the on the bosonic submanifolds $\mathrm{AdS}_4 \equiv \frac{\mathrm{Sp}(4,\mathbb{R})}{\mathrm{SO}(1,3)}$, $\mathbb{P}^3 \equiv \frac{\mathrm{SO}(6)}{\mathrm{U}(3)}$ of (3.2) and 1-forms on the fermionic subspace of (3.2). Let us denote by B^{ab}, B^a, and by $\mathcal{B}^{\alpha\beta}$, $\mathcal{B}^{\alpha}$ the connections and vielbein on the two bosonic subspaces, respectively. The supergauge completion is finally accomplished by expressing the p-forms satisfying the rheonomic parametrization of the FDA in the mini-superspace. This amounts to expressing them in terms of the 1-forms on (3.2). The final expression of the $D = 10$ fields will involve not only the bosonic 1-forms B^{ab}, B^a, $\mathcal{B}^{\alpha\beta}$, $\mathcal{B}^{\alpha}$, but also the Killing spinors on the background. The latter play indeed a spacial role in this analysis since they can be identified with the fundamental harmonics of the cosets $\mathrm{SO}(2,3)/SO(1,3)$ and $\mathrm{SO}(6)/U(3)$, respectively, [32]. Before writing the explicit solution we need to discuss the Killing spinors on the $\mathrm{AdS}_4 \times \mathbb{P}^3$ background.

A. Killing spinors of the AdS_4 manifold

As anticipated, on of the main items for the construction of the supergauge completion is given by the Killing spinors of anti-de Sitter space. They can be constructed in terms of the coset representative $\mathrm{L_B}$, namely, in terms of the fundamental harmonic of the coset $\mathrm{SO}(2,3)/SO(1,3)$.

The defining equation is given by:

$$\nabla^{\mathrm{Sp}(4)}\chi_x \equiv \left(d - \frac{1}{4}B^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 B^a\right)\chi_x = 0 \quad (3.5)$$

and states that the Killing spinor is a covariantly constant section of the $\mathfrak{sp}(4,\mathbb{R})$ bundle defined over AdS_4. This bundle is flat since the vanishing of the $\mathfrak{sp}(4,\mathbb{R})$ curvature is nothing else but the Maurer-Cartan equation of $\mathfrak{sp}(4,\mathbb{R})$ and hence corresponds to the structural equations of the AdS_4 manifold. We are therefore guaranteed that there exists a basis of four linearly independent sections of such a bundle, namely, four linearly independent solutions of Eq. (3.5) which we can normalize as follows:

$$\bar{\chi}_x\gamma_5\chi_y = \epsilon_{xy}. \quad (3.6)$$

The 1-forms on AdS_4 are defined in terms of $\mathrm{L_B}$ as follows:

$$-\frac{1}{4}B^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 B^a = \Delta_B = \mathrm{L_B^{-1}}d\mathrm{L_B}. \quad (3.7)$$

It follows that the inverse matrix $\mathrm{L_B^{-1}}$ satisfies the equation:

$$(d + \Delta_B)\mathrm{L_B^{-1}} = 0. \quad (3.8)$$

Regarding the first index y of the matrix $(\mathrm{L_B^{-1}})^y{}_x$ as the spinor index acted on by the connection Δ_B and the second index x as the labeling enumerating the Killing spinors, Eq. (3.8) is identical with Eq. (3.5) and hence we have explicitly constructed its four independent solutions. In order to achieve the desired normalization (3.6) it suffices to multiply by a phase factor $\exp[-\mathrm{i}\frac{1}{4}\pi]$, namely, it suffices to set:

$$\chi^y_{(x)} = \exp\left[-\mathrm{i}\frac{1}{4}\pi\right](\mathrm{L_B^{-1}})^y{}_x. \quad (3.9)$$

In this way the four Killing spinors fulfill the Majorana condition, having chosen a representation of the $D = 4$ Clifford algebra in which $\mathcal{C} = i\gamma_0$ (see Appendix B for conventions on spinors). Furthermore since $\mathrm{L_B^{-1}}$ is symplectic it satisfies the defining relation

$$\mathrm{L_B^{-1}}\mathcal{C}\gamma_5\mathrm{L_B} = \mathcal{C}\gamma_5 \quad (3.10)$$

which implies (3.6).

B. Explicit construction of $\mathbb{P}^3$ geometry

The complex three-fold $\mathbb{P}^3$ is Kähler. Indeed the existence of the Kähler 2-form is one of the essential items in constructing the solution ansatz.

Let us begin by discussing all the relevant geometric structures of $\mathbb{P}^3$. We need now to construct the explicit form of the internal manifold geometry, in particular, the spin connection, the vielbein and the Kähler 2-form. This is fairly easy, since $\mathbb{P}^3$ is a coset manifold:

$$\mathbb{P}^3 = \frac{\mathrm{SU}(4)}{\mathrm{SU}(3) \times \mathrm{U}(1)} \quad (3.11)$$

so that everything is defined in terms of structure constants of the $\mathfrak{su}(4)$ Lie algebra. The quickest way to introduce these structure constants and their chosen normalization is by writing the Maurer-Cartan equations. We do this introducing already the splitting:

$$\mathfrak{su}(4) = \mathbb{H} \oplus \mathbb{K} \quad (3.12)$$

between the subalgebra $\mathbb{H} \equiv \mathfrak{su}(3) \times \mathfrak{u}(1)$ and the complementary orthogonal subspace $\mathbb{K}$ which is tangent to the coset manifold. Hence we name $H^i (i = 1, \ldots, 9)$ a basis of 1-form generators of $\mathbb{H}$ and $K^\alpha (\alpha = 1, \ldots, 6)$ a basis of 1-form generators of $\mathbb{K}$. With these notation the Maurer-Cartan equations defining the structure constants of $\mathfrak{su}(4)$ have the following form:

$$\begin{aligned} dK^\alpha + \mathcal{B}^{\alpha\beta} \wedge K^\gamma\delta_{\beta\gamma} &= 0 \\ d\mathcal{B}^{\alpha\beta} + \mathcal{B}^{\alpha\gamma} \wedge \mathcal{B}^{\delta\beta}\delta_{\gamma\delta} - \mathcal{X}^{\alpha\beta}{}_{\gamma\delta}K^\gamma \wedge K^\delta &= 0, \end{aligned} \quad (3.13)$$

where:

(1) the antisymmetric 1-form valued matrix $B^{\alpha\beta}$ is parametrized by the 9 generators of the $\mathfrak{u}(3)$ subalgebra of $\mathfrak{so}(6)$ in the following way:

$$\mathcal{B}^{\alpha\beta} = \begin{pmatrix} 0 & H^9 & -H^8 & H^1 + H^2 & H^6 & -H^5 \\ -H^9 & 0 & H^7 & H^6 & H^1 + H^3 & H^4 \\ H^8 & -H^7 & 0 & -H^5 & H^4 & H^2 + H^3 \\ -H^1 - H^2 & -H^6 & H^5 & 0 & H^9 & -H^8 \\ -H^6 & -H^1 - H^3 & -H^4 & -H^9 & 0 & H^7 \\ H^5 & -H^4 & -H^2 - H^3 & H^8 & -H^7 & 0 \end{pmatrix} \tag{3.14}$$

(2) the symbol $\mathcal{X}^{\alpha\beta}{}_{\gamma\delta}$ denotes the following constant, 4-index tensor:

$$\mathcal{X}^{\alpha\beta}{}_{\gamma\delta} \equiv (\delta^{\alpha\beta}_{\gamma\delta} + \mathcal{K}^{\alpha\beta}\mathcal{K}^{\gamma\delta} + \mathcal{K}^{\alpha}{}_{\gamma}\mathcal{K}^{\beta}{}_{\delta}) \tag{3.15}$$

(3) the symbol $\mathcal{K}^{\alpha\beta}$ denotes the entries of the following antisymmetric matrix:

$$\mathcal{K} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.16}$$

The Maurer Cartan Eqs. (3.13) can be reinterpreted as the structural equations of the $\mathbb{P}^3$ 6-dimensional manifold. It suffices to identify the antisymmetric 1-form valued matrix $\mathcal{B}^{\alpha\beta}$ with the spin connection and identify the vielbein $\mathcal{B}^{\alpha}$ with the coset generators K^{α}, modulo a scale factor λ

$$\mathcal{B}^{\alpha} = \frac{1}{\lambda} K^{\alpha} \tag{3.17}$$

With these identifications the first of Eqs. (3.13) becomes the vanishing torsion equation, while the second singles out the Riemann tensor as proportional to the tensor $\mathcal{X}^{\alpha\beta}{}_{\gamma\delta}$ of Eq. (3.15). Indeed we can write:

$$\mathcal{R}^{\alpha\beta} = d\mathcal{B}^{\alpha\beta} + \mathcal{B}^{\alpha\gamma} \wedge \mathcal{B}^{\delta\beta}\delta_{\gamma\delta} = \mathcal{R}^{\alpha\beta}{}_{\gamma\delta}\mathcal{B}^{\gamma} \wedge \mathcal{B}^{\delta}, \tag{3.18}$$

where:

$$\mathcal{R}^{\alpha\beta}{}_{\gamma\delta} = \lambda^2 \mathcal{X}^{\alpha\beta}{}_{\gamma\delta} \tag{3.19}$$

Using the above Riemann tensor we immediately retrieve the explicit form of the Ricci tensor:

$$\mathrm{Ric}_{\alpha\beta} = 4\lambda^2 \eta_{\alpha\beta}. \tag{3.20}$$

For later convenience in discussing the compactification ansatz it is convenient to rename the scale factor as follows:

$$\lambda = 2e. \tag{3.21}$$

In this way we obtain:

$$\mathrm{Ric}_{\alpha\beta} = 16e^2 \eta_{\alpha\beta}, \tag{3.22}$$

which will be recognized as one of the field equations of type IIA supergravity.

Let us now come to the interpretation of the matrix $\mathcal{K}$. This matrix is immediately identified as encoding the intrinsic components of the Kähler 2-form. Indeed $\mathcal{K}$ is the unique antisymmetric matrix which, within the fundamental 6-dimensional representation of the $\mathfrak{so}(6) \sim \mathfrak{su}(4)$ Lie algebra, commutes with the entire subalgebra $\mathfrak{u}(3) \subset \mathfrak{su}(4)$. Hence $\mathcal{K}$ generates the U(1) subgroup of U(3) and this guarantees that the Kähler 2-form will be closed and coclosed as it should be. Indeed it is sufficient to set:

$$\hat{\mathcal{K}} = \mathcal{K}_{\alpha\beta}\mathcal{B}^{\alpha} \wedge \mathcal{B}^{\beta} \tag{3.23}$$

namely:

$$\hat{\mathcal{K}} = -2(\mathcal{B}^1 \wedge \mathcal{B}^4 + \mathcal{B}^2 \wedge \mathcal{B}^5 + \mathcal{B}^3 \wedge \mathcal{B}^6) \tag{3.24}$$

and we obtain that the 2-form $\hat{\mathcal{K}}$ is closed and coclosed:

$$d\hat{\mathcal{K}} = 0, \qquad d^{\star}\hat{\mathcal{K}} = 0. \tag{3.25}$$

Let us also note that the antisymmetric matrix $\mathcal{K}$ satisfies the following identities:

$$\mathcal{K}^2 = -1_{6\times 6} \qquad 8\mathcal{K}_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta\tau\sigma}\mathcal{K}^{\gamma\delta}\mathcal{K}^{\tau\sigma}. \tag{3.26}$$

Using the $\mathfrak{so}(6)$ Clifford Algebra defined in Appendix A 1 we define the following spinorial operators:

$$\mathcal{W} = \mathcal{K}_{\alpha\beta}\tau^{\alpha\beta}; \qquad \mathcal{P} = \mathcal{W}\tau_7 \tag{3.27}$$

and we can verify that the matrix $\mathcal{P}$ satisfies the following algebraic equations:

$$\mathcal{P}^2 + 4\mathcal{P} - 12 \times \mathbf{1} = 0 \tag{3.28}$$

whose roots are 2 and -6. Indeed in the chosen τ-matrix basis the matrix $\mathcal{P}$ is diagonal with the following explicit form:

$$\mathcal{P} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{pmatrix}. \tag{3.29}$$

Let us also introduce the following matrix valued 1-form:

$$\mathcal{Q} \equiv \left(\frac{3}{2}\mathbf{1} + \frac{1}{4}\mathcal{P}\right)\tau_\alpha \mathcal{B}^\alpha, \tag{3.30}$$

whose explicit form in the chosen basis is the following one:

$$\mathcal{Q} = \begin{pmatrix} 0 & 2\mathcal{B}^3 & -2\mathcal{B}^2 & 0 & -2\mathcal{B}^6 & 2\mathcal{B}^5 & -2\mathcal{B}^4 & 2\mathcal{B}^1 \\ -2\mathcal{B}^3 & 0 & 2\mathcal{B}^1 & 2\mathcal{B}^6 & 0 & -2\mathcal{B}^4 & -2\mathcal{B}^5 & 2\mathcal{B}^2 \\ 2\mathcal{B}^2 & -2\mathcal{B}^1 & 0 & -2\mathcal{B}^5 & 2\mathcal{B}^4 & 0 & -2\mathcal{B}^6 & 2\mathcal{B}^3 \\ 0 & -2\mathcal{B}^6 & 2\mathcal{B}^5 & 0 & -2\mathcal{B}^3 & 2\mathcal{B}^2 & 2\mathcal{B}^1 & 2\mathcal{B}^4 \\ 2\mathcal{B}^6 & 0 & -2\mathcal{B}^4 & 2\mathcal{B}^3 & 0 & -2\mathcal{B}^1 & 2\mathcal{B}^2 & 2\mathcal{B}^5 \\ -2\mathcal{B}^5 & 2\mathcal{B}^4 & 0 & -2\mathcal{B}^2 & 2\mathcal{B}^1 & 0 & 2\mathcal{B}^3 & 2\mathcal{B}^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.31}$$

and let us consider the following Killing spinor equation:

$$\mathcal{D}\eta + e\mathcal{Q}\eta = 0 \tag{3.32}$$

where, by definition:

$$\mathcal{D} = d - \frac{1}{4}\mathcal{B}^{\alpha\beta}\tau_{\alpha\beta} \tag{3.33}$$

denotes the $\mathfrak{so}(6)$ covariant differential of spinors defined over the $\mathbb{P}^3$ manifold. The connection $\mathcal{Q}$ is closed with respect to the spin connection

$$\Omega = -\frac{1}{4}\mathcal{B}^{\alpha\beta}\tau_{\alpha\beta} \tag{3.34}$$

since we have:

$$\mathcal{D}\mathcal{Q} \equiv d\mathcal{Q} + e^2\Omega \wedge \mathcal{Q} + \mathcal{Q} \wedge \Omega = 0 \tag{3.35}$$

as it can be explicitly checked. The above result follows because the matrix $\mathcal{K}_{\alpha\beta}$ commutes with all the generators of $\mathfrak{u}(3)$. In view of Eq. (3.35) the integrability of the Killing (3.32) becomes the following one:

$$\mathrm{Hol}\,\eta = 0, \tag{3.36}$$

where we have defined the holonomy 2-form:

$$\mathrm{Hol} \equiv (\mathcal{D}^2 + e^2\mathcal{Q}\wedge\mathcal{Q}) = \left(-\frac{1}{4}\mathcal{R}^{\alpha\beta}\tau_{\alpha\beta} + e^2\mathcal{Q}\wedge\mathcal{Q}\right) \tag{3.37}$$

and $\mathcal{R}^{\alpha\beta}$ denotes the curvature 2-form (3.18). Explicit evaluation of the holonomy 2-form yields the following result.

$$\mathrm{Hol} = e^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 8[\mathcal{B}^2\wedge\mathcal{B}^6 - \mathcal{B}^3\wedge\mathcal{B}^5] & 8\mathcal{B}^5\wedge\mathcal{B}^6 - 8\mathcal{B}^2\wedge\mathcal{B}^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8\mathcal{B}^3\wedge\mathcal{B}^4 - 8\mathcal{B}^1\wedge\mathcal{B}^6 & 8[\mathcal{B}^1\wedge\mathcal{B}^3 - \mathcal{B}^4\wedge\mathcal{B}^6] \\ 0 & 0 & 0 & 0 & 0 & 0 & 8[\mathcal{B}^1\wedge\mathcal{B}^5 - \mathcal{B}^2\wedge\mathcal{B}^4] & 8\mathcal{B}^4\wedge\mathcal{B}^5 - 8\mathcal{B}^1\wedge\mathcal{B}^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8[\mathcal{B}^2\wedge\mathcal{B}^3 - \mathcal{B}^5\wedge\mathcal{B}^6] & 8[\mathcal{B}^2\wedge\mathcal{B}^6 - \mathcal{B}^3\wedge\mathcal{B}^5] \\ 0 & 0 & 0 & 0 & 0 & 0 & 8\mathcal{B}^4\wedge\mathcal{B}^6 - 8\mathcal{B}^1\wedge\mathcal{B}^3 & 8\mathcal{B}^3\wedge\mathcal{B}^4 - 8\mathcal{B}^1\wedge\mathcal{B}^6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8[\mathcal{B}^1\wedge\mathcal{B}^2 - \mathcal{B}^4\wedge\mathcal{B}^5] & 8[\mathcal{B}^1\wedge\mathcal{B}^5 - \mathcal{B}^2\wedge\mathcal{B}^4] \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8\hat{\mathcal{K}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 8\hat{\mathcal{K}} & 0 \end{pmatrix}. \tag{3.38}$$

It is evident by inspection that the holonomy 2-form vanishes on the subspace of spinors that belong to the eigenspace of eigenvalue 2 of the operator $\mathcal{P}$. In the chosen basis this eigenspace is spanned by all those spinors whose last two components are zero and on such spinors the operator Hol vanishes.

Let us now connect these geometric structures to the compactification ansatz.

C. The compactification ansatz

As usual we denote with Latin indices those in the direction of 4-space and with Greek indices those in the direction of the internal 6-space. Let us also adopt the notation: B^a for the AdS_4 vielbein just as $\mathcal{B}^\alpha$ is the vielbein of the Kähler three-fold described in the previous section.[1] With these notations the Kaluza-Klein ansatz is the following one:

$$\begin{aligned} \mathcal{G}_{\underline{ab}} &= \begin{cases} 2e\exp[-\varphi_0]\mathcal{K}_{\alpha\beta} \\ 0 \text{ otherwise} \end{cases} \\ \mathcal{G}_{\underline{a_1a_2a_3a_4}} &= \begin{cases} -e\exp[-\varphi_0]\epsilon_{a_1a_2a_3a_4} \\ 0 \text{ otherwise} \end{cases} \\ \mathcal{H}_{\underline{a_1a_2a_3}} &= 0 \\ \varphi &= \varphi_0 = \text{constant} \\ V^a &= B^a \\ V^\alpha &= \mathcal{B}^\alpha \\ \omega^{ab} &= B^{ab} \\ \omega^{\alpha\beta} &= \mathcal{B}^{\alpha\beta}, \end{aligned} \tag{3.39}$$

where B^a, B^{ab} respectively denote the vielbein and the spin connection of AdS_4, satisfying the following structural equations:

$$\begin{aligned} 0 &= dB^a - B^{ab}\wedge B^c\eta_{bc} \\ dB^{ab} - B^{ac}\wedge B^{db}\eta_{cd} &= -16e^2B^a\wedge B^b \\ &\Downarrow \\ \mathrm{Ric}_{ab} &= -24e^2\eta_{ab} \end{aligned} \tag{3.40}$$

while $\mathcal{B}^\alpha$ and $\mathcal{B}^{\alpha\beta}$ are the analogous data for the internal $\mathbb{P}^3$ manifold:

$$\begin{aligned} 0 &= d\mathcal{B}^\alpha - \mathcal{B}^{\alpha\beta}\wedge\mathcal{B}^\gamma\eta_{\beta\gamma} \\ d\mathcal{B}^{\alpha\beta} - \mathcal{B}^{\alpha\gamma}\wedge\mathcal{B}^{\delta\beta}\eta_{\gamma\delta} &= -R^{\alpha\beta}{}_{\gamma\delta}\mathcal{B}^\gamma\wedge\mathcal{B}^\delta \\ &\Downarrow \\ \mathrm{Ric}_{\alpha\beta} &= 16e^2\eta_{\alpha\beta} \end{aligned} \tag{3.41}$$

whose geometry we described in the previous section.

With these normalizations we can check that the dilaton Eq. (2.31) and the Einstein Eq. (2.26), are satisfied upon insertion of the above Kaluza-Klein ansatz. All the other equations are satisfied thanks to the fact that the Kähler form $\hat{\mathcal{K}}$ is closed and coclosed: Eq. (3.25)

D. Killing spinors on $\mathbb{P}^3$

The next task we are faced with is to determine the equation for the Killing spinors on the chosen background, which by construction is a solution of supergravity equations.

[1] This formulation is analogue to the one used in the case of M-theory compactifications [33,34].

Following a standard procedure we recall that the vacuum has been defined by choosing certain values for the bosonic fields and setting all the fermionic ones equal to zero:

$$\psi_{L/R|\underline{\mu}} = 0 \qquad \chi_{L/R} = 0 \qquad \rho_{L/R|\underline{ab}} = 0. \tag{3.42}$$

The equation for the Killing spinors will be obtained by imposing that the parameter of supersymmetry preserves the vanishing values of the fermionic fields once the specific values of the bosonic ones is substituted into the expression for the supersymmetry (SUSY) rules, namely, into the rheonomic parametrizations.

To implement these conditions we begin by choosing a well adapted basis for the $d = 11$ gamma matrices. This is done by setting:

$$\Gamma^{\underline{a}} = \begin{cases} \Gamma^a = \gamma^a\otimes\mathbf{1} \\ \Gamma^\alpha = \gamma^5\otimes\tau^\alpha \\ \Gamma^{11} = \mathrm{i}\gamma^5\otimes\tau^7 \end{cases}. \tag{3.43}$$

Next we consider the tensors and the matrices introduced in Eqs. (2.10), (2.11), (2.12), and (2.13). In the chosen background we find:

$$\begin{aligned} &\mathcal{M}_{\alpha\beta} = \frac{1}{4}e\mathcal{K}_{\alpha\beta}; \qquad \mathcal{M}_{abcd} = \frac{1}{16}e\epsilon_{abcd} \\ &\mathcal{N}_0 = 0; \qquad \mathcal{N}_{\alpha\beta} = \frac{1}{2}e\mathcal{K}_{\alpha\beta}; \\ &\mathcal{N}_{abcd} = -\frac{1}{24}e\epsilon_{abcd}, \end{aligned} \tag{3.44}$$

all the other components of the above matrices being zero. Hence in terms of the operators introduced in the previous section we find:

$$\begin{aligned} \mathcal{M}_\pm &= \mathrm{i}e\left(\mp\frac{1}{4}\mathbf{1}\otimes\mathcal{W} - \frac{3}{2}\mathrm{i}\gamma_5\otimes\mathbf{1}\right) \\ \mathcal{N}_\pm^{(\mathrm{even})} &= e\left(\frac{1}{2}\mathbf{1}\otimes\mathcal{W}\mp\mathrm{i}\gamma_5\otimes\mathbf{1}\right) \\ \mathcal{N}_\pm^{(\mathrm{odd})} &= 0. \end{aligned} \tag{3.45}$$

It is now convenient to rewrite the Killing spinor condition in a non chiral basis introducing a supersymmetry parameter of the following form:

$$\epsilon = \epsilon_L + \epsilon_R. \tag{3.46}$$

In this basis the matrices $\mathcal{M}$ and $\mathcal{N}^{(\mathrm{even})}$ read

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_+\frac{1}{2}(11+\Gamma^{11}) + \mathcal{M}_-\frac{1}{2}(11-\Gamma^{11}) \\ &= -\frac{i}{8}e^\varphi G_{\underline{ab}}\Gamma^{\underline{ab}}\Gamma^{11} - \frac{i}{16}e^\varphi G_{\underline{abcd}}\Gamma^{\underline{abcd}} \\ &= \frac{e}{4}\gamma_5\otimes(\mathcal{W}\tau_7 + 611), \end{aligned} \tag{3.47}$$

$$\begin{aligned}\mathcal{N}^{(\mathrm{even})} &= \mathcal{N}_+^{(\mathrm{even})}\frac{1}{2}(11+\Gamma^{11}) + \mathcal{N}_-^{(\mathrm{even})}\frac{1}{2}(11-\Gamma^{11}) \\ &= \frac{1}{4}e^{\varphi}G_{\underline{ab}}\Gamma^{\underline{ab}} + \frac{1}{24}e^{\varphi}G_{\underline{abcd}}\Gamma^{\underline{abcd}} \\ &= \frac{e}{2}11\otimes(\mathcal{W}+2\tau_7). \end{aligned} \tag{3.48}$$

Upon use of this parameter the Killing spinor equation coming from the gravitino rheonomic parametrization (2.21) takes the following form:

$$\mathcal{D}\epsilon = -\mathcal{M}\Gamma_{\underline{a}}V^{\underline{a}}\epsilon, \tag{3.49}$$

while the Killing spinor equation coming from the dilatino rheonomic parametrization is as follows:

$$0 = \mathcal{N}^{(\mathrm{even})}\epsilon. \tag{3.50}$$

Let us now insert these results into the Killing spinor equations and let us take a tensor product representation for the Killing spinor:

$$\epsilon = \varepsilon \otimes \eta, \tag{3.51}$$

where ε is a 4-component $d=4$ spinor and η is an 8-component $d=6$ spinor.

With these inputs Eq. (3.49) becomes:

$$\begin{aligned} 0 = {}& \mathcal{D}_{[4]}\varepsilon\otimes\eta - e\gamma_a\gamma_5 B^a\varepsilon\otimes\left(\frac{3}{2}+\frac{1}{4}\mathcal{P}\right)\eta \\ &+ \varepsilon\otimes\left[\mathcal{D}_{[6]} + e\left(\frac{3}{2}+\frac{1}{4}\mathcal{P}\right)\tau_\alpha\mathcal{B}^\alpha\right]\eta \end{aligned} \tag{3.52}$$

while Eq. (3.50) takes the form:

$$0 = \varepsilon\otimes\left(\frac{1}{2}\mathcal{W}+\tau_7\right)\eta. \tag{3.53}$$

Let us now recall that Eq. (3.32) is integrable on the eigenspace of eigenvalue 2 of the $\mathcal{P}$-operator. Then Eq. (3.52) is satisfied if:

$$\begin{aligned} (\mathcal{D}_{[4]} - 2e\gamma_a\gamma_5 B^a)\varepsilon = 0 \qquad \mathcal{P}\eta = 2\eta \\ (\mathcal{D}_{[6]} + e\mathcal{Q})\eta = 0. \end{aligned} \tag{3.54}$$

The first of the above equation is the correct equation for Killing spinors in AdS_4. It emerges if the eigenvalue of $\mathcal{P}$ is 2. The second and the third are the already studied integrable equation for six Killing spinors out of eight. It should now be that the dilatino Eq. (3.53) is satisfied on the eigenspace of eigenvalue 2, which is indeed the case:

$$\mathcal{P}\eta = 2\eta \Rightarrow \left(\frac{1}{2}\mathcal{W}+\tau_7\right)\eta = 0. \tag{3.55}$$

E. Gauge completion in mini superspace

As a necessary ingredient of our construction let η_A ($A = 1,\dots,6$) denote a complete and orthonormal basis of solutions the internal Killing spinor equation, namely:

$$\begin{aligned} \mathcal{P}\eta_A = 2\eta_A \qquad (\mathcal{D}_{[6]} + e\mathcal{Q})\eta_A = 0 \\ \eta_A^T\eta_B = \delta_{AB}; \qquad A, B = A = 1,\dots,6. \end{aligned} \tag{3.56}$$

On the other hand let χ_x denote a basis of solutions of the Killing spinor equation on AdS_4-space, namely (3.5), normalized as in Eq. (3.6). Furthermore let us recall the matrix K defining the intrinsic components of the Kähler 2-form.

In terms of these objects we can satisfy the rheonomic parametrizations of the 2-forms spanning the $d=10$ superPoincaré subalgebra of the FDA with the following position[2]:

$$\Psi = \chi_x\otimes\eta_A\Phi^{x|A} \tag{3.57}$$

$$V^a = B^a - \frac{1}{8e}\bar{\chi}_x\gamma^a\chi_y\Delta^{xy} \tag{3.58}$$

$$V^\alpha = \mathcal{B}^\alpha - \frac{1}{8}\eta_A^T\tau^\alpha\eta_B\mathcal{A}^{AB} \tag{3.59}$$

$$\omega^{ab} = B^{ab} + \frac{1}{2}\bar{\chi}_x\gamma^{ab}\gamma_5\chi_y\Delta^{xy} \tag{3.60}$$

$$\omega^{\alpha\beta} = \mathcal{B}^{\alpha\beta} + \frac{e}{4}\eta_A^T\tau^{\alpha\beta}\eta_B\mathcal{A}^{AB} - \frac{e}{4}\mathcal{K}^{\alpha\beta}\mathcal{K}_{AB}\mathcal{A}^{AB}. \tag{3.61}$$

The proof that the above ansatz satisfies the rheonomic parametrizations is by direct evaluation upon use of the following crucial spinor identities.

Let us define

$$\mathcal{U} = \left(\frac{3}{2}\mathbf{1} + \frac{1}{4}\mathcal{P}\right). \tag{3.62}$$

We can verify that:

$$(\eta_A\tau^\alpha\mathcal{U}\tau^\alpha\eta_B - \eta_A\tau^{\alpha\beta}\eta_B)\mathcal{A}^{AB} = \mathcal{K}^{\alpha\beta}\mathcal{K}_{AB}\mathcal{A}^{AB}. \tag{3.63}$$

Furthermore, naming:

$$\Delta\mathcal{B}^\alpha = -\frac{1}{8}\eta_A^T\tau^\alpha\eta_B\mathcal{A}^{AB} \tag{3.64}$$

$$\Delta\omega^{\alpha\beta} = \frac{e}{4}\eta_A^T\tau^{\alpha\beta}\eta_B\mathcal{A}^{AB} - \frac{e}{4}\mathcal{K}^{\alpha\beta}\mathcal{K}_{AB}\mathcal{A}^{AB} \tag{3.65}$$

we obtain:

$$-\Delta\omega^{\alpha\beta}\wedge\Delta\mathcal{B}^\beta = \frac{e}{8}\eta_A^T\tau^\alpha\eta_B\mathcal{A}^{AC}\wedge\mathcal{A}^{CB}. \tag{3.66}$$

[2]With respect to the results obtained in [35] for the mini-superspace extension of M-theory configuration everything is identical in Eqs. (3.57), (3.58), (3.59), and (3.60) except the obvious reduction of the index range of $(\alpha, \beta, \dots)$ from 7 to 6-values. The only difference is in Eq. (3.61) where the last contribution proportional to the Kähler form is an essential novelty of this new type of compactification.

These identities together with the $d = 4$ spinor identities (A11) and (A12) suffice to verify that the above ansatz satisfies the required equations.

F. Gauge completion of the $\mathbf{B}^{[2]}$ form

The next task in order to write the explicit form of the pure spinor sigma-model is the derivation of the explicit expression for the $\mathbf{B}^{[2]}$ form. When this is done we will be able to write the complete Green Schwarz action in explicit form.

There is an ansatz for $\mathbf{B}^{[2]}$ which is the following one:

$$\mathbf{B}^{[2]} = \alpha \bar{\chi}_x \chi_y \bar{\eta}_A \tau_7 \eta_B \Phi_A^x \wedge \Phi_B^y. \tag{3.67}$$

By explicit evaluation we verify that with

$$\alpha = \frac{1}{4e}. \tag{3.68}$$

The rheonomic parametrization of the H-field strength is satisfied, namely:

$$d\mathbf{B}^{[2]} = -\mathrm{i}\bar{\psi} \wedge \Gamma_{\underline{a}} \Gamma_{11} \psi \wedge V^{\underline{a}}. \tag{3.69}$$

G. Rewriting the mini-superspace gauge completion as MC forms on the complete supercoset

Next, following the procedure introduced in [32], we rewrite the mini-superspace extension of the bosonic solution solely in terms of Maurer Cartan forms on the supercoset (3.2). Let the graded matrix $\mathbb{L} \in \mathrm{Osp}(6|4)$ be the coset representative of the coset $\mathcal{M}^{10|24}$, such that the Maurer Cartan form Σ can be identified as:

$$\Sigma = \mathbb{L}^{-1} d\mathbb{L}. \tag{3.70}$$

Let us now factorize $\mathbb{L}$ as in [32]:

$$\mathbb{L} = \mathbb{L}_F \mathbb{L}_B, \tag{3.71}$$

where $\mathbb{L}_F$ is a coset representative for the coset:

$$\frac{\mathrm{Osp}(6|4)}{\mathrm{SO}(6) \times \mathrm{Sp}(4, \mathbb{R})} \ni \mathbb{L}_F, \tag{3.72}$$

just in Eq. (3.72) but $\mathbb{L}_B$ rather than being the $\mathrm{Osp}(6|4)$ embedding of a coset representative of just AdS_4, is the embedding of a coset representative of $\mathrm{AdS}_4 \times \mathbb{P}^3$, namely:

$$\mathbb{L}_B = \begin{pmatrix} \mathrm{L}_{\mathrm{AdS}_4} & 0 \\ 0 & \mathrm{L}_{\mathbb{P}^3} \end{pmatrix}; \quad \frac{\mathrm{Sp}(4, \mathbb{R})}{\mathrm{SO}(1, 3)} \ni \mathrm{L}_{\mathrm{AdS}_4}; \quad \frac{\mathrm{SO}(6)}{\mathrm{U}(3)} \ni \mathrm{L}_{\mathbb{P}^3}. \tag{3.73}$$

In this way we find:

$$\Sigma = \mathbb{L}_B^{-1} \Sigma_F \mathbb{L}_B + \mathbb{L}_B^{-1} d\mathbb{L}_B. \tag{3.74}$$

Let us now write the explicit form of Σ_F, as in [32]:

$$\Sigma_F = \begin{pmatrix} \Delta_F & \Phi_A \\ 4\mathrm{i}e\bar{\Phi}\gamma_5 & -e\tilde{\mathcal{A}}_{AB} \end{pmatrix}, \tag{3.75}$$

where Φ_A is a Majorana-spinor valued fermionic 1-form and where Δ_F is an $\mathfrak{sp}(4, \mathbb{R})$ Lie algebra valued 1-form presented as a 4×4 matrix. Both Φ_A as Δ_F and $\tilde{\mathcal{A}}_{AB}$ depend only on the fermionic θ coordinates and differentials.

On the other hand we have:

$$\mathbb{L}_B^{-1} d\mathbb{L}_B = \begin{pmatrix} \Delta_{\mathrm{AdS}_4} & 0 \\ 0 & \mathcal{A}_{\mathbb{P}^3} \end{pmatrix}, \tag{3.76}$$

where the Δ_{AdS_4} is also an $\mathfrak{sp}(4, \mathbb{R})$ Lie algebra valued 1-form presented as a 4×4 matrix, but it depends only on the bosonic coordinates x^μ of the anti-de Sitter space AdS_4. In the same way $\mathcal{A}_{\mathbb{P}^3}$ is an $\mathfrak{su}(4)$ Lie algebra element presented as an $\mathfrak{so}(6)$ antisymmetric matrix in six-dimensions. It depends only on the bosonic coordinates y^α of the internal $\mathbb{P}^3$ manifold. According to Eq. (3.7), we can write:

$$\Delta_{\mathrm{AdS}_4} = -\frac{1}{4} B^{ab} \gamma_{ab} - 2e\gamma_a \gamma_5 B^a, \tag{3.77}$$

where $\{B^{ab}, B^a\}$ are, respectively, the spin-connection and the vielbein of AdS_4.

Similarly, using the inversion formula (B3) presented in appendix we can write:

$$\mathcal{A}_{\mathbb{P}^3} = \left(-2\mathcal{B}^\alpha \bar{\tau}_\alpha + \frac{1}{4e} \mathcal{B}^{\alpha\beta} \bar{\tau}_{\alpha\beta} - \frac{1}{4e} \mathcal{B}^{\alpha\beta} \mathcal{K}_{\alpha\beta} K\right), \tag{3.78}$$

where $\{\mathcal{B}^{\alpha\beta}, \mathcal{B}^\alpha\}$ are the connection and vielbein of the internal coset manifold $\mathbb{P}^3$.

Relying once again on the inversion formulas discussed in the appendix we conclude that we can rewrite Eqs. (3.57), (3.58), (3.59), (3.60), and (3.61) as follows:

$$\Psi^{x|A} = \Phi^{x|A} \tag{3.79}$$

$$V^a = E^a \tag{3.80}$$

$$V^\alpha = E^\alpha \tag{3.81}$$

$$\omega^{ab} = E^{ab} \tag{3.82}$$

$$\omega^{\alpha\beta} = E^{\alpha\beta}, \tag{3.83}$$

where the objects introduced above are the MC forms on the supercoset (3.2) according to:

$$\Sigma = \mathbb{L}^{-1} d\mathbb{L} = \begin{pmatrix} -\frac{1}{4}E^{ab}\gamma_{ab} - 2e\gamma_a\gamma_5 E^a & \Phi \\ 4ie\bar{\Phi}\gamma_5 & 2eE^\alpha\bar{\tau}_\alpha - \frac{1}{4}\mathcal{B}^{\alpha\beta}\bar{\tau}_{\alpha\beta} + \frac{1}{4}E^{\alpha\beta}\mathcal{K}_{\alpha\beta}K \end{pmatrix}. \tag{3.84}$$

Consequently the gauge completion of the $\mathbf{B}^{[2]}$ form becomes:

$$\mathbf{B}^{[2]} = \frac{1}{4e}\bar{\Phi}(1 \otimes \bar{\tau}_7) \wedge \Phi. \tag{3.85}$$

IV. PURE SPINORS FOR Osp(6|4)

In the present section, we show that the number of independent pure spinor components obtained by solving the pure spinor constraint in the present background matches correctly the number of anticommuting θ's. This implies that, at least formally (since it must be proved in detail) the number of bosonic and fermionic fields match leading to a conformal invariant theory. However, as is known, this is not sufficient for having a conformal invariant theory since all loop contributions to the Weyl anomaly should cancel. This can be guaranteed only by symmetry reasons and for the vanishing of one-loop contribution.

Nevertheless, we study the pure spinor equations adapted to the present background and we will see that the number of the independent components of the pure spinors is equal to 14 (since we have an interacting theory with RR fields we cannot distinguish between left- and right-movers). We recall the form of the pure spinor constraints for type IIA theory

$$\bar{\lambda}\Gamma_{\underline{a}}\lambda = 0, \qquad \bar{\lambda}\Gamma_{\underline{a}}\Gamma^{11}\lambda V^{\underline{a}} = 0, \tag{4.1}$$

$$\bar{\lambda}\Gamma_{[\underline{ab}]}\lambda V^{\underline{a}} V^{\underline{b}} = 0, \qquad \bar{\lambda}\Gamma^{11}\lambda = 0, \tag{4.2}$$

where we have combined the 16-component spinors λ_1 and λ_2 into a 32-component Dirac spinor λ. These equations are valid for any background and we have shown in [30] the number of independent components for the pure spinors matches the number of pure spinor in the Berkovits' ''background-independent'' constraints. However, in the present setting we can adapt the constraints to the specific background and, in particular, we choose to embed the vielbein $V^{\underline{a}}$ using his equation of motion in the momentum $\Pi^{\underline{a}}_{\pm} e^{\pm}$ and thus simplifying the constraints as follows

$$\begin{aligned} \bar{\lambda}\Gamma_a\lambda = 0, &\qquad a = 1, \ldots, 4, \\ \bar{\lambda}\Gamma_\alpha\lambda = 0, &\qquad \alpha = 1, \ldots, 6, \end{aligned} \tag{4.3}$$

$$\bar{\lambda}\Gamma_\pm\Gamma^{11}\lambda = 0, \qquad \bar{\lambda}\Gamma_{+-}\lambda = 0, \qquad \bar{\lambda}\Gamma^{11}\lambda = 0. \tag{4.4}$$

For $\Gamma_\pm$ we use the combination $\Gamma_1 \pm \Gamma_3$.

Now, we can insert the decomposition of λ on the basis of Killing spinors

$$\lambda = \chi_x \otimes \eta_A \Lambda^{x|A}, \tag{4.5}$$

where, as usual, χ_x are the AdS_4-Killing spinors and η_A are the $\mathbb{CP}^3$ Killing spinors. The free parameters $\Lambda^{x|A}$ are the components the pure spinors. Notice that the index x runs over the four independent AdS-Killing spinor basis and the index A runs over the six values of vector representation of SO(6). Therefore, we have in total 24 independent degrees of freedom to solve (4.3). The number of equations is independent of the background, but the number of independent degrees of freedom is reduced from 32 to 24 and therefore, we need to explore the existence of the solution.

Using the decomposition of the Gamma matrices provided in (3.43) and the normalizations of the Killing spinors $\chi_x C\gamma_5\chi_y = \epsilon_{xy}$ and $\eta_A\eta_B = \delta_{AB}$, Eqs. (4.3) read

$$\begin{aligned} (\chi_x C\gamma_a\chi_y)\delta_{AB}\Lambda^{x|A}\Lambda^{y|B} = 0, \\ (\chi_x C\gamma_5\chi_y)\eta_A\tau^\alpha\eta_B\Lambda^{x|A}\Lambda^{y|B} = 0, \end{aligned} \tag{4.6}$$

$$(\chi_x C\gamma_5\chi_y)\eta_A\tau^7\eta_B\Lambda^{x|A}\Lambda^{y|B} = 0, \tag{4.7}$$

$$\begin{aligned} (\chi_x C\gamma_5\gamma_\pm\chi_y)\eta_A\tau^7\eta_B\Lambda^{x|A}\Lambda^{y|B} = 0, \\ (\chi_x C\gamma_{+-}\chi_y)\delta_{AB}\Lambda^{x|A}\Lambda^{y|B} = 0. \end{aligned} \tag{4.8}$$

where C is charge conjugation matrix.

To solve these equations is convenient to adopt a new basis. Since we already know the solution in the basis when the spinor Λ is decomposed as follows

$$\begin{aligned} \lambda_1 = \phi_+ \otimes \zeta_1^+ + \phi_- \otimes \zeta_1^-, \\ \lambda_2 = \phi_+ \otimes \zeta_2^- + \phi_- \otimes \zeta_2^+ \end{aligned} \tag{4.9}$$

where:

$$\begin{aligned} \phi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; &\qquad \phi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \zeta_A^+ = \begin{pmatrix} 0 \\ \omega_A^+ \end{pmatrix}; &\qquad \zeta_A^- = \begin{pmatrix} \omega_A^- \\ 0 \end{pmatrix}, \end{aligned} \tag{4.10}$$

where $\omega_A^\pm$ are 8-dimensional vectors. In writing Eqs. (4.10) we have observed that the unique component of $\phi_\pm$ can always be reabsorbed in the normalization of $\omega_A^\pm$ and hence set to one. Thus, we have to express the entries of the rectangular matrix $\Lambda^{x|A}$ in terms of $\omega_A^\pm$ ($A = 1, 2$) and this can be done by combining λ_1 and λ_2 in a single 32-dimensional pure spinor and projecting it on the basis formed by $\chi_x \otimes \eta_A$ (where we left A running over 8 values) and we get the relation

$$\Lambda^{x|A} = \begin{pmatrix} \omega_{2,1}^- & \ldots & \omega_{2,8}^- \\ -i\omega_{1,1}^+ & \ldots & -i\omega_{1,8}^+ \\ -i\omega_{1,1}^- & \ldots & -i\omega_{1,8}^- \\ \omega_{2,1}^+ & \ldots & \omega_{2,8}^+ \end{pmatrix}. \tag{4.11}$$

In order to reduce the number of components to the necessary 24 ones, we will set the last components $\omega^{\pm}_{A,7}$ and $\omega^{\pm}_{A,8}$ to zero. In order to check if this is possible it is convenient first to exploit all gauge symmetries.

We recall that λ_A are solutions of the constraints if the components $\omega^{\pm}_{A}$ are decomposed in the following way

$$\begin{aligned} \omega^+_1 &= (\varpi^\alpha, 0) & \omega^-_2 &= (\pi^\alpha, 0) \\ \omega^-_1 &= (a^{\alpha\beta\gamma}\chi_\beta\varpi_\gamma, \chi\cdot\varpi) & \omega^+_2 &= (a^{\alpha\beta\gamma}\xi_\beta\pi_\gamma, \xi\cdot\pi) \end{aligned} \tag{4.12}$$

in terms of 7-component fields ϖ^α, π^α, ξ^α, χ^α satisfying the constraints

$$\varpi\cdot\varpi = 0 \tag{4.13}$$

$$\pi\cdot\pi = 0 \tag{4.14}$$

$$a^{\alpha\beta\gamma}\chi_\alpha\pi_\beta\varpi_\gamma = 0 \tag{4.15}$$

$$a^{\alpha\beta\gamma}\xi_\alpha\pi_\beta\varpi_\gamma = 0. \tag{4.16}$$

Here $a^{\alpha\beta\gamma}$ is the totally-antisymmetric invariant tensor for G_2 group. Notice that constraints (4.13), (4.14), (4.15), and (4.16) are invariant under the gauge symmetry

$$\begin{aligned} \chi_\alpha &\rightarrow \chi_\alpha + x_1\pi_\alpha + x_2\varpi_\alpha, \\ \xi_\alpha &\rightarrow \xi_\alpha + x_3\pi_\alpha + x_4\varpi_\alpha. \end{aligned} \tag{4.17}$$

On the other side, the decomposition (4.12) is not invariant under the symmetries parametrized by x_1 and x_4. So, there are only two gauge symmetries generated by x_2 and x_3 which can be used to set some components of χ_α and ξ_α to zero.

In order to reduce the number of independent degrees of freedom from 32 to 24, we set ϖ^7 and π^7 to zero, this condition, together with (4.13) and (4.14), implies that ω^+_1 and ω^-_2 have, respectively, 5 and 5 independent degrees of freedom. In addition, we impose the equations

$$\chi\cdot\varpi = 0, \qquad a^{7\beta\gamma}\chi_\beta\varpi_\gamma = 0, \tag{4.18}$$

$$\xi\cdot\pi = 0, \qquad a^{7\beta\gamma}\xi_\beta\pi_\gamma = 0. \tag{4.19}$$

such that the 7th and the 8th components of $\Lambda^{x|A}$ are zero. Together with constraints (4.15) and (4.16), they can be solved in terms of 3 components of χ_α and 3 components ξ_α. This reduces the number of unfixed components from 14 to 8. Using the gauge symmetries (4.17), we can lower them to 6 unfixed components. Finally, observe that there are two additional gauge symmetries generated by the constraints $\pi^7 = 0$ and $\varpi^7 = 0$ which reduce the number of unfixed parameters for χ_α and ξ_α to 4. The total counting of the pure spinor conditions, in the space of 24 components of the matrix $\Lambda^{x|A}$, is exactly 14 (5 for ϖ, 5 for π, 2 for χ and 2 for ξ), which is the correct number of degrees of freedom in order to cancel the total central charge. Indeed, we have 10 from the boson $x^{\underline{a}}$, 24 for θ's and the bosons Λ which are 14 cancel the total charge.

In addition, we can compute the number of the conjugate fields for the θ and for w and using the constraints and the gauge symmetry it is easy to perform the same computations as in [30] to see that the number matches again.

V. ACTION

Following the notations of [26] the complete action of Pure Spinor superstrings on Type IIA backgrounds is the sum of two parts, the Green-Schwarz action plus the gauge-fixing action containing the pure spinor sector:

$$\mathcal{A}^{\mathrm{IIA}}_{\mathrm{PS}} = \int \mathcal{L}_{\mathrm{GS}} + \int \mathcal{L}^{\mathrm{IIA}}_{gf}. \tag{5.1}$$

The GS action is written as follows

$$\mathcal{L}_{\mathrm{GS}} = \Big(\Pi^{\underline{a}}_{+} V^{\underline{b}}\eta_{\underline{ab}} \wedge e^+ - \Pi^{\underline{a}}_{-} V^{\underline{b}}\eta_{\underline{ab}} \wedge e^- + \frac{1}{2}\Pi^{\underline{a}}_{i}\Pi^{\underline{b}}_{j}\eta^{ij}\eta_{\underline{ab}} e^+ \wedge e^-\Big) + \frac{1}{2}\mathbf{B}^{[2]}, \tag{5.2}$$

where $\Pi^{\underline{a}}_{\pm}$ are auxiliary fields whose field equations identify them with the pullback of the target-space vielbein $V^{\underline{a}}$ on the world sheet, respectively, along the *zweibein* e^+ and e^-. η_{ij} and $\eta_{\underline{ab}}$ are the Minkowskian flat metrics, respectively, on the world sheet and on the 10d target space. The variation in the *zweibein* yields the Virasoro constraints. The background geometry of the world sheet encoded in the reference frame $e^{\pm}$ is treated classically [36,37].

The gauge-fixing terms of the string-action is written in [26] as:

$$\begin{aligned} \mathcal{L}^{\mathrm{IIA}}_{gf} = {}& \bar{\mathbf{d}}_+\psi_R \wedge e^+ + \bar{\mathbf{d}}_-\psi_L \wedge e^- + \frac{\mathrm{i}}{2}\bar{\mathbf{d}}_+\mathcal{M}_-\mathbf{d}_- \\ & + \bar{w}_+\mathcal{D}\lambda_R \wedge e^+ + \bar{w}_-\mathcal{D}\lambda_L \wedge e^- \\ & - \frac{\mathrm{i}}{2}\bar{w}_+(\mathcal{S}_R\mathcal{M}_-)\mathbf{d}_- + \frac{\mathrm{i}}{2}\bar{\mathbf{d}}_+(\mathcal{S}_L\mathcal{M}_-)w_- \\ & - \frac{\mathrm{i}}{2}\bar{w}_+(\mathcal{S}_R\mathcal{S}_L\mathcal{M}_-)w_- + \frac{\mathrm{i}}{2}\bar{w}_+\mathcal{M}_-\{\mathcal{S}_L, \mathcal{S}_R\}w_-. \end{aligned} \tag{5.3}$$

The operators $\mathcal{S}_{L/R}$ represent the components of the BRST operator $\mathcal{S}$ which are parametrized by the left/right components of the pure spinor λ. The subscript $\pm$ on the spinor matrices refer to their action on fermions with left/right chirality, respectively. The last term is generated by the nonvanishing the $\mathcal{S}_L\mathcal{S}_R$-piece of the action in [26]. With reference to [26], we note that on the considered background the operator $\hat{\mathcal{S}}_{L/R}$ coincide with $\mathcal{S}_{L/R}$ since $\mathcal{H}^{abc}$ field strength vanishes in this case.

The bosonic background corresponding to the $\mathrm{AdS}_4 \times \mathbb{P}^3$ solution of Type IIA theory is characterized by the values of the background fields displayed in Eq. (3.39). The spinor matrices $\mathcal{M}$ and $\mathcal{N}^{(\mathrm{even})}$, encoding the RR field-strengths, are given in Eqs. (3.47) and (3.48) respectively. The matrix $\mathcal{M}$ in the present background is constant and, therefore we can eliminate the auxiliary fields $\mathbf{d}_\pm$ and write the complete quadratic part of the action in terms of the MC forms. We start from the first two lines of (5.3)

$$\mathcal{L}^{\mathrm{IIA}}_{gf,2} = \bar{\mathbf{d}}_+ \psi_R \wedge e^+ + \bar{\mathbf{d}}_- \psi_L \wedge e^- + \frac{\mathrm{i}}{2}\bar{\mathbf{d}}_+ \mathcal{M}_- \mathbf{d}_- e^+ \wedge e^-. \quad (5.4)$$

We use the decomposition of the gravitinos

$$\Psi = \psi_+ e^+ + \psi_- e^- = \chi_x \otimes \eta_A (\Phi_+^{xA} e^+ + \Phi_-^{xA} e^-),$$

where the 1-form is pullback onto the world sheet, then (5.4) yields

$$\mathcal{L}^{\mathrm{IIA}}_{gf,2} = \Big(-\mathbf{d}_+^T \frac{C(1-\Gamma_{11})}{2}\psi_- + \mathbf{d}_-^T \frac{C(1+\Gamma_{11})}{2}\psi_+ + \frac{\mathrm{i}}{2}\mathbf{d}_+^T C\mathcal{M}_- \mathbf{d}_-\Big) e^+ \wedge e^-. \quad (5.5)$$

By eliminating the d's, we have

$$\mathcal{L}^{\mathrm{IIA}}_{gf,2} = -2\mathrm{i}\psi_+^T \frac{C(1-\Gamma_{11})}{2}\mathcal{M}_-^{-1}\frac{(1-\Gamma_{11})}{2}\psi_-. \quad (5.6)$$

and after some simple algebra, one gets

$$\mathcal{L}^{\mathrm{IIA}}_{gf,2} = -\frac{1}{2e}\Phi_+^T (C_4 \otimes \bar{\tau}^7 + \mathrm{i}C_4\gamma^5 \otimes 11_6)\Phi_-. \quad (5.7)$$

Finally summing the $B^{[2]}$ part and the contribution of the ghost fields we have the quadratic part of the fermionic action

$$\mathcal{L}^{\mathrm{IIA}}_{gf,2} - \frac{1}{e}\Phi_+^T\Big(\frac{1}{4}C_4 \otimes \bar{\tau}^7 - \frac{\mathrm{i}}{2}C_4\gamma^5 \otimes 11_6\Big)\Phi_- e^+ \wedge e^- + \Big(\frac{1}{2}w_-^T (C_4 \otimes 11_6 - \gamma^5 \otimes \bar{\tau}^7)\nabla_+ \lambda - \frac{1}{2}w_+^T (C_4 \otimes 11_6 + \gamma^5 \otimes \bar{\tau}^7)\nabla_- \lambda\Big) e^+ \wedge e^-. \quad (5.8)$$

Notice that the matrices $(C_4 \otimes 11_6 \pm \gamma^5 \otimes \bar{\tau}^7)$ are projectors and by using the result of the appendix B, $\bar{\tau}^7_{AB} = \bar{\eta}_A \tau^7 \eta_B = K_{AB}$, we see that the projectors couple the 4-d chirality to the eigenspaces of K_{AB}.

The third line of Eq. (5.3) vanishes on our background by showing that

$$\mathcal{S}_{L/R}\mathcal{M} = \mathcal{S}_R \mathcal{S}_L \mathcal{M} = 0.$$

Using the formulas in [26] one can easily verify that $\mathcal{S}\mathcal{M} = 0$ since the BRST transformation of the RR field strengths $G_{\underline{ab}}, G_{\underline{abcd}}$ vanishes as a consequence of the fact that, on our background, $\chi = \mathcal{D}_{\underline{a}}\chi = \rho_{\underline{ab}} = 0$. The vanishing of $\mathcal{S}_R\mathcal{S}_L\mathcal{M} = 0$, on the other hand, follows from the properties $\mathcal{S}\chi = \mathcal{S}\mathcal{D}_{\underline{a}}\chi = \mathcal{S}\rho_{\underline{ab}} = 0$, which must hold for consistency and which can be recast, on our background, in the following way:

$$\mathcal{S}\chi = \mathcal{N}\lambda = 0, \qquad \mathcal{S}\mathcal{D}_{\underline{a}}\chi = -\mathcal{N}\mathcal{M}\Gamma_{\underline{a}}\lambda = 0,$$
$$\mathcal{S}\rho_{\underline{ab}} = \Big(\mathcal{M}\Gamma_{[\underline{a}}\mathcal{M}\Gamma_{\underline{b}]} - \frac{1}{4}R_{\underline{ab},\underline{cd}}\Gamma^{\underline{cd}}\Big)\lambda = 0.$$

The above equations are satisfied in virtue of the ansatz (4.5) and the Killing spinor Eqs. (3.49) and (3.50).

The last line can be computed and we get

$$\mathcal{L}^{\mathrm{IIA}}_{gf,4} = \frac{1}{4}\bar{w}_+ \mathcal{M}_- \Gamma_{\underline{ab}} w_- \bar{\lambda}_L \Gamma^{[\underline{a}}\mathcal{M}_+\Gamma^{\underline{b}]}\lambda_R. \quad (5.9)$$

By simple algebra, (5.9) can be decomposed in terms of the eigenspaces of $\mathcal{K}_{AB}$ and of given chiralities so as to get the expected form of the action

$$\mathcal{L}^{\mathrm{IIA}}_{gf,4} = R^{ab,cd}N_{ab,+}N_{cd,-} + R^I{}_K{}^J{}_L N_{I,+}{}^K N_{J,-}{}^L, \quad (5.10)$$

where $R^{ab,cd}$ is the AdS_4 Riemann tensor and $R^I{}_K{}^J{}_L$ is the Riemann tensor for $\mathbb{P}^3$. The bilinears $N_{ab}, N_{I,+}{}^K$ are the Lorentz generators of SO(1,3) and of U(3) of the subgroup of the coset $\mathrm{Osp}(6|4)/\mathrm{SO}(1,3) \times \mathrm{U}(3)$. They can be written compactly in $4 \oplus 6$ notation as follows

$$\begin{aligned} N_{\underline{ab},+} &\equiv \bar{w}_+ \Gamma_{ab}\lambda_R \\ &= -\frac{i}{8}(\bar{w}_{I,+}(\mathbf{1}+\gamma_5)\gamma_{ab}\lambda^I + \bar{w}_-^I(\mathbf{1}-\gamma_5)\gamma_{ab}\lambda_I) \\ N_{\underline{ab},-} &\equiv \bar{w}_- \Gamma_{\underline{ab}}\lambda_L \\ &= -\frac{i}{8}(\bar{w}_-^I(\mathbf{1}+\gamma_5)\gamma_{ab}\lambda_I + \bar{w}_{I,-}(\mathbf{1}-\gamma_5)\gamma_{ab}\lambda^I). \end{aligned} \quad (5.11)$$

Notice that the specific form of the action is dictated by the invariance under the gauge symmetry of the subgroup $\mathrm{SO}(1,3) \times \mathrm{U}(3)$ and by the pure spinor conditions. By using the decomposition as in [12] it is easy to perform the Fierz identities. Even if the result is written in a different notation, the equivalence with [13] can be easily checked.

VI. CONCLUSION

We have shown how to derive the pure spinor sigma model for the background $\mathrm{AdS}_4 \times \mathbb{P}^3$. Using the formulation provided in [26], we have specified all tensors appearing in the general action and we have compared with the formulation derived in [12]. The action is the classical starting point form where to compute higher order corrections in α'. Of course, one can repeat the work done in the case of $\mathrm{AdS}_5 \times S^5$ and check the conformal invariance. We leave this work to a future work.

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APPENDIX A: $D = 6$ AND $D = 4$ GAMMA MATRIX BASES

In the discussion of the $\mathrm{AdS}_4 \times \mathbb{P}^3$ compactification we need to consider the decomposition of the $D = 10$ gamma matrix algebra into the tensor product of the $\mathfrak{so}(6)$ Clifford algebra times that of $\mathfrak{so}(1, 3)$. In this section we discuss and explicit basis for the $\mathfrak{so}(6)$ gamma matrix algebra using that of $\mathfrak{so}(7)$. Conventionally we identify the 7-matrix τ_7 with the chirality matrix in $d = 6$.

1. $D = 6$ Clifford algebra

In this paper, the indices $\alpha, \beta, \ldots$ run on six values and denote the vector indices of $\mathfrak{so}(6)$. In order to discuss the gamma matrix basis we introduce $\mathfrak{so}(7)$ indices

$$\bar{\alpha} = \alpha, 7 \tag{A1}$$

which run on seven values and we define the Clifford algebra with negative metric:

$$\{\tau_{\bar{\alpha}}, \tau_{\bar{\beta}}\} = -\delta_{\overline{\alpha\beta}} \tag{A2}$$

This algebra is satisfied by the following, real, antisymmetric matrices:

$$\tau_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tau_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \tau_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tau_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}; \quad \tau_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \tag{A3}$$

$$\tau_7 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

2. $D = 4$ γ-matrix basis and spinor identities

In this section we construct a basis of $\mathfrak{so}(1, 3)$ gamma matrices such that it explicitly realizes the isomorphism $\mathfrak{so}(2, 3) \sim \mathfrak{sp}(4, \mathbb{R})$ with the conventions used in the main text. Naming σ_i the standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A4}$$

we realize the $\mathfrak{so}(1,3)$ Clifford algebra:

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}; \qquad \eta_{ab} = \mathrm{diag}(+,-,-,-) \tag{A5}$$

by setting:

$$\begin{aligned} &\gamma_0 = \sigma_2 \otimes \mathbf{1}; \qquad \gamma_1 = \mathrm{i}\sigma_3 \otimes \sigma_1 \qquad \gamma_2 = \mathrm{i}\sigma_1 \otimes \mathbf{1}; \\ &\gamma_3 = \mathrm{i}\sigma_3 \otimes \sigma_3 \qquad \gamma_5 = \sigma_3 \otimes \sigma_2; \qquad \mathcal{C} = \mathrm{i}\sigma_2 \otimes \mathbf{1}, \end{aligned} \tag{A.6}$$

where γ_5 is the chirality matrix and $\mathcal{C}$ is the charge conjugation matrix. From the general theory (see for instance Eqs. (2.2) and (2.3) of [12]) we see that the antisymmetric matrix entering the definition of the orthosymplectic algebra, namely $\mathcal{C}\gamma_5$ is the following one:

$$\mathcal{C} = \mathrm{i}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{C}\gamma_5 = \epsilon = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{A7}$$

namely it is proportional, through an overall i-factor, to a real completely off-diagonal matrix. On the other hand all the generators of the $\mathfrak{so}(2,3)$ Lie algebra, *i.e.* γ_{ab} and $\gamma_a\gamma_5$ are real, symplectic 4×4 matrices. Indeed we have

$$\begin{aligned}
&\gamma_{01} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \qquad \gamma_{02} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&\gamma_{12} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_{13} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
&\gamma_{23} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}; \qquad \gamma_{34} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
&\gamma_0\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \qquad \gamma_1\gamma_5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&\gamma_2\gamma_5 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \qquad \gamma_3\gamma_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned} \tag{A8}$$

On the other hand we find that $\mathcal{C}\gamma_0 = \mathrm{i}\mathbf{1}$. Hence the Majorana condition becomes:

$$\mathrm{i}\,\psi = \psi^\star \tag{A9}$$

so that a Majorana spinor is just a real spinor multiplied by an overall phase $\exp[-i\frac{\pi}{4}]$.

These conventions being xed let χ_x $(x = 1, \ldots, 4)$ be a set of (commuting) Majorana spinors normalized in the following way:

$$\begin{aligned} \chi_x &= \mathcal{C}\bar{\chi}_x^T; && \text{Majorana condition} \\ \bar{\chi}_x\gamma_5\chi_y &= \mathrm{i}(\mathcal{C}\gamma_5)_{xy}; && \text{symplectic normal basis.} \end{aligned} \tag{A10}$$

Then by explicit evaluation we can verify the following Fierz identity:

$$\frac{1}{2}\gamma^{ab}\chi_z\bar{\chi}_x\gamma_5\gamma_{ab}\chi_y - \gamma_a\gamma_5\chi_z\bar{\chi}_x\gamma_a\chi_y = -2\mathrm{i}[(C\gamma_5)_{zx}\chi_y + (C\gamma_5)_{zy}\chi_x]. \tag{A11}$$

Another identity which we can prove by direct evaluation is the following one:

$$\bar{\chi}_x\gamma_5\gamma_{ab}\chi_y\bar{\chi}_z\gamma^b\chi_t - \bar{\chi}_z\gamma_5\gamma_{ab}\chi_t\bar{\chi}_x\gamma^b\chi_y = \mathrm{i}(\bar{\chi}_x\gamma_a\chi_t(\mathcal{C}\gamma_5)_{yz} + \bar{\chi}_y\gamma_a\chi_t(\mathcal{C}\gamma_5)_{xz} + \bar{\chi}_x\gamma_a\chi_z(\mathcal{C}\gamma_5)_{yt} + \bar{\chi}_y\gamma_a\chi_z(\mathcal{C}\gamma_5)_{xt}). \tag{A12}$$

Finally let us mention some relevant formulas for the derivation of the compactification. With the above conventions we find:

$$\gamma_0\gamma_1\gamma_2\gamma_3 = \mathrm{i}\gamma_5 \tag{A13}$$

and if we x the convention:

$$\epsilon_{0123} = +1 \tag{A14}$$

we obtain:

$$\frac{1}{24}\epsilon^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d = -\mathrm{i}\gamma_5. \tag{A15}$$

APPENDIX B: AN $\mathfrak{so}(6)$ INVERSION FORMULA

In order to discuss the conversion of supergravity forms into MC forms of the supercoset a key role is played by an inversion formula which we utilize in the main text and we discuss in this appendix. Let us define the following set of 6×6 matrices:

$$\bar{\tau}^{\alpha}_{AB} \equiv \eta_A^T\tau^{\alpha}\eta_B \qquad \bar{\tau}^{\alpha\beta}_{AB} = \eta_A^T\tau^{\alpha\beta}\eta_B$$
$$K_{AB} = \mathcal{K}_{AB} = \frac{1}{2}\mathcal{K}_{\alpha\beta}\bar{\tau}^{\alpha\beta}_{AB}. \tag{B1}$$

where η_A are the 6 killing internal killing spinors and τ denote the 1-index and 2-index $\mathfrak{so}(6)$ gamma-matrices. By construction the barred $\bar{\tau}$.s are antisymmetric 6×6 matrices, hence $\mathfrak{so}(6)$ generators in the fundamental representation just as the Kähler form K. Counting these matrices we find that they are $6 + 15 + 1$, namely, 22, which is too much as a set of independent generators of $\mathfrak{so}(6)$. This means that there must be linear dependences. By calculating traces of these matrices we find that the 6 matrices $\bar{\tau}^{\alpha}$ are linear independent and orthogonal to the 15, $\bar{\tau}^{\alpha\beta}$, and to the unique K while among these latter 16 matrices only 9 are linear independent.

This observation is important for the following reason. When we write the following formulas:

$$\Delta\mathcal{B}^{\alpha} = -\frac{1}{8}\bar{\tau}^{\alpha}_{AB}\mathcal{A}^{AB}$$
$$\Delta\mathcal{B}^{\alpha\beta} = \frac{e}{4}\bar{\tau}^{\alpha\beta}_{AB}\mathcal{A}^{AB} - \frac{e}{4}\mathcal{K}^{\alpha\beta}K_{AB}\mathcal{A}^{AB} \tag{B2}$$

we are actually decomposing the $\mathfrak{so}(6)$ connection $\mathcal{A}^{AB}$ along an over-complete basis of $15 + 6 = 21$ generators of $\mathfrak{so}(6)$, which is obviously a well defined operation.

It is interesting to establish the inverse formula, namely, to express the original connection $\mathcal{A}^{AB}$ in terms of the over-complete set of objects ΔB^{α} and $\Delta B^{\alpha\beta}$. The inverse formula can be established by means of direct calculation in the explicit τ-matrix basis we have chosen and we find what follows:

$$\mathcal{A}_{AB} = \left(-2\Delta\mathcal{B}^{\alpha}\bar{\tau}_{\alpha} + \frac{1}{4e}\Delta\mathcal{B}^{\alpha\beta}\bar{\tau}_{\alpha\beta} - \frac{1}{4e}\Delta B^{\alpha\beta}\mathcal{K}_{\alpha\beta}K\right)_{AB}. \tag{B3}$$

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