Stochastic comparisons of series and parallel systems with randomized independent components

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Abstract

Consider a series or parallel system of independent components and assume that the components are randomly chosen from two different batches, with the components of the first batch being more reliable than those of the second. In this note it is shown that the reliability of the system increases, in usual stochastic order sense, as the random number of components chosen from the first batch increases in increasing convex order. As a consequence, we establish a result analogous to the Parrondo's paradox, which shows that randomness in the number of components extracted from the two batches improves the reliability of the series system.

Keywords: Series systems, parallel systems, stochastic orders, Parrondo's paradox.

MSC: 62N05; 90B25; 60E15

1. Introduction and preliminaries

Design and optimization of complex systems have been extensively considered in reliability literature. On this topic, the paper by Kuo and Rajendra Prasad [9] provides an useful overview of methods developed for solving optimization problems, and provides a comprehensive list of references. Some notable papers among them include Baxter and Harche [4], Chern [5], Malon [12], Prasad and Raghavachari [15], which contains illustrative results on the optimal allocation of components to series-parallel reliability systems.

In particular, many papers deal with stochastic comparisons of lifetimes as a tool to compare, or to provide, improvements in systems' reliability. For example, interesting results concerning stochastic comparisons between random lifetimes of a series system with redundant components and two allocation choices are due to Li and Hu [10], and Valdés and Zequeira [20]. Comparisons of lifetimes of more general coherent systems, such as k-out-of-n systems, are presented for instance in Shaked and Shanthikumar [16] and Singh and Vijayasree [19]. Furthermore, the classical book of Barlow and Proschan [2] contains various examples of bounds of the reliability of series and parallel systems obtained via comparisons based on suitable stochastic orders.

The problem considered in this note is related to previous papers oriented to compare minima and maxima of i.i.d. random variables when the size of the sequences is random (see Bartoszewicz [3], Li and Zuo [11], and Shaked and Wong [18]). The novelty of our approach relies on the assumption that each sequence is formed by two different types of random variables, where the size of each subsequence is random and the total number of components in the system is fixed.

Specifically, consider two different batches B_X and B_Y of components, whose corresponding random lifetimes are denoted by $\{X_i; i = 1, \dots, m\}$ and $\{Y_i; i = 1, \dots, m\}$. Given two independent random variables X and Y, we assume that $X_1, X_2, \dots$ are i.i.d. and identically distributed as X, and that $Y_1, Y_2, \dots$ are i.i.d. and identically distributed as Y. Also, consider a series or parallel system having a fixed number m of components performing a similar task, and assume that k of the m components are taken from the batch B_X, while the remaining $m-k$ components are taken from the batch B_Y. Hence, by setting

$$\Pi_k = \begin{cases} \{Y_1, \dots, Y_m\} & \text{if } k = 0 \\ \{X_1, \dots, X_k, Y_{k+1}, \dots, Y_m\} & \text{if } k = 1, \dots, m-1 \\ \{X_1, \dots, X_m\}, & \text{if } k = m, \end{cases}$$

the lifetimes of the series and the parallel systems can be respectively expressed as

$$S_k = \min\{\Pi_k\} \quad \text{and} \quad P_k = \max\{\Pi_k\}, \qquad k = 0, 1, \dots, m. \tag{1}$$

We suppose that the components of the batch B_X have higher reliability, in the usual stochastic order (defined below), than the components of the batch B_Y (which, for example, are built in a different production center having lower performances, or come from a set of used components). Let us now assume that the composition of the series system and of the parallel system is randomized. Precisely, denote by K, with support in $\{0, 1, \dots, m\}$, the random number of components taken from the batch B_X, whereas the remaining $m - K$ components are taken from the batch B_Y. It is quite intuitive, and easy to prove, that if K increases in the usual stochastic order then the reliability of the series or parallel system increases in the same stochastic sense.

From the operational point of view, the knowledge of which batch contains the components with higher reliability would suggest to take as many components as possible from that batch. In the absence of such knowledge, then the optimal choice is not obvious. A possible choice is to select half of the components from each batch, this ensuring that half of the components have high reliability. This is also suggested by what is commonly done in risk theory, i.e. acting with diversification as a tool that minimizes risks. However, in this paper we aim to show that this choice is not always optimal, and in addition we provide suitable conditions to improve the reliability of the system.

More formally, we purpose to establish that the reliability of the series (parallel) system increases when the random number K of components taken from the batch B_X increases in the increasing convex (increasing concave) order, which is weaker that the usual stochastic order. The main relevance of this result is that, as clearly explained later, random variables having the same expectation can be compared in the increasing convex (increasing concave) order, while this is not allowed in the usual stochastic order (unless the random variables are identically distributed).

In what follows, we briefly review the definition of the stochastic orders that will be used throughout the paper to compare random lifetimes or numbers of components. As usual in reliability literature, throughout the paper the terms 'increasing' and 'decreasing' stand for 'non-decreasing' and 'non-increasing', respectively. Moreover, we shall adopt the following notation: $u \wedge v = \min\{u, v\}$ and $u \vee v = \max\{u, v\}$. For any random variable Z we shall denote its distribution function by $F_Z(x) = \mathbb{P}(Z \leq x)$ and its survival function by $\overline{F}_Z(x) = \mathbb{P}(Z > x)$.

Definition 1. *Given two random variables X and Y we say that X is greater than Y in the usual stochastic order [convex order, increasing convex order, increasing concave order] (denoted by $X \geq_{st} [\geq_{cx}, \geq_{icx}, \geq_{icv}]\ Y$) iff $\mathbb{E}[\phi(X)] \geq \mathbb{E}[\phi(Y)]$ for all increasing [convex, increasing convex, increasing concave] functions ϕ for which the expectations exist.*

Details, properties and a list of potential applications in reliability of these stochastic orders may be found, for example, in Müller and Stoyan [13] or Shaked and Shanthikumar [17].

We recall that comparison of component lifetimes based on the usual stochastic order implies the comparison between their reliability, being $X \geq_{st} Y$ equivalent to $\overline{F}_X(t) \geq \overline{F}_Y(t)$ for all $t \in \mathbb{R}$. Also, observe that both the usual stochastic order and the convex order strictly imply the increasing convex order, and that $X \geq_{icx} Y$ ($X \leq_{icv} Y$) is equivalent to $X \geq_{cx} Y$ whenever X and Y have equal finite means (see Theorem 4.A.35 of Shaked and Shanthikumar [17]). Hence, use of the increasing convex order instead of the usual stochastic order allows one to assess a wider range of cases and applications. For instance, two random variables having equal means are not ordered in the usual stochastic order sense unless they are identically distributed (see Theorem 1.A.8 of Shaked and Shanthikumar [17]) whereas they can be ordered in the increasing convex order sense.

2. Series systems

In this section we consider the case of series systems. Let us denote by S_K the mixture of the family of lifetimes $\{S_0, S_1, \dots, S_m\}$ defined in (1) with respect to the random number K of components chosen from batch B_X, so that S_K is distributed as S_k if $S = k$. Clearly, the survival function of S_K is:

$$\overline{F}_{S_K}(t) = \sum_{k=0}^{m} \overline{F}_X^k(t)\, \overline{F}_Y^{m-k}(t)\, \mathbb{P}(K = k), \qquad t \geq 0. \tag{2}$$

Assume that the lifetimes of components coming from batch B_X are greater, in usual stochastic order sense, than the lifetimes of components coming from batch B_Y, i.e. $X \geq_{st} Y$. Due to Theorem 1.A.3(b) of Shaked and Shanthikumar [17], in this case the sequence of lifetimes $\{S_0, S_1, \dots, S_m\}$ is such that $S_{k-1} \leq_{st} S_k$ for all $k = 1, \dots, m$. Hence, from Theorem 1.A.6 of the same reference the following statement immediately follows, where K_1 and K_2 are random variables with support in $\{0, 1, \dots, m\}$ denoting the number of components coming from batch B_X in two different instances.

Proposition 1. *Let $X \geq_{st} Y$. If $K_1 \leq_{st} K_2$ then $S_{K_1} \leq_{st} S_{K_2}$.*

Example 1. Let us consider two series systems, each having m components chosen according to identical and independent Bernoulli trials. Precisely, each component of system S_{K_i} is taken either from batch B_X or from batch B_Y with probability p_i and $1 - p_i$, respectively, for $i = 1, 2$. Then, K_i has a binomial distribution and the survival function of S_{K_i} is given by

$$\overline{F}_{S_{K_i}}(t) = \left[p_i \overline{F}_X(t) + (1 - p_i)\overline{F}_Y(t)\right]^m, \qquad t \geq 0, \ \ i = 1, 2.$$

It is thus not hard to verify that if $p_1 \le p_2$ then $K_1 \le_{st} K_2$, and if in addition $X \ge_{st} Y$ then $S_{K_1} \le_{st} S_{K_2}$, in agreement with the result stated in Proposition 1.

Hereafter we will show that condition $S_{K_1} \le_{st} S_{K_2}$ can be proved also under the weaker assumption $K_1 \le_{icx} K_2$. To this aim, we first obtain two preliminary results.

Lemma 1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing function, and let $X \ge_{st} Y$. Then the function $h(u) = \mathbb{E}[\phi(u \wedge X) - \phi(u \wedge Y)]$ is increasing in $u \in \mathbb{R}$.*

Proof. Let $u_1 \le u_2$, and consider the function $l(x) = \phi(u_1 \wedge x) - \phi(u_2 \wedge x)$, with $x \in \mathbb{R}$. Since the function $l(x)$ is decreasing in x for any $u_1 \le u_2$, assumption $X \ge_{st} Y$ implies

$$\mathbb{E}[l(X)] = \int_{\mathbb{R}} [\phi(u_1 \wedge x) - \phi(u_2 \wedge x)] \mathrm{d}F_X(x) \le \int_{\mathbb{R}} [\phi(u_1 \wedge x) - \phi(u_2 \wedge x)] \mathrm{d}F_Y(x) = \mathbb{E}[l(Y)],$$

so that

$$h(u_1) = \mathbb{E}[\phi(u_1 \wedge X) - \phi(u_1 \wedge Y)] \le \mathbb{E}[\phi(u_2 \wedge X) - \phi(u_2 \wedge Y)] = h(u_2),$$

and the assertion follows. $\square$

Example 2. Let X and Y be exponentially distributed random variables with parameters λ and μ, respectively. Let $\lambda \le \mu$, so that $X \ge_{st} Y$. In agreement with Lemma 1, for $\phi(x) = x$, the function $h(u) = \mathbb{E}[(u \wedge X) - (u \wedge Y)]$ is increasing in $u \in \mathbb{R}$, since $h(u) = 0$ for $u < 0$ and $h(u) = \frac{1}{\lambda}\left(1 - \mathrm{e}^{-\lambda u}\right) - \frac{1}{\mu}\left(1 - \mathrm{e}^{-\mu u}\right)$ for $u \ge 0$.

Lemma 2. *Let $X \ge_{st} Y$. Then the sequence of lifetimes $\{S_0, S_1, \ldots, S_m\}$ defined in (1) is such that $\eta(k) := \mathbb{E}[\phi(S_k)]$ is increasing and convex in $k = 1, \ldots, m$ for every increasing function $\phi : \mathbb{R} \to \mathbb{R}$.*

Proof. Let ϕ be an increasing real function. Since $S_{k-1} \le_{st} S_k$ for all $k = 1, \ldots, m$, and since ϕ is increasing, it follows that $\eta(k-1) = \mathbb{E}[\phi(S_{k-1})] \le \mathbb{E}[\phi(S_k)] = \eta(k)$ for all $k = 1, \ldots, m$, i.e. $\eta(k)$ is increasing in k. To prove that $\eta(k)$ is convex, let us verify that

$$\eta(k+1) - \eta(k) \ge \eta(k) - \eta(k-1) \quad \text{for all } k = 1, \ldots, m-1. \tag{3}$$

Recalling (1), Eq. (3) is equivalent to

$$\begin{aligned}
&\mathbb{E}\big[\phi\big(X_1 \wedge \cdots \wedge X_k \wedge X_{k+1} \wedge Y_{k+2} \wedge \cdots \wedge Y_m\big) \\
&\qquad - \phi\big(X_1 \wedge \cdots \wedge X_k \wedge Y_{k+1} \wedge Y_{k+2} \wedge \cdots \wedge Y_m\big)\big] \\
&\ge \mathbb{E}\big[\phi\big(X_1 \wedge \cdots \wedge X_{k-1} \wedge X_k \wedge Y_{k+1} \wedge \cdots \wedge Y_m\big) \\
&\qquad - \phi\big(X_1 \wedge \cdots \wedge X_{k-1} \wedge Y_k \wedge Y_{k+1} \wedge \cdots \wedge Y_m\big)\big].
\end{aligned}$$

By setting $Z_1 = X_1 \wedge \cdots \wedge X_k \wedge Y_{k+2} \wedge \cdots \wedge Y_m$ and $Z_2 = X_1 \wedge \cdots \wedge X_{k-1} \wedge Y_{k+1} \wedge \cdots \wedge Y_m$, the above inequality becomes

$$\int_{\mathbb{R}} \mathbb{E}[\phi(u \wedge X_{k+1}) - \phi(u \wedge Y_{k+1})] \mathrm{d}F_{Z_1}(u) \geq \int_{\mathbb{R}} \mathbb{E}[\phi(X_k \wedge u) - \phi(Y_k \wedge u)] \mathrm{d}F_{Z_2}(u). \quad (4)$$

Now consider the function

$$h(u) = \mathbb{E}\big[\phi\big(u \wedge X) - \phi(u \wedge Y)\big] = \mathbb{E}[\phi(u \wedge X_{k+1}) - \phi(u \wedge Y_{k+1})] = \mathbb{E}[\phi(X_k \wedge u) - \phi(Y_k \wedge u)],$$

where the equalities follow from the assumption that the sequences of X_i's and of Y_j's are both formed by i.i.d. random variables. Inequality (4) is equivalent to

$$\mathbb{E}[h(Z_1)] \geq \mathbb{E}[h(Z_2)].$$

This last inequality is satisfied since $Z_1 \geq_{st} Z_2$, being $X_k \geq_{st} Y_{k+1}$, and since $h(u)$ is increasing due to Lemma 1 and assumption $X \geq_{st} Y$. Thus, inequality (3) is also satisfied, and the assertion follows. □

The main theorem of this section immediately follows.

Theorem 1. *Let $X \geq_{st} Y$, and let the sequence of lifetimes $\{S_0, S_1, \ldots, S_m\}$ be defined as in (1). Then*

$$K_1 \leq_{icx} K_2 \quad \Rightarrow \quad S_{K_1} \leq_{st} S_{K_2}.$$

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an arbitrary increasing function. By Lemma 2 the function $\eta(k) = \mathbb{E}[\phi(S_k)]$ is increasing and convex. Thus

$$\mathbb{E}[\phi(S_{K_1})] = \mathbb{E}[\mathbb{E}[\phi(S_{K_1})|K_1]] = \mathbb{E}[\eta(K_1)] \leq \mathbb{E}[\eta(K_2)] = \mathbb{E}[\mathbb{E}[\phi(S_{K_2})|K_2]] = \mathbb{E}[\phi(S_{K_2})],$$

where the inequality is due to assumption $K_1 \leq_{icx} K_2$. The assertion then follows. □

We notice that assumption $K_1 \leq_{st} K_2$ of Proposition 1 has been weakened to $K_1 \leq_{icx} K_2$ in Theorem 1. Recalling that random numbers having the same expectation cannot be compared in usual stochastic order, the usefulness of Theorem 1 becomes clear in the following example.

Example 3. Consider the family of discrete random variables $\{X(\epsilon); 0 \leq \epsilon \leq 1/n\}$ having support $\{1, 2, \ldots, n\}$, with $n \geq 4$, and such that

$$\mathbb{P}[X(\epsilon) = k] = \begin{cases} 1/n + \epsilon, & k = 1, n \\ 1/n - \epsilon, & k = 2, n-1 \\ 1/n, & k = 3, 4, \ldots, n-2. \end{cases}$$

It is not hard to see that $X(\epsilon_1) \leq_{icx} X(\epsilon_2)$ for $0 \leq \epsilon_1 < \epsilon_2 \leq 1/n$, since

$$\sum_{j \geq k} \mathbb{P}[X(\epsilon_1) = j] \leq \sum_{j \geq k} \mathbb{P}[X(\epsilon_2) = j] \qquad \forall k \in \{1, 2, \dots, n\}.$$

However, we have $\mathbb{E}[X(\epsilon)] = (n+1)/2$ for all $\epsilon \in [0, 1/n]$, so that the random variables $X(\epsilon)$ are not ordered in the usual stochastic order sense. It follows that $K_i =_{st} X(\epsilon_i)$, $i = 1, 2$, satisfy the assumption $K_1 \leq_{icx} K_2$ of Theorem 1 for $0 \leq \epsilon_1 < \epsilon_2 \leq 1/n$, whereas they do not fulfill the hypothesis $K_1 \leq_{st} K_2$ of Proposition 1.

Making use of the result given in Theorem 1, hereafter we provide an upper bound and a lower bound on the survival function (2) when m is even.

Corollary 1. *Let $X \geq_{st} Y$, and let the sequence of lifetimes $\{S_0, S_1, \dots, S_m\}$ be defined as in (1), with $m = 2n$, and let K have support in $\{0, 1, \dots, 2n\}$ with $\mathbb{E}(K) = n$. Then, for all $t \geq 0$ we have*

$$\overline{F}_X^n(t)\, \overline{F}_Y^n(t) \leq \overline{F}_{S_K}(t) \leq \frac{1}{2}\left[\overline{F}_X^{2n}(t) + \overline{F}_Y^{2n}(t)\right].$$

Proof. It immediately follows from Theorem 1, by recalling the properties of usual stochastic order, and by noting that $K_1 \leq_{icx} K \leq_{icx} K_2$, where $K_1 = n$ almost surely (a.s.) and K_2 takes values 0 and $2n$ with equal probabilities. $\square$

Another application of Theorem 1 will be given in Section 4.

3. Parallel systems

In this section we deal with the case of parallel systems. Similarly to the approach taken in Section 2, we now denote by P_K the mixture of the family of lifetimes $\{P_0, P_1, \dots, P_m\}$ defined in (1) with respect to the random number K of components chosen from batch B_X, and thus P_K is distributed as P_k for $K = k$. The distribution function of P_K is thus:

$$F_{P_K}(t) = \sum_{k=0}^{m} F_X^k(t)\, F_Y^{m-k}(t)\, \mathbb{P}(K = k), \qquad t \geq 0. \tag{5}$$

We assume that the lifetimes of components coming from batch B_X are greater, in the usual stochastic order sense, than the lifetimes of components coming from batch B_Y, i.e. $X \geq_{st} Y$. Hence, as for the case of series system it is easy to verify that $P_{k-1} \leq_{st} P_k$ for all $k = 1, \dots, m$ and to prove the following result.

Proposition 2. *Let $X \geq_{st} Y$. If $K_1 \leq_{st} K_2$ then $P_{K_1} \leq_{st} P_{K_2}$.*

Example 4. Examine two parallel systems, each formed by m components chosen according to identical and independent Bernoulli trials. As for Example 1, each component of system S_{K_i} is taken either from batch B_X or from batch B_Y with probability p_i and $1-p_i$, respectively, for $i = 1, 2$. Hence, K_i has a binomial distribution and the distribution function of P_{K_i} is:

$$F_{P_{K_i}}(t) = [p_i F_X(t) + (1-p_i) F_Y(t)]^m, \qquad t \in \mathbb{R},\ i = 1, 2.$$

If $p_1 \leq p_2$ we have $K_1 \leq_{st} K_2$, and if in addition $X \geq_{st} Y$ then $P_{K_1} \leq_{st} P_{K_2}$, according to Proposition 2.

Even for parallel systems it is possible to prove that the stochastic inequality between the mixtures also holds under a weaker ordering between K_1 and K_2, namely $K_1 \leq_{icv} K_2$. Again, we need some preliminary results stated in the two following lemmas.

Lemma 3. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an increasing function, and let $X \geq_{st} Y$. Then the function $g(u) = \mathbb{E}[\phi(u \vee X) - \phi(u \vee Y)]$ is decreasing in $u \in \mathbb{R}$.*

Proof. For $u_1 \leq u_2$, let us define the function $\ell(x) = \phi(u_1 \vee x) - \phi(u_2 \vee x)$, $x \in \mathbb{R}$. It is easy to verify that $\ell(x)$ is increasing in x for any $u_1 \leq u_2$. Hence, since $X \geq_{st} Y$ we have

$$\mathbb{E}[\ell(X)] = \int_{\mathbb{R}} [\phi(u_1 \vee x) - \phi(u_2 \vee x)] \mathrm{d}F_X(x) \geq \int_{\mathbb{R}} [\phi(u_1 \vee x) - \phi(u_2 \vee x)] \mathrm{d}F_Y(x) = \mathbb{E}[\ell(Y)].$$

This yields

$$g(u_1) = \mathbb{E}[\phi(u_1 \vee X) - \phi(u_1 \vee Y)] \geq \mathbb{E}[\phi(u_2 \vee X) - \phi(u_2 \vee Y)] = g(u_2).$$

The proof is thus completed. $\square$

Example 5. Let X and Y be exponentially distributed as in Example 2, so that $X \geq_{st} Y$. In agreement with Lemma 3, for $\phi(x) = x$, we have that function $g(u) = \mathbb{E}[(u \vee X) - (u \vee Y)]$ is decreasing in $u \in \mathbb{R}$, since $g(u) = \frac{1}{\lambda} - \frac{1}{\mu}$ for $u < 0$ and $g(u) = \frac{1}{\lambda} \mathrm{e}^{-\lambda u} - \frac{1}{\mu} \mathrm{e}^{-\mu u}$ for $u \geq 0$.

Lemma 4. *Let $X \geq_{st} Y$. Then the sequence of lifetimes $\{P_0, P_1, \ldots, P_m\}$ defined in (1) is such that $\xi(k) = \mathbb{E}[\phi(P_k)]$ is increasing and concave for every increasing function $\phi : \mathbb{R} \to \mathbb{R}$.*

Proof. Let ϕ be an increasing real function. Since $P_{k-1} \leq_{st} P_k$ for all $k = 1, \ldots, m$, and since ϕ is increasing, it follows that $\xi(k-1) = \mathbb{E}[\phi(P_{k-1})] \leq \mathbb{E}[\phi(P_k)] = \xi(k)$ for all $k = 1, \ldots, m$, i.e. $\xi(k)$ is increasing. To prove that $\xi(k)$ is concave, let us verify that

$$\xi(k+1) - \xi(k) \leq \xi(k) - \xi(k-1) \quad \text{for all } k = 1, \ldots, m-1, \tag{6}$$

i.e., that

$$\begin{aligned}
&\mathbb{E}\big[\phi\big(X_1 \vee \cdots \vee X_k \vee X_{k+1} \vee Y_{k+2} \vee \cdots \vee Y_m\big) \\
&\qquad -\phi\big(X_1 \vee \cdots \vee X_k \vee Y_{k+1} \vee Y_{k+2} \vee \cdots \vee Y_m\big)\big] \\
&\leq \mathbb{E}\big[\phi\big(X_1 \vee \cdots \vee X_{k-1} \vee X_k \vee Y_{k+1} \vee \cdots \vee Y_m\big) \\
&\qquad -\phi\big(X_1 \vee \cdots \vee X_{k-1} \vee Y_k \vee Y_{k+1} \vee \cdots \vee Y_m\big)\big].
\end{aligned}$$

Letting $Z_1 = X_1 \vee \cdots \vee X_k \vee Y_{k+2} \vee \cdots \vee Y_m$ and $Z_2 = X_1 \vee \cdots \vee X_{k-1} \vee Y_{k+1} \vee \cdots \vee Y_m$, the above inequality is equivalent to

$$\int_{\mathbb{R}} \mathbb{E}[\phi(u \vee X_{k+1}) - \phi(u \vee Y_{k+1})] \mathrm{d}F_{Z_1}(u) \leq \int_{\mathbb{R}} \mathbb{E}[\phi(X_k \vee u) - \phi(Y_k \vee u)] \mathrm{d}F_{Z_2}(u).$$

Now consider the function

$$\psi(u) = \mathbb{E}\big[\phi\big(u \vee X\big) - \phi(u \vee Y)\big] = \mathbb{E}[\phi(u \vee X_{k+1}) - \phi(u \vee Y_{k+1})] = \mathbb{E}[\phi(X_k \vee u) - \phi(Y_k \vee u)],$$

where the equalities follow from the assumption of independence and identical distribution for all X_i's and for all Y_j's. Due to Lemma 3 and assumption $X \geq_{st} Y$ the function $\psi(u)$ is decreasing. Moreover, note that the inequality above is equivalent to

$$\mathbb{E}[\psi(Z_1)] \leq \mathbb{E}[\psi(Z_2)].$$

This last inequality is satisfied since $Z_1 \geq_{st} Z_2$, being $X_k \geq_{st} Y_{k+1}$, and since $\psi(u)$ is decreasing. Hence, also inequality (6) is satisfied, thus completing the proof. □

We are now able to state the main result of this section.

Theorem 2. *Let $X \geq_{st} Y$, and let the sequence of lifetimes $\{P_0, P_1, \ldots, P_m\}$ be defined as in (1). Then*

$$K_1 \leq_{icv} K_2 \quad \Rightarrow \quad P_{K_1} \leq_{st} P_{K_2}.$$

Proof. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an arbitrary increasing function. By Lemma 4 the function $\xi(k) = \mathbb{E}[\phi(S_k)]$ is increasing and concave. Hence,

$$\mathbb{E}[\phi(S_{K_1})] = \mathbb{E}[\mathbb{E}[\phi(S_{K_1})|K_1]] = \mathbb{E}[\xi(K_1)] \leq \mathbb{E}[\xi(K_2)] = \mathbb{E}[\mathbb{E}[\phi(S_{K_2})|K_2]] = \mathbb{E}[\phi(S_{K_2})],$$

where the inequality is due to assumption $K_1 \leq_{icv} K_2$. The assertion finally follows. □

We remark that comments similar to those made about the usefulness of Theorem 1 can also be made about Theorem 2, due to the strict analogies between icx and icv orders. Indeed, the following bounds to the distribution function (5) hold when m is even.

Corollary 2. *Let $X \geq_{st} Y$, and let the sequence of lifetimes $\{P_0, P_1, \ldots, P_m\}$ be defined as in (1), with $m = 2n$, and let K have support in $\{0, 1, \ldots, 2n\}$ with $\mathbb{E}(K) = n$. Then, for all $t \geq 0$ we have*

$$\frac{1}{2}\left[F_X^{2n}(t) + F_Y^{2n}(t)\right] \leq F_{P_K}(t) \leq F_X^n(t)\, F_Y^n(t).$$

The proof is omitted, being similar to that of Corollary 1.

4. A Parrondo's type paradox

Recently, the Parrondo's paradox has attracted the attention of several researchers. Its simplest formulation is based on two fair gambling games, both of which having a negative expectation. Precisely, when the two games are played individually, for each of them the player is expected to lose. However, when the games are played in an alternating random sequence, the resulting compound game is a winning game, with a positive expectation (see, for instance, Abbott [1] and references therein). The Parrondo's paradox is thus centered on the counterintuitive result that the right combination of several losing strategies can be turned into a winning strategy. We remark that this is not a contradiction. The explanatory key is that the games are not independent. Thus it is not surprising that a suitable combination of non-independent losing games may produce a winning game.

A suitable version of Parrondo's paradox in the field of reliability theory has been presented recently by Di Crescenzo [6]. It involves independent pairs of systems in series with two independent components, where the units of the first system are less reliable than those of the second. It is shown in Di Crescenzo [6] that, if the first system can be modified by a random choice of its components, allowing each of the two units to be randomly chosen from a set of components identical to those in the series, then its reliability can be improved, obtaining a system more reliable the the second one, although the original one was worse. An extension to the case of dependent components has been treated recently in Di Crescenzo and Pellerey [7]. See also Navarro and Spizzichino [14], where stochastic comparisons of series and parallel systems with vectors of component lifetimes sharing the same copula are studied.

It should be pointed out that the results given in the present paper have a narrow intersection with the contribution of Di Crescenzo and Pellerey [7]. First of all, papers [6] and [7] deal with 2-unit series and parallel systems, whereas the systems we examine here have arbitrary fixed sizes. Moreover, the single common case involves random variables $K_1 = 1$ a.s. and K_2 having binomial distribution with parameters $n = 2$ and $p = 1/2$. A similar comment holds for Navarro and Spizzichino [14]. In the latter paper, under suitable assumptions about the dependence between the component lifetimes, it is shown that the heterogeneity of the random lifetimes decreases (increases) the overall reliability of the series (parallel) system. However, the relevant difference is that in the present contribution the number of components of a given batch is assumed to be random.

According to Harmer and Abbott [8] randomness or "noise" in physical systems is often considered to yields a deleterious effect. However, the rapidly growing fields of stochastic resonance, Brownian ratchets and stochastic molecular motors have brought the increasing realization that noise can play a constructive role. This paper provides an illustration that the presence of randomness in the construction of coherent systems may improve the system reliability in some suitable criteria. The main result presented in Section 2 can thus be viewed as a new alternative version of Parrondo's paradox in the field of reliability theory. In fact, as an immediate corollary of Theorem 1, we can show that the reliability of a series system constituted by an even number $2n$ of components coming from different batches can be improved by randomly choosing the number of components coming from the batches. Assume that it is impossible to verify which one of the two batches contains the better performing components. Indeed, let us consider a series system constituted by a number n of components taken from the batch B_X and by n components taken from a different batch B_Y, and assume that X and Y are stochastically ordered (in any one of the two directions). Now assume that the fixed number n of components taken from B_X can be replaced (through any kind of randomization procedure) by a random number K_2 of components of the same type satisfying the constraint $\mathbb{E}[K_2] = n$. The remaining $n - K_2$ components are taken from B_Y. Thus, in this case, letting $K_1 = n$ a.s., by Theorem 3.A.24 in Shaked and Shanthikumar [17] it follows $K_1 \leq_{cx} K_2$, and therefore also $K_1 \leq_{icx} K_2$. As a consequence, if in addition we assume $X \geq_{st} Y$, then from Theorem 1 we immediately obtain that the reliability of the resulting system is improved, i.e. $S_{K_1} \leq_{st} S_{K_2}$. We notice that, by symmetry, an analogous result can be proved also under the assumption $Y \geq_{st} X$. Thus, roughly speaking, we get that if the fixed number n of components taken from any one of the two batches is replaced by a random number having the same mean, then the reliability of the overall system improves in usual stochastic order sense.

The above stochastic structure can be reformulated as follows. Consider a series system formed by 2 subsystems in series, each having n components. The components of the first subsystem are taken from batch B_X, whereas those of the second subsystem are taken from batch B_Y. Let K_0 denote the number of components of the whole system whose lifetimes are distributed as X; hence, $K_0 = n$ a.s.

Assume now that the components of the system can be changed according to the following randomizing procedure: We perform $2j$ independent Bernoulli trials by flipping $2j$ unbiased coins ($j = 1, \ldots, n$). For each of the first j coins giving 'head' a component of the first subsystem is changed into another component taken from batch B_Y, whereas for the last j coins giving 'head' a component of the second subsystem is changed into another component taken from batch B_X. Of course, we assume that all lifetimes are independent. Denote by K_j the random number of components of the system whose lifetimes are distributed as X after the randomizing procedure has been performed. Clearly, for $j = 0, 1, \ldots, n$ we have

$$K_j = K_0 + \sum_{i=1}^{j} U_i - \sum_{i=1}^{j} V_i, \tag{7}$$

where the U_i's and V_i's are i.i.d. random variables having Bernoulli distribution with parameter $1/2$. Thus, $K_j - n + j$ has binomial distribution with parameters $(2j, 1/2)$, so that

$$\mathbb{P}(K_j = k) = \binom{2j}{k-n+j} \left(\frac{1}{2}\right)^{2j}, \qquad n - j \le k \le n + j$$

and

$$\mathbb{E}(K_j) = n, \qquad \mathbb{V}ar(K_j) = \frac{j}{2} \qquad (j = 0, 1, \ldots, n).$$

Hence, the mean number of components whose lifetimes are distributed as X is constant, whereas the randomness becomes greater as j increases. Now consider the mixture S_{K_j} of the family of lifetimes $\{S_{n-j}, S_{n-j+1}, \ldots, S_{n+j}\}$ defined as in (1) with respect to the random number K_j. Recalling (2) it follows that the survival function of S_{K_j}, for $j = 0, 1, \ldots, n$, is given by

$$\mathbb{P}(S_{K_j} > t) = \left[\overline{F}_X(t)\overline{F}_Y(t)\right]^{n-j} \left[\frac{\overline{F}_X(t) + \overline{F}_Y(t)}{2}\right]^{2j}, \qquad t \ge 0. \tag{8}$$

We have $K_j \not\le_{st} K_{j+1}$ but $K_j \le_{cx} K_{j+1}$ for $j = 0, 1, \ldots, n-1$ (by (7) and Theorem 3.A.34 in Shaked and Shanthikumar [17]), so that $K_j \le_{icx} K_{j+1}$. Hence, if $X \ge_{st} Y$ then from Theorem 1 we obtain $S_{K_j} \le_{st} S_{K_{j+1}}$ for $j = 0, 1, \ldots, n-1$. This means that the insertion of randomness in the above scheme produces an effect that improves the reliability of the overall system, similarly as for the Parrondo's paradox.

Finally, it should be pointed out that the opposite conclusion is obtained when dealing with parallel systems. In fact, introducing randomness in the number of components coming from one of the two batches, the reliability of a parallel system decreases due to Theorem 2. Thus, for parallel systems where it is impossible to recognize which one of the batches contains the better performing components, the maximal reliability is obtained by taking exactly half of the components from the first batch and the remaining half from the second one.

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