Gray identities, Chern connections and integrability

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11 May 2024

GRAY IDENTITIES, CANONICAL CONNECTION AND INTEGRABILITY

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ABSTRACT. We characterize quasi Kähler manifolds whose curvature tensor associated to the canonical Hermitian connection satisfies the first Bianchi identity. This condition is related with the third Gray identity and in the almost Kähler case implies the integrability. Our main tool is the existence of generalized holomorphic frames introduced by the second author previously. By using such frames we also give a simpler and shorter proof of a Theorem of Goldberg. Furthermore we study almost Hermitian structures having the curvature tensor associated to the canonical Hermitian connection equal to zero. We show some explicit examples of quasi Kähler structures on the Iwasawa manifold having the Hermitian curvature vanishing and the Riemann curvature tensor satisfying the second Gray identity.

1. INTRODUCTION

Quasi Kähler and almost Kähler manifolds are special classes of almost Hermitian manifolds and can be considered as natural generalizations of Kähler manifolds to the context of almost symplectic and symplectic manifolds. It is well known that if (M, ω) is a (almost) symplectic manifold, then there always exists an almost complex structure J compatible with ω. Furthermore the choice of such an almost complex structure is unique up to homotopy. Hence quasi Kähler and almost Kähler structures can be consider as a tool to study (almost) symplectic manifolds.

The interplay between the integrability of almost Hermitian structures and the curvature has been largely studied in the last years (see e.g. [3], [11] and the references therein). One of the most important results in this topics is due to Goldberg. Indeed Goldberg in [10] proved that if the Riemann curvature tensor of an almost Kähler metric g satisfies the first Gray condition, i.e. if it commutes with the almost complex structure, then g is a Kähler metric. Gray's conditions were introduced in [9] and consist of some formulae involving the curvature tensor of an almost Hermitian metric and the associated almost complex structure. The Goldberg theorem has been further generalized to the following formula:

$$(1) \qquad\qquad s_* - s = \|\nabla \omega\|^2,$$

where s and s_* are the scalar curvature and the $*$-scalar curvature associated to an almost Kähler structure (g, J, ω) respectively (see e.g. [3]). The classical proof of this result is based on the Weitzenböck decomposition.

Another important curvature tensor in almost Hermitian geometry is the *Hermitian curvature tensor* $\widetilde{R}$. This tensor is defined as the curvature of the unique Hermitian connection $\widetilde{\nabla}$ whose torsion has $(1,1)$-part vanishing.

In [6] de Bartolomeis and Tomassini proved that a quasi Kähler manifold always admits a special complex frame. This result has been improved by the second author in [17] introducing generalized normal holomorphic frames. Such frames have been further taken into account in [18] to prove that if the holomorphic bisectional curvature associated to an almost Kähler metric g and the holomorphic bisectional curvature associated to the canonical connection coincides, then g is a Kähler metric. This result is not trivial, since the Hermitian curvature tensor does not necessary satisfy the first Bianchi identity.

As first result of this paper we give a new proof of formula (1). Our proof is elementary and does not make use of the Weitzenböck decomposition, but of the existence of generalized normal holomorphic frames only. Sections 3, 4 are dedicated to the study of the Hermitian curvature tensor in quasi Kähler and almost Kähler manifolds. We show that in the quasi Kähler case this curvature tensor satisfies the first Bianchi identity if and only if the curvature of g satisfies both the third Gray condition and another special identity involving the derivative of the Nijenhuis tensor. Namely,

Theorem 1.1. *Let (M, g, J, ω) be a quasi Kähler manifold. The Hermitian curvature tensor $\widetilde{R}$ satisfies the first Bianchi identity*

$$(2) \qquad \underset{X,Y,Z}{\mathfrak{S}}\, \widetilde{R}(X, Y, Z, \cdot) = 0\,, \quad \text{for every } X, Y, Z \in \Gamma(TM)$$

if and only if the following conditions hold:

1. *the curvature tensor R associated to g satisfies the third Gray identity*

   $$R(\overline{Z}_1, Z_2, Z_3, Z_4) = 0\,, \quad \text{for every } Z_1, Z_2, Z_3, Z_4 \in \Gamma(T^{1,0}M)\,;$$

2. *we have*

   $$R(Z_1, Z_2, \overline{Z}_3, \overline{Z}_4) = \frac{1}{4} F(\overline{Z}_3, Z_1, Z_2, \overline{Z}_4)$$

   for every $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T^{1,0}M)$, where F is the tensor

   $$F(X, Y, Z, W) := g((\nabla_X N)(Y, Z), W)\,,$$

   ∇ *is the Levi-Civita connection of g and N denotes the Nijenhuis tensor.*

The previous theorem allows us to prove the following

Corollary 1.2. *Let (M, g, J, ω) be an almost Kähler manifold. Assume that the Hermitian curvature tensor associated to (g, J) satisfies the first Bianchi identity (2), then (M, g, J, ω) is a Kähler manifold.*

In section 4 we study almost Hermitian manifolds whose Hermitian curvature tensor vanishes. By corollary 1.2 this condition forces a 4-dimensional quasi Kähler structure to be Kähler. In higher dimensions things work differently even in the compact case. We show that it is possible to construct examples of strictly quasi Kähler nilmanifolds having the Hermitian curvature equal to zero.

The study of the tensor $\widetilde{R}$ is also related with a conjecture of Donaldson. Indeed, $\widetilde{R}$ has been recently taken into account by Tosatti, Weinkove and Yau in [15] to study a Donaldson's conjecture stated in [5]. More precisely, they proved that if (M, ω) is a symplectic manifold, J is an almost complex structure tamed by ω and $\mathcal{R}(g, J)$ denotes the tensor

$$(3) \qquad \mathcal{R}_{i\bar{j}k\bar{l}}(g, J) := \widetilde{R}^{j}_{ik\bar{l}} + 4N^{r}_{\bar{l}j}\overline{N^{i}_{\bar{r}k}},$$

where g is the metric associated to (ω, J) and N is the Nijenhuis tensor of J, then condition $\mathcal{R}(g, J) \geq 0$ implies that the Donaldson's conjecture holds.

It is important to observe that in the examples described in section 4 the tensor $\mathcal{R}(g, J)$ vanishes.

NOTATION. Given a differential manifold M, TM denotes its tangent bundle. If a vector bundle F is fixed, then $\Gamma(F)$ denotes the vector space of the relative smooth sections. If Z_i is a complex vector field on a manifold M, then we usually write $Z_{\bar{i}}$ instead of $\overline{Z}_i$. The cyclic sum is denoted with the symbol $\mathfrak{S}$.

2. REVIEW

2.1. **Almost Hermitian manifolds.** Let M be a $2n$-dimensional manifold. An *almost complex structure* on M is an endomorphism J of TM satisfying $J^2 = -\text{Id}$. An almost complex structure J is said to be *integrable* if the Nijenhuis tensor

$$N(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \text{ for } X, Y \in \Gamma(TM)$$

vanishes everywhere. In view of the celebrated Newlander-Nirenberg Theorem (see [12]), J is integrable if and only if it is induced by a system of holomorphic coordinates. Any almost complex structure on M induces a natural splitting of the complexified tangent bundle into

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ and $T^{0,1}M$ are the eigenspaces relatively to i and $-i$, respectively. Consequently the vector bundle $\wedge^p M \otimes \mathbb{C}$ of complex p-forms on M splits as

$$\wedge^p M \otimes \mathbb{C} = \bigoplus_{r+s=p} \wedge^{r,s} M.$$

Since

$$d(\Gamma(\wedge^{r,s}M)) \subseteq \Gamma(\wedge^{r+2,s-1}M \oplus \wedge^{r+1,s}M \oplus \wedge^{r,s+1}M \oplus \wedge^{r-1,s+2}M),$$

then the exterior derivative splits as

$$d = A + \partial + \overline{\partial} + \overline{A}.$$

It is well known that J is integrable if and only if $A = 0$. Furthermore, it can be useful to observe that the Nijenhuis tensor satisfies

$$(4) \qquad N(Z_1, Z_2) \in \Gamma(T^{0,1}M), \quad N(Z_1, \overline{Z}_2) = 0$$

for every $Z_1, Z_2 \in \Gamma(T^{1,0}M)$. A Riemannian metric g on (M, J) is said to be J-Hermitian if it is preserved by J. In this case the pair (g, J) is called an *almost Hermitian structure*. Any almost Hermitian structure (g, J) induces a natural almost symplectic structure $\omega(\cdot, \cdot) := g(J\cdot, \cdot)$.

Definition 2.1. *The triple (g, J, ω) is called:*

- *a* quasi Kähler *structure if $\overline{\partial}\omega = (\mathrm{d}\omega)^{1,2} = 0$;*
- *an* almost Kähler *structure if $\mathrm{d}\omega = 0$.*

On the other hand, if ω is a non-degenerate 2-form on an almost complex manifold (M, J), the we say that J is *tamed* by ω if

$$\omega(X, JX) > 0, \quad \text{for all } X \neq 0.$$

In this case we can define a Riemannian metric g by

$$g(X, Y) := \frac{1}{2}(\omega(X, JY) + \omega(Y, JX)).$$

The following lemma will be useful in the sequel (see e.g. [13, 17])

Lemma 2.2. *Let (M, g, J, ω) be an almost Hermitian manifold and let ∇ be the Levi-Civita connection associated to g. Then the following facts hold:*

a. *The form ω is quasi Kähler if and only if*

$$(5) \qquad \nabla_{\overline{Z}_1} Z_2 \in \Gamma(T^{1,0}M), \quad \text{for all } Z_1, Z_2 \in \Gamma(T^{1,0}M);$$

b. *The form ω is almost Kähler if and only if it is quasi Kähler and the Nijenhuis tensor of J satisfies*

$$(6) \qquad g(\nabla_{Z_1} Z_2, Z_3) = \frac{1}{4} g(N(Z_2, Z_3), Z_1)), \quad \text{for all } Z_1, Z_2, Z_3 \in \Gamma(T^{1,0}M).$$

Proof. It is well known that for an almost Hermitian structure (g, J, ω) the following fundamental relation holds

$$(7) \qquad 2g((\nabla_X J)Y, Z) = \mathrm{d}\omega(X, JY, JZ) - \mathrm{d}\omega(X, Y, Z) + g(N(Y, Z), JX),$$

for every $X, Y, Z \in \Gamma(TM)$. The items a. and b. can be obtained just by considering the complex extension of (7). □

2.2. The canonical connection. A linear connection on an almost Hermitian manifold (M, g, J) is called *Hermitian* if it preserves g and J. Any almost Hermitian manifold admits a canonical Hermitian connection $\widetilde{\nabla}$, which is characterized by the following properties

$$\widetilde{\nabla} g = 0, \quad \widetilde{\nabla} J = 0, \quad \mathrm{Tor}(\widetilde{\nabla})^{1,1} = 0,$$

where $\mathrm{Tor}(\widetilde{\nabla})^{1,1}$ denotes the $(1,1)$-part of the torsion of $\widetilde{\nabla}$. In the special case of a quasi Kähler structure, $\widetilde{\nabla}$ is given by

$$\widetilde{\nabla} = \nabla - \frac{1}{2} J \nabla J,$$

where ∇ is the Levi-Civita connection of g (see for instance [8]). We will call $\widetilde{\nabla}$ simply the *canonical connection*. The connection $\widetilde{\nabla}$ induces the *Hermitian curvature tensor*

$$\widetilde{R}(X, Y, Z, W) = g(\widetilde{\nabla}_X \widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y \widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]} Z, W).$$

Since $\widetilde{\nabla}$ preserves g, one has

$$\widetilde{R}(X,Y,Z,W) = -\widetilde{R}(Y,X,Z,W) = -\widetilde{R}(X,Y,W,Z)\,.$$

Note that since $\widetilde{\nabla}$ has torsion, in general $\widetilde{R}$ does not satisfies the first Bianchi identity (2). Moreover in general we don't have $\widetilde{R}(X,Y,Z,W) = \widetilde{R}(Z,W,X,Y)$.

2.3. The Gray conditions.
In [9] Gray considered some special classes of almost Hermitian manifolds characterized by some identities involving the curvature tensor.

Definition 2.3. Let (M,g,J) be an almost Hermtian manifold and let R be the curvature tensor of g. Then R is said to satisfy

- the *first Gray identity* (G$_1$) if $R(Z_1, Z_2, \cdot, \cdot) = 0$;
- the *second Gray identity* (G$_2$) if $R(Z_1, Z_2, Z_3, Z_4) = R(\overline{Z}_1, Z_2, Z_3, Z_4) = 0$;
- the *third Gray identity* (G$_3$) if $R(\overline{Z}_1, Z_2, Z_3, Z_4) = 0$;

for every $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T^{1,0}M)$.

Clearly one has

$$(\mathrm{G}_1) \Longrightarrow (\mathrm{G}_2) \Longrightarrow (\mathrm{G}_3)$$

and that the curvature tensor of a Kähler manifold satisfies (G$_1$). Furthermore, in view of a Theorem of Goldberg (see [10]), any almost Kähler manifold whose curvature tensor satisfies (G$_1$) is a genuine Kähler manifold. The same can not be claimed for the condition (G$_2$). Indeed in dimension greater than 6 there exist examples of compact strictly almost Kähler manifolds whose curvature tensor satisfies (G$_2$) (see [4]). In dimension 4 there is a different behavior since we have the following theorem due to Apostolov, Armstrong and Drăghici:

Theorem 2.4 ([2], Theorem 2). *In dimension 4 there are no compact strictly almost Kähler manifolds whose curvature tensor satisfies* (G$_3$).

2.4. Generalized normal holomorphic frames.
Let (M,g,J,ω) be a $2n$-dimensional almost Hermitian manifold. Denote by ∇ the Levi-Civita connection associated to the metric g, by R the curvature tensors associated to ∇ and by N the Nijenhuis tensor of J.

Definition 2.5. Let o be an arbitrary point in M. A *generalized normal holomorphic frame* (or shortly a *g.n.h.f*) around o is a local $(1,0)$-complex frame $\{Z_1, \ldots, Z_n\}$ satisfying the following properties:

a. $\nabla_i Z_{\overline{j}}(o) = 0$;

b. $\nabla_i Z_j(o)$ is of type $(0,1)$;

c. $g_{i\overline{j}}(o) = \delta_{ij}$, $\mathrm{d}g_{i\overline{j}}(o) = 0$;

d. $\nabla_i \nabla_{\overline{j}} Z_k(o) = 0$;

for every $i, j, k = 1, \ldots, n$.

We can recall the following

Theorem 2.6 ([17], Theorem 1). *The following facts are equivalent*

a. ω *is a quasi Kähler form;*

b. *Any point o in M admits a generalized normal holomorphic frame.*

The following lemma, whose proof is similar to the one of Theorem 3.3 of [18], will be useful in the sequel

Lemma 2.7. *Let F be the smooth tensor on M defined by*

$$F(X, Y, Z, W) := g((\nabla_X N)(Y, Z), W) \quad for \ X, Y, Z, W \in \Gamma(TM) \,.$$

Consider an arbitrary point o of M and let $\{Z_1, \ldots, Z_n\}$ be a g.n.h.f. around o. Then

$$F_{\bar{i}jk\bar{l}}(o) = 4g([Z_j, Z_k], \nabla_{\bar{i}} Z_{\bar{l}})(o)$$

for every $i, j, k, l = 1, \ldots, n$.

The next result is a slight improvement of Theorem 3.3 of [18] and can be viewed as a corollary of Lemma 2.7

Theorem 2.8. *Let (M, g, J, ω) be a quasi Kähler manifold and assume that the Nijenhuis tensor of J satisfies*

$$(8) \qquad \underset{X,Y,Z}{\mathfrak{S}} \nabla_X N(Y, Z) = 0, \quad \forall \, X, Y, Z \in \Gamma(TM) \,,$$

then J is integrable.

Proof. Let $o \in M$ and let $\{Z_1, \ldots, Z_n\}$ be a g.n.h.f. around o. By (4), we have

$$N_{i\bar{k}}(o) = 0 \,; \quad N_{ik}(o) \in T_o^{0,1} M \,, \quad \text{for every } i, j = 1, \ldots, n \,.$$

Furthermore, by the properties of the g.n.h.f., we have

$$\underset{\bar{i},j,k}{\mathfrak{S}} (\nabla_{\bar{i}} N)(Z_j, Z_k)(o) = \nabla_{\bar{i}}(N(Z_j, Z_k))(o) \,.$$

Hence equation (8) implies $(\nabla_{\bar{i}} N)_{jk} = 0$ which, in view of Lemma 2.7, is equivalent to $N = 0$. $\qquad\qquad\square$

A direct computation gives the following

Proposition 2.9. *The components of the curvature tensor with respect to a g.n.h.f. $\{Z_1, \ldots, Z_n\}$ around a point o write as*

$$R_{i\bar{j}k\bar{l}}(o) = -g(\nabla_{\bar{j}} \nabla_i Z_k, Z_{\bar{l}})(o) \,;$$
$$R_{\bar{i}jkl}(o) = g(\nabla_{\bar{i}} \nabla_j Z_k, Z_l)(o) \,;$$
$$R_{\bar{i}\bar{j}kl}(o) = -g(\nabla_{[Z_{\bar{i}}, Z_{\bar{j}}]} Z_k, Z_l)(o) \,;$$
$$R_{ijkl}(o) = g(\nabla_i \nabla_j Z_k, Z_l)(o) - g(\nabla_j \nabla_i Z_k, Z_l)(o) \,.$$

2.5. **Proof of formula** (1). The aim of this section is to give an alternative proof of formula (1) without use the Weitzenböck decomposition:

Proof of formula (1). Let (M, g, J, ω) be an almost Kähler manifold. First of all we recall the definition of the $*$-Ricci tensor and the $*$-scalar curvature

$$r_*(X, Y) := \sum_{i=1}^{2n} R(JX, JX_i, X_i, Y) \,, \quad s_* := \sum_{i=1}^{2n} r_*(X_i, X_i) \,,$$

where $\{X_1, \ldots, X_{2n}\}$ is an arbitrary orthonormal frame on M. It is easy to see that in complex coordinates the scalar curvature and the $*$-scalar curvature write as

$$s = 2 \sum_{i,j=1}^n \{R_{i\bar{j}j\bar{i}} - R_{ij\overline{ij}}\} \,, \quad s_* = 2 \sum_{i,j=1}^n \{R_{i\bar{j}j\bar{i}} + R_{ij\overline{ij}}\} \,,$$

being $\{Z_1, \ldots, Z_n\}$ an arbitrary unitary $(1,0)$-frame on M. In particular

$$s_* - s = 4 \sum_{i,j=1}^{n} R_{ij\overline{ij}}$$

and formula (1) can be rewritten as

$$\sum_{i,j=1}^{n} R_{ij\overline{ij}} = \frac{1}{4} \|\nabla\omega\|^2 \,.$$

Fix an arbitrary point o of M and let $\{Z_1, \ldots, Z_n\}$ be a g.n.h.f. around o. Since $\nabla_i Z_j(o) \in T_o^{0,1}M$, then $N_{ij}(o) = -4[Z_i, Z_j](o)$; hence formula (6) reads at o as

$$g([Z_i, Z_j], Z_l)(o) = -g(\nabla_l Z_i, Z_j)(o) \,.$$

Since $\{Z_1, \ldots, Z_n\}$ is an unitary frame we have

$$[Z_i, Z_j](o) = -\sum_{l=1}^{n} \Gamma_{li}^{\overline{j}}(o) \, Z_{\overline{l}}(o) \,,$$

where $\Gamma_{li}^{\overline{j}} := g(\nabla_l Z_i, Z_j)$. Furthermore we have

$$R_{ij\overline{ij}}(o) = -g(\nabla_{[Z_i, Z_j]} Z_{\overline{i}}, Z_{\overline{j}})(o) = \sum_{l=1}^{n} \Gamma_{li}^{\overline{j}} g(\nabla_{\overline{l}} Z_{\overline{i}}, Z_{\overline{j}})(o)$$

$$= \sum_{l=1}^{n} \Gamma_{li}^{\overline{j}}(o) \Gamma_{\overline{li}}^{j}(o) = \sum_{l=1}^{n} |\Gamma_{li}^{\overline{j}}|^2(o) \,.$$

Hence

$$\sum_{i,j=1}^{n} R_{ij\overline{ij}}(o) = \sum_{l,i,j=1}^{n} |\Gamma_{li}^{\overline{j}}|^2(o)$$

and the claim follows since $(\nabla_Z \omega)(X, Y) = \frac{1}{2} g(N(X,Y), JZ)$. $\qquad\square$

Condition (1) is related to the subspace $\mathcal{W}_4$ described in [16, pag. 372] (see also [7] where $\mathcal{W}_4 = \mathcal{C}_4$). Indeed, by using Lemma 4.5 at page 371 in [16] it is easy to see that the projection $R^{\mathcal{W}_4}$ of R to $\mathcal{W}_4$ is given by

$$R^{\mathcal{W}_4} = \frac{(s - s_*)}{16n(n-1)} = \frac{1}{4n(n-1)} \sum_{i,j=1}^{n} R_{ij\overline{ij}} = \frac{1}{16n(n-1)} \|\nabla\omega\|^2 \,.$$

3. The first Bianchi identity for the Hermitian curvature

In this section we are going to prove Theorem 1.1 and its Corollary 1.2.

Let $\widetilde{\nabla}$ be the canonical connection associated to a quasi Kähler structure (g, J, ω) on a $2n$-dimensional manifold M. We have

Lemma 3.1. Let Z_1, Z_2 be two arbitrary $(1,0)$-vector fields on M. Then

$$\widetilde{\nabla}_{Z_1} Z_2 \in \Gamma(T^{1,0}M) \,, \quad \widetilde{\nabla}_{\overline{Z}_1} Z_2 = \nabla_{\overline{Z}_1} Z_2 \in \Gamma(T^{1,0}M) \,.$$

Proof. It is enough to consider the definition of $\widetilde{\nabla}$ and to apply Lemma 2.2. $\qquad\square$

As a direct consequence of Lemma 3.1 we have the following

Proposition 3.2. *Let $\{Z_1, \ldots, Z_n\}$ be an arbitrary $(1,0)$-frame on M and let $\widetilde{R}$ be the Hermitian curvature tensor of M. Then*

1. $\widetilde{R}_{ijk\bar{l}} = R_{ijk\bar{l}}$;
2. $\widetilde{R}_{\overline{ij}kl} = \widetilde{R}_{ijkl} = \widetilde{R}_{i\bar{j}kl} = 0$.

Lemma 3.3. *Let o be an arbitrary point of M and let $\{Z_1, \ldots, Z_n\}$ be a g.n.h.f. around o. Then*

$$\widetilde{\nabla}_i Z_j(o) = 0, \quad \widetilde{\nabla}_{\bar{i}} Z_j(o) = 0, \quad \text{for any } i, j = 1, \ldots, n,$$

i.e. the canonical connection acts on generalized normal holomorphic frames in quasi Kähler manifolds as the Levi-Civita connection acts on normal holomorphic frames in Kähler manifolds.

Proof. Let $\{Z_1, \ldots, Z_n\}$ be a g.n.h.f. around o. Since $\nabla_i Z_j(o) \in T_o^{0,1}M$, then we have

$$\begin{aligned}
\widetilde{\nabla}_i Z_j(o) =& \frac{1}{2}\{\nabla_i Z_j - J\nabla_i J Z_j\}(o) = \frac{1}{2}\nabla_i Z_j(o) - \mathrm{i}\frac{1}{2}J\nabla_i Z_j(o) \\
=& \frac{1}{2}\nabla_i Z_j(o) - \frac{1}{2}\nabla_i Z_j(o) = 0.
\end{aligned}$$

Moreover since $\nabla_{\bar{i}} Z_j(o) = 0$, we have

$$\widetilde{\nabla}_{\bar{i}} Z_j(o) = \frac{1}{2}\{\nabla_{\bar{i}} Z_j - J\nabla_{\bar{i}} J Z_j\}(o) = \frac{1}{2}\nabla_{\bar{i}} Z_j(o) - \mathrm{i}\frac{1}{2}J\nabla_{\bar{i}} Z_j(o) = 0$$

and the claim follows. $\qquad\square$

We have the following

Proposition 3.4. *The components of the Hermitian curvature tensor $\widetilde{R}$ with respect to a g.n.h.f. $\{Z_1, \ldots, Z_n\}$ around a point o write as*

1. $\widetilde{R}_{i\bar{j}k\bar{l}}(o) = R_{i\bar{j}k\bar{l}}(o) - g(\nabla_i Z_k, \nabla_{\bar{j}} Z_{\bar{l}})(o)$;
2. $\widetilde{R}_{ijk\bar{l}}(o) = R_{ijk\bar{l}}(o)$;
3. $\widetilde{R}_{\overline{ij}kl}(o) = \widetilde{R}_{ijkl}(o) = \widetilde{R}_{i\bar{j}kl}(o) = 0$.

Proof. The items 2. and 3. come from Proposition 3.2. The proof the first identity can be obtained as follows:
By definition of $\widetilde{R}$ and the equation $[Z_i, Z_{\bar{j}}](o) = 0$, we have

$$\begin{aligned}
\widetilde{R}_{i\bar{j}k\bar{l}}(o) =& g(\widetilde{\nabla}_i \widetilde{\nabla}_{\bar{j}} Z_k - \widetilde{\nabla}_{\bar{j}} \widetilde{\nabla}_i Z_k - \widetilde{\nabla}_{[Z_i, Z_{\bar{j}}]} Z_k, Z_{\bar{l}})(o) \\
=& g(\widetilde{\nabla}_i \widetilde{\nabla}_{\bar{j}} Z_k - \widetilde{\nabla}_{\bar{j}} \widetilde{\nabla}_i Z_k, Z_{\bar{l}})(o).
\end{aligned}$$

Applying Lemma 3.1 and Lemma 3.3, we get

$$\begin{aligned}
\widetilde{R}_{i\bar{j}k\bar{l}}(o) =& g(\widetilde{\nabla}_i \widetilde{\nabla}_{\bar{j}} Z_k - \widetilde{\nabla}_{\bar{j}} \widetilde{\nabla}_i Z_k, Z_{\bar{l}})(o) \\
=& g(\widetilde{\nabla}_i \nabla_{\bar{j}} Z_k, Z_{\bar{l}})(o) - g(\widetilde{\nabla}_{\bar{j}} \widetilde{\nabla}_i Z_k, Z_{\bar{l}})(o) \\
=& Z_i g(\nabla_{\bar{j}} Z_k, Z_{\bar{l}})(o) - g(\nabla_{\bar{j}} Z_k, \widetilde{\nabla}_i Z_{\bar{l}})(o) - Z_{\bar{j}} g(\widetilde{\nabla}_i Z_k, Z_{\bar{l}})(o) \\
& + g(\widetilde{\nabla}_i Z_k, \widetilde{\nabla}_{\bar{j}} Z_{\bar{l}})(o) \\
=& Z_i g(\nabla_{\bar{j}} Z_k, Z_{\bar{l}})(o) - Z_{\bar{j}} g(\widetilde{\nabla}_i Z_k, Z_{\bar{l}})(o).
\end{aligned}$$

Finally, taking into account Lemma 2.2 and that ∇ and $\widetilde{\nabla}$ preserve g, we obtain

$$\begin{aligned}
\widetilde{R}_{i\bar{j}k\bar{l}}(o) =& Z_i g(\nabla_{\bar{j}}Z_k, Z_{\bar{l}})(o) - Z_{\bar{j}}g(\widetilde{\nabla}_i Z_k, Z_{\bar{l}})(o) \\
=& g(\nabla_i \nabla_{\bar{j}}Z_k, Z_{\bar{l}})(o) + g(\nabla_{\bar{j}}Z_k, \nabla_i Z_{\bar{l}})(o) \\
& - Z_{\bar{j}}Z_i g_{k\bar{l}}(o) + Z_{\bar{j}}g(Z_k, \widetilde{\nabla}_i Z_{\bar{l}})(o) \\
=& - Z_{\bar{j}}Z_i g_{k\bar{l}}(o) + Z_{\bar{j}}g(Z_k, \widetilde{\nabla}_i Z_{\bar{l}})(o) \\
=& - Z_{\bar{j}}g(\nabla_i Z_k, Z_{\bar{l}})(o) - Z_{\bar{j}}g(Z_k, \nabla_i Z_{\bar{l}})(o) \\
& - Z_{\bar{j}}g(Z_k, \nabla_i Z_{\bar{l}})(o) \\
=& - g(\nabla_{\bar{j}}\nabla_i Z_k, Z_{\bar{l}})(o) - g(\nabla_i Z_k, \nabla_{\bar{j}}Z_{\bar{l}})(o) \\
& - g(\nabla_{\bar{j}}Z_k, \nabla_i Z_{\bar{l}})(o) - g(Z_k, \nabla_{\bar{j}}\nabla_i Z_{\bar{l}})(o) \\
=& R_{i\bar{j}k\bar{l}}(o) - g(\nabla_i Z_k, \nabla_{\bar{j}}Z_{\bar{l}})(o),
\end{aligned}$$

i.e.

$$\widetilde{R}_{i\bar{j}k\bar{l}}(o) = R_{i\bar{j}k\bar{l}}(o) - g(\nabla_i Z_k, \nabla_{\bar{j}}Z_{\bar{l}})(o),$$

and the claim follows. $\qquad\square$

Now we are ready to prove Theorem 1.1:

Proof of Theorem 1.1. Let $o \in M$ be an arbitrary point and let $\{Z_1, \ldots, Z_n\}$ be a g.n.h.f. around o. By Proposition 3.4, we have

$$\underset{i,j,k}{\mathfrak{S}} \widetilde{R}_{ijkl}(o) = \underset{i,j,k}{\mathfrak{S}} \widetilde{R}_{ijk\bar{l}}(o) = 0.$$

Moreover

$$\underset{i,\bar{j},k}{\mathfrak{S}} \widetilde{R}_{i\bar{j}kl}(o) = R_{ki\bar{j}l}(o). \tag{9}$$

Furthermore

$$\begin{aligned}
\underset{i,\bar{j},k}{\mathfrak{S}} \widetilde{R}_{i\bar{j}k\bar{l}}(o) =& \widetilde{R}_{i\bar{j}k\bar{l}}(o) + \widetilde{R}_{ki\bar{j}\bar{l}}(o) + \widetilde{R}_{\bar{j}ki\bar{l}}(o) \\
=& \widetilde{R}_{i\bar{j}k\bar{l}}(o) + \widetilde{R}_{\bar{j}ki\bar{l}}(o) \\
=& R_{i\bar{j}k\bar{l}}(o) + R_{\bar{j}ki\bar{l}}(o) - g(\nabla_i Z_k, \nabla_{\bar{j}}Z_{\bar{l}})(o) + g(\nabla_k Z_i, \nabla_{\bar{j}}Z_{\bar{l}})(o) \\
=& - R_{ki\bar{j}\bar{l}}(o) - g([Z_i, Z_k], \nabla_{\bar{j}}Z_{\bar{l}})(o),
\end{aligned}$$

i.e.

$$\underset{i,\bar{j},k}{\mathfrak{S}} \widetilde{R}_{i\bar{j}k\bar{l}}(o) = R_{ik\bar{j}\bar{l}}(o) - g([Z_i, Z_k], \nabla_{\bar{j}}Z_{\bar{l}})(o). \tag{10}$$

Hence the Hermitian curvature $\widetilde{R}$ satisfies the first Bianchi identity at o if and only if the following equations hold:

$$R_{ki\bar{j}l}(o) = 0; \tag{11}$$

$$R_{ik\bar{j}\bar{l}}(o) - g([Z_i, Z_k], \nabla_{\bar{j}}Z_{\bar{l}})(o) = 0. \tag{12}$$

Equation (11) is the third Gray condition, while, in view of Lemma 2.7, equation (12) is satisfied if and only if

$$R(Z_1, Z_2, \overline{Z}_3, \overline{Z}_4) = \frac{1}{4}g((\nabla_{\overline{Z}_3}N)(Z_1, Z_2), \overline{Z}_4)$$

for every $Z_1, Z_2, Z_3, Z_4 \in \Gamma(T^{1,0}M)$. $\qquad\square$

Now we can prove Corollary 1.2.

Proof of Corollary 1.2. Assume that (M, g, J, ω) is an almost Kähler manifold and let $\widetilde{R}$ be the Hermitian curvature of (g, J). Fix an arbitrary point o of M, consider a g.n.h.f. $\{Z_1, \ldots, Z_n\}$ around o and assume that $\widetilde{R}$ satisfies the first Bianchi identity. Then, in view of Theorem 1.1, we have

$$0 = R_{ik\bar{j}\bar{l}}(o) - g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o) = -g(\nabla_{[Z_i, Z_k]} Z_{\bar{j}}, Z_{\bar{l}})(o) - g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o),$$

i.e.

(13) $$g(\nabla_{[Z_i, Z_k]} Z_{\bar{j}}, Z_{\bar{l}})(o) = -g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o).$$

In particular

$$g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o) = -g([Z_i, Z_k], \nabla_{\bar{l}} Z_{\bar{j}})(o),$$

i.e. $g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o)$ is skew-symmetric with respect to the indexes $\bar{j}, \bar{l}$. In view of formula (6), we have

$$\begin{aligned} g(\nabla_{[Z_i, Z_k]} Z_{\bar{j}}, Z_{\bar{l}})(o) &= \frac{1}{4} g(N_{\bar{j}\bar{l}}, [Z_i, Z_k])(o) = -g([Z_{\bar{j}}, Z_{\bar{l}}], [Z_i, Z_k])(o) \\ &= -2g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o) \end{aligned}$$

Hence equation (13) implies

$$g([Z_i, Z_k], \nabla_{\bar{j}} Z_{\bar{l}})(o) = 0$$

which forces J to be integrable. $\qquad\square$

4. The condition $\widetilde{R} = 0$ in quasi Kähler manifolds

In this section we investigate the case $\widetilde{R} = 0$. We start by considering the following preliminar

Lemma 4.1. *Let (M, g, J, ω) be a quasi Kähler manifold. Then the following are equivalent:*

1. *the curvature tensor of the canonical connection associated to (g, J) vanishes;*

2. *every $o \in M$ admits an open neighborhood U and a complex unitary $(1,0)$-frame $\{Z_1, \ldots, Z_n\}$ on U such that*
$$\nabla_i Z_j \in \Gamma(T^{0,1}U), \quad \nabla_{\bar{i}} Z_j = 0, \quad i, j = 1, \ldots, n.$$

Proof. The condition $\widetilde{R} = 0$ is equivalent to require that every point o of M admits an open neighborhood U equipped with a complex unitary $(1,0)$-frame $\{Z_1, \ldots, Z_n\}$ such that

(14) $$\widetilde{\nabla}_i Z_j = 0, \quad \widetilde{\nabla}_{\bar{i}} Z_j = 0, \quad i, j = 1, \ldots, n.$$

Since

$$\widetilde{\nabla}_i Z_j = 0 = \frac{1}{2} \nabla_i Z_j - \frac{1}{2} J \nabla_i J Z_j = \frac{1}{2} \nabla_i Z_j - \frac{1}{2} i J \nabla_i Z_j;$$

and

$$\widetilde{\nabla}_{\bar{i}} Z_j = 0 = \frac{1}{2} \nabla_{\bar{i}} Z_j - \frac{1}{2} J \nabla_{\bar{i}} J Z_j = \frac{1}{2} \nabla_{\bar{i}} Z_j - \frac{1}{2} i J \nabla_{\bar{i}} Z_j,$$

then (14) is equivalent to require that $\nabla_i Z_j, \nabla_{\bar{i}} Z_j \in \Gamma(T^{0,1}U)$ for every $i, j = 1, \ldots, n$. By the assumption on M to be quasi Kähler we have $\nabla_{\bar{i}} Z_j = 0$. $\qquad\square$

Remark 4.2. Note that the second item of the previous Lemma in particular implies that if g is an $\widetilde{R}$-flat quasi Kähler metric, then we can always find a local unitary $(1,0)$-coframe $\{\zeta_1, \ldots, \zeta_n\}$ such that

$$\partial \zeta_i = \overline{\partial} \zeta_i = 0, \quad i = 1, \ldots, n.$$

We recall that a 4-dimensional quasi Kähler manifold is always almost Kähler. Hence, in view of Theorem 1.1, if a 4-dimensional quasi Kähler manifold has $\widetilde{R} = 0$ then it is Kähler. In greater dimension things work differently:

Theorem 4.3. *There exists a quasi Kähler structure (g_0, J_0, ω_0) on the Iwasawa manifold with the following properties:*

    1. the Hermitian curvature of (g_0, J_0) vanishes;

    2. the Riemann curvature of g_0 satisfies the second Gray identity (G_2).

Proof. Let G be the *complex Heisenberg* group

$$G := \left\{ \left(\begin{array}{ccc} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{array} \right) : z_i \in \mathbb{C}, \mathrm{i} = 1, 2, 3 \right\}$$

and let M be the compact manifold $M = G/\Gamma$, where Γ is the co-compact lattice of G formed by the matrices with integral entries. Then M is the *Iwasawa manifold*. It is well known that M admits a global frame $\mathcal{B} = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ satisfying the following structure equations

$$[X_1, X_2] = X_3, \quad [X_4, X_5] = -X_3 \quad [X_2, X_4] = X_6, \quad [X_5, X_1] = X_6.$$

Let J_0 be the almost complex structure defined on the basis $\mathcal{B}$ by

$$J_0 X_1 = X_4, \qquad J_0 X_2 = X_5, \qquad J_0 X_3 = X_6,$$
$$J_0 X_4 = -X_1, \quad J_0 X_5 = -X_2, \quad J_0 X_6 = -X_3,$$

let g_0 be the J_0-almost Hermitian metric

$$g_0 = \sum_{i=1}^{6} \alpha_i \otimes \alpha_i,$$

and let

$$\omega_0 := \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6,$$

being $\{\alpha_1, \ldots, \alpha_6\}$ the dual frame of $\mathcal{B}$. Then (g_0, J_0, ω_0) is a quasi Kähler structure on M.

The almost complex structure J_0 induces the $(1,0)$-frame

$$Z_1 = X_1 - \mathrm{i}\, X_4, \quad Z_2 = X_2 - \mathrm{i}\, X_5, \quad Z_3 = X_3 - \mathrm{i}\, X_6.$$

Clearly

$$[Z_1, Z_2] = 2\, Z_{\overline{3}}, \quad [Z_{\overline{1}}, Z_{\overline{2}}] = 2\, Z_3$$

and all other brackets involving the vectors of the frame vanish. Furthermore, a direct computation gives $\nabla_{\overline{i}} Z_j = 0$, for $i, j = 1, 2, 3$ and

$$\nabla_1 Z_1 = 0, \qquad \nabla_2 Z_1 = -Z_{\overline{3}}, \qquad \nabla_3 Z_1 = Z_{\overline{2}},$$
$$\nabla_1 Z_2 = Z_{\overline{3}}, \qquad \nabla_2 Z_2 = 0, \qquad\qquad \nabla_3 Z_2 = Z_{\overline{1}},$$
$$\nabla_1 Z_3 = -Z_{\overline{2}}, \quad \nabla_2 Z_3 = Z_{\overline{1}}, \qquad \nabla_3 Z_3 = 0$$

where ∇ is the Levi-Civita connection associated to g_0. Hence $\nabla_i Z_j \in \Gamma(T^{0,1}M)$ and in view of Lemma 4.1 the Hermitian curvature tensor of (g_0, J_0) vanishes.

Furthermore a straightforward application of our formulae yields that the curvature tensor associated to g_0 satisfies the second Gray identity.   □

Remark 4.4. The almost Hermitian structure J_0 described in the proof of the above theorem corresponds to the almost complex structure denoted by J_3 [1].

The Iwasawa manifold is (in some fashion) the unique example of a 6-dimensional non-Kähler almost complex nilmanifold admitting a quasi Kähler $\widetilde{R}$-flat metric. More precisely we have the following

Theorem 4.5. *Let (G, J) be a 6-dimensional Lie group equipped with a left-invariant non-integrable almost complex structure admitting a J-compatible quasi Kähler metric g with vanishing Hermitian curvature tensor. Then the Lie algebra of G endowed with the almost complex structure induced by J is isomorphic as complex Lie algebra to the one of the complex Heisenberg group equipped with the almost complex structure induced by J_0.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. In view of Lemma 4.1 there exists a complex $(1,0)$-frame $\{Z_1, Z_2, Z_3\}$ on $\mathfrak{g}$ such that

$$[Z_i, Z_j] = \sum_{k=1}^{3} A_{ij}^{\overline{k}} Z_{\overline{k}}, \quad [Z_i, Z_{\overline{j}}] = 0, \quad i, j = 1, 2, 3.$$

Since J is by hypothesis non-integrable, there exists at least a bracket different from zero. We may assume

$$[Z_1, Z_2] \neq 0.$$

Now we observe that $A_{12}^{\overline{3}} \neq 0$. Indeed, if by contradiction $A_{12}^{\overline{3}} = 0$, then

$$[Z_1, Z_2] = A_{12}^{\overline{1}} Z_{\overline{1}} + A_{12}^{\overline{2}} Z_{\overline{2}}$$

and by the Jacobi identity

$$0 = [[Z_1, Z_2], Z_{\overline{1}}] = -A_{12}^{\overline{2}} [Z_{\overline{1}}, Z_{\overline{2}}],$$
$$0 = [[Z_1, Z_2], Z_{\overline{2}}] = -A_{12}^{\overline{1}} [Z_{\overline{1}}, Z_{\overline{2}}]$$

which implies $[Z_1, Z_2] = 0$. Hence $A_{12}^{\overline{3}}$ has to be different from zero and, consequently,

$$W_1 := Z_1, \quad W_2 = Z_2 \quad W_3 := \frac{1}{A_{12}^{\overline{3}}} (Z_3 - A_{12}^{1} Z_1 - A_{12}^{2} Z_2)$$

is a $(1,0)$-frame on $(\mathfrak{g}, J)$. Such a frame satisfies

$$[W_1, W_2] = W_{\overline{3}}.$$

Finally, using again the Jacobi identity, we get

$$0 = [[W_1, W_2], W_{\overline{1}}] = -[W_{\overline{2}}, W_{\overline{3}}],$$
$$0 = [[W_1, W_2], W_{\overline{2}}] = -[W_{\overline{1}}, W_{\overline{3}}],$$

i.e.

$$[W_2, W_3] = [W_1, W_3] = 0$$

which ends the proof.   □

It is possible to find some non-equivalent quasi Kähler structures on the Iwasawa manifold having $\widetilde{R} = 0$. For instance we have the following example

Example 4.6. It easy to show that the Iwasawa manifold M admits a global coframe $\{\alpha_1, \ldots, \alpha_6\}$ satisfying the following structure equations

$$\mathrm{d}\,\alpha_1 = \mathrm{d}\,\alpha_3 = -\alpha_1 \wedge \alpha_2 + \alpha_4 \wedge \alpha_5 - \alpha_2 \wedge \alpha_3 + \alpha_5 \wedge \alpha_6\,;$$
$$\mathrm{d}\,\alpha_2 = \mathrm{d}\,\alpha_5 = 0\,;$$
$$\mathrm{d}\,\alpha_4 = \mathrm{d}\,\alpha_6 = -\alpha_2 \wedge \alpha_4 + \alpha_1 \wedge \alpha_5 - \alpha_3 \wedge \alpha_5 + \alpha_2 \wedge \alpha_6\,.$$

Let $\{X_1, \ldots, X_6\}$ be the frame dual to $\{\alpha_1, \ldots, \alpha_6\}$ and consider the almost complex structure J on M defined on $\{X_1, \ldots, X_6\}$ by

$$JX_1 = X_4\,, \qquad JX_2 = X_5\,, \qquad JX_3 = X_6\,,$$
$$JX_4 = -X_1\,, \quad JX_5 = -X_2\,, \quad JX_6 = -X_3\,.$$

Let

$$\omega := \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_5 + \alpha_3 \wedge \alpha_6\,;$$

then a direct computation gives that ω is a $\overline{\partial}$-closed form compatible with J. The basis $\{X_1, \ldots X_6\}$ induces the complex $(1,0)$-frame

$$Z_1 = X_1 - \mathrm{i}\,X_4\,, \quad Z_2 = X_2 - \mathrm{i}\,X_5\,, \quad Z_3 = X_3 - \mathrm{i}\,X_6\,.$$

One easily gets

$$[Z_1, Z_2] = 2(Z_{\overline{1}} + Z_{\overline{3}})\,, \quad [Z_2, Z_3] = 2(Z_{\overline{1}} + Z_{\overline{3}})\,, \quad [Z_1, Z_3] = 0\,.$$

Since $[Z_i, Z_{\overline{j}}] = 0$ and (g, J, ω) is a quasi Kähler structure, in view of Lemma 2.2 we have

$$\nabla_{\overline{i}} Z_j = 0\,,$$

being ∇ the Levi-Civita connection associated to the metric g. Furthermore a direct computation gives

$$\nabla_1 Z_1 = -2Z_{\overline{2}}\,, \quad \nabla_2 Z_1 = -2Z_{\overline{3}}\,, \quad \nabla_3 Z_1 = 0\,,$$
$$\nabla_1 Z_2 = 2Z_{\overline{1}}\,, \qquad \nabla_2 Z_2 = 0\,, \qquad \nabla_3 Z_2 = -2Z_{\overline{3}}\,,$$
$$\nabla_1 Z_3 = 2Z_{\overline{1}}\,, \qquad \nabla_2 Z_3 = 2Z_{\overline{1}}\,, \qquad \nabla_3 Z_3 = 2Z_{\overline{2}}\,;$$

hence

$$\nabla_i Z_j \in \Gamma(T^{0,1}M)\,, \quad \text{for every } i, j = 1, 2, 3\,.$$

By Lemma (4.1), we get that the Hermitian curvature tensor of g vanishes. Also in this case a straightforward computation gives that the curvature tensor of the metric g satisfies the second Gray identity (G_2).

Remark 4.7. In the quasi Kähler case the condition $\widetilde{R} = 0$ implies that the tensor $\mathcal{R}(g, J)$ described by (3) vanishes. Hence it is very natural to take into account the following problem:

- Does there exist a symplectic form ω' on the Iwasawa manifold taming the almost complex structure J_0 and such that the pair (ω', J_0) induces an $\widetilde{R}$-flat quasi Kähler structure on M ?

(This problem was suggested us by Valentino Tosatti). The answer is negative. In order to show this we fix a quasi Kähler $\widetilde{R}$-flat metric g on the Iwasawa manifold M compatible with J_0. Then we can find a global unitary $(1,0)$-coframe $\{\zeta_1, \zeta_2, \zeta_3\}$ such that

$$(15) \qquad\qquad \mathrm{d}\zeta_1 = \mathrm{d}\zeta_2 = 0\,, \quad \mathrm{d}\zeta_3 = -\zeta_{\overline{1}} \wedge \zeta_{\overline{2}}\,.$$

Assume that there exists a symplectic structure ω' taming J_0 and such that the pair (ω', J_0) induces the metric g. Then one necessary has

$$\omega' = \omega + \beta + \overline{\beta}$$

ω being the quasi Kähler form associated to g and β a complex form of type $(2,0)$. Equation $\mathrm{d}\omega' = 0$ reads in terms of ω and β as

$$\begin{cases} A\omega + \partial\beta = 0 \\ \overline{\partial}\beta + A\overline{\beta} = 0 \,. \end{cases}$$

We can write $\beta = a\zeta_{12} + b\zeta_{23} + c\zeta_{13}$, where a, b, c are smooth functions on M. Taking into account equations (15), one has

$$\overline{\partial}\beta = \sum_{r=1}^{3} \zeta_{\overline{r}}(a)\zeta_{12\overline{r}} + \zeta_{\overline{r}}(b)\zeta_{23\overline{r}} + \zeta_{\overline{r}}(c)\zeta_{13\overline{r}} \,,$$
$$A\overline{\beta} = \overline{b}\,\zeta_{12\overline{2}} + \overline{c}\,\zeta_{12\overline{1}}$$

Hence equation $\overline{\partial}\beta + A\overline{\beta} = 0$ readily implies that b, c are holomorphic functions on M and that the map a satisfies

$$\zeta_{\overline{1}}(a) = \overline{c}\,, \quad \zeta_{\overline{2}}(a) = \overline{b}\,, \quad \zeta_{\overline{3}}(a) = 0 \,.$$

Since M is compact, b and c have to be constant. In particular one has $\partial\overline{\partial}a = 0$ and, consequently, also a has to be constant. Since the components of β are constant, one has $\partial\beta = \overline{\partial}\beta = 0$ and this condition contradicts equation $A\omega + \partial\beta = 0$.

In view of Remark 4.2, require that a quasi Kähler metric g locally admits a complex unitary $(1,0)$-frame $\{\zeta_1, \ldots, \zeta_n\}$ satisfying

$$\partial\zeta_i = \overline{\partial}\zeta_i = 0\,, \quad i = 1, \ldots, n$$

is a bit less than require that the Hermitian curvature tensor of g vanishes. Hence it is rather natural to wonder if an almost Kähler structure can admit such a coframe. The answer is negative, since we have the following

Proposition 4.8. *Let (M, g, J, ω) be an almost Kähler manifold. Assume that M admits a global unitary $(1,0)$-coframe $\{\zeta_1, \ldots, \zeta_n\}$ satisfying*

$$\partial\zeta_i = \overline{\partial}\zeta_i = 0\,, \quad i = 1, \ldots, n \,.$$

Then M is Kähler.

Proof. Assume that such a coframe exists and let $\{Z_1, \ldots, Z_n\}$ be the dual frame. Then we have

$$[Z_i, Z_{\overline{j}}] = 0\,, \quad [Z_i, Z_j] \in \Gamma(T^{0,1}M)\,, \quad i, j = 1, \ldots, n \,.$$

In particular, we can write

$$[Z_i, Z_j] = \sum_{k=1}^{n} A_{ij}^{\overline{k}} Z_{\overline{k}}$$

and the Nijenhuis tensor of J satisfies

$$N(Z_i, Z_j) = -4 \sum_{k=1}^{n} A_{ij}^{\overline{k}} Z_{\overline{k}} \,.$$

Now we recall that the Nijenhuis tensor of an almost Kähler manifold always satisfies

$$\underset{X,Y,Z}{\mathfrak{S}}\, g(N(X,Y),Z) = 0\,.$$

This formula in our case reads as

$$(16) \qquad A_{ij}^{\overline{k}} + A_{ki}^{\overline{j}} + A_{jk}^{\overline{i}} = 0\,, \quad 1 \le i,j,k \le n\,.$$

Since the brackets of the form $[Z_i, Z_{\overline{j}}]$ vanish, then the Jacobi identity in terms of Z_i's reads as

$$[[Z_i, Z_j], Z_{\overline{r}}] = 0\,, \quad 1 \le i,j,r \le n\,,$$

i.e.

$$0 = [[Z_i, Z_j], Z_{\overline{r}}] = \sum_{k=1}^{n}[A_{ij}^{\overline{k}} Z_{\overline{k}}, Z_{\overline{r}}] = -\sum_{k=1}^{n} Z_{\overline{r}}(A_{ij}^{\overline{k}})\, Z_{\overline{k}} + \sum_{k,s=1}^{n} A_{ij}^{\overline{k}} \overline{A}_{kr}^{\overline{s}} Z_s\,.$$

In particular one has

$$(17) \qquad \sum_{k=1}^{n} A_{ij}^{\overline{k}} \overline{A}_{kr}^{\overline{s}} = 0\,, \quad 1 \le i,j,s,r \le n\,.$$

Using equations (16) and (17), we get

$$0 = \sum_{k=1}^{n} A_{ij}^{\overline{k}} \overline{A}_{ki}^{\overline{j}} = -\sum_{k=1}^{n}\{A_{ij}^{\overline{k}} \overline{A}_{ij}^{\overline{k}} - A_{ij}^{\overline{k}} \overline{A}_{kj}^{\overline{i}}\} = -\sum_{k=1}^{n} |A_{ij}^{\overline{k}}|^2$$

which forces (M, g, J, ω) to be a Kähler manifold. $\qquad\square$

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