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# LINKING SOLUTIONS FOR $p$-LAPLACE EQUATIONS WITH NONLINEARITY AT CRITICAL GROWTH

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Abstract. Under a suitable condition on $n$ and $p$, the quasilinear equation at critical growth $-\Delta_p u = \lambda|u|^{p-2}u + |u|^{p^*-2}\,u$ is shown to admit a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ for any $\lambda \geq \lambda_1$. Nonstandard linking structures, for the associated functional, are recognized.

## 1. Introduction and main results

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, let $1 < p < n$ and let $\lambda \in \mathbb{R}$. We are interested in the existence of nontrivial solutions $u$ for the quasilinear problem

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}\,u + |u|^{p^*-2}\,u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where $\Delta_p u := \mathrm{div}\,(|\nabla u|^{p-2}\,\nabla u)$ denotes the $p$-Laplace operator and $p^* := np/(n-p)$ the critical Sobolev exponent for the embedding of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$. Let us also set

$$S = \inf\left\{\frac{\displaystyle\int_{\mathbb{R}^n}|\nabla u|^p\,dx}{\left(\displaystyle\int_{\mathbb{R}^n}|u|^{p^*}\,dx\right)^{p/p^*}}\,:\; u \in C_c^\infty(\mathbb{R}^n)\setminus\{0\}\right\},$$

$$\lambda_1 = \min\left\{\frac{\displaystyle\int_\Omega|\nabla u|^p\,dx}{\displaystyle\int_\Omega|u|^p\,dx}\,:\; u \in W_0^{1,p}(\Omega)\setminus\{0\}\right\}$$

and denote by $\varphi_1 \in W_0^{1,p}(\Omega)$ a positive solution of $-\Delta_p u = \lambda_1|u|^{p-2}\,u$ (see Lindqvist [16]).

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After the seminal paper of Brezis-Nirenberg [4], many works have been devoted to problems at critical growth, mainly when $p = 2$. In particular, let us recall that, according to the main result of [4], problem (1.1) admits a positive solution $u$ for any $\lambda \in ]0, \lambda_1[$, provided that $p = 2$ and $n \geq 4$. The result has been extended by Egnell, Garcia Azorero-Peral Alonso, Guedda-Veron [9, 12, 14], who have proved that problem (1.1) admits a positive solution $u$ for any $\lambda \in ]0, \lambda_1[$, provided that $p > 1$ and $n \geq p^2$. Such a solution $u$ can be obtained via the Mountain pass theorem of Ambrosetti-Rabinowitz [1] applied to the $C^1$-functional $f : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined as

$$f(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{p} \int_\Omega |u|^p \, dx - \frac{1}{p^*} \int_\Omega |u|^{p^*} \, dx \tag{1.2}$$

and satisfies

$$0 < f(u) < \frac{1}{n} S^{n/p} \, . \tag{1.3}$$

On the other hand, it is known [4, 9, 14] that, if $\Omega$ is star-shaped and with smooth boundary, then problem (1.1) has no nontrivial solution $u$ for any $\lambda \leq 0$.

When $\lambda \geq \lambda_1$, it is still meaningful to look for nontrivial solutions $u$, but the situation is quite different in the two cases $p = 2$ and $p \neq 2$. If $p = 2$, it has been proved by Capozzi-Fortunato-Palmieri [5] that problem (1.1) has a nontrivial solution $u$ in each of the following cases:

($a$) $\lambda \geq \lambda_1$ and $n \geq 5$;

($b$) $\lambda > \lambda_1$, $\lambda \notin \sigma(-\Delta_2)$ and $n \geq 4$;

(see also Gazzola-Ruf [13, Corollary 1]). Such a solution can be obtained via the Linking theorem of Rabinowitz (see e.g. [19, Theorem 5.3]) applied to the functional $f$ and still satisfies (1.3).

On the other hand, when $p \neq 2$ there is in general no direct sum decomposition of $W_0^{1,p}(\Omega)$, which allows to recognize a linking structure in a standard way. To our knowledge, the only workable situation amounts to the fact that, if $\Omega$ is connected and we set

$$\underline{\lambda}_2 = \sup\left\{\min\left\{\int_\Omega |\nabla u|^p\,dx:\ u \in Y\,,\ \int_\Omega |u|^p\,dx = 1\right\}:\right.$$
$$\left. W_0^{1,p}(\Omega) = (\mathbb{R}\varphi_1) \oplus Y \text{ with } Y \text{ closed in } W_0^{1,p}(\Omega)\right\},$$

then $\underline{\lambda}_2 > \lambda_1$ and, for every $b < \underline{\lambda}_2$, there exists a decomposition

$$W_0^{1,p}(\Omega) = (\mathbb{R}\varphi_1) \oplus Y$$

such that

$$\int_\Omega |\nabla u|^p\,dx \,=\, \lambda_1 \int_\Omega |u|^p\,dx\,, \qquad \forall u \in \mathbb{R}\varphi_1\,,$$
$$\int_\Omega |\nabla u|^p\,dx \,\geq\, b \int_\Omega |u|^p\,dx\,, \qquad \forall u \in Y\,.$$

Taking advantage of this fact, Arioli-Gazzola [2] have proved that, for any $p > 1$, problem (1.1) has a nontrivial solution $u$ in each of the following cases:

($a$) $\lambda_1 \leq \lambda < \underline{\lambda}_2$ and $\frac{n^2}{n+1} > p^2$;

($b$) $\lambda_1 < \lambda < \underline{\lambda}_2$ and $n \geq p^2$.

Such a solution is still obtained via the classical Linking theorem and satisfies (1.3).

Our purpose is to provide a complete extension to the case $p > 1$ of the mentioned result of Capozzi-Fortunato-Palmieri. Because of the lack of decompositions by linear subspaces, we will apply the results of our recent paper [7], which provide an extension of the Linking theorem with linear subspaces substituted by cones. In the line of the case ($a$), we prove the following:

**Theorem 1.1.** *Assume that*

(1.4) $$\Omega \text{ has } C^{1,\overline{\alpha}} \text{ boundary for some } \overline{\alpha} \in ]0,1[$$

*and that*

$$\frac{n^3 + p^3}{n^2 + n} > p^2 \,. \tag{1.5}$$

*Then problem* (1.1) *has a nontrivial solution* $u$ *satisfying* (1.3) *for every* $\lambda \geq \lambda_1$.

By the way, we also improve the condition on $n$ and $p$ of Arioli-Gazzola, as (1.5) is equivalent to

$$\frac{n^2 + \frac{p^3}{n}}{n+1} > p^2 \,.$$

This is due to a different concentration technique on points "moving to the boundary" of $\Omega$, rather than at a fixed interior point (the key information is contained in Lemma 3.2).

Still in the line of $(a)$, we also prove the following results:

**Theorem 1.2.** *Assume that*

$$\frac{n^2}{n+1} > p^2 \,. \tag{1.6}$$

*Then problem* (1.1) *has a nontrivial solution* $u$ *satisfying* (1.3) *for every* $\lambda \geq \lambda_1$.

In other words, under the condition of Arioli-Gazzola, the result holds for any $\lambda \geq \lambda_1$, without any smoothness assumption on the boundary of $\Omega$.

**Theorem 1.3.** *Assume that* $\Omega$ *is a ball and that* (1.6) *holds. Then problem* (1.1) *has a nontrivial radial solution* $u$ *satisfying* (1.3) *for every* $\lambda \geq \lambda_1$.

A comparison between Theorems 1.1 and 1.3 raises an interesting question: if $\Omega$ is a ball and

$$\frac{n^2}{n+1} \leq p^2 < \frac{n^3 + p^3}{n^2 + n} \,,$$

what about the existence of a nontrivial *radial* solution $u$ satisfying (1.3), say, when $\lambda = \lambda_1$? A (negative) answer could come from an extension of the result of Arioli-Gazzola-Grunau-Sassone [3].

In order to state our results in the line of $(b)$, let us set, according to [6, 7, 17, 18],

$$\lambda_m = \inf\left\{\sup_{u\in A} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx} :\ A \subseteq W_0^{1,p}(\Omega)\setminus\{0\},\ A \text{ is symmetric and } \operatorname{Index}(A) \geq m\right\}, \tag{1.7}$$

where Index is the $\mathbb{Z}_2$-cohomological index of Fadell-Rabinowitz [10, 11]. Then it is well-known that $(\lambda_m)$ is a nondecreasing divergent sequence and $\lambda_1$ is the same as before, while $\underline{\lambda}_2 \leq \lambda_2$. Moreover, in the case $p = 2$ we have $\{\lambda_m :\ m \geq 1\} = \sigma(-\Delta_2)$, but for $p \neq 2$ it is only known that the equation $-\Delta_p u = \lambda_m |u|^{p-2}\,u$ admits a nontrivial solution $u$ for any $m \geq 1$.

We prove the following result:

**Theorem 1.4.** *If $n \geq p^2$, then problem* (1.1) *has a nontrivial solution $u$ satisfying* (1.3) *for every $\lambda > \lambda_1$ with $\lambda \notin \{\lambda_m :\ m \geq 1\}$.*

If $\Omega$ is a ball, let

$$\lambda_m^{(r)} = \inf\left\{\sup_{u\in A} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u|^p\,dx} :\ A \subseteq W_{0,r}^{1,p}(\Omega)\setminus\{0\},\ A \text{ is symmetric and } \operatorname{Index}(A) \geq m\right\},$$

where $W_{0,r}^{1,p}(\Omega)$ denotes the corresponding Sobolev space of radial functions. From the results of [16] it follows that $\lambda_1^{(r)} = \lambda_1$. Then we have

**Theorem 1.5.** *Assume that $\Omega$ is a ball. If $n \geq p^2$, then problem* (1.1) *has a nontrivial radial solution $u$ satisfying* (1.3) *for every $\lambda > \lambda_1$ with $\lambda \notin \left\{\lambda_m^{(r)} :\ m \geq 1\right\}$.*

In the next section we recall and prove some preliminary facts, while in Section 3 we prove the results we have stated in the Introduction.

## 2. Linking over cones

First of all, let us recall from [7] a generalization of the Linking theorem in which linear subspaces are substituted by symmetric cones.

**Theorem 2.1.** *Let $X$ be a Banach space and let $f : X \longrightarrow \mathbb{R}$ be a function of class $C^1$. Let $X_-, X_+$ be two symmetric cones in $X$ such that $X_+$ is closed in $X$,*

$$X_- \cap X_+ = \{0\}\,,$$

$$\mathrm{Index}\,(X_- \setminus \{0\}) = \mathrm{Index}\,(X \setminus X_+) < \infty\,.$$

*Let also $e \in X \setminus X_-$, $0 < r_+ < r_-$,*

$$S_+ = \{v \in X_+ :\ \|v\| = r_+\}\ ,$$

$$Q = \{te + w :\ t \geq 0\,,\ w \in X_-\,,\ \|te + w\| \leq r_-\}\ ,$$

$$P = \{w \in X_- :\ \|w\| \leq r_-\} \cup \{te + w :\ t \geq 0\,,\ w \in X_-\,,\ \|te + w\| = r_-\}$$

*be such that*

$$\sup_P f < \inf_{S_+} f\,, \qquad \sup_Q f < +\infty\,.$$

*Define*

$$c = \inf_{\eta \in \mathcal{N}}\ \sup f(\eta(Q \times \{1\}))\,,$$

*where $\mathcal{N}$ is the set of deformations $\eta : Q \times [0,1] \longrightarrow X$ with $\eta(P \times [0,1]) \cap S_+ = \emptyset$.*

*Then we have*

$$\inf_{S_+} f \leq c \leq \sup_Q f \tag{2.1}$$

*and there exists a sequence $(u_k)$ in $X$ with $\|f'(u_k)\| \to 0$ and $f(u_k) \to c$.*

*Proof.* From [7, Theorem 2.2 and Corollary 2.9] it follows that (2.1) holds. If, by contradiction, there is no sequence $(u_k)$ as required, then there exists $\sigma > 0$ such that $\|f'(u)\| \geq \sigma$ whenever $c - \sigma \leq f(u) \leq c + \sigma$. In particular, $f$ satisfies $(PS)_c$ and from

[7, Theorem 2.2 and Corollary 2.9] we deduce that $c$ is a critical value of $f$, whence a contradiction. $\square$

Assume now that $\Omega$ is a bounded open subset of $\mathbb{R}^n$ and that $1 < p < \infty$. If we define $\lambda_m$ according to (1.7), by [7, Theorem 3.2] the following holds:

**Theorem 2.2.** *If $m \geq 1$ is such that $\lambda_m < \lambda_{m+1}$, then we have*

$$\begin{aligned}\mathrm{Index}&\left(\left\{u \in W_0^{1,p}(\Omega)\setminus\{0\} : \quad \int_\Omega |\nabla u|^p\,dx \leq \lambda_m \int_\Omega |u|^p\,dx\right\}\right)\\ &= \mathrm{Index}\left(\left\{u \in W_0^{1,p}(\Omega) : \quad \int_\Omega |\nabla u|^p\,dx < \lambda_{m+1} \int_\Omega |u|^p\,dx\right\}\right) = m\,.\end{aligned}$$

In view of the application of Theorem 2.1, the simplest choice is

$$X_+ = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p\,dx \geq \lambda_{m+1} \int_\Omega |u|^p\,dx\right\}, \tag{2.2}$$

while $X_-$ could be defined as

$$\left\{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p\,dx \leq \lambda_m \int_\Omega |u|^p\,dx\right\}. \tag{2.3}$$

The next result asserts that as $X_-$ we can also choose a smaller cone, with better regularity properties. Let us set $\|u\| = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$ for every $u \in W_0^{1,p}(\Omega)$ and denote by $\|\ \|_q$ the usual norm in $L^q(\Omega)$. We also set $M = \left\{u \in W_0^{1,p}(\Omega) :\ \int_\Omega |u|^p\,dx = 1\right\}$ and denote by $\mathrm{B}_\varrho(x)$ the open ball of center $x$ and radius $\varrho$.

**Theorem 2.3.** *Let $m \geq 1$ be such that $\lambda_m < \lambda_{m+1}$. Then there exists a symmetric cone $X_-$ in $W_0^{1,p}(\Omega)$ such that $X_-$ is closed in $L^p(\Omega)$ and:*

(*a*) *we have*

$$X_- \subseteq \left\{u \in W_0^{1,p}(\Omega) : \quad \int_\Omega |\nabla u|^p\,dx \leq \lambda_m \int_\Omega |u|^p\,dx\right\} \cap L^\infty(\Omega) \cap C^{1,\alpha}_{loc}(\Omega)\,;$$

(*b*) *$X_- \cap M$ is bounded in $L^\infty(\Omega)$ and in $C^{1,\alpha}_{loc}(\Omega)$;*

(*c*) *$X_- \cap M$ is strongly compact in $W_0^{1,p}(\Omega)$ and in $C^1(\Omega)$;*

(*d*) *we have* $\mathrm{Index}\,(X_- \setminus \{0\}) = m$.

*Moreover, if* $\Omega$ *satisfies* (1.4)*, we have that* $X_- \cap M$ *is bounded in* $C^{1,\alpha}(\overline{\Omega})$*, for some* $\alpha \in ]0,1[$*, and strongly compact in* $C^1(\overline{\Omega})$*.*

*Proof.* We only prove the case $1 < p < n$. The case $p \geq n$ can be treated with minor modifications.

Let $\widetilde{X}_-$ be the symmetric cone defined in (2.3). Then $M \cap \widetilde{X}_-$ is a symmetric subset of $W_0^{1,p}(\Omega) \setminus \{0\}$ with Index $\left(M \cap \widetilde{X}_-\right) = m$, being an odd deformation retract of $\widetilde{X}_- \setminus \{0\}$. Moreover, $M \cap \widetilde{X}_-$ is strongly compact in $L^p(\Omega)$.

Let us recall that, for every $w \in L^q(\Omega)$ with $q \geq (p^*)'$, there exists one and only one $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u = w$. Moreover, if $q \neq n/p$, we have $u \in L^{\beta(q)}(\Omega)$ and $\|u\|_{\beta(q)}^{p-1} \leq c(\Omega, p, q)\|w\|_q$, where

$$\beta(q) = \begin{cases} \dfrac{n(p-1)q}{n-pq} & \text{if } q < n/p\,, \\ \infty & \text{if } q > n/p\,. \end{cases}$$

(see e.g. [14, Propositions 1.2 and 1.3]).

In particular, for every $w \in L^q(\Omega)$ with $q/(p-1) \geq (p^*)'$, there exists one and only one $u \in W_0^{1,p}(\Omega)$ such that $-\Delta_p u = |w|^{p-2}w$. Moreover, if $q/(p-1) \neq n/p$, we have $u \in L^{\gamma(q)}(\Omega)$ and $\|u\|_{\gamma(q)} \leq \tilde{c}(\Omega, p, q)\|w\|_q$, where

$$\gamma(q) = \begin{cases} \dfrac{nq}{n(p-1)-pq} & \text{if } q/(p-1) < n/p\,, \\ \infty & \text{if } q/(p-1) > n/p\,. \end{cases}$$

For every $w \in M$, let $J(w) \in M$ be defined as $J(w) = u/\|u\|_p$, where $u \in W_0^{1,p}(\Omega)$ is the solution of $-\Delta_p u = |w|^{p-2}w$. Then it is easily seen that there exists $k \geq 2$ such that $J^{k-1}(M)$ is bounded in $L^\infty(\Omega)$. By [8, 15, 20] it follows that $J^k(M)$ is also bounded in $C^{1,\alpha}_{loc}(\Omega)$, or even in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in ]0,1[$, if $\Omega$ satisfies (1.4).

Moreover, we have

$$\begin{aligned} \int_\Omega |w|^{p-2}w\left(\frac{u}{\|\nabla u\|_p^p}\right) dx &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\left(\frac{u}{\|\nabla u\|_p^p}\right) dx = 1 \\ &= \int_\Omega |w|^p\, dx = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w\, dx \leq \|\nabla u\|_p^{p-1}\|\nabla w\|_p\,, \end{aligned}$$

which implies, by the convexity of $\| \ \|_p^p$,

$$\left\| \frac{u}{\|\nabla u\|_p^p} \right\|_p^p \geq \|w\|_p^p + p \int_\Omega |w|^{p-2} w \left( \frac{u}{\|\nabla u\|_p^p} - w \right) dx = 1 \,,$$

hence

$$\left\| \nabla \left( \frac{u}{\|u\|_p} \right) \right\|_p \leq \frac{1}{\|\nabla u\|_p^{p-1}} \leq \|\nabla w\|_p \,,$$

namely $\|\nabla (J(w))\|_p \leq \|\nabla w\|_p$.

Therefore $J^k(M \cap \widetilde{X}_-)$ is a bounded subset of $L^\infty(\Omega)$ and of $C^{1,\alpha}_{loc}(\Omega)$ (resp. $C^{1,\alpha}(\overline{\Omega})$) with $J^k(M \cap \widetilde{X}_-) \subseteq M \cap \widetilde{X}_-$. Since $J$ is odd and continuous from the topology of $L^p(\Omega)$ to that of $W^{1,p}_0(\Omega)$, it follows that

$$\text{Index}\left(J^k(M \cap \widetilde{X}_-)\right) = \text{Index}\left(M \cap \widetilde{X}_-\right) = m$$

and that $J^k(M \cap \widetilde{X}_-)$ is strongly compact in $W^{1,p}_0(\Omega)$. By the boundedness in $C^{1,\alpha}_{loc}(\Omega)$, the set $J^k(M \cap \widetilde{X}_-)$ is also strongly compact in $C^1(\Omega)$ (or even in $C^1(\overline{\Omega})$, if we have the boundedness in $C^{1,\alpha}(\overline{\Omega})$). Now, if we set

$$X_- = \left\{ tu : \ t \geq 0 \,, \ u \in J^k(M \cap \widetilde{X}_-) \right\} \,,$$

it is clear that $X_-$ is a symmetric cone in $W^{1,p}_0(\Omega)$ satisfying $(a)$–$(d)$. Since $J^k(M \cap \widetilde{X}_-)$ is compact in $W^{1,p}_0(\Omega)$ with $0 \notin J^k(M \cap \widetilde{X}_-)$, we also have that $X_-$ is closed in $L^p(\Omega)$. □

## 3. Proof of the main results

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and let $p > 1$ with $p^2 \leq n$. For every $\varepsilon > 0$ we set, as in [2],

$$u^*_\varepsilon(x) = \frac{c(n,p)\, \varepsilon^{\frac{n-p}{p(p-1)}}}{\left(\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}} \,,$$

where $c(n,p) > 0$ is such that

$$\int_{\mathbb{R}^n} |\nabla u^*_\varepsilon|^p \, dx = \int_{\mathbb{R}^n} |u^*_\varepsilon|^{p^*} \, dx = S^{n/p} \,.$$

Up to a different parametrization with respect to $\varepsilon$, the family $(u^*_\varepsilon)$ is the same of [9, 12, 14]. Let also $\eta : \mathbb{R} \longrightarrow [0,1]$ be a $C^\infty$-function such that $\eta(s) = 1$ for $s \leq 1/4$ and

$\eta(s) = 0$ for $s \geq 1/2$. For every $\varepsilon, \varrho > 0$, we set

$$u_{\varrho,\varepsilon}(x) = \eta\left(\frac{|x|}{\varrho}\right) u^*_\varepsilon(x)\,.$$

**Lemma 3.1.** *There exist $C, \sigma > 0$ such that*

$$\int_{\mathbb{R}^n} |\nabla u_{\varrho,\varepsilon}|^p \, dx \leq S^{n/p} + C(\varepsilon/\varrho)^{\frac{n-p}{p-1}}\,, \tag{3.1}$$

$$\int_{\mathbb{R}^n} |u_{\varrho,\varepsilon}|^{p^*} \, dx \geq S^{n/p} - C(\varepsilon/\varrho)^{\frac{n}{p-1}}\,, \tag{3.2}$$

$$\int_{\mathbb{R}^n} |u_{\varrho,\varepsilon}|^p \, dx \geq \begin{cases} \sigma\varepsilon^p - C\varrho^p(\varepsilon/\varrho)^{\frac{n-p}{p-1}} & \text{if } n > p^2\,, \\ \sigma\varepsilon^p \log(\varrho/\varepsilon) - C\varepsilon^p & \text{if } n = p^2\,, \end{cases} \tag{3.3}$$

*for every $\varrho, \varepsilon > 0$.*

*Proof.* Formulae (3.1) and (3.2) can be found in [2]. Formula (3.3) is similar. Let us prove it for reader's convenience. Since

$$u_{\varrho,\varepsilon}(\varrho x) = \varrho^{-\frac{n-p}{p}} u_{1,\varepsilon/\varrho}(x)\,,$$

we have

$$\int_{\mathbb{R}^n} |u_{\varrho,\varepsilon}(x)|^p \, dx = \varrho^n \int_{\mathbb{R}^n} |u_{\varrho,\varepsilon}(\varrho y)|^p \, dy = \varrho^p \int_{\mathbb{R}^n} |u_{1,\varepsilon/\varrho}(y)|^p \, dy\,.$$

On the other hand, it is well known (see e.g. [14]) that

$$\int_{\mathbb{R}^n} |u_{1,\varepsilon}(y)|^p \, dy \geq \begin{cases} \sigma\varepsilon^p - C\varepsilon^{\frac{n-p}{p-1}} & \text{if } n > p^2\,, \\ \sigma\varepsilon^p \log(1/\varepsilon) - C\varepsilon^p & \text{if } n = p^2\,. \end{cases}$$

Then formula (3.3) easily follows. □

Now let $\overline{x} \in \Omega$ and $R > 0$ be such that $\mathrm{B}_R(\overline{x}) \subseteq \Omega$ and $\partial\,\mathrm{B}_R(\overline{x}) \cap \partial\Omega \neq \emptyset$. If $x_0 \in \partial\,\mathrm{B}_R(\overline{x}) \cap \partial\Omega$ and

$$x_\varrho = x_0 + \varrho\,\frac{\overline{x} - x_0}{|\overline{x} - x_0|}\,,$$

we have that $|x_\varrho - x_0| = \varrho$ and $\mathrm{B}_\varrho(x_\varrho) \subseteq \Omega$ for every $\varrho \in ]0, R]$.

Let $\vartheta : \mathbb{R} \to [0,1]$ be a $C^\infty$-function such that $\vartheta(s) = 0$ for $s \leq 1/2$ and $\vartheta(s) = 1$ for $s \geq 1$. Let also $m \geq 1$ with $\lambda_m < \lambda_{m+1}$, let $X_+$ be as in (2.2), $X_-$ be as in Theorem 2.3

and let

$$e_{\varrho,\varepsilon}(x) = u_{\varrho,\varepsilon}(x - x_\varrho)\,,$$

$$v_\varrho(x) = \vartheta\left(\frac{|x - x_\varrho|}{\varrho}\right) v(x) \qquad \text{for every } v \in X_-\,,$$

$$X_-^\varrho = \{v_\varrho:\ v \in X_-\}\,.$$

Of course, $X_-^\varrho$ also is a symmetric cone in $W_0^{1,p}(\Omega)$.

**Lemma 3.2.** *Assume that $\Omega$ satisfies (1.4). Then there exists $C > 0$ such that*

$$\int_\Omega |v_\varrho|^p\,dx \geq \int_\Omega |v|^p\,dx - C\varrho^{n+p}\left(\int_\Omega |v|^{p^*}\,dx\right)^{p/p^*}\,, \tag{3.4}$$

$$\int_\Omega |v_\varrho|^{p^*}\,dx \geq \int_\Omega |v|^{p^*}\,dx - C\varrho^{n+p^*}\int_\Omega |v|^{p^*}\,dx\,, \tag{3.5}$$

$$\int_\Omega |\nabla v_\varrho|^p\,dx \leq \int_\Omega |\nabla v|^p\,dx + C\varrho^{n}\left(\int_\Omega |v|^{p^*}\,dx\right)^{p/p^*}\,, \tag{3.6}$$

*for every $v \in X_-$ and $\varrho \in ]0,R]$.*

*Moreover, there exists $\varrho_0 \in ]0,R]$ such that*

$$e_{\varrho,\varepsilon} \notin X_-^\varrho \text{ and } X_-^\varrho \text{ is closed in } L^p(\Omega)\,,$$

$$X_-^\varrho \cap X_+ = \{0\}\,, \qquad \mathrm{Index}\,(X_-^\varrho \setminus \{0\}) = \mathrm{Index}\left(W_0^{1,p}(\Omega) \setminus X_+\right) = m\,,$$

*for every $\varrho \in ]0,\varrho_0]$ and $\varepsilon > 0$.*

*Proof.* Since $\Omega$ is smooth enough, according to Theorem 2.3 there exists $C > 0$ such that

$$\begin{cases} v(x_0) = 0 \\ \|v\|_\infty + \|\nabla v\|_\infty \leq C\|v\|_p \end{cases} \qquad \text{for every } v \in X_-\,. \tag{3.7}$$

For every $v \in X_-$ and $\varrho \in ]0,R]$, we have

$$\int_\Omega |v_\varrho|^p\,dx \geq \int_\Omega |v|^p\,dx - \mathcal{L}^n\left(\mathrm{B}_\varrho(x_\varrho)\right)\ \sup_{\mathrm{B}_\varrho(x_\varrho)} |v|^p\,.$$

On the other hand, since $v(x_0) = 0$ it holds

$$\sup_{\mathrm{B}_\varrho(x_\varrho)} |v| \leq 2\varrho\|\nabla v\|_\infty\,.$$

Then (3.4) easily follows. The proof of (3.5) is similar. We also have

$$\int_\Omega |\nabla v_\varrho|^p\,dx \leq \int_\Omega |\nabla v|^p\,dx + C\mathcal{L}^n\left(\mathrm{B}_\varrho\left(x_\varrho\right)\right)\left(\sup_{\mathrm{B}_\varrho(x_\varrho)} |\nabla v|^p + \varrho^{-p}\sup_{\mathrm{B}_\varrho(x_\varrho)} |v|^p\right)\,,$$

whence assertion (3.6).

From (3.4), (3.6) and (3.7) it follows that

$$\int_\Omega |\nabla v_\varrho|^p\,dx \leq \frac{1}{2}\left(\lambda_m + \lambda_{m+1}\right)\int_\Omega |v_\varrho|^p\,dx\,,$$

provided that $\varrho$ is small enough. Therefore $X_-^\varrho \cap X_+ = \{0\}$. Moreover, for every $v \in X_-$ we have

$$\begin{aligned}\int_\Omega |v|^p\,dx &\leq \left(\mathcal{L}^n\left(\mathrm{B}_\varrho\left(x_\varrho\right)\right)\right)^{1-\frac{p}{p^*}}\left(\int_\Omega |v|^{p^*}\,dx\right)^{\frac{p}{p^*}} + \int_{\Omega\setminus \mathrm{B}_\varrho(x_\varrho)} |v|^p\,dx \\ &\leq S^{-1}\left(\mathcal{L}^n\left(\mathrm{B}_\varrho\left(x_\varrho\right)\right)\right)^{1-\frac{p}{p^*}}\int_\Omega |\nabla v|^p\,dx + \int_{\Omega\setminus \mathrm{B}_\varrho(x_\varrho)} |v|^p\,dx \\ &\leq S^{-1}\lambda_m\left(\mathcal{L}^n\left(\mathrm{B}_\varrho\left(x_\varrho\right)\right)\right)^{1-\frac{p}{p^*}}\int_\Omega |v|^p\,dx + \int_{\Omega\setminus \mathrm{B}_\varrho(x_\varrho)} |v|^p\,dx\,.\end{aligned}$$

If $\varrho$ is small enough, we get

$$\int_\Omega |v|^p\,dx \leq C\int_{\Omega\setminus \mathrm{B}_\varrho(x_\varrho)} |v|^p\,dx \qquad \text{for every } v \in X_-\,.$$

First of all, it follows that $e_{\varrho,\varepsilon} \notin X_-^\varrho$ and that we have $v_\varrho = 0$ only for $v = 0$. Since $\{v \mapsto v_\varrho\}$ is continuous and odd from $X_- \setminus \{0\}$ to $X_-^\varrho \setminus \{0\}$, it follows

$$\mathrm{Index}\left(X_-^\varrho \setminus \{0\}\right) \geq \mathrm{Index}\left(X_- \setminus \{0\}\right) = \mathrm{Index}\left(W_0^{1,p}(\Omega) \setminus X_+\right) = m\,.$$

Actually, equality holds, as $X_-^\varrho \setminus \{0\} \subseteq W_0^{1,p}(\Omega) \setminus X_+$. Finally, let $(v^{(k)})$ be a sequence in $X_-$ with $(v_\varrho^{(k)})$ convergent to some $z$ in $L^p(\Omega)$. Then $(v^{(k)})$ is bounded in $L^p(\Omega \setminus \mathrm{B}_\varrho\left(x_\varrho\right))$, hence in $L^p(\Omega)$, hence in $W_0^{1,p}(\Omega)$. Up to a subsequence, $(v^{(k)})$ is $L^p(\Omega)$-convergent to some element of $X_-$, whence $z \in X_-^\varrho$. $\square$

Now let $f : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ be the functional defined in (1.2).

**Lemma 3.3.** *Assume that* $\Omega$ *satisfies* (1.4) *and that* (1.5) *holds. Let* $m \geq 1$ *be such that* $\lambda_m < \lambda_{m+1}$, $\lambda_m \leq \lambda$ *and let* $X_-$ *be as in Theorem 2.3. Then there exist* $\delta > 0$ *and two* *sequences $\varepsilon_k \to 0^+$ and $\varrho_k \to 0^+$ with $\varepsilon_k/\varrho_k \to 0^+$ such that*

$$\sup\Big\{ f\left(te_{\varrho_k,\varepsilon_k} + w\right) : \ t \geq 0\,, \ w \in X_-^{\varrho_k} \Big\} \leq \frac{1}{n}\, S^{n/p}\left(1 - \delta\varepsilon_k^p\right)^{n/p}$$

*for every $k \in \mathbb{N}$.*

*Proof.* Since $X_-^\varrho$ is a cone, it is easily seen that

$$\begin{aligned}
&\sup\Big\{ f\left(te_{\varrho,\varepsilon} + w\right) : \ t \geq 0\,, \ w \in X_-^{\varrho} \Big\} \\
&\qquad = \frac{1}{n}\left[\sup\left\{ \frac{\|\nabla(e_{\varrho,\varepsilon} + w)\|_p^p - \lambda\|e_{\varrho,\varepsilon} + w\|_p^p}{\|e_{\varrho,\varepsilon} + w\|_{p^*}^p} : \ w \in X_-^\varrho \right\}\right]^{n/p} \\
&\qquad = \frac{1}{n}\left[\sup\left\{ \frac{\left(\|\nabla e_{\varrho,\varepsilon}\|_p^p - \lambda\|e_{\varrho,\varepsilon}\|_p^p\right) + \left(\|\nabla w\|_p^p - \lambda\|w\|_p^p\right)}{\left(\|e_{\varrho,\varepsilon}\|_{p^*}^{p^*} + \|w\|_{p^*}^{p^*}\right)^{p/p^*}} : \ w \in X_-^\varrho \right\}\right]^{n/p},
\end{aligned}$$

as supt $(e_{\varrho,\varepsilon}) \cap$ supt $(w)$ is negligible. Writing $w = v_\varrho$ with $v \in X_-$, the assertion we need to prove takes the form

$$\sup\left\{ \frac{\left(\|\nabla e_{\varrho,\varepsilon}\|_p^p - \lambda\|e_{\varrho,\varepsilon}\|_p^p\right) + \left(\|\nabla v_\varrho\|_p^p - \lambda\|v_\varrho\|_p^p\right)}{\left(\|e_{\varrho,\varepsilon}\|_{p^*}^{p^*} + \|v_\varrho\|_{p^*}^{p^*}\right)^{p/p^*}} : \ v \in X_- \right\} \leq S\left(1 - \delta\varepsilon^p\right).$$

On the other hand, by Lemmas 3.1, 3.2 and the fact that $\lambda_m \leq \lambda$, we have

$$\begin{aligned}
&\frac{\left(\|\nabla e_{\varrho,\varepsilon}\|_p^p - \lambda\|e_{\varrho,\varepsilon}\|_p^p\right) + \left(\|\nabla v_\varrho\|_p^p - \lambda\|v_\varrho\|_p^p\right)}{\left(\|e_{\varrho,\varepsilon}\|_{p^*}^{p^*} + \|v_\varrho\|_{p^*}^{p^*}\right)^{p/p^*}} \\
&\qquad \leq \frac{\left(S^{\frac{n}{p}} + C\left(\frac{\varepsilon}{\varrho}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon^p + \lambda C\varrho^p\left(\frac{\varepsilon}{\varrho}\right)^{\frac{n-p}{p-1}}\right) + \left(C\varrho^n\|v\|_{p^*}^p - \lambda C\varrho^{n+p}\|v\|_{p^*}^p\right)}{\left(S^{\frac{n}{p}} - C\left(\frac{\varepsilon}{\varrho}\right)^{\frac{n}{p-1}} + \|v\|_{p^*}^{p^*} - C\varrho^{n+p^*}\|v\|_{p^*}^{p^*}\right)^{p/p^*}}.
\end{aligned}$$

Now, let $\varepsilon_k \to 0^+$ and let $\varrho_k = \mu\varepsilon_k^{p^2/n^2}$ with $\mu > 0$ small enough, which will be determined later. We need to show that, for every sequence $(v_k)$ in $X_-$,

$$\varepsilon_k^{-p}\left[\frac{\left(S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + \lambda C\varrho_k^p\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}\right) + \left(C\varrho_k^n\|v_k\|_{p^*}^p - \lambda C\varrho_k^{n+p}\|v_k\|_{p^*}^p\right)}{S\left(S^{\frac{n}{p}} - C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n}{p-1}} + \|v_k\|_{p^*}^{p^*} - C\varrho_k^{n+p^*}\|v_k\|_{p^*}^{p^*}\right)^{p/p^*}} - 1\right]$$

has strictly negative upper limit as $k \to \infty$. Up to subsequences, it is enough to consider the three cases:

(*i*) $\|v_k\|_{p^*} \to +\infty$,

(ii) $\|v_k\|_{p^*} \to \ell \in ]0, +\infty[$,

(iii) $\|v_k\|_{p^*} \to 0$.

In case (i) we get

$$\frac{\left(S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + \lambda C\varrho_k^p\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}\right) + \left(C\varrho_k^n\|v_k\|_{p^*}^p - \lambda C\varrho_k^{n+p}\|v_k\|_{p^*}^p\right)}{S\left(S^{\frac{n}{p}} - C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n}{p-1}} + \|v_k\|_{p^*}^{p^*} - C\varrho_k^{n+p^*}\|v_k\|_{p^*}^{p^*}\right)^{p/p^*}} \to 0\,,$$

while in case (ii) we obtain

$$\frac{\left(S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + \lambda C\varrho_k^p\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}\right) + \left(C\varrho_k^n\|v_k\|_{p^*}^p - \lambda C\varrho_k^{n+p}\|v_k\|_{p^*}^p\right)}{S\left(S^{\frac{n}{p}} - C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n}{p-1}} + \|v_k\|_{p^*}^{p^*} - C\varrho_k^{n+p^*}\|v_k\|_{p^*}^{p^*}\right)^{p/p^*}}$$

$$\to \frac{S^{\frac{n}{p}}}{S\left(S^{\frac{n}{p}} + \ell^{p^*}\right)^{p/p^*}} < 1\,.$$

In both cases, the assertion easily follows. In case (iii), it is equivalent to consider, neglecting higher order terms, the upper limit of

$$\varepsilon_k^{-p}\left[\frac{S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + C\varrho_k^n\|v_k\|_{p^*}^p}{S\left(S^{\frac{n}{p}} + \|v_k\|_{p^*}^{p^*}\right)^{p/p^*}} - 1\right].$$

Since there exists $a > 0$ such that

$$\left(S^{\frac{n}{p}} + \|v_k\|_{p^*}^{p^*}\right)^{p/p^*} \geq S^{\frac{n}{p^*}} + a\|v_k\|_{p^*}^{p^*}\,,$$

we have

$$\begin{aligned}
&\varepsilon_k^{-p}\left[\frac{S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + C\varrho_k^n\|v_k\|_{p^*}^p}{S\left(S^{\frac{n}{p}} + \|v_k\|_{p^*}^{p^*}\right)^{p/p^*}} - 1\right] \\
&\qquad \leq \varepsilon_k^{-p}\left[\frac{S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + C\varrho_k^n\|v_k\|_{p^*}^p}{SS^{\frac{n}{p^*}} + aS\|v_k\|_{p^*}^{p^*}} - 1\right] \\
&\qquad\qquad = \varepsilon_k^{-p}\,\frac{C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + C\varrho_k^n\|v_k\|_{p^*}^p - aS\|v_k\|_{p^*}^{p^*}}{S^{\frac{n}{p}} + aS\|v_k\|_{p^*}^{p^*}}\,.
\end{aligned}$$

By Young's inequality, there exists $C_1 > 0$ such that

$$C\varrho_k^n \|v_k\|_{p^*}^p \le C_1 \varrho_k^{\frac{np^*}{p^*-p}} + aS\|v_k\|_{p^*}^{p^*} = C_1 \varrho_k^{\frac{n^2}{p}} + aS\|v_k\|_{p^*}^{p^*}\,.$$

It follows

$$\varepsilon_k^{-p}\left[\frac{S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + C\varrho_k^n\|v_k\|_{p^*}^p}{S\left(S^{\frac{n}{p}} + \|v_k\|_{p^*}^{p^*}\right)^{p/p^*}} - 1\right] \le \varepsilon_k^{-p}\,\frac{C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\sigma\varepsilon_k^p + C_1\varrho_k^{\frac{n^2}{p}}}{S^{\frac{n}{p}} + aS\|v_k\|_{p^*}^{p^*}}\,.$$

If we choose $\mu > 0$ small enough to guarantee that

$$C_1\,\varrho_k^{\frac{n^2}{p}} = C_1\,\mu^{\frac{n^2}{p}}\,\varepsilon_k^p \le \frac{1}{2}\,\lambda\sigma\varepsilon_k^p\,,$$

it only remains to control the term $\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}$ by requiring

$$\frac{n-p}{p-1} - \frac{p^2}{n^2}\,\frac{n-p}{p-1} > p\,.$$

This is exactly assumption (1.5) and the assertion follows. $\square$

*Proof of Theorem 1.1.* Let $m \ge 1$ be such that $\lambda_m \le \lambda < \lambda_{m+1}$, let $X_+$ be as in (2.2) and $X_-$ be as in Theorem 2.3.

Since $\lambda < \lambda_{m+1}$, there exist $r_+, \alpha > 0$ such that $f(u) \ge \alpha$ for every $u \in X_+$ with $\|u\| = r_+$. On the other hand, since $\lambda \ge \lambda_m$, by Lemma 3.2 we also have, for every $v \in X_-$,

$$f(v_\varrho) \le \frac{C}{p}\,\varrho^n\|v\|_{p^*}^p - \frac{\lambda}{p}\,C\varrho^{n+p}\|v\|_{p^*}^p - \frac{1}{p^*}\,\|v\|_{p^*}^{p^*} + \frac{C}{p^*}\,\varrho^{n+p^*}\|v\|_{p^*}^{p^*} \le \frac{1}{2}\,\alpha - \frac{1}{2p^*}\,\|v\|_{p^*}^{p^*}$$

if $\varrho > 0$ is small enough. Combining this fact with Lemmas 3.2 and 3.3, we see that there exist $\varepsilon, \varrho, \delta > 0$ such that $e_{\varrho,\varepsilon} \notin X_-^\varrho$, $X_-^\varrho$ is closed in $L^p(\Omega)$ and

$$X_-^\varrho \cap X_+ = \{0\}\,, \qquad \mathrm{Index}\,(X_-^\varrho \setminus \{0\}) = \mathrm{Index}\left(W_0^{1,p}(\Omega) \setminus X_+\right) = m\,,$$

$$\sup\Big\{f\,(te_{\varrho,\varepsilon} + w):\ t \ge 0\,,\ w \in X_-^\varrho\Big\} \le \frac{1}{n}\,S^{n/p}\,(1 - \delta\varepsilon^p)^{n/p}\,,$$

$$\sup\Big\{f(w):\ w \in X_-^\varrho\Big\} \le \frac{1}{2}\,\alpha\,.$$

Since $X_-^\varrho$ is closed in $L^p(\Omega)$, we have

$$\|te_{\varrho,\varepsilon}\|_{p^*} + \|w\|_{p^*} \le b\|te_{\varrho,\varepsilon} + w\|_{p^*} \qquad \text{for every } t \in \mathbb{R} \text{ and } w \in X_-^\varrho$$

for some $b > 0$ (see also [7]). It follows that

$$f(u) \to -\infty \qquad \text{whenever } \|u\| \to \infty \text{ with } u \in \mathbb{R}e_{\varrho,\varepsilon} + X_-^\varrho\,.$$

In particular, there exists $r_- > r_+$ such that $f(u) \leq 0$ whenever $u \in \mathbb{R}e_{\varrho,\varepsilon} + X_-^\varrho$ with $\|u\| = r_-$.

From Theorem 2.1 we deduce that $f$ admits a Palais-Smale sequence at a level $c$ with $0 < c < \frac{1}{n}\,S^{n/p}$. On the other hand, by [14, Theorem 3.4] $f$ satisfies the Palais-Smale condition at such a level. Then $f$ admits a critical point $u$ with

$$0 < f(u) < \frac{1}{n}\,S^{n/p}\,.$$

Of course, $u$ is a nontrivial weak solution of (1.1). □

*Proof of Theorem 1.2.* Since the general lines of the argument are the same, we only point out the changes. This time, given $\mathrm{B}_{2R}(\overline{x}) \subseteq \Omega$, we set as in [2]

$$e_{\varrho,\varepsilon}(x) = u_{\varrho,\varepsilon}(x - \overline{x})\,,$$

$$v_\varrho(x) = \vartheta\left(\frac{|x-\overline{x}|}{\varrho}\right) v(x) \qquad \text{for every } v \in X_-\,.$$

Without any assumption on $\partial\Omega$, we know from Theorem 2.3 that

$$\sup_{\mathrm{B}_R(\overline{x})} |\nabla v| + \sup_{\mathrm{B}_R(\overline{x})} |v| \leq C\|v\|_p \qquad \text{for every } v \in X_-\,.$$

Then Lemma 3.2 holds with (3.4), (3.5) and (3.6) substituted by

$$\int_\Omega |v_\varrho|^p\,dx \geq \int_\Omega |v|^p\,dx - C\varrho^n \left(\int_\Omega |v|^{p^*}\,dx\right)^{p/p^*}\,, \tag{3.8}$$

$$\int_\Omega |v_\varrho|^{p^*}\,dx \geq \int_\Omega |v|^{p^*}\,dx - C\varrho^n \int_\Omega |v|^{p^*}\,dx\,, \tag{3.9}$$

$$\int_\Omega |\nabla v_\varrho|^p\,dx \leq \int_\Omega |\nabla v|^p\,dx + C\varrho^{n-p} \left(\int_\Omega |v|^{p^*}\,dx\right)^{p/p^*}\,. \tag{3.10}$$

In the proof of Lemma 3.3, only case $(iii)$ needs some adaptation. Neglecting higher order terms, we have to consider the upper limit of

$$\varepsilon_k^{-p}\left[\frac{S^{\frac{n}{p}}+C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}-\lambda\sigma\varepsilon_k^p+C\varrho_k^{n-p}\|v_k\|_{p^*}^p}{S\left(S^{\frac{n}{p}}+\|v_k\|_{p^*}^{p^*}\right)^{p/p^*}}-1\right],$$

which is less than or equal to

$$\varepsilon_k^{-p}\,\frac{C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}-\lambda\sigma\varepsilon_k^p+C\varrho_k^{n-p}\|v_k\|_{p^*}^p-aS\|v_k\|_{p^*}^{p^*}}{S^{\frac{n}{p}}+aS\|v_k\|_{p^*}^{p^*}}.$$

By Young's inequality, there exists $C_1>0$ such that

$$C\varrho_k^{n-p}\|v_k\|_{p^*}^p\le C_1\varrho_k^{\frac{(n-p)p^*}{p^*-p}}+aS\|v_k\|_{p^*}^{p^*}=C_1\varrho_k^{\frac{(n-p)n}{p}}+aS\|v_k\|_{p^*}^{p^*},$$

whence

$$\varepsilon_k^{-p}\left[\frac{S^{\frac{n}{p}}+C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}-\lambda\sigma\varepsilon_k^p+C\varrho_k^{n-p}\|v_k\|_{p^*}^p}{S\left(S^{\frac{n}{p}}+\|v_k\|_{p^*}^{p^*}\right)^{p/p^*}}-1\right]$$
$$\le\varepsilon_k^{-p}\,\frac{C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}-\lambda\sigma\varepsilon_k^p+C_1\varrho_k^{\frac{(n-p)n}{p}}}{S^{\frac{n}{p}}+aS\|v_k\|_{p^*}^{p^*}}.$$

Here we choose $\varrho_k=\mu\varepsilon_k^{p^2/(n-p)n}$ with $\mu>0$ small enough to guarantee that

$$C_1\,\varrho_k^{\frac{(n-p)n}{p}}=C_1\,\mu^{\frac{(n-p)n}{p}}\,\varepsilon_k^p\le\frac{1}{2}\,\lambda\sigma\varepsilon_k^p.$$

In the end, to control the term $\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}}$, we have to require that

$$\frac{n-p}{p-1}-\frac{p^2}{(n-p)n}\,\frac{n-p}{p-1}>p.$$

This is exactly assumption (1.6) and the assertion follows. □

*Proof of Theorem 1.3.* We follow step by step the proof of Theorem 1.2, taking as $\overline{x}$ the center of $\Omega$ and working in the space of radial functions. It is easily seen that the proof of Theorem 2.3 and all the other constructions are compatible with radiality. Then the assertion follows in a standard way. □

*Proof of Theorem 1.4.* Again the proof is similar to that of Theorem 1.2. We only point out the changes, concerning case $(iii)$ in the proof of Lemma 3.3.

First of all, now $\lambda_m < \lambda < \lambda_{m+1}$. Since

$$\begin{aligned}\|\nabla v_\varrho\|_p^p - \lambda\|v_\varrho\|_p^p &\leq \|\nabla v\|_p^p - \lambda\|v\|_p^p + C\varrho^{n-p}\|v\|_{p^*}^p + C\varrho^n\|v\|_{p^*}^p \\ &\leq -S\lambda_m^{-1}(\lambda-\lambda_m)\|v\|_{p*}^p + C\varrho^{n-p}\|v\|_{p^*}^p + C\varrho^n\|v\|_{p^*}^p\,,\end{aligned}$$

up to higher order terms, we have to consider the upper limit of

$$\varepsilon_k^{-p}\left[\frac{S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\|e_{\varrho_k,\varepsilon_k}\|_p^p - S\lambda_m^{-1}(\lambda-\lambda_m)\|v_k\|_{p*}^p + C\varrho_k^{n-p}\|v_k\|_{p^*}^p}{S\left(S^{\frac{n}{p}} + \|v_k\|_{p^*}^{p^*}\right)^{p/p^*}} - 1\right].$$

In turn, it is enough to argue on the upper limit of

$$\varepsilon_k^{-p}\left[\frac{S^{\frac{n}{p}} + C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\|e_{\varrho_k,\varepsilon_k}\|_p^p}{SS^{\frac{n}{p^*}}} - 1\right] = \varepsilon_k^{-p}\,\frac{C\left(\frac{\varepsilon_k}{\varrho_k}\right)^{\frac{n-p}{p-1}} - \lambda\|e_{\varrho_k,\varepsilon_k}\|_p^p}{S^{\frac{n}{p}}}\,.$$

Now, in both cases $n > p^2$ and $n = p^2$, it is easily seen that, for every sequence $\varepsilon_k \to 0^+$, there exists some sequence $(\varrho_k)$, going to 0 slowly enough, which guarantees the result. □

*Proof of Theorem 1.5.* It is enough to repeat the proof of Theorem 1.4 in the setting of radial functions. □

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