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MULTIVARIATE AGING WITH ARCHIMEDEAN DEPENDENCE STRUCTURES IN HIGH DIMENSIONS

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Abstract. Bivariate aging notions for a vector $\mathbf{X}$ of lifetimes based on stochastic comparisons between $\mathbf{X}$ and $\mathbf{X}_t$, where $\mathbf{X}_t$ is the multivariate residual lifetime after time $t > 0$, have been studied in Pellerey (2008) under the assumption that the dependence structure in $\mathbf{X}$ is described by an Archimedean survival copula. Similar stochastic comparisons between $\mathbf{X}_t$ and $\mathbf{X}_{t+s}$, for all $t; s > 0$, were considered in Mulero and Pellerey (2010). In this paper, these results are generalized and extended to the multivariate case. Two illustrative examples are also provided.

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1. Introduction

Let X be a random variable, and for each real $t \in \{q : P\{X > q\} > 0\}$ let $X_t = [X - t \big| X > t]$ denote a random variable whose distribution function (df) is the same as the conditional df of $X - t$ given that $X > t$. When X is the lifetime of a device, then X_t can be interpreted as the residual lifetime of the device at time t, given that the device is alive at time t.

Several characterizations of aging notions for items, components, or individuals, by means of stochastic comparisons between the residual lifetimes X_0, X_t, and X_{t+s}, with $t, t + s \in \{q : P(X > q) > 0\}$, have been considered and studied in the literature. These characterizations serve a few purposes; for example, they can be used to provide bounds for the df's of lifetimes of complex systems having components that satisfy these notions, or to solve optimization problems dealing with allocation of components in a system, and they also throw a new light of understanding on the intrinsic meaning of the aging notions involved (see, e.g., Barlow and Proschan, 1981, or Lai and Xie, 2006, and references therein).

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Among others, two of the most important notions of aging are the *Increasing Failure Rate* (IFR) and *Decreasing Failure Rate* (DFR) properties, which are satisfied by lifetimes having absolutely continuous df's and increasing, or decreasing, hazard rate functions, or, equivalently, by lifetimes X satisfying $X_t \leq_{st} [\geq_{st}] X_{t+s}$ for all $t, s \geq 0$, where the $\leq_{st}$ denotes the usual stochastic order, defined later (see, for example, Barlow and Proschan, 1891, for this equivalence).

Since in most systems the dependency between lifetimes of components is an unavoidable assumption, several generalizations of these notions have been extensively considered in the literature. For example, the dynamic multivariate increasing failure rate is a known extension that was introduced in Shaked and Shanthikumar (1991), while different other multivariate IFR notions have been defined more recently in Durante et al. (2010), or Arias-Nicols et al. (2009). In particular, a simple extension of the IFR property in the multivariate setting was introduced in Mulero and Pellerey (2010), generalizing the inequalities above to the bivariate case.

For similar purposes, it has been found useful to compare lifetimes, or residual lifetimes of individuals, by means of the usual stochastic order, i.e., to verify whenever $X \leq_{st} Y$, or whenever $X_t \leq_{st} Y_t$ for all $t \geq 0$. This last case is equivalent to $X \leq_h Y$, where $\leq_h$ stands for hazard rate order, since, as well-known and easy to verify, the inequality $X_t \leq_{st} Y_t$ for all $t \geq 0$ is satisfied if and only if the hazard rate of X is smaller than that of Y (assuming absolute continuity). Inequalities of this kind have been considered in different applied problems in reliability to compare lifetimes of systems, or in decision theory to compare risks (see, e.g., Shaked and Shantikumar, 2007). For this order also, several generalizations to the multivariate setting have been defined and studied in the literature (see, again, Shaked and Shantikumar, 2007, or Hu et al., 2003). Among others, multivariate stochastic orders defined in Section 2.

When considering dependence between lifetimes, one of most useful tools to describe and investigate such dependence is the notion of copula, or of survival copula. In fact, there are two important reasons that have made copulas very popular in modelling system's lifetimes: firstly, parameters of dependency between components can be chosen distinctly from the parameters of the components' df's; secondly, no restrictions should be given on choosing the df's of the system's components. A prominent class of copulas that is very popular in applications is the class of Archimedean copulas, that have a very close relation with Laplace transforms, as noticed by Marshall and Olkin (1988). A detailed description of Archimedean copulas may be found in Genest and Rivest (1993), Joe (1997) or Nelsen (2006).

The purpose of this paper is to provide conditions for vectors of lifetimes to satisfy the multivariate IFR properties introduced in Mulero and Pellerey (2010), or to be comparable in the multivariate stochastic orders defined in the same paper, under the assumption that the dependence structure in the components of the random vectors is described by an Archimedean survival copula. Since only the bivariate case has been considered in Mulero and Pellerey (2010), the results described here are more general, dealing with vectors of dimensions higher than two.

In the next section, Archimedean copulas, multivariate notions of aging (such as the multivariate IFR notion) and some multivariate stochastic orderings, are recalled. In Section 3, some conditions under which random vectors whose dependence structure is described by Archimedean copulas satisfy these notions or stochastic orders are presented. Finally, in Section 4, two illustrative examples are provided.

Some conventions and notations that are used throughout the paper are given in the following. The notation $=_{st}$ means equality in law. For any random variable (or vector) X and an event A, $[X\,|A]$ denotes a random variable whose df is the conditional df of X given A. Throughout this paper we write "increasing" instead of "non-decreasing" and "decreasing" instead of "non-increasing". Given two real valued vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, the notation $\mathbf{x} \le [<]\ \mathbf{y}$ means $x_i \le [<]\ y_i\ \forall i = 1, \ldots, n$. Also, a function $\phi : \Re^n \to \Re$ is said to be increasing if $\mathbf{x} \le \mathbf{y}$ implies $\phi(\mathbf{x}) \le \phi(\mathbf{y})$. Finally, we will denote with $d \in \{2, 3, \ldots\}$ the dimension of the vectors considered.

2. Some Preliminaries and auxiliary results

The following two multivariate generalizations of the usual stochastic order are well known (see Shaked and Shanthikumar, 2007, for related properties, equivalent definitions and applications). Considered two multivariate random vectors, $\mathbf{X}$ and $\mathbf{Y}$, we say that

- $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the *usual stochastic order* (denoted by $\mathbf{X} \le_{st} \mathbf{Y}$) if, and only if, $E[h(\mathbf{X})] \le E[h(\mathbf{Y})]$ for every increasing function $h : \Re^d \to \Re$ provided that the two expectations exist;
- $\mathbf{X}$ is smaller than $\mathbf{Y}$ in the *lower orthant order* ($\mathbf{X} \le_{lo} \mathbf{Y}$) if, and only if, $F_{\mathbf{X}}(x_1, \ldots, x_d) \ge F_{\mathbf{Y}}(x_1, \ldots, x_d)$ for all $(x_1, \ldots, x_d) \in \Re^d$.

It is a well known fact that $\mathbf{X} \le_{st} \mathbf{Y} \ \Rightarrow\ \mathbf{X} \le_{lo} \mathbf{Y}$, while the opposite is not true.

Given a vector $\mathbf{X} = (X_1, \ldots, X_d)$ of lifetimes, let

$$\mathbf{X}_t = [(X_1 - t, \ldots, X_d - t)|X_1 > t, \ldots, X_d > t]$$

be the multivariate residual lifetimes at time $t \ge 0$. Mulero and Pellerey (2010) introduced a bivariate generalization of the IFR notion, mentioning that it can be extended to any dimension, by the stochastic inequalities

$$\mathbf{X}_{t+s} \le_{st} [\ge_{st}]\ \mathbf{X}_t \ \textit{for all}\ t, s \ge 0, \tag{2.1}$$

and

$$\mathbf{X}_{t+s} \le_{lo} [\ge_{lo}]\ \mathbf{X}_t \ \textit{for all}\ t, s \ge 0, \tag{2.2}$$

respectively. We will denote the class of multivariate lifetimes that satisfy (2.1) by $\mathcal{A}^{+}_{FR}$ $[\mathcal{A}^{-}_{FR}]$, and the class of multivariate lifetimes that satisfy (2.2) by $\mathcal{A}^{w+}_{FR}$ $[\mathcal{A}^{w-}_{FR}]$ (here w means weakly).

Examples of application of these multivariate aging notions in reliability theory may be provided in the analysis of coherent systems, which they are often considered to describe the structure and the performance of complex systems. Recall that a system is said to be *coherent* whenever every component is relevant (i.e., it affects the working or failure of the system) and the structure function is monotone in every component (i.e., replacing a failed component by a working component cannot cause a working system to fail). For example, k-out-of-n systems, and series and parallel systems in particular, are coherent systems (see Esary and Marshall, 1970, or Barlow and Proschan, 1981, for a detailed introduction to coherent systems and related properties and applications). Also recall that for any coherent system having d components, the relationship between the vector $\mathbf{X}$ of component's lifetimes and the system's lifetime $T_{\mathbf{X}}$ is described by the relation $T_{\mathbf{X}} = \tau(\mathbf{X})$, where the *coherent life function* $\tau : \Re^d \to \Re$ is increasing and $\tau(t_1 - s, \ldots, t_d - s) = \tau(t_1, \ldots, t_d) - s$ for every $s \geq 0$ and $t_i \geq s$. Consider now any coherent system having coherent life function τ. By the above mentioned properties of coherent life functions, and by Theorem 6.B.16(a) in Shaked and Shanthikumar (2007), it can be immediately observed that if $\mathbf{X}$ satisfies the property $\mathcal{A}^{+}_{FR}$ $[\mathcal{A}^{-}_{FR}]$, then $T_{\mathbf{X}_t} = \tau(\mathbf{X}_t)$ is stochastically decreasing [increasing] in the set $\{t \geq 0 : X_i > t, \forall i = 1, \ldots, d\}$, i.e., the residual lifetime of the system stochastically decreases [increases] along time conditioning on the fact that all its components are in a working state at time t.

Two examples of application of the weaker properties $\mathcal{A}^{w+}_{FR}$ and $\mathcal{A}^{w-}_{FR}$ are the following. Let $\mathbf{X}$ and $\mathbf{Y}$ be two vectors of lifetimes, and observe that $\mathbf{X} \leq_{lo} \mathbf{Y} \Rightarrow E[\mathbf{X}] \leq E[\mathbf{Y}]$, where $E[\mathbf{X}]$ and $E[\mathbf{Y}]$ denote the corresponding vectors of the expected lifetimes. Thus, if $\mathbf{X}$ satisfies $\mathcal{A}^{w+}_{FR}$ $[\mathcal{A}^{w-}_{FR}]$, then the vector $E[\mathbf{X}_t] = (E[X_1 - t|X_i > t], \ldots, E[X_d - t|X_i > t])$ of the expected residual lifetimes is decreasing [increasing] in the set $\{t \geq 0 : X_i > t,\ \forall i = 1, \ldots, d\}$. Moreover, it is interesting to observe that $\mathbf{X} \leq_{lo} \mathbf{Y} \Rightarrow X_i \leq_{st} Y_i,\ \forall i = 1, \ldots, d$, i.e., corresponding univariate components of vectors ordered in the lower orthant sense are ordered in the usual stochastic order. Thus, if $\mathbf{X}$ satisfies the property $\mathcal{A}^{w+}_{FR}$ $[\mathcal{A}^{w-}_{FR}]$, then every marginal residual lifetime $[X_k - t|X_i > t,\ \forall i = 1, \ldots, d]$ is stochastically increasing [decreasing] in t. Consider, for example, d different items working together and performing the same task, having dependent lifetimes. If the vector $\mathbf{X}$ of their lifetimes satisfies the property $\mathcal{A}^{w+}_{FR}$ $[\mathcal{A}^{w-}_{FR}]$, then the residual lifetime of each one of the items stochastically increases along time, if all the items do not fail.

Conditions such that a vector $\mathbf{X}$ of lifetimes satisfies the multivariate IFR notions, defined above, will be provided when the dependence between the lifetimes is described by an Archimedean survival copula. Recall that a *copula* associated to a multivariate df F is a df $C : [0, 1]^d \mapsto [0, 1]$ satisfying: $F(\mathbf{x}) = C(F_1(x_1), \ldots, F_d(x_d))$, where the F_i's, for

$1 \le i \le d$, are the univariate marginal df's. Similarly, a *survival copula* associated to a multivariate df F is a df $C : [0,1]^d \mapsto [0,1]$ satisfying: $\overline{F}(\mathbf{x}) = C(\overline{F}_1(x_1), \ldots, \overline{F}_d(x_d))$, where the $\overline{F}_i$'s, for $1 \le i \le d$, are the univariate survival functions.

Also recall that a function $\psi : \Re_+ \mapsto [0,1]$ is called *d-alternating* if $(-1)^k \psi^{(k)} \ge 0$ for $k \in \{1, \ldots, d\}$, and if it is d-alternating for all $d \in \{1, 2, \ldots\}$, it is called *completely monotone*. A copula C_ψ is called an Archimedean copula if

$$C_\psi(u_1, \ldots, u_d) = \psi\left(\sum_{i=1}^{d} \psi^{-1}(u_i)\right), \tag{2.3}$$

where $\psi : \Re_+ \mapsto [0,1]$ is an d-alternating function such that $\psi(0) = 1$, and $\lim_{x\to\infty} \psi(x) = 0$. Here, ψ is called the generator function of the copula (see Joe, 1997, or Nelsen, 2006, for more details). Whenever the generator of an Archimedean copula is completely monotone, the Archimedean copula (2.3) can be written as

$$C_\psi(u_1, \ldots, u_d) = \int_0^\infty \prod_{i=1}^{d} G^\alpha(u_i) dM_\psi(\alpha), \tag{2.4}$$

where $G(x) = \exp\Big(-\psi^{-1}(x)\Big)$ and $M_\psi(.)$ is the df of a positive random variable having Laplace transform ψ (see Joe, 1997, page 93). In a similar manner, we can write the survival function of a vector $\mathbf{X} = (X_1, \ldots, X_d)$ having an Archimedean survival copula in the form

$$\overline{F}_\psi(x_1, \ldots, x_d) = \psi\left(\sum_{i=1}^{d} \psi^{-1}(\overline{F}_i(x_i))\right), \tag{2.5}$$

where $\overline{F}_i' s$ are the marginal survival functions of the random variables $X_i, i = 1, \ldots, d$.

3. Main Results

In this section, we assume that $\mathbf{X} = (X_1, \ldots, X_d)$ is a random vector with multivariate survival function defined as in (2.5), where ψ is a d-alternating function, having univariate survival functions and density functions $\overline{F}_i$ and f_i, respectively. The following result provides some conditions for comparing multivariate residual lifetimes at different ages t and $t + s$ in the lower orthant order.

Theorem 3.1. *If* $\frac{(-1)^d \psi^{(d)}(t)}{\psi(t)}$ *is a decreasing [increasing] function of* t, $\psi^{-1}\Big(\overline{F}_i(x_i)\Big)$ *is a concave [convex] function and* $(-1)^d \psi^{(d)}(t)$ *is a log-concave [log-convex] function of* t, *then* $\mathbf{X} \in \mathcal{A}_{FR}^{w-}$ *[*$\mathbf{X} \in \mathcal{A}_{FR}^{w+}$*].*

Proof. We give the proof for the case that $\frac{(-1)^d \psi^{(d)}(t)}{\psi(t)}$ is decreasing, $\psi^{-1}\Big(\overline{F}_i(x_i)\Big)$ is concave and $(-1)^d \psi^{(d)}(t)$ is log-concave. The proof for the alternative case is similar. We should show that $F_{t+s}(\mathbf{x}) \le F_t(\mathbf{x})$, for all $t, s \ge 0$, where $F_t(\mathbf{x})$ denotes the df of $\mathbf{X}_t$. We have

$$F_t(\mathbf{x}) = \frac{P(t < X_1 < t + x_1, \ldots, t < X_d < t + x_d)}{P(X_1 > t, \ldots, X_d > t)}.$$

Now, since $f(x_1,\ldots,x_d) = (-1)^d \frac{\partial^d}{\partial x_1 \cdots \partial x_d}\overline{F}(x_1,\ldots,x_d)$ and

$$(-1)^d \frac{\partial^d}{\partial x_1 \cdots \partial x_d}\overline{F}(x_1,\ldots,x_d) = (-1)^d \psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(x_i)\big)\Big)\prod_{i=1}^d \frac{-f_i(x_i)}{\psi'\Big(\psi^{-1}\big(\overline{F}_i(x_i)\big)\Big)},$$

$F_t(\mathbf{x})$ equals the ratio

$$\frac{\int_t^{t+x_1}\cdots\int_t^{t+x_d}(-1)^d\psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(y_i)\big)\Big)\prod_{i=1}^d \frac{-f_i(y_i)}{\psi'\Big(\psi^{-1}\big(\overline{F}_i(y_i)\big)\Big)}dy_1\cdots dy_d}{\psi\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(t)\big)\Big)} \tag{3.6}$$

$$= \frac{\int_0^{x_1}\cdots\int_0^{x_d}(-1)^d\psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(y_i+t)\big)\Big)\prod_{i=1}^d \frac{-f_i(y_i+t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_i(y_i+t)\big)\Big)}dy_1\cdots dy_d}{\psi\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(t)\big)\Big)}.$$

Thus, by assumption, $F_{t+s}(\mathbf{x}) \le F_t(\mathbf{x})$ if we have

$$\frac{(-1)^d\psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(y_i+t+s)\big)\Big)\prod_{i=1}^d \frac{-f_i(y_i+t+s)}{\psi'\Big(\psi^{-1}\big(\overline{F}_i(y_i+t+s)\big)\Big)}}{\psi\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(t+s)\big)\Big)} \tag{3.7}$$

$$\le \frac{(-1)^d\psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(y_i+t)\big)\Big)\prod_{i=1}^d \frac{-f_i(y_i+t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_i(y_i+t)\big)\Big)}}{\psi\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(t)\big)\Big)},$$

or, equivalently, if the last term is a decreasing function of t. By the assumption, we have that $\psi^{-1}\big(\overline{F}_i(x_i)\big)$ is a concave function of x_i, thus

$$\prod_{i=1}^d \frac{-f_i(y_i+t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_i(y_i+t)\big)\Big)}$$

is a decreasing function of t. So, it is sufficient to show that the function

$$h(t) := \frac{(-1)^d\psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(y_i+t)\big)\Big)}{\psi\Big(\sum_{i=1}^d \psi^{-1}\big(\overline{F}_i(t)\big)\Big)}. \tag{3.8}$$

is decreasing or equivalently, $g(t) = \left(\psi\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)\right)^2 h'(t)$ is non-positive. Clearly, we have

$$g(t) = (-1)^d\psi^{(d+1)}\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(y_i+t)\right)\right)\sum_{i=1}^{d}\frac{-f_i(y_i+t)}{\psi'\left(\psi^{-1}\left(\overline{F}_i(y_i+t)\right)\right)}\psi\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)$$
$$-(-1)^d\psi^{(d)}\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(y_i+t)\right)\right)\psi'\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)\sum_{i=1}^{d}\frac{-f_i(t)}{\psi'\left(\psi^{-1}\left(\overline{F}_i(t)\right)\right)}.$$

Since $\ln\left((-1)^d\psi^{(d)}(t)\right)$ is a concave function of t, in view of the fact that $\psi^{-1}\left(\overline{F}_i(y_i+t)\right)$ is increasing and $\frac{(-1)^d\psi^{(d)}(t)}{\psi(t)}$ is decreasing in t, we obtain

$$\begin{aligned}
&\frac{(-1)^d\psi^{(d+1)}\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(y_i+t)\right)\right)}{(-1)^d\psi^{(d)}\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(y_i+t)\right)\right)}\\
&\leq \frac{(-1)^d\psi^{(d+1)}\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)}{(-1)^d\psi^{(d)}\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)}\\
&\leq \frac{\psi'\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)}{\psi\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)}.
\end{aligned}$$

This obviously implies that $g(t) \leq 0$ because $0 \leq \frac{-f_i(y_i+t)}{\psi'\left(\psi^{-1}\left(\overline{F}_i(y_i+t)\right)\right)} \leq \frac{-f_i(t)}{\psi'\left(\psi^{-1}\left(\overline{F}_i(t)\right)\right)}$ and, thus, the proof is completed. □

The particular case in which ψ is a completely monotone function is considered in the following Theorem; In this case, under appropriate conditions on the marginals, we show that $\mathbf{X} \in \mathcal{A}_{FR}^{w-}$.

Theorem 3.2. *If $\psi(t)$ is a completely monotone function and $\psi^{-1}\left(\overline{F}_i(x_i)\right)$ is a concave function, then $\mathbf{X} \in \mathcal{A}_{FR}^{w-}$.*

Proof. With similar argument as in the proof of the Theorem 3.1, it is sufficient to show that $h(t)$ in (3.8) is a decreasing function of t. Observe that, by (2.4), $(-1)^d\psi^{(d)}(t) = \int_0^\infty \alpha^d e^{-\alpha t}dM_\psi(\alpha)$, so that

$$h(t) = \frac{(-1)^d\psi^{(d)}\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(y_i+t)\right)\right)}{\psi\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)} = \frac{\alpha^{d-1}\int_0^\infty \alpha e^{-\alpha\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(y_i+t)\right)}dM_\psi(\alpha)}{\psi\left(\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(t)\right)\right)}$$

Moreover, since

$$-\psi'\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)=\int_0^\infty \alpha e^{-\alpha\sum_{i=1}^{d}\psi^{-1}\left(\overline{F}_i(y_i+t)\right)}dM_\psi(\alpha),$$

then $h(t)$ is a decreasing function of t if

$$-k(t):=\frac{-\psi'\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)}{\psi\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(t)\Big)\Big)}$$

is a decreasing function of t.

Observe that $\Big(\psi\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(t)\Big)\Big)\Big)^2 k'(t)$ is equal to

$$\psi''\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)\psi\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(t)\Big)\Big)\sum_{i=1}^{d}\frac{-f_i(y_i+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)}$$

$$-\psi'\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(t)\Big)\Big)\psi'\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)\sum_{i=1}^{d}\frac{-f_i(t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_i(t)\Big)\Big)}. \tag{3.9}$$

Since the function $Q(t):=\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)$ is an increasing function of t, for any $y_i, i=1,\ldots,d$, its derivative

$$Q'(t)=\sum_{i=1}^{d}\frac{-f_i(y_i+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)}$$

is positive, and also decreasing in t. Moreover, $\psi(t), -\psi'(t)$ and $\psi''(t)$ are decreasing functions of t. Then, it follows that, since $y_i>0$ for $i=1,\ldots,d$, (3.9) is greater than or equal to

$$\psi''\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(t+y_i)\Big)\Big)\psi\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(t+y_i)\Big)\Big)\sum_{i=1}^{d}\frac{-f_i(t+y_i)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_i(t+y_i)\Big)\Big)}$$

$$-\psi'\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(t+y_i)\Big)\Big)\psi'\Big(\sum_{i=1}^{d}\psi^{-1}\Big(\overline{F}_i(t+y_i)\Big)\Big)\sum_{i=1}^{d}\frac{-f_i(t+y_i)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_i(t+y_i)\Big)\Big)}.$$

Therefore, $h(t)$ is a decreasing function of t if $\psi''(t)\psi(t)-\Big(\psi'(t)\Big)^2$ is non-negative. Recalling that $\psi''(t)=E(X^2e^{-tX})$, $\psi'(t)=E(Xe^{-tX})$ and $\psi(t)=E(e^{-tX})$, for a random variable having df M_ψ, it follows that $\psi''(t)\psi(t)-\Big(\psi'(t)\Big)^2$ reduces to $E(X^2e^{-tX})E(e^{-tX})-\Big(E(Xe^{-tX})\Big)^2$, which is positive by Cauchy-Schwarz inequality. Thus $h(t)$ is decreasing in t, and this completes the proof. □

Next, we consider conditions for the stronger aging notion $\mathbf{X} \in \mathcal{A}_{FR}^-$ $[\mathbf{X} \in \mathcal{A}_{FR}^+]$. To this aim, two preliminary results are needed. The first one follows reasoning as in the proof of Theorem 2.8 in Müller and Scarsini (2005).

Lemma 3.3. *$(-1)^{d-1}\psi^{(d-1)}(t)$ is a log convex [log concave] function if, and only if, $(-1)^i\psi^{(i)}(t)$ is log convex [log concave] for any $i = 1, \ldots, d-1$.*

The second preliminary result is Theorem 6.B.3 in Shaked and Shanthikumar (2007), which describes conditions under which a random vector $\mathbf{X}$ can be compared in the usual stochastic order with respect to another vector $\mathbf{Y}$

Lemma 3.4. *If $X_1 \leq_{st} Y_1$ and if, for $k = 1, \ldots, d$,*

$$[X_k | X_{k-1} = x_{k-1}, \ldots, X_1 = x_1] \leq_{st} [Y_k | Y_{k-1} = y_{k-1}, \ldots, Y_1 = y_1] \tag{3.10}$$

whenever $x_j \leq y_j$, $j = 1, \ldots, k-1$, then $\mathbf{X} \leq_{st} \mathbf{Y}$.

Conditions for the comparison $\mathbf{X}_t \leq_{st} \mathbf{X}_{t+s}$ are described in the next result.

Theorem 3.5. *If $(-1)^{d-1}\psi^{(d-1)}(t)$ is a log convex [log concave] function of t, $\psi^{-1}\Big(\overline{F}_i(x_i)\Big)$ is a concave [convex] function and $\bar{F}_1$ is log convex [log concave], then $\mathbf{X} \in \mathcal{A}_{FR}^-$ [$\mathbf{X} \in \mathcal{A}_{FR}^+$].*

Proof. We prove the statement outside of the brackets, the other being similar. Let $\mathbf{X}_t =_{st} (\tilde{X}_1^t, \ldots, \tilde{X}_d^t)$ be a vector having df $F_t(\mathbf{x})$. From Lemma 3.4, the inequality $\mathbf{X}_t \leq_{st} \mathbf{X}_{t+s}$ is satisfied if $\tilde{X}_1^t \leq_{st} \tilde{X}_1^{t+s}$ and if, for $k = 1, \ldots, d$,

$$[\tilde{X}_k^t |\ \tilde{X}_{k-1}^t = x_{k-1}, \ldots, \tilde{X}_1^t = x_1] \leq_{st} [\tilde{X}_k^{t+s} |\ \tilde{X}_{k-1}^{t+s} = y_{k-1}, \ldots, \tilde{X}_1^{t+s} = y_1] \tag{3.11}$$

holds whenever $x_j \leq y_j$ for $j = 1, \ldots, k-1$. Let us observe that the condition $\tilde{X}_1^t \leq_{st} \tilde{X}_1^{t+s}$ is equivalent to $\frac{\bar{F}_1(x+t)}{\bar{F}_1(t)} \leq \frac{\bar{F}_1(x+t+s)}{\bar{F}_1(t+s)}$, which means that $\bar{F}_1$ is log convex. Moreover,

$$P(\tilde{X}_k^t \leq x_k | \tilde{X}_{k-1}^t = x_{k-1}, \ldots, \tilde{X}_1^t = x_1) = \frac{\frac{\partial^k}{\partial x_1 \cdots \partial x_k} F_t^{1,\ldots,k}(x_1, \ldots, x_k)}{\frac{\partial^{k-1}}{\partial x_1 \cdots \partial x_{k-1}} F_t^{1,\ldots,k-1}(x_1, \ldots, x_{k-1})}, \tag{3.12}$$

where

$$\begin{aligned}
F_t^{1,\dots,i}(x_1,\dots,x_i) &= \int_0^{x_1}\cdots\int_0^{x_i}\int_0^{\infty}\cdots\int_0^{\infty}\frac{(-1)^d\psi^{(d)}\Big(\sum_{j=1}^d\psi^{-1}\Big(\overline{F}_j(y_j+t)\Big)\Big)}{\psi\Big(\sum_{j=1}^d\psi^{-1}\Big(\overline{F}_j(t+s)\Big)\Big)} \\
&\quad\cdot\prod_{j=1}^d\frac{-f_j(y_j+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_j(y_j+t)\Big)\Big)}dy_1\cdots dy_d \\
&= \int_0^{x_1}\cdots\int_0^{x_i}(-1)^i\psi^{(i)}\Big(\sum_{j=1}^i\psi^{-1}\Big(\overline{F}_j(y_j+t)\Big)+\sum_{j=i+1}^d\psi^{-1}\Big(\overline{F}_j(t)\Big)\Big) \\
&\quad\cdot\frac{\prod_{j=1}^i\frac{-f_j(y_j+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_j(y_j+t)\Big)\Big)}}{\psi\Big(\sum_{j=1}^d\psi^{-1}\Big(\overline{F}_j(t+s)\Big)\Big)}dy_1\cdots dy_i.
\end{aligned}$$

Thus, (3.12) is equal to

$$\frac{\prod_{j=1}^{k-1}\frac{-f_j(x_j+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_j(x_j+t)\Big)\Big)}}{\psi\Big(\sum_{j=1}^d\psi^{-1}\Big(\overline{F}_j(t)\Big)\Big)}\cdot\frac{\psi\Big(\sum_{j=1}^d\psi^{-1}\Big(\overline{F}_j(t)\Big)\Big)}{\prod_{j=1}^{k-1}\frac{-f_j(x_j+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_j(x_j+t)\Big)\Big)}}\cdot$$

$$\cdot\int_0^{x_k}\frac{(-1)^k\psi^{(k)}\Big(\sum_{j=1}^k\psi^{-1}\Big(\overline{F}_j(x_j+t)\Big)+\sum_{j=k+1}^d\psi^{-1}\Big(\overline{F}_j(t)\Big)\Big)\frac{-f_k(x+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_k(x+t)\Big)\Big)}}{(-1)^{k-1}\psi^{(k-1)}\Big(\sum_{j=1}^{k-1}\psi^{-1}\Big(\overline{F}_j(x_j+t)\Big)+\sum_{j=i}^d\psi^{-1}\Big(\overline{F}_j(t)\Big)\Big)}dx.$$

Let $K_1(t,x_1,\dots,x_{k-1})=\sum_{j=1}^{k-1}\psi^{-1}\Big(\overline{F}_j(x_j+t)\Big)+\sum_{j=k+1}^d\psi^{-1}\Big(\overline{F}_j(t)\Big)$. Then (3.12) is equal to

$$\frac{(-1)^{k-1}\psi^{(k-1)}\Big(K_1(t,x_1,\dots,x_{k-1})+\psi^{-1}\Big(\overline{F}_k(x_k+t)\Big)\Big)}{(-1)^{k-1}\psi^{(k-1)}\Big(K_1(t,x_1,\dots,x_{k-1})+\psi^{-1}\Big(\overline{F}_k(t)\Big)\Big)}-1. \tag{3.13}$$

Therefore, (3.11) holds if, and only if, (3.13) is a decreasing function of t and of x_j, for $j=1,\dots,k-1$. Since $K_1(t,x_1,\dots,x_{k-1})$ is an increasing function of $x_1,\dots,x_{k-1}$, (3.13) is a decreasing function of x_j for $j=1,\dots,k-1$ if, and only if, $\frac{(-1)^{k-1}\psi^{(k-1)}(y+z)}{(-1)^{k-1}\psi^{(k-1)}(y)}$ is an increasing function of y for any positive constant z. This holds, equivalently, if

$(-1)^{k-1}\psi^{(k-1)}$ is log convex. Also, (3.13) is a decreasing function of t if

$$(-1)^k\psi^{(k)}\Big(K_2(t)+\psi^{-1}\big(\overline{F}_k(x+t)\big)\Big)(-1)^{k-1}\psi^{(k-1)}\Big(K_2(t)+\psi^{-1}\big(\overline{F}_k(t)\big)\Big)\cdot$$
$$\Big(\sum_{j=1}^{k-1}\frac{-f_j(x_j+t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_j(x_j+t)\big)\Big)}+\frac{-f_j(x+t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_j(x+t)\big)\Big)}+\sum_{j=k+1}^{d}\frac{-f_j(t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_j(t)\big)\Big)}\Big)$$
$$-(-1)^k\psi^{(k-1)}\Big(K_2(t)+\psi^{-1}\big(\overline{F}_k(x+t)\big)\Big)(-1)^{k-1}\psi^{(k)}\Big(K_2(t)+\psi^{-1}\big(\overline{F}_k(t)\big)\Big)\cdot$$
$$\Big(\sum_{j=1}^{k-1}\frac{-f_j(x_j+t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_j(x_j+t)\big)\Big)}+\sum_{j=i}^{d}\frac{-f_j(t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_j(t)\big)\Big)}\Big)\le 0,$$

where $K_2(t):=K_1(t,x_1,\ldots,x_{k-1})$. Therefore, (3.13) is a decreasing function of t, if

$$\frac{(-1)^k\psi^{(k)}\Big(K_2(t)+\psi^{-1}\big(\overline{F}_k(x+t)\big)\Big)}{(-1)^{k-1}\psi^{(k-1)}\Big(K_2(t)+\psi^{-1}\big(\overline{F}_k(x+t)\big)\Big)}\quad\text{and}\quad\frac{f_j(x+t)}{\psi'\Big(\psi^{-1}\big(\overline{F}_j(x+t)\big)\Big)}$$

are two decreasing functions of x. Since $\psi^{-1}(\bar{F}(x))$ is an increasing function, the first quotient is a decreasing function of x if $-\frac{\partial}{\partial x}\log\Big((-1)^{k-1}\psi^{(k-1)}(x)\Big)$ is a decreasing function of x. Therefore, (3.13) is a decreasing function of t if $\psi^{-1}(\bar{F}(x))$ is concave and $\log\Big((-1)^{k-1}\psi^{(k-1)}(x)\Big)$ is convex. From Lemma 3.3, this holds if, and only if, $(-1)^{d-1}\psi^{(d-1)}(x)$, is log convex. This completes the proof. ☐

Corollary 3.6. *If ψ is a completely monotone function and, for all $i=1,\ldots,d$, the composition $\psi^{-1}(\overline{F}_i)$ is a concave function and $\bar{F}_1$ is log convex, then $\mathbf{X}\in\mathcal{A}^-_{FR}$.*

Proof. Since ψ is a completely monotone function, then $(-1)^d\psi^{(d)}(u)=E(X^de^{-uX})$ for some variable X. Hence, by Cauchy-Schwarz inequality, $E(X^{d+2}e^{-uX})E(X^de^{-uX})-\Big(E(X^{d+1}e^{-uX})\Big)^2\ge 0$, which means that $(-1)^d\psi^{(d)}(u)$ is log convex. The assertion now follows from previous results. ☐

We conclude the section presenting conditions for the comparison in the lower orthant sense between the vectors of the residual lifetimes corresponding to two vectors $\mathbf{X}=(X_1,\ldots,X_d)$ and $\mathbf{Y}=(Y_1,\ldots,Y_d)$ defined as in (2.5). Let $\mathbf{X}$ and $\mathbf{Y}$ have the same generator function ψ and marginal survival functions, and density functions, $\overline{F}_i, f_i$ and $\overline{G}_i, g_i$, respectively.

Theorem 3.7. *Let $S_i(t)=\psi^{-1}(\bar{F}_i(t+y_i))-\psi^{-1}(\bar{G}_i(t+y_i))$, $i=1,\ldots,d$ and assume that $\psi^{(d+1)}(t)$ exists for any $t\ge 0$. If the function $(-1)^d\psi^{(d)}(t)$ [$\frac{(-1)^d\psi^{(d)}(t+s)}{\psi(t)}$; for any $s\in\Re$] is an increasing [decreasing] function of t, and if, for every $i=1,\ldots,d$, $X_i\ge_{st}$ [$\le_{st}$] Y_i and the function $S_i(t)$ is decreasing [increasing], then, $\mathbf{X}_t\ge_{lo}$ [$\le_{lo}$] $\mathbf{Y}_t$ for every $t\ge 0$.*

Proof. We prove the case $X_t \geq_{lo} Y_t$ for every t; the other case is proved similarly. Fix $t \geq 0$, and let $F_{\mathbf{X},t}(\mathbf{x})$ and $G_{\mathbf{Y},t}(\mathbf{x})$ be the df's of the residual lifetimes $\mathbf{X}$ and $\mathbf{Y}$ at time t, respectively. We should prove that $F_{\mathbf{X},t}(\mathbf{y}) \leq G_{\mathbf{Y},t}(\mathbf{y})$ for any $\mathbf{y} = (y_1, \ldots, y_d)$. By (3.6), it is sufficient to prove that

$$\frac{(-1)^d \psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big) \prod_{i=1}^d \frac{-f_i(y_i+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)}}{\psi\Big(\sum_{i=1}^d \psi^{-1}\Big(\overline{F}_i(t)\Big)\Big)}$$

$$\leq \frac{(-1)^d \psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\Big(\overline{G}_i(y_i+t)\Big)\Big) \prod_{i=1}^d \frac{-g_i(y_i+t)}{\psi'\Big(\psi^{-1}\Big(\overline{G}_i(y_i+t)\Big)\Big)}}{\psi\Big(\sum_{i=1}^d \psi^{-1}\Big(\overline{G}_i(t)\Big)\Big)}.$$

Since $S_i(t)$ is a decreasing function of t for every $i = 1, \ldots, d$, then

$$\prod_{i=1}^d \frac{-f_i(y_i+t)}{\psi'\Big(\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)} \leq \prod_{i=1}^d \frac{-g_i(y_i+t)}{\psi'\Big(\psi^{-1}\Big(\overline{G}_i(y_i+t)\Big)\Big)}.$$

Hence, what we actually should prove is that

$$\frac{(-1)^d \psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\Big(\overline{F}_i(y_i+t)\Big)\Big)}{\psi\Big(\sum_{i=1}^d \psi^{-1}\Big(\overline{F}_i(t)\Big)\Big)} \leq \frac{(-1)^d \psi^{(d)}\Big(\sum_{i=1}^d \psi^{-1}\Big(\overline{G}_i(y_i+t)\Big)\Big)}{\psi\Big(\sum_{i=1}^d \psi^{-1}\Big(\overline{G}_i(t)\Big)\Big)}$$

Let, $t_1 = \sum_{i=1}^d \psi^{-1}\Big(\overline{F}_i(t)\Big)$ and $t_2 = \sum_{i=1}^d \psi^{-1}\Big(\overline{G}_i(t)\Big)$. Obviously, by $X_i \geq_{st} Y_i$, we have $t_1 \leq t_2$. Clearly, there exist s_1 and s_2 such that

$$\sum_{i=1}^d \psi^{-1}\Big(\overline{F}_i(y_i+t)\Big) = t_1 + s_1,$$

and

$$\sum_{i=1}^d \psi^{-1}\Big(\overline{G}_i(y_i+t)\Big) = t_2 + s_2.$$

Therefore,

$$\begin{aligned} s_1 - s_2 &= \left(\sum_{i=1}^d \psi^{-1}\Big(\overline{F}_i(y_i+t)\Big) - \sum_{i=1}^d \psi^{-1}\Big(\overline{G}_i(y_i+t)\Big) - \sum_{i=1}^d \psi^{-1}\Big(\overline{F}_i(t)\Big) + \sum_{i=1}^d \psi^{-1}\Big(\overline{G}_i(t)\Big)\right) \\ &= \sum_{i=1}^d \left(\psi^{-1}\Big(\overline{F}_i(y_i+t)\Big) - \psi^{-1}\Big(\overline{G}_i(y_i+t)\Big) - \psi^{-1}\Big(\overline{F}_i(t)\Big) + \psi^{-1}\Big(\overline{G}_i(t)\Big)\right) \leq 0, \end{aligned}$$

because $S_i(t) = \psi^{-1}\Big(\overline{F}_i(y_i+t)\Big) - \psi^{-1}\Big(\overline{G}_i(y_i+t)\Big)$ is a decreasing function of t, for all $i = 1, \ldots, d$. By assumption, $(-1)^d \psi^{(d)}(t+s)$ is an increasing function of t. Thus, clearly,

$\frac{(-1)^d\psi^{(d)}(t+s)}{\psi(t)}$ is also increasing in t, for any $s \in \Re$. Therefore, we have

$$\frac{(-1)^d\psi^{(d)}\Big(t_1+s_1\Big)}{\psi\Big(t_1\Big)} \leq \frac{(-1)^d\psi^{(d)}\Big(t_2+s_1\Big)}{\psi\Big(t_2\Big)} \leq \frac{(-1)^d\psi^{(d)}\Big(t_2+s_2\Big)}{\psi\Big(t_2\Big)},$$

This completes the proof. □

The corollary below follows from Theorems 3.2 and 3.7.

Corollary 3.8. *If $X_i \leq_{st} Y_i$ for $i = 1, \ldots, d$, if ψ is completely monotone and if $\psi^{-1}(\bar{F}_i(t+y_i)) - \psi^{-1}(\bar{G}_i(t+y_i))$ is an increasing function of t for every $i = 1, \ldots, d$, then $\mathbf{X}_t \leq_{lo} \mathbf{Y}_t$ for every $t \geq 0$.*

An example for the application of the previous corollary follows immediately from the definition of lower orthant order. Consider two parallel systems, each one composed by d components, and assume that lifetimes of the components depend on a common environmental random parameter having Laplace transform ψ, so that the two corresponding vectors of component's lifetimes $\mathbf{X}$ and $\mathbf{Y}$ have joint survivals

$$\overline{F}(x_1, \ldots, x_d) = \psi\left(\sum_{i=1}^{d} \psi^{-1}(\overline{F}_i(x_i))\right) \quad \text{and} \quad \overline{G}(y_1, \ldots, y_d) = \psi\left(\sum_{i=1}^{d} \psi^{-1}(\overline{G}_i(y_i))\right),$$

respectively, where $\overline{F}_i$ and $\overline{G}_i$ denote the marginal survival functions of the random variables $X_i, i = 1, \ldots, d$ and $Y_i, i = 1, \ldots, d$. Since $\mathbf{X}_t \leq_{lo} \mathbf{Y}_t$ clearly implies

$$P[X_1 - t \leq s, \ldots, X_d - t \leq s | X_i > t, \forall i] \geq P[Y_1 - t \leq s, \ldots, Y_d - t \leq s | X_i > t, \forall i]$$

for all $t, s \geq 0$, if the assumptions of Corollary 3.8 are satisfied, then

$$P[\max(X_1, \ldots, X_d) \leq t + s | X_i > t, \forall i] \geq P[\max(Y_1, \ldots, Y_d) \leq t + s | Y_i > t, \forall i]$$

for all $t, s \geq 0$, i.e., the residual lifetime of the first parallel system will remain smaller in univariate stochastic order than those of the second parallel system whenever all the components in the two systems are in the working state.

4. Some examples

The following example illustrates same cases where the assumptions of Theorems 3.1, 3.5 and 3.7 may be satisfied.

Example 4.1. *Let us consider the example already considered in Remark 2.13 in Müller and Scarsini (2005) with some modifications. Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{2\pi} e^{-t^2/2} dt$ be the standard normal df and $\psi : \Re_+ \to \Re$ be such that $\psi^{(2)}(x) = ce^{-g(x)}$ with*

$$g(x) := \begin{cases} -2\Phi(x) - 1, & x > a; \\ \alpha x + \beta, & x \leq a, \end{cases}$$

where $c^{-1} = \int_0^\infty e^{-g(x)}dx$*, a is a large constant, and* α *and* β *are such that g is continuously differentiable in a, so that* $g : \Re_+ \to \Re$ *is concave and* $\log(-\psi^{(2)})$ *is convex. Let* $\psi_2(x) = \psi^{(2)}(x)$ *and* $\psi_d = \frac{\int_x^\infty \psi_{d-1}(t)dt}{\int_0^\infty \psi_{d-1}(t)dt}$*. It is easy to verify that* ψ_d *is a d-alternating function. Thus, we have*

$$\begin{aligned}
\frac{(-1)^d\psi_d^{(d)}(x)}{\psi_d(x)} &= (-1)^d \frac{-\psi_{d-1}^{(d-1)}(x)}{\int_0^\infty \psi_{d-1}(t)dt\psi_d(x)} \\
= \quad \cdots &= (-1)^d \frac{(-1)^{d-2}\psi_2^{(2)}(x)}{\int_0^\infty \psi_{d-1}(t)dt \cdots \int_0^\infty \psi_2(t)dt\psi_d(x)} \\
= \quad & \frac{ce^{-g(x)}}{\int_0^\infty \psi_{d-1}(t)dt \cdots \int_0^\infty \psi_2(t)dt\psi_d(x)} = \begin{cases} A(x)e^{2\Phi(x)+1}, & x > a; \\ A(x)e^{-\alpha(x)-\beta}, & x \le a. \end{cases}
\end{aligned}$$

where $\phi = \Phi'$ *and where* $A(x) = [\int_0^\infty \psi_{d-1}(t)dt \cdots \int_0^\infty \psi_2(t)dt\psi_d(x)]^{-1}$ *is increasing in x. Observe that for negative* α*, the function* $e^{-\alpha(x)-\beta}$ *and* $e^{2\Phi(x)+1}$ *are increasing. Therefore, for sufficiently large a the ratio* $\frac{(-1)^d\psi_d^{(d)}(x)}{\psi_d(x)}$ *is an increasing and* $(-1)^d\psi_d^{(d)}(x)$ *is a log convex function of x, which can be used in Theorems 3.1 and 3.7. Let* $g(x) = \psi^{-1}(\bar{F}_i(x))$*. Thus, if* $f'(x) > 0$*, then*

$$g''(x) = \frac{-f'(x)\psi'(\psi^{-1}(\bar{F}_i(x))) - f^2(x)\frac{\psi''(\psi^{-1}(\bar{F}_i(x)))}{\psi'(\psi^{-1}(\bar{F}_i(x)))}}{(\psi'(\psi^{-1}(\bar{F}_i(x))))^2}$$

is positive. In this case, X_i*'s can be distributed as truncated normal standard on* $(-\infty, 0)$*, truncated Cauchy standard on* $(-\infty, 0)$*, truncated logistic standard on* $(-\infty, 0)$ *or truncated standard double-exponential on* $(-\infty, 0)$*. Also, in this case, if we redefine* $\psi'(x) = -ce^{-g(x)}$*, then we have* $(-1)^{(d-1)}\psi^{d-1}$ *is log concave and the assumptions of Theorem 3.5 are satisfied. Now, let* $g(x)$ *be as*

$$g(x) := \begin{cases} 2\Phi(x) - 1, & x \le a; \\ \alpha x + \beta, & x > a. \end{cases}$$

Then, this time $(-1)^{(d-1)}\psi^{d-1}$ *would be log convex. Thus the opposite assumptions in Theorem 3.5 are satisfied.*

The following example illustrates that the assumptions of Theorem 3.2 and Corollary 3.6 may be satisfied.

Example 4.2. *Let us now restrict our attention to the case when* C_ψ *is a Clayton survival copula, i.e.,* $\mathbf{X}$ *is a multivariate vector having joint survival function defined as in (2.5), where* $\psi(s) = (1+s)^{-\theta}$ *for* $s > 0$ *and some positive constants* θ *(see Clayton, 1978). Relevance of Clayton copulas has been pointed out, for example, in Javid (2009). Let* $g(x) = \psi^{-1}(\bar{F}(x)) = (\bar{F}(x))^{-1/\theta} - 1$*. Thus,*

$$\begin{aligned}
g''(x) &= \frac{1}{\theta}(1+\frac{1}{\theta})(\bar{F}(x))^{-\frac{1}{\theta}-2}f^2(x) + \frac{1}{\theta}(\bar{F}(x))^{-\frac{1}{\theta}-1}f'(x) \\
&= \frac{1}{\theta}(\bar{F}(x))^{-\frac{1}{\theta}-2}((1+\frac{1}{\theta})f^2(x) + \bar{F}(x)f'(x)) .
\end{aligned}$$

Now, let $h_\theta(x) = (1+\frac{1}{\theta})f^2(x) + \bar{F}(x)f'(x)$. *Clearly,* $\psi^{-1}(\bar{F}(x))$ *is concave if, and only if,* $h_\theta(x)$ *is negative. Assume that* $K(x) = \log(\bar{F}(x))$ *so that,*

$$K''(x) = -\frac{f'(x)\bar{F}(x) + f^2(x)}{\bar{F}^2(x)} \geq -\frac{h_\theta(x)}{\bar{F}^2(x)}.$$

Thus, if $h_\theta(x)$ *is negative,* $K''(x)$ *is positive, which yields that* $\bar{F}(x)$ *is log convex. Therefore, if* $h_\theta(x)$ *is negative the conditions of Theorem 3.2 and Corollary 3.6 are satisfied.*

If we choose

$$F(x) = \frac{e^{bx_0^{-a}} - e^{bx^{-a}}}{e^{bx_0^{-a}} - 1}, \quad x \in (x_0, \infty), \ a, b, x_0 > 0,$$

the required conditions hold, because in this case, $h_\theta(x)$ *equals*

$$\begin{aligned} & \frac{1}{e^{bx_0^{-a}} - 1}\Big((1+\frac{1}{\theta})(abx^{-a-1}e^{bx^{-a}})^2 - (e^{bx^{-a}} - 1)\Big(ab(a+1)x^{-a-2}e^{bx^{-a}} + (abx^{-a-1})^2 e^{bx^{-a}}\Big)\Big) \\ = \ & \frac{1}{e^{bx_0^{-a}} - 1}(abx^{-a-1}e^{bx^{-a}})^2\Big(1 + \frac{1}{\theta} - (1 - e^{-bx^{-a}})(\frac{a+1}{ab}x^a + 1)\Big). \end{aligned}$$

Now, we can select a, b, θ *and* x_0 *such that* $h_\theta(x) < 0$ *for* $x > x_0$. *For example, let* $a = b = 1$ *and* $\theta \geq 2$. *Then we have*

$$\begin{aligned} h_\theta(x) &= \frac{1}{e^{x_0^{-1}} - 1}(x^{-2}e^{x^{-1}})^2\Big(1 + \frac{1}{\theta} - (1 - e^{-x^{-1}})(2x+1)\Big) \\ &\leq \frac{1}{e^{x_0^{-1}} - 1}(x^{-2}e^{x^{-1}})^2\Big(1.5 + e^{-x^{-1}} + 2xe^{-x^{-1}} - 2x - 1\Big). \end{aligned}$$

But, as we see in Figure 1, the last function is negative for $x > x_0 = 0.268558$. *Therefore,* $h_\theta(x)$ *is negative for* $x > x_0$.

FIGURE 1. Plot of the function $h_\theta(x) = 0.5 + e^{-x^{-1}} + 2xe^{-x^{-1}} - 2x$ for $x \in (0, 5)$. This function is a decreasing function and converges to -0.5 as $x \to \infty$.

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