Negative aging and stochastic comparisons of residual lifetimes in multivariate frailty models

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Abstract

Consider two random vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ whose distributions are defined according to the multivariate frailty approach, and let $\mathbf{X}_{k,\mathbf{t}} = [\mathbf{X}_k - \mathbf{t}|\mathbf{X}_k > \mathbf{t}]$, $k = 1, 2$, be the corresponding vectors of residual lifetimes at $\mathbf{t} = (t_1, \ldots, t_n), t_i \in \mathbb{R}, i = 1, \ldots, n$. Conditions for multivariate stochastic comparisons of random vectors described by the frailty approach have been recently presented in Misra, Gupta and Gupta (2009), "Stochastic comparisons of multivariate frailty models", *Journal of Statistical Planning and Inference*, **139**, 2084–2090. Here we prosecute their study, providing sufficient conditions for the stochastic comparison $\mathbf{X}_{1,\mathbf{t}} \leq_{st} \mathbf{X}_{2,\mathbf{t}}$, where $\mathbf{t}$ is an arbitrary vector in $\mathbb{R}^n$. Sufficient conditions for the stochastic comparisons $\mathbf{X}_{i,\mathbf{t}} \leq_{st} \mathbf{X}_{i,\mathbf{t}+\mathbf{v}}$, where $\mathbf{t}$ is as above and $\mathbf{v}$ is a vector with non-negative components, are presented too.

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1 Introduction

The frailty approach is commonly used in reliability theory and survival analysis to model the dependence between subjects or components; according to this model the frailty (an unobservable random variable that describes environmental factors) acts simultaneously on the hazard functions of the lifetimes. In details, for fixed $k = 1, 2$ the vector $\mathbf{X}_k = (X_{k,1}, \ldots, X_{k,n})$ is said to be described by a multivariate frailty model if its joint survival function is defined as

$$\overline{F}_{\mathbf{X}_k}(t_1, \ldots, t_n) = \mathbb{P}[X_{k,1} > t_1, \ldots, X_{k,n} > t_n] = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i)\right)^{\Theta_k}\right],\ t_i \in \mathbb{R}^+, \quad (1.1)$$

where Θ_k is an environmental random frailty taking values in $\mathbb{R}^+$ and $\overline{G}_{k,i}$ is any survival function, commonly called *baseline survival function* of $X_{k,i}$ (and, of course, different from the survival function of $X_{k,i}$ unless $\Theta_k = 1$ a.s.). For a detailed description of frailty models and their applications we refer the reader to Hougaard (2000). Note that, commonly, frailty models are used to describe vectors of non–independent lifetimes, but, actually, non–negativity of variables $X_{k,i}$ is not required in subsequent sections.

Recall that given two random vectors (or variables) $\mathbf{X}_1$ and $\mathbf{X}_2$, then $\mathbf{X}_1$ is said to be smaller than $\mathbf{X}_2$ in the *usual stochastic order* (denoted $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$) iff $\mathbf{E}[\phi(\mathbf{X}_1)] \leq \mathbf{E}[\phi(\mathbf{X}_2)]$ for every increasing function ϕ such that the expectations exist (see Shaked and Shanthikumar (2007) for details, properties and applications of the usual stochastic order). Also, recall that, in the univariate case, $X_1 \leq_{st} X_2$ iff $\overline{F}_{X_1}(t) \leq \overline{F}_{X_2}(t)$ for all $t \in \mathbb{R}$.

Interesting conditions for stochastic comparisons between two vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ defined as above have been recently shown in Misra et al. (2009). In particular, in Misra et al. (2009) it is shown that $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$ whenever $\overline{G}_{1,i} = \overline{G}_{2,i}$ for all $i = 1, \ldots, n$ and $\Theta_2 \leq_{st} \Theta_1$, where $\leq_{st}$ is the usual stochastic order.

In Section 2 we provide an alternative sufficient condition for $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$, and we describe an immediate consequence of this result in comparisons of corresponding vectors of residual lifetimes at multivariate times $\mathbf{t} \in \mathbb{R}^n$. In particular, we show that the inequality $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$ follows also from a different stochastic inequality between the random frailties Θ_1 and Θ_2, called here $\leq_{n-Lt-r}$, whose definition is the following.

Definition 1.1. *Given two non-negative random variables Θ_1 and Θ_2 we say that Θ_2 is smaller than Θ_1 in the n–Laplace transform–ratio order (shortly $\Theta_2 \leq_{n-Lt-r} \Theta_1$), with $n \in \mathbb{N}^+$, iff the ratio*

$$\frac{\mathbf{E}[\Theta_1^{n-1}\exp(-s\Theta_1)]}{\mathbf{E}[\Theta_2^{n-1}\exp(-s\Theta_2)]}$$

is decreasing in $s \in \mathbb{R}^+$.

In Section 4 some of the relationships between the $\leq_{n-Lt-r}$ order and other well-known univariate stochastic orders will be mentioned; for the moment just observe that these orders do not imply, nor are implied by, the $\leq_{st}$ order, and that $\Theta_2 \leq_{n-Lt-r} \Theta_1$

holds iff the ratio

$$\frac{W_1^{(n-1)}(s)}{W_2^{(n-1)}(s)}$$

is decreasing in s, where

$$W_k(s) = \mathbf{E}[\exp(-s\Theta_k)] = \int_0^{\infty} \exp(-s\theta)dH_k(\theta), \ \ s \in \mathbb{R}^+, \tag{1.2}$$

where $W_k^{(n-1)}$ is the derivative of order $n-1$ of W_k (with $W_k^{(0)} = W_k$) and where H_k is the cumulative distribution of Θ_k, $k = 1, 2$. Moreover, observe that, in particular, the order $\leq_{1-Lt-r}$ is equivalent to the *Laplace transform ratio order* ($\leq_{Lt-r}$) studied in Shaked and Wong (1997), while $\leq_{2-Lt-r}$ is equivalent to the *differentiated Laplace transform ratio order* ($\leq_{d-lt-r}$) recently defined and in Li et al. (2009).

Finally, in Section 3 we will describe a second application of the main result, providing conditions for comparisons between vectors of residual lifetimes from the same vector $\mathbf{X}_1$, i.e., providing conditions for comparisons $\mathbf{X}_{1,\mathbf{t}} \leq_{st} \mathbf{X}_{2,\mathbf{t}+\mathbf{v}}$, where $\mathbf{v}$ is a vector with non-negative components.

Some conventions and notations that are used throughout the paper are given previously. Notation $=_{st}$ means equality in law. For any random variable (or vector) X and an event A, $[X\,|A]$ denotes a random variable whose distribution is the conditional distribution of X given A. Throughout this paper we write "increasing" instead of "non-decreasing" and "decreasing" instead of "non-increasing". Given two real valued vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, the notation $\mathbf{x} \leq [<]\ \mathbf{y}$ means $x_i \leq [<]\ y_i\ \forall i = 1, \ldots, n$. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be increasing if $\mathbf{x} \leq \mathbf{y}$ implies $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$. Finally, we will denote with $\widetilde{X}_{k,i}$ the random variable whose survival function is the baseline survival function $\overline{G}_{k,i}$, for $k = 1, 2$ and $i = 1, \ldots, n$.

2 Comparison of residual lifetimes

Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two random vectors having joint survival functions defined as in (1.1); the main result of this section describes conditions for the usual stochastic order between the corresponding vectors $\mathbf{X}_{k,\mathbf{t}} = [\mathbf{X}_k - \mathbf{t}|\mathbf{X}_k > \mathbf{t}]$ for every vector $\mathbf{t} \in \mathbb{R}^n$.

Three preliminary results are needed. The proof of the first two easily follows from standard Total Positivity techniques (see Karlin, 1968, for definitions, main properties and details on Total Positivity theory).

Lemma 2.1. *Let the survival functions W_k, with $k = 1, 2$, be defined as in (1.2). Then*

$$\frac{W_k^{(n-1)}(s+z)}{W_k^{(n-1)}(s)}$$

is increasing in $s \in \mathbb{R}^+$ for every $z \in \mathbb{R}^+$ and $n \geq 1$.

Proof. First observe that the assertion holds iff for every $n \geq 1$ the ratio

$$\frac{W_k^{(n-1)}(s)}{W_k^{(n)}(s)} \tag{2.1}$$

is decreasing in $s \in \mathbb{R}^+$. Denote

$$W_k^{(n)}(s) = (-1)^n \widetilde{W}_k^{(n)}(s) = (-1)^n \int_0^\infty a(n,\theta) b(s,\theta) dH_k(\theta),$$

where $a(n,\theta) = \theta^n$ and $b(s,\theta) = \exp(-s\theta)$. It is easy to verify that $a(n,\theta)$ is TP_2 (*totally positive of order 2*), while $b(s,\theta)$ is RR_2 (*reverse regular of order 2*). Thus by the Basic Composition Formula it follows that $\widetilde{W}_k^{(n)}(s)$ is RR_2 in (n,s), i.e., that the ratio $\widetilde{W}_k^{(n-1)}(s)/\widetilde{W}_k^{(n)}(s)$ is increasing in s. The assertion now follows observing that

$$\frac{W_k^{(n-1)}(s)}{W_k^{(n)}(s)} = - \frac{\widetilde{W}_k^{(n-1)}(s)}{\widetilde{W}_k^{(n)}(s)}.$$

□

The second preliminary result describes the relationships among the $\leq_{n-Lt-r}$ orders.

Lemma 2.2. *Let $\Theta_2 \leq_{n-Lt-r} \Theta_1$. Then $\Theta_2 \leq_{i-Lt-r} \Theta_1$ for every $i = 1, \ldots, n-1$, and, in particular,*

$$\mathbf{E}[\exp(-s\Theta_2)] \geq \mathbf{E}[\exp(-s\Theta_1)] \quad \textit{for all } s \in \mathbb{R}^+$$

Proof. Again using the Basic Composition Formula it is easy to verify that when the ratio $W_1^{(i)}(s)/W_2^{(i)}(s)$ is decreasing then also

$$\frac{\int_s^\infty W_1^{(i)}(z)dz}{\int_s^\infty W_2^{(i)}(z)dz} = \frac{W_1^{(i-1)}(s)}{W_2^{(i-1)}(s)}$$

is decreasing in s. In particular, also $W_1(s)/W_2(s)$ is decreasing in s, thus

$$1 = \frac{W_1(0)}{W_2(0)} \geq \frac{W_1(s)}{W_2(s)} = \frac{\mathbf{E}[\exp(-s\Theta_1)]}{\mathbf{E}[\exp(-s\Theta_2)]} \; .$$

□

The third preliminary result is stated as Theorem 6.B.4 in Shaked and Shanthikumar (2007). For it, recall that a random vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ is said to be conditionally increasing in sequence (shortly CIS) if, for $i = 2, \ldots, n$,

$$[Y_i | Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}] \leq_{st} [Y_i | Y_1 = y'_1, \ldots, Y_{i-1} = y'_{i-1}]$$

for all $y_j \leq y'_j$, $j = 1, \ldots, i-1$, where $[Y_i | Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}]$ denotes the conditional distribution of Y_i given $Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}$ for all $y_1, \ldots, y_{i-1} \in \mathbb{R}$.

Lemma 2.3. *Let* $\mathbf{Y}_1 = (Y_{1,1}, \ldots, Y_{1,n})$ *and* $\mathbf{Y}_2 = (Y_{2,1}, \ldots, Y_{2,n})$ *be two random vectors such that* $\mathbf{Y}_1$*, or* $\mathbf{Y}_2$*, is CIS. Then the stochastic inequality* $\mathbf{Y}_1 \leq_{st} \mathbf{Y}_2$ *holds if:*

(i) $Y_{1,1} \leq_{st} Y_{2,1}$*;*

(ii) $[Y_{1,i}|Y_{1,1} = t_1, \ldots, Y_{1,i-1} = t_{i-1}] \leq_{st} [Y_{2,i}|Y_{2,1} = t_1, \ldots, Y_{2,i-1} = t_{i-1}]$ $\forall i = 2, \ldots, n$ *and* $t_j \geq 0,$ *with* $j = 1, \ldots, i-1$*.*

The following main result describes new conditions for the usual stochastic comparison between two multivariate frailty models. Recall that $\widetilde{X}_{k,i}$ denotes the random variable whose survival function is the baseline survival function $\overline{G}_{k,i}$.

Theorem 2.1. *Let the* n*–dimensional vectors* $\mathbf{X}_k$*, with* $k = 1, 2$*, have survival functions defined as in (1.1). If:*

(a) $\Theta_2 \leq_{n-Lt-r} \Theta_1$*;*

(b) $\widetilde{X}_{1,i} \leq_{st} \widetilde{X}_{2,i}$ $\forall i = 1, \ldots, n$*,*

then $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$*.*

Proof. Let us consider a vector $\mathbf{Y} = (Y_1, \ldots, Y_n)$ having joint survival function

$$\overline{F}_{\mathbf{Y}}(t_1, \ldots, t_n) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{2,i}(t_i)\right)^{\Theta_1}\right], \ t_i \in \mathbb{R}.$$

First we prove that $\mathbf{Y} \leq_{st} \mathbf{X}_2$. For it, observe that the joint survival function of $\mathbf{X}_2$ can be written as

$$\overline{F}_{\mathbf{X}_2}(t_1, \ldots, t_n) = W_2\left(-\sum_{i=1}^n \ln \overline{G}_{2,i}(t_i)\right).$$

Observing that the survival functions of the margins $X_{2,i}$ are

$$\overline{F}_{X_{2,i}}(t_i) = W_2(-\ln \overline{G}_{2,i}(t_i)),$$

while their inverses are

$$\overline{F}^{-1}_{X_{2,i}}(u_i) = \overline{G}^{-1}_{2,i}(\exp(-W_2^{-1}(u_i))),$$

one can verify that the survival copula of $\mathbf{X}_2$ is Archimedean, i.e., that

$$\overline{F}_{\mathbf{X}_2}(\overline{F}^{-1}_{X_{2,1}}(u_1), \ldots, \overline{F}^{-1}_{X_{2,n}}(u_n)) = W_2\left(\sum_{i=1}^n W_2^{-1}(u_i)\right)$$

for all $u_1, \ldots, u_n \in [0, 1]$.

It should observed that the survival copula of $\mathbf{X}_2$ does not depend on the baseline distributions $\overline{G}_{2,i}$, but only on the random frailty Θ_2. Similarly, the survival copulas of vectors $\mathbf{X}_1$ and $\mathbf{Y}$ depend only on the random frailty Θ_1, and therefore $\mathbf{X}_1$ and $\mathbf{Y}$ have the same survival copula.

Since by Lemma 2.1 the ratio $W_2^{(n-1)}(s+z)/W_2^{(n-1)}(s)$ is decreasing in s, we can apply Theorem 2.8 in Müller and Scarsini (2005), which states that in this case $\mathbf{X}_2$ satisfies the CIS property[1]. Thus, in order to prove that $\mathbf{Y} \leq_{st} \mathbf{X}_2$ it suffices to verity that assumptions (i) and (ii) in Lemma 2.3 are satisfied (letting $\mathbf{Y} := \mathbf{Y}_1$ and $\mathbf{X}_2 := \mathbf{Y}_2$). Note that, for all $t_1 \in \mathbb{R}$,

$$\begin{aligned}
\overline{F}_{Y_1}(t_1) = \mathbf{E}[\overline{G}_{2,1}(t_1)^{\Theta_1}] &= \mathbf{E}[\exp(\Theta_1 \ln \overline{G}_{2,1}(t_1))] \\
&\leq \mathbf{E}[\exp(\Theta_2 \ln \overline{G}_{2,1}(t_1))] = \mathbf{E}[\overline{G}_{2,1}(t_1)^{\Theta_2}] = \overline{F}_{X_{2,1}}(t_1),
\end{aligned}$$

where the inequality follows from assumption (a) and Lemma 2.2. Thus (i) in Lemma 2.3 holds.

Moreover, for all $i = 1, \ldots, n$ and $t_j \in \mathbb{R}, j = 1, \ldots, i$, it holds

$$\begin{aligned}
\overline{F}_{Y_i|Y_1=t_1,\ldots,Y_{i-1}=t_{i-1}}(t_i) &= \int_{t_i}^{\infty} f_{Y_i|Y_1=t_1,\ldots,Y_{i-1}=t_{i-1}}(u)du \\
&= \int_{t_i}^{\infty} \frac{\int_0^\infty \theta^i g_{2,i}(u)\overline{G}_{2,i}^{\theta-1}(u) \prod_{j=1}^{i-1} g_{2,j}(t_j)\overline{G}_{2,j}^{\theta-1}(t_j)dH_1(\theta)}{\int_0^\infty \theta^{i-1} \prod_{j=1}^{i-1} g_{2,j}(t_j)\overline{G}_{2,j}^{\theta-1}(t_j)dH_1(\theta)} du \\
&= \frac{\int_0^\infty \theta^{i-1}\overline{G}_{2,i}^{\theta}(t_i) \prod_{j=1}^{i-1} \overline{G}_{2,j}^{\theta}(t_j)dH_1(\theta)}{\int_0^\infty \theta^{i-1} \prod_{j=1}^{i-1} \overline{G}_{2,j}^{\theta}(t_j)dH_1(\theta)} \\
&= \frac{W_1^{(i-1)}(-\ln \overline{G}_{2,i}(t_i) - \sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))}{W_1^{(i-1)}(-\sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))} \\
&\leq \frac{W_2^{(i-1)}(-\ln \overline{G}_{2,i}(t_i) - \sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))}{W_2^{(i-1)}(-\sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))} \\
&= \overline{F}_{X_{2,i}|X_{2,1}=t_1,\ldots,X_{2,i-1}=t_{i-1}}(t_i),
\end{aligned}$$

where, again, the inequality follows from assumption (a). Thus, also assumption (ii) in Lemma 2.3 is satisfied. We can then assert that $\mathbf{Y} \leq_{st} \mathbf{X}_2$.

Now observe that, by Theorem 6.B.14 in Shaked and Shanthikumar (2007), it holds $\mathbf{X}_1 \leq_{st} \mathbf{Y}$, having the vectors $\mathbf{X}_1$ and $\mathbf{Y}$ the same copula (as mentioned before) and stochastically ordered margins (by assertion (b) and closure of usual stochastic order with respect to mixtures).

The main assertion now follows from $\mathbf{X}_1 \leq_{st} \mathbf{Y} \leq_{st} \mathbf{X}_2$. $\square$

Under an assumption stronger than (b) of Theorem 2.1 it is possible to get a stronger comparison between $\mathbf{X}_1$ and $\mathbf{X}_2$, which involves the vectors of their residual lifetimes.

Theorem 2.2. *Let the vectors $\mathbf{X}_k$, with $k = 1, 2$, have survival functions defined as in (1.1). If:*

(a) $\Theta_2 \leq_{n-Lt-r} \Theta_1$;

(b) $\widetilde{X}_{1,i} =_{st} \widetilde{X}_{2,i}\ \forall i = 1, \ldots, n$;

then $\mathbf{X}_{1,\mathbf{t}} \leq_{st} \mathbf{X}_{2,\mathbf{t}}$ *for every vector* $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$.

[1] Actually, the vector $\mathbf{X}_2$ also satisfies the stronger positive dependence notion MTP_2, as follows from Application 3.2 in Khaledi and Kochar (2001).

Proof. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be an arbitrary vector with non-negative components. Note that

$$\begin{aligned}
\overline{F}_{\mathbf{X}_{k,\mathbf{t}}}(\mathbf{u}) &= \frac{\overline{F}_k(\mathbf{t}+\mathbf{u})}{\overline{F}_k(\mathbf{t})} = \frac{\int_0^\infty \left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i+u_i)\right)^\theta dH_k(\theta)}{\int_0^\infty \left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i)\right)^\theta dH_k(\theta)} \\
&= \frac{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j+u_j)]\} dH_k(\theta)}{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)} \\
&= \int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln(\frac{\overline{G}_{k,j}(t_j+u_j)}{\overline{G}_{k,j}(t_j)})]\} \frac{\exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)}{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)} \\
&= \int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln(\frac{\overline{G}_{k,j}(t_j+u_j)}{\overline{G}_{k,j}(t_j)})]\} d\widetilde{H}_k(\theta).
\end{aligned}$$

Thus, $\mathbf{X}_{k,\mathbf{t}}$ has joint survival function which can be expressed as

$$\overline{F}_{\mathbf{X}_{k,\mathbf{t}}}(\mathbf{u}) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{k,i,t_i}(u_i)\right)^{\widetilde{\Theta}_k}\right]$$

where

$$\overline{G}_{k,i,t_i}(u_i) = \frac{\overline{G}_{k,j}(t_j+u_j)}{\overline{G}_{k,j}(t_j)} \tag{2.2}$$

and where $\widetilde{\Theta}_k$ has distribution $\widetilde{H}_k$ defined as

$$\widetilde{H}_k(\theta) = \frac{\int_0^\theta \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\tau)}{\int_0^\infty \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\tau)} .$$

Thus, also,

$$\mathbf{E}[\exp(-s\widetilde{\Theta}_k)] = \frac{\mathbf{E}[\exp(-(s+\widetilde{t}_k)\Theta_k)]}{\mathbf{E}[\exp(-\widetilde{t}_k\Theta_k)]},$$

where $\widetilde{t}_k = -\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)$. Note that $\widetilde{t}_1 = \widetilde{t}_2$ by assumption (b).

Let us now denote $\widetilde{W}_{k,\mathbf{t}}(s) = \mathbf{E}[\exp(-s\widetilde{\Theta}_k)]$. It holds

$$\begin{aligned}
\frac{\widetilde{W}_{1,\mathbf{t}}^{(n-1)}(s)}{\widetilde{W}_{2,\mathbf{t}}^{(n-1)}(s)} &= \frac{\mathbf{E}[\exp(-\widetilde{t}_2\Theta_2)]}{\mathbf{E}[\exp(-\widetilde{t}_1\Theta_1)]} \cdot \frac{W_1^{(n-1)}(s+\widetilde{t}_1)}{W_2^{(n-1)}(s+\widetilde{t}_2)} \\
&= \frac{\mathbf{E}[\exp(-\widetilde{t}_2\Theta_2)]}{\mathbf{E}[\exp(-\widetilde{t}_1\Theta_1)]} \cdot \frac{W_1^{(n-1)}(s+\widetilde{t}_1)}{W_2^{(n-1)}(s+\widetilde{t}_1)}.
\end{aligned}$$

Since $\frac{W_1^{(n-1)}(s+\widetilde{t}_1)}{W_2^{(n-1)}(s+\widetilde{t}_1)}$ is decreasing in s by assumption (a), it holds $\widetilde{\Theta}_2 \leq_{n-Lt-r} \widetilde{\Theta}_1$. Moreover, denoted with $\widetilde{X}_{k,i,t_i}$ the random lifetimes having survival functions defined as in (2.2), from assumption (b) obviously follows that $\overline{G}_{1,i,t_i}(u_i) \leq \overline{G}_{2,i,t_i}(u_i)$ for all $u_i \in \mathbb{R}^+$ and $i = 1, \ldots, n$, i.e., $\widetilde{X}_{1,i,t_i} \leq_{st} \widetilde{X}_{2,i,t_i}$ $\forall i = 1, \ldots, n$.

Thus one can apply Theorem 2.1 to $\mathbf{X}_{1,\mathbf{t}}$ and $\mathbf{X}_{2,\mathbf{t}}$, getting the assertion. □

3 On negative aging of frailty models

In the literature one can find several characterizations of aging notions for univariate non-negative variables by means of stochastic comparisons between the residual lifetimes $X_t = [X - t \big| X > t]$ (see, e.g, Barlow and Proschan, 1975). Among others, the following negative aging notion is well–known: the random lifetime X is said to be *Decreasing in Failure Rate* (shortly DFR) iff

$$X_t \le_{st} X_{t+v} \text{ for all } t, v \ge 0. \tag{3.1}$$

Different multivariate generalizations of this aging property have been suggested. Some of them are based on alternative characterizations of univariate DFR distributions (see, e.g., Bassan and Spizzichino, 2005, or Shaked and Shanthikumar, 1991), while others have the shortcoming that they do not order the lifetime vectors in the sense of usual stochastic order $\le_{st}$ as (3.1) does in one dimension (see Barlow and Proschan, 1975, or Block and Savits, 1981). On the other hand, the following natural multivariate generalization of inequality (3.1) has been considered in Mulero and Pellerey (2009): a vector of lifetimes $\mathbf{X}$ is said to be *multivariate DFR* if

$$\mathbf{X_t} \le_{st} \mathbf{X_{t+v}} \tag{3.2}$$

holds for all vectors $\mathbf{t}$ and $\mathbf{v}$ having non-negative components. It should be pointed out that such a notion is actually weaker than the multivariate DFR notion considered in Arjas (1981) and further studied in Shaked and Shanthikumar (1988), whose definition is based on more general conditioning.

Using arguments similar to those in the proof of Theorem 2.2 it is possible to prove the following result, which describes conditions for inequality (3.2). Here the vector $\mathbf{X}_1$ does not need to have non-negative components.

Theorem 3.1. *Let the n–dimensional vector $\mathbf{X}_1$ have joint survival function defined as in (1.1). Then $\mathbf{X}_{1,\mathbf{t}} \le_{st} \mathbf{X}_{1,\mathbf{t+v}}$ holds for every $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$ and every non-negative $\mathbf{v} = (v_1, \ldots, v_n)$ if for all $i = 1, \ldots, n$ the variable $\widetilde{X}_{1,i}$ has decreasing hazard rate, i.e., if $\widetilde{X}_{1,i,t_i} \le_{st} \widetilde{X}_{1,i,t_i+v_i}$ $\forall t_i \in \mathbb{R}$ and $v_i \in \mathbb{R}^+$.*

Proof. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be any vector with non-negative components. Note that, as shown in the proof of Theorem 2.2,

$$\overline{F}_{\mathbf{X}_{1,\mathbf{t}}}(\mathbf{u}) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{1,i,t_i}(u_i)\right)^{\widetilde{\Theta}_{\mathbf{t}}}\right] \text{ and } \overline{F}_{\mathbf{X}_{1,\mathbf{t+v}}}(\mathbf{u}) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{1,i,t_i+v_i}(u_i)\right)^{\widetilde{\Theta}_{\mathbf{t+v}}}\right]$$

where $\widetilde{\Theta}_{\mathbf{t}}$ and $\widetilde{\Theta}_{\mathbf{t+v}}$ have distribution $\widetilde{H}_{\mathbf{t}}$ and $\widetilde{H}_{\mathbf{t+v}}$, respectively, defined as

$$\widetilde{H}_{\mathbf{t}}(\theta) = \frac{\int_0^\theta \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{1,j}(t_j)]\} dH_{\mathbf{t}}(\tau)}{\int_0^\infty \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{1,j}(t_j)]\} dH_k(\tau)}$$

and

$$\widetilde{H}_{\mathbf{t}+\mathbf{v}}(\theta)=\frac{\int_0^\theta \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{1,j}(t_j+v_j)]\}dH_{\mathbf{t}+\mathbf{v}}(\tau)}{\int_0^\infty \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{1,j}(t_j+v_j)]\}dH_k(\tau)}.$$

Thus

$$\mathbf{E}[\exp(-s\widetilde{\Theta}_{\mathbf{t}})]=\frac{\mathbf{E}[\exp(-(s+\widetilde{t})\Theta_1)]}{\mathbf{E}[\exp(-\widetilde{t}\Theta_1)]}$$

and

$$\mathbf{E}[\exp(-s\widetilde{\Theta}_{\mathbf{t}+\mathbf{v}})]=\frac{\mathbf{E}[\exp(-(s+\widetilde{t}_v)\Theta_1)]}{\mathbf{E}[\exp(-\widetilde{t}_v\Theta_1)]},$$

where $\widetilde{t}=-\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)$ and $\widetilde{t}_v=-\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j+v_j)$.

Let us denote with

$$\widetilde{W}_{1,\mathbf{t}}(s)=\mathbf{E}[\exp(-s\widetilde{\Theta}_{\mathbf{t}})] \text{ and } \widetilde{W}_{1,\mathbf{t}+\mathbf{v}}(s)=\mathbf{E}[\exp(-s\widetilde{\Theta}_{\mathbf{t}+\mathbf{v}})]$$

the Laplace transforms of $\widetilde{H}_{\mathbf{t}}$ and $\widetilde{H}_{\mathbf{t}+\mathbf{v}}$, respectively.

It holds

$$\frac{\widetilde{W}_{1,\mathbf{t}}^{(n-1)}(s)}{\widetilde{W}_{1,\mathbf{t}+\mathbf{v}}^{(n-1)}(s)}=\frac{\mathbf{E}[\exp(-\widetilde{t}_v\Theta_1)]}{\mathbf{E}[\exp(-\widetilde{t}\Theta_1)]}\cdot\frac{W_1^{(n-1)}(s+\widetilde{t})}{W_1^{(n-1)}(s+\widetilde{t}_v)}.$$

It is easy to verify that the ratio $\frac{W_1^{(n-1)}(s+\widetilde{t})}{W_1^{(n-1)}(s+\widetilde{t}_v)}$ is decreasing in s because of Lemma 2.1 and inequality $\widetilde{t}\le\widetilde{t}_v$. Thus $\widetilde{\Theta}_{\mathbf{t}+\mathbf{v}}\le_{n-Lt-r}\widetilde{\Theta}_{\mathbf{t}}$.

Moreover, from the assumption on the variables $\widetilde{X}_{1,i}$ easily follows that $\widetilde{X}_{1,i,t_i}\le_{st}\widetilde{X}_{1,i,t_i+v_i}$, i.e., that $\overline{G}_{1,i,t_i}(u_i)\le_{st}\overline{G}_{1,i,t_i+v_i}(u_i)$ for all $u_i\in\mathbb{R}^+$ and $i=1,\dots,n$.

Thus the assertion follows applying Theorem 2.1. □

This result is not surprising, in particular if compared with similar conditions reported in literature for other notions of negative multivariate aging (see, e.g., Spizzichino and Torrisi, 2001).

4 The Laplace transform – likelihood ratio order

The $\le_{n-Lt-r}$ orders have been never considered before in general in the literature. However, the particular case $\le_{1-Lt-r}$ is equivalent to the *Laplace transform ratio order* $\le_{Lt-r}$ defined and studied in Shaked and Wong (1997), and further considered in Bartoszewicz (1999), who derived some of its characterizations and established inequalities for negative moments of ordered random variables. Also, the $\le_{2-Lt-r}$ order is the same as the *differentiated Laplace transform ratio order* recently defined and in Li et al. (2009), where a complete study on its properties and applications is provided.

Like the orders mentioned above, the orders $\le_{n-Lt-r}$ do not imply the usual stochastic order $\le_{st}$. To prove it, it suffices to consider the variables Θ_1 and Θ_2 having discrete densities f_{Θ_k} defined as

$$f_{\Theta_1}(t) = \begin{cases} 0.2 & \text{if } t = 1 \\ 0.4 & \text{if } t = 2 \\ 0.4 & \text{if } t = 2.9 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_{\Theta_2}(t) = \begin{cases} 0.3 & \text{if } t = 1 \\ 0.4 & \text{if } t = 2 \\ 0.3 & \text{if } t = 3 \\ 0 & \text{otherwise.} \end{cases}$$

With some straightforward calculation it is easy to verify that $\Theta_2 \leq_{2-Lt-r} \Theta_1$, while the usual stochastic order between Θ_1 and Θ_2 is not satisfied, since their survival functions do intersect. Moreover, the usual stochastic order does not imply the n–Laplace transform–likelihood ratio orders, since it does not imply the $\leq_{Lt-r}$ order (see Shaked and Wong, 1997).

A second example of variables that are ordered in *2-Lt-r* sense but not in usual stochastic order is for $\Theta_1 \sim U[0,3]$ and $\Theta_2 \sim U[1,2]$. These two variables are not ordered in usual stochastic order because their survival functions do intersect (and neither are ordered in the stronger *likelihood ratio order*, as one can verify), however it holds $\Theta_2 \leq_{2-Lt-r} \Theta_1$ being the ratio

$$\frac{W_1^{(1)}(s)}{W_2^{(1)}(s)} = \frac{-3e^{-3s}s + (1 - e^{-3s})}{3[(e^{-s} - 2e^{-2s})s + (e^{-s} - e^{-2s})]}$$

decreasing in $s \geq 0$.

An example where two non-negative variables are ordered in $\leq_{n-Lt-r}$ order for every value of n is described in the following proposition. Here, $Ga(\alpha, \lambda)$ denotes the gamma distribution with shape parameter α and scale parameter λ.

Proposition 4.1. *Let $\Theta_1 \sim Ga(\alpha_1, \lambda_1)$ and $\Theta_2 \sim Ga(\alpha_2, \lambda_2)$. Then $\Theta_2 \leq_{n-Lt-r} \Theta_1$ for every $n \geq 0$ whenever $\alpha_1 \geq \alpha_2$ and $\lambda_1 \leq \lambda_2$.*

Proof. It is well-known that if $\Theta \sim Ga(\alpha, \lambda)$, its associated Laplace transform is given by

$$W(s) = \lambda^{\alpha}(\lambda + s)^{-\alpha},$$

and that its derivative of order n is given by

$$\begin{aligned} W^{(n-1)}(s) &= (-1)^{n-1}\frac{(\alpha + n - 1)!}{(\alpha - 1)!}\lambda^{\alpha}(\lambda + s)^{-(\alpha+n-1)} \\ &= -(\alpha + n - 1)(\lambda + s)^{-1}W^{(n-2)}(s) \end{aligned} \tag{4.1}$$

Therefore,

$$\begin{aligned} \frac{W_1^{(n-1)}(s)}{W_2^{(n-1)}(s)} &= C_{\alpha_1,\alpha_2,\lambda_1,\lambda_2,n}\frac{(\lambda_1 + s)^{-\alpha_1-n}}{(\lambda_2 + s)^{-\alpha_2-n}} \\ &= C_{\alpha_1,\alpha_2,\lambda_1,\lambda_2,n}(\lambda_2 + s)^{\alpha_2-\alpha_1}\left(\frac{\lambda_2 + s}{\lambda_1 + s}\right)^{n+\alpha_1}, \end{aligned} \tag{4.2}$$

where $C_{\alpha_1,\alpha_2,\lambda_1,\lambda_2,n}$ does not depend on s. It is easy to see that this ratio is decreasing in s if and only if $\alpha_1 \geq \alpha_2$ and $\lambda_1 \leq \lambda_2$. □

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