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# EMBEDDING ALMOST-COMPLEX MANIFOLDS IN ALMOST-COMPLEX EUCLIDEAN SPACES 

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#### Abstract

We show that any compact almost-complex manifold $(M, J)$ of complex dimension $m$ can be pseudo-holomorphically embedded in $\mathbb{R}^{6 m}$ equipped with a suitable almost-complex structure $\widetilde{J}$.


KEYWORDS: embedding, almost-complex structure, manifold, pseudo-holomorphic embedding.
AMS Classification: 32Q60, 32H02.

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## 1. Introduction

An almost-complex structure on a $2 n$-dimensional smooth manifold $M$ is a tensor $J \in$ $\operatorname{End}(T M)$ such that $J^{2}=-\mathrm{id}$. If $M$ is oriented we say that $J$ is positive if the orientation induced by $J$ on $M$ agrees with the given one. An almost-complex structure is called integrable if it is induced by a holomorphic atlas. In dimension two any almost-complex structure is integrable, while in higher dimension this is far from true. A smooth map $f: N \rightarrow M$ between two almost-complex manifolds $\left(N, J^{\prime}\right),(M, J)$ is called pseudo-holomorphic if $J \circ T f=T f \circ J^{\prime}$, where $T f: T N \rightarrow T M$ is the tangent map of $f$. When the map $f$ is an embedding, $\left(N, J^{\prime}\right)$ is said to be an almost-complex submanifold of $(M, J)$. In this case we can identify $N$ with its image $f(N) \subset M$ and the almost-complex structure $J^{\prime}$ with the restriction of $J$ to $T N \cong$ $T(f(N)) \subset T M$.

If we equip $\mathbb{R}^{2 n}$ with the canonical complex structure, that is to say $\mathbb{R}^{2 n} \cong \mathbb{C}^{n}$, then it does not admit any compact complex submanifold (by the maximum principle). Thus, it is a very natural problem to ascertain if it is possible to find compact complex manifolds pseudo-holomorphically embedded in $\mathbb{R}^{2 n}$ equipped with an integrable or non-integrable almost-complex structure.

In [2] Calabi and Eckmann constructed the first examples of compact, simply connected complex manifolds $M_{p, q}$ which are not algebraic. Topologically $M_{p, q}$ is the product $S^{2 p+1} \times S^{2 q+1}$. Then by deleting a point on each factor one obtains a complex structure $J$ on $\mathbb{R}^{2 p+2 q+2}$. In section 5 of [2] it was shown that when $p, q>1$ there exists a complex torus as a complex submanifold of $\left(\mathbb{R}^{2 p+2 q+2}, J\right)[2$, p. 499]. It follows that the Calabi-Eckmann complex structure

[^0]$J$ on $\mathbb{R}^{2 n}$ cannot be tamed by any symplectic form and in particular cannot be Kähler. Calabi and Eckmann also observed that the only holomorphic functions on $\left(\mathbb{R}^{2 p+2 q+2}, J\right)$ are the constants answering negatively to a question raised by Bochner about the uniformization of complex structures on $\mathbb{R}^{2 n}$. In [1] Bryant constructed pseudo-holomorphic non-constant maps $\varphi: M^{2} \rightarrow S^{6}$ for any compact Riemann surface $M^{2}$, where $S^{6}$ is equipped with the almostcomplex structure induced by the octonion multiplication. These maps realize compact Riemann surfaces as pseudo-holomorphic singular curves in $S^{6}$.

In [3] was showed that any almost-complex torus $\mathbb{T}^{n}=\mathbb{R}^{2 n} / \Lambda$ can be pseudo-holomorphically embedded into $\left(\mathbb{R}^{4 n}, J_{\Lambda}\right)$ for a suitable almost-complex structure $J_{\Lambda}$. It follows that any compact Riemann surface can be realized as a pseudo-holomorphic curve of some $\left(\mathbb{R}^{2 n}, J\right)$, where $J$ is a suitable almost-complex structure.

In this paper we prove the following general theorem.
Theorem 1. Any compact almost-complex manifold $(M, J)$ of real dimension $2 m$ can be pseudoholomorphically embedded in $\left(\mathbb{R}^{6 m}, \widetilde{J}\right)$ for a suitable positive almost-complex structure $\widetilde{J}$.

In particular, any compact Riemann surface can be realized as a pseudo-holomorphic curve in $\left(\mathbb{R}^{6}, \widetilde{J}\right)$. In $[3]$ was shown that the torus is the only compact Riemann surface that can be pseudo-holomorphically embedded in $\left(\mathbb{R}^{4}, \widetilde{J}\right)$ for some $\widetilde{J}$.

## 2. Preliminaries

The space of positive linear complex structures on $\mathbb{R}^{2 n}$ is diffeomorphic to the homogeneus space $\widetilde{\mathfrak{J}}(n)=G L^{+}(2 n, \mathbb{R}) / G L(n, \mathbb{C})$ and is homotopy equivalent to $\mathfrak{J}(n)=S O(2 n) / U(n)$. So, an almost-complex structure $J$ on $\mathbb{R}^{2 n}$ can be regarded as a smooth map $J: \mathbb{R}^{2 n} \rightarrow \widetilde{\mathfrak{J}}(n)$.
Lemma 2. Let $M \subset \mathbb{R}^{2 n}$ be a closed submanifold and let $J: M \rightarrow \widetilde{\mathfrak{J}}(n)$ be a smooth map. Then there exists a smooth extension $\widetilde{J}: \mathbb{R}^{2 n} \rightarrow \widetilde{\mathfrak{J}}(n)$ if and only if $J$ is homotopic to a constant.
Proof. The 'only if' part follows immediately from the fact that $\mathbb{R}^{2 n}$ is contractible.
Let us prove the 'if' part. Consider a smooth homotopy $H: M \times[0,1] \rightarrow \widetilde{\mathfrak{J}}(2 n)$ such that $H_{0}(x)=J_{0}$ for all $x \in M$, and $H_{1}=J$ where $H_{t}(x)=H(x, t)$ and $J_{0} \in \widetilde{\mathfrak{J}}(n)$. We can extend $H$ to $\mathbb{R}^{2 n} \times\{0\} \subset \mathbb{R}^{2 n} \times[0,1]$ by setting $H(x, 0)=J_{0}$ for any $x \in \mathbb{R}^{2 n}$. By the homotopy extension property [4, Chapter 0] there exists $\widetilde{H}: \mathbb{R}^{2 n} \times[0,1] \rightarrow \widetilde{\mathfrak{J}}(n)$ which extends $H$. We conclude the proof by setting $\widetilde{J}=\widetilde{H}_{1}$.

Let $(M, J)$ be an almost-complex manifold. The strategy to prove Theorem 1 will be to choose an arbitrary embedding $f: M \hookrightarrow \mathbb{R}^{6 m}$, which exists for the weak Whitney embedding theorem, and to show that $J$ extends to the pullback $f^{*}\left(T \mathbb{R}^{6 m}\right)$ and this extension is null-homotopic.

Consider the standard filtration $S O(1) \subset S O(2) \subset \cdots$. Since $S O(n-1)$ contains the ( $n-2$ )skeleton of $S O(n)$ (because the standard fibration $S O(n) \rightarrow S^{n-1}$ ) it follows that the $k$-skeleton of $S O(n)$ is contained on $S O(k+1)$ for $0 \leq k \leq n-2$.

Since $S O(n) \subset U(n)$ it follows that $U(n)$ contains the $(n-1)$-skeleton of $S O(2 n)$ for $n \geq 1$. Then the homomorphism induced by the inclusion $i_{*}: \pi_{j}(U(n)) \rightarrow \pi_{j}(S O(2 n))$ is an isomorphism for $j \leq n-2$ and is an epimorphism for $j=n-1$.

From the homotopy exact sequence of the fibre bundle $S O(2 n) \rightarrow \mathfrak{J}(n)$ given by the projection map it follows that $\pi_{j}(\widetilde{\mathfrak{J}}(n)) \cong \pi_{j}(\mathfrak{J}(n)) \cong 0$ for $j \leq n-1$.
Definition 3. A space $X$ is said to be $n$-connected if $\pi_{j}(X) \cong 0$ for all $j \leq n$.
In particular, 0-connected means path-connected.
From the above considerations we have that $\widetilde{\mathfrak{J}}(n)$ is $(n-1)$-connected. The following proposition is well-known in the theory of CW-complexes.

Proposition 4. If $X$ is n-connected then any map $Y \rightarrow X$ defined on a $C W$-complex $Y$ of dimension $\leq n$ is homotopic to a constant.

Also the following proposition is standard, and we give only the idea of the proof.
Proposition 5. Let $\xi: E \rightarrow M$ be an oriented real vector bundle of rank $2 k$ over an m-manifold $M$. If $k \geq m$ then $\xi$ admits a positive complex structure.
Proof. Consider the bundle $\xi^{\mathfrak{J}}: \widetilde{\mathfrak{J}}(E) \rightarrow M$ with fibre $\widetilde{\mathfrak{J}}(k)$ induced by $\xi$. Namely, for any $p \in M$ the fibre of $\xi^{\mathfrak{J}}$ over $p$ is the space of positive linear complex structures on $\xi^{-1}(p)$. Since $\widetilde{\mathfrak{J}}(k)$ is $(k-1)$-connected, it follows that $\xi^{\mathfrak{J}}$ admits a section if $k \geq m$, see [7, Part III]. This section is a positive complex structure on $\xi$.

Let $f: M \rightarrow \mathbb{R}^{N}$ be an immersion. The normal bundle $\nu_{f}(M)$ is, as usual, the orthogonal complement of $T M$ in $f^{*}\left(T \mathbb{R}^{N}\right)$, that is to say:

$$
f^{*}\left(T \mathbb{R}^{N}\right)=T M \oplus \nu_{f}(M)
$$

If $M$ is oriented then the normal bundle can be equipped with a canonical orientation, namely that which makes the splitting of $f^{*}\left(T \mathbb{R}^{N}\right)$ into a Whitney sum of oriented fibre bundles, where $\mathbb{R}^{N}$ is considered with the standard orientation.

## 3. Proof of the main results

Theorem 6. Let $M \subset \mathbb{R}^{2 n}$ be a submanifold of even dimension endowed with an almostcomplex structure J. If the normal bundle of $M$ in $\mathbb{R}^{2 n}$ admits a positive complex structure with respect to the canonical orientation, then for any $k \geq \max \left(0, \operatorname{dim}_{\mathbb{R}} M-n+1\right)$ there exists an almost-complex structure $\tilde{J}$ on $\mathbb{R}^{2 n} \times \mathbb{R}^{2 k}$ such that $M \times\{0\} \subset \mathbb{R}^{2 n} \times \mathbb{R}^{2 k}$ is an almost-complex submanifold.

Proof. Let us choose a positive complex structure on the normal bundle of $M$. Then by taking the Whitney sum with the almost-complex structure on $M$ we get a complex structure on $\left(T \mathbb{R}^{2 n}\right)_{\mid M}$. So we obtain a smooth map $J: M \rightarrow \widetilde{\mathfrak{J}}(n)$.

In view of Lemma 2 our target is to get a $J$ null-homotopic. This is so if $\operatorname{dim}_{\mathbb{R}} M \leq n-1$ because $\widetilde{\mathfrak{J}}(n)$ is $(n-1)$-connected and Proposition 4.

If $\operatorname{dim}_{\mathbb{R}} M>n-1$ we take the product $\mathbb{R}^{2 n} \times \mathbb{R}^{2 k}$, where $\mathbb{R}^{2 k}$ is endowed with the standard complex structure, and we embed $M$ as $M \times\{0\}$. We get a complex structure on the normal bundle of $M$ in $\mathbb{R}^{2 n} \times \mathbb{R}^{2 k}$ in the obvious way. So we obtain a map $J_{k}: M \rightarrow \widetilde{\mathfrak{J}}(n+k)$. It follows that $J_{k}$ is homotopic to a constant if $k \geq \operatorname{dim}_{\mathbb{R}} M-n+1$. In this case $J_{k}$ extends on $\mathbb{R}^{2 n} \times \mathbb{R}^{2 k}$ by Lemma 2.

It follows that if $(M, J)$ is contained in $\mathbb{C}^{n}$ with a complex normal bundle and if $n \geq$ $2 \operatorname{dim}_{\mathbb{C}} M+1$, then there is a positive almost-complex structure $\widetilde{J}$ on $\mathbb{C}^{n}$ which makes $(M, J)$ an almost-complex submanifold of $\left(\mathbb{C}^{n}, \widetilde{J}\right)$.

Proof of Theorem 1. Let $f: M \hookrightarrow \mathbb{R}^{6 m}$ be any embedding. The normal bundle $\nu_{f}(M)$ has rank $4 m$ and is orientable. By Proposition 5 there is a complex structure on the normal bundle and then we conclude by an application of Theorem 6 with $k=0$.

In some cases we can construct an embedding in an euclidean space of lower dimension. Recall that an s-inverse of the tangent bundle $T M$ is a vector bundle $\xi$ such that $T M \oplus \xi$ is a trivial vector bundle. Observe that if $f: M \rightarrow \mathbb{R}^{N}$ is an immersion then the normal bundle $\nu_{f}(M)$ is a real s-inverse of the tangent bundle $T M$. The converse also holds and is a Theorem of Hirsch [5]. Namely,

Theorem 7. (Hirsch [5]) Any s-inverse of TM is the normal bundle of some immersion $f$ : $M \rightarrow \mathbb{R}^{N}$.

Let $\xi$ be a complex s-inverse of $(T M, J)$ of complex rank $k$, namely $T M \oplus \xi$ is trivial as a complex vector bundle. Now Hirsch's Theorem 7 imply that there exists an immersion $f: M \rightarrow$ $\mathbb{R}^{2(m+k)}$ such that $\xi$ is isomorphic to $\nu_{f}(M)$ as real vector bundles. So $\nu_{f}(M)$ carries a complex structure.

Up to a product with some $\mathbb{R}^{2 h}$, we can assume that $k \geq m+1$, and then $f$ is regularly homotopic, namely homotopic through immersions, to an embedding $f_{1}: M \rightarrow \mathbb{R}^{2(m+k)}$. It follows that $\nu_{f_{1}}(M) \cong \nu_{f}(M)$ carries a complex structure. Now apply Theorem 6 to get $\widetilde{J}$.

If the rank of $\xi$ satisfies $m+1 \leq k \leq 2 m-1$ we get a pseudo-holomorphic embedding in an euclidean space of complex dimension $m+k<3 m$.

Let $\left(S^{6}, J\right)$ be the six dimensional sphere equipped with the standard almost-complex structure $J$ obtained from the octonion multiplication. Theorem 1 imply that $\left(S^{6}, J\right)$ can be pseudoholomorphically embedded in $\left(\mathbb{R}^{18}, \widetilde{J}\right)$ for a suitable positive almost-complex structure $\widetilde{J}$. Using the existence of a low dimensional s-inverse of $\left(T S^{6}, J\right)$ we have the following result:

Corollary 8. The almost-complex sphere $\left(S^{6}, J\right)$ can be pseudo-holomorphically embedded in $\left(\mathbb{R}^{14}, \widetilde{J}\right)$ for a suitable positive almost-complex structure $\widetilde{J}$.

Proof. Since $S^{6}$ is embedded in $\mathbb{R}^{8}$ with trivial normal bundle we conclude by an application of Theorem 6 with $k=3$.

Notice that $\left(S^{6}, J\right)$ can not be pseudo-holomorphically embedded in $\left(\mathbb{R}^{12}, \widetilde{J}\right)$. In fact, the Euler class of the normal bundle of any embedding of $S^{6}$ in $\mathbb{R}^{12}$ is zero by a theorem of Whitney, see [6, p. 138]. On the other hand, if $S^{6}$ is contained pseudo-holomorphically in $\left(\mathbb{R}^{12}, \widetilde{J}\right)$, by a straightforward computation with the Chern class, we obtain for the Euler class $e\left(\nu\left(S^{6}\right)\right)=c_{3}\left(\nu\left(S^{6}\right)\right)=-2 \lambda \neq 0$, which is a contradiction, where $\lambda \in H^{6}\left(S^{6}\right)$ is the standard generator.

We conclude with a question. Since our construction is essentially homotopy-theoretic, we are unable to control the integrability of the almost-complex structure $\widetilde{J}$ of Theorem 1 . So the following question is very natural.

Question 9. Let $(M, J)$ be an integrable complex manifold. Is there an embedding of $(M, J)$ into an integrable $\left(\mathbb{R}^{2 n}, \widetilde{J}\right)$ ?

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