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Abstract

Different sufficient conditions for stochastic comparisons between random vectors have been described in the literature. In particular, conditions for the comparison of random vectors having the same copula, i.e., the same dependence structure, may be found in Müller and Scarsini (2001). Here we provide conditions for the comparison, in the usual stochastic order sense and in other weaker stochastic orders, of two time transformed exponential bivariate lifetimes having different copulas. Some examples of applications are provided too.

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1 Introduction and preliminaries

Let $\mathbf{X} = (X_1, X_2)$ be a pair of exchangeable lifetimes. The vector $\mathbf{X}$ is said to be defined via a *Time Transformed Exponential Model* (shortly, TTE model) if its joint survival function $\overline{F}$ can be written as

$$\overline{F}(t,s) = W(R(t) + R(s)),\ t,s \geq 0, \tag{1.1}$$

for a suitable one-dimensional, continuous, convex and strictly decreasing survival function W and for a suitable continuous and strictly increasing function $R : [0,+\infty) \to [0,+\infty)$ such that $R(0) = 0$ and $\lim_{t\to\infty} R(t) = \infty$. We will write in this case that $\mathbf{X} \sim TTE(W,R)$. Observe that, for a TTE model, the marginals X_1 and X_2 are exchangeable and have survival functions $\overline{G}(t) = \overline{F}(t,0) = W(R(t)), t \geq 0$.

TTE models have been recently considered in literature as an appropriate way to describe bivariate lifetimes (see Bassan and Spizzichino (2005b) and references therein): their main characteristic is that they "separate", in a sense, dependence properties (based on W) and aging (based on R). TTE models include relevant cases of dependent bivariate lifetimes, like independent or Schur constant laws (take $W(x) = W_\lambda(x) = \exp(-\lambda x)$ and $R(t) = t$, respectively), and can be derived for example from frailty models (see Marshall and Olkin, 1988, or Oakes, 1989). In fact, in the frailty approach it is assumed that X_1 and X_2 are independent conditionally on some random environmental factor Θ, having conditional survival marginals $\overline{G}_\theta(t) = \mathbb{P}[X_i > t|\Theta = \theta] = \overline{H}(t)^\theta$ for some survival function $\overline{H}$. Thus, for this model,

$$\begin{aligned}\overline{F}(t,s) &= \mathbf{E}[\overline{H}(t)^\Theta \overline{H}(s)^\Theta] = \mathbf{E}[\exp(\Theta(\ln \overline{H}(t))) \exp(\Theta(\ln \overline{H}(s)))] \\ &= W(-\ln \overline{H}(t) - \ln \overline{H}(s)) = W(R(t) + R(s)), \quad t,s \geq 0,\end{aligned}$$

where $W(x)$ is the Laplace transform of the density of Θ, i.e.,

$$W(x) = \mathbf{E}[\exp(-x\Theta)],\ x \geq 0, \quad \text{and} \quad R(t) = -\ln \overline{H}(t),\ t \geq 0. \tag{1.2}$$

In this paper we will provide simple conditions to compare, in different stochastic ways, two bivariate lifetimes $\mathbf{X} \sim TTE(W_\mathbf{X}, R_\mathbf{X})$ and $\mathbf{Y} \sim TTE(W_\mathbf{Y}, R_\mathbf{Y})$. These conditions are essentially based on comparisons, again in some stochastic sense, between $W_\mathbf{X}$ and $W_\mathbf{Y}$ (or, better, between the univariate variables X^* and Y^* having survival functions $W_\mathbf{X}$ and $W_\mathbf{Y}$, respectively). Thus, in contrast to the results presented in Müller and Scarsini (2001), we will provide here simple conditions for the stochastic comparison between bivariate lifetimes having different copulas. Some examples of applications will be also provided.

Some preliminary definitions and results should be recalled in order to describe the main statements.

First, we recall that the copula of a random vector $\mathbf{X} = (X_1, X_2)$ is an useful tool to describe the structure of dependence between its components, and it is defined by

$$C(u,v) = F(G^{-1}(u), G^{-1}(v)), \quad u,v \in [0,1],$$

where G is the cumulative distribution function of the marginals X_i. We also recall the notion of survival copula, that similarly describes the dependence between the components of the random vector, but considering the survival function $\overline{G}$ of the marginals X_i instead of their cumulative distribution G:

$$K(u,v) = \overline{F}(\overline{G}^{-1}(u), \overline{G}^{-1}(v)), \quad u,v \in [0,1].$$

Further details, properties and applications of these two notions may be found in Nelsen (1999).

Among copulas, particularly interesting is the class of Archimedean copulas: a copula is said to be *Archimedean* if it can be written as

$$C(u,v) = \phi^{-1}(\phi(u) + \phi(v)) \ \forall u,v \in [0,1] \tag{1.3}$$

for a suitable decreasing and convex function $\phi : [0,1] \to [0,1]$ such that $\phi(1) = 0$ (and similarly for survival copulas). The function ϕ is usually called the *generator* of the Archimedean copula C. As pointed out in Nelsen (1999), many standard bivariate distributions (such as the ones in Gumbel, Frank, Clayton and Ali-Mikhail-Haq families) are special cases of this class. We also refer the reader to Müller and Scarsini (2005) or Bassan and Spizzichino (2005a), and references therein, for details, properties and recent applications of Archimedean copulas.

It is interesting to observe, and easy to verify, that if $\mathbf{X} \sim TTE(W,R)$ then

$$K(u,v) = W(W^{-1}(u) + W^{-1}(v)),$$

i.e., its survival copula K is Archimedean with generator W^{-1}. Viceversa, bivariate survival functions $\bar{F}$ that admit a (strict) Archimedean survival copula can be written in the form as in (1.1), i.e., they can be defined via a $TTE(W,R)$ model for suitable functions W and R.

Also, considered the vector $\mathbf{X}_t = [(X_1 - t, X_2 - t) | X_1 > t, X_2 > t]$ of the residual lifetimes at time $t \geq 0$, then $\mathbf{X}_t \sim TTE(W_t, R_t)$, i.e., it has joint survival function $\overline{F}_t(x,y)$ given by

$$\overline{F}_t(x,y) = W_t(R_t(x) + R_t(y)) \tag{1.4}$$

where

$$W_t(x) = \frac{W(2R(t) + x)}{W(2R(t))}, \tag{1.5}$$

and where

$$R_t(x) = R(t+x) - R(t), \tag{1.6}$$

for $t, x \geq 0$.

Thus, the survival copula of $\mathbf{X}_t$ is defined by

$$K_t(u,v) = W_t(W_t^{-1}(u) + W_t^{-1}(v)), \ u,v \in [0,1],$$

where W_t is defined as in (1.5).

Now we recall the definitions of the stochastic orders considered throughout the paper. Further details, equivalent definitions and applications may be found in Shaked and Shanthikumar (2007) and Kaas et al. (2001).

Definition 1.1. *Let X and Y be two nonnegative random variables. Then*

a) X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\mathbf{E}[\phi(X)] \leq \mathbf{E}[\phi(Y)]$ for all increasing function $\phi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist, or, equivalently, if $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all $t \in \mathbb{R}$, where $\overline{F}_X$ and $\overline{F}_Y$ are the survival functions of X and Y, respectively..

b) X is said to be smaller than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if they both are absolutely continuous and the ratio $\frac{f_Y(t)}{f_X(t)}$ is an increasing function in t, where f_X and f_Y are the density functions of X and Y, respectively.

c) X is said to be smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$) if the ratio $\frac{\overline{F}_Y(t)}{\overline{F}_X(t)}$ is an increasing function in t.

d) X is said to be smaller than Y in the increasing convex [increasing concave] order (denoted by $X \leq_{icx\ [icv]} Y$) if $\mathbf{E}[\phi(X)] \leq \mathbf{E}[\phi(Y)]$ for all increasing convex [increasing concave] functions $\phi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

Definition 1.2. *Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors. Then*

a) $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the usual multivariate stochastic order (denoted by $\mathbf{X} \leq_{st} \mathbf{Y}$) if $\mathbf{E}[\phi(\mathbf{X})] \leq \mathbf{E}[\phi(\mathbf{Y})]$ for all increasing function $\phi : \mathbb{R}^n \to \mathbb{R}$ for which the expectations exist.

b) $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the upper orthant order (denoted by $\mathbf{X} \leq_{uo} \mathbf{Y}$) if $\overline{F}_X(x_1, \dots, x_n) \leq \overline{F}_Y(x_1, \dots, x_n)$ for all $\mathbf{x} = (x_1, \dots, x_n)$, where $\overline{F}_X$ and $\overline{F}_Y$ are the survival functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively, or, equivalently, if $\mathbf{E}[\prod_{i=1}^n g_i(X_i)] \leq \mathbf{E}[\prod_{i=1}^n g_i(Y_i)]$ for every collection $\{g_1, \dots, g_n\}$ of univariate nonnegative increasing functions for which the expectations exist.

c) $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the lower orthant order (denoted by $\mathbf{X} \leq_{lo} \mathbf{Y}$) if $F_X(x_1, \dots, x_n) \geq G_Y(x_1, \dots, x_n)$ for all $\mathbf{x} = (x_1, \dots, x_n)$, where F_X and F_Y are the distribution functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively, or, equivalently, if $\mathbf{E}[\prod_{i=1}^n h_i(X_i)] \geq \mathbf{E}[\prod_{i=1}^n h_i(Y_i)]$ for every collection $\{h_1, \dots, h_n\}$ of univariate nonnegative decreasing functions for which the expectations exist.

d) $\mathbf{X}$ is said to be smaller than $\mathbf{Y}$ in the upper orthant-convex order ($\mathbf{X} \leq_{uo-cx} \mathbf{Y}$) if $\mathbf{E}[\prod_{i=1}^n g_i(X_i)] \leq \mathbf{E}[\prod_{i=1}^n g_i(Y_i)]$ for every collection $\{g_1, \dots, g_n\}$ of univariate nonnegative increasing convex functions, for which the expectations exist.

e) $\mathbf{X}$ *is said to be smaller than* $\mathbf{Y}$ *in the lower orthant-concave order (*$\mathbf{X} \leq_{lo-cv} \mathbf{Y}$*) if* $\mathbf{E}[\prod_{i=1}^n h_i(X_i)] \leq \mathbf{E}[\prod_{i=1}^n h_i(Y_i)]$ *for every collection* $\{h_1, \ldots, h_n\}$ *of univariate nonnegative increasing functions such that* h_i *is concave on the union of the supports of* X_i *and* Y_i*,* $i = 1, \ldots, n$*, for which the expectations exist.*

We recall that the multivariate usual stochastic order implies all the orders $\leq_{uo}$, $\leq_{lo}$,$\leq_{uo-cx}$ and $\leq_{lo-cv}$, but not viceversa.

We also recall a positive dependence notion that will be mentioned along this paper (see again Shaked and Shanthikumar, 2007, for details).

Definition 1.3. *A random vector* $\mathbf{X} = (X_1, \ldots, X_n)$ *is said to be conditionally increasing in sequence (CIS) if, for* $i = 2, \ldots, n$*,*

$$[X_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq_{st} [X_i | X_1 = x_1', \ldots, X_{i-1} = x_{i-1}']$$

for all $x_j \leq x_j'$*,* $j = 1, \ldots, i-1$*, where* $[X_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}]$ *denotes the conditional distribution of* X_i *given* $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}$ *for all* $x_1, \ldots, x_{i-1} \in \mathbb{R}$*.*

Note that, in the bivariate case, $\mathbf{X} = (X_1, X_2)$ is CIS if $[X_2 | X_1 = u_1] \leq_{st} [X_2 | X_1 = u_2]$ for all $u_1 \leq u_2$ (i.e., if X_2 is stochastically increasing in X_1). One of the reasons of interest in the CIS property is due to the following statement (Shaked and Shanthikumar, 2007, Theorem 6.B.4).

Lemma 1.1. *Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be two n-dimensional random vectors. If either* $\mathbf{X}$ *or* $\mathbf{Y}$ *is CIS and*

a) $X_1 \leq_{st} Y_1$,

b) $[X_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq_{st} [Y_i | Y_1 = x_1, \ldots, Y_{i-1} = x_{i-1}]$ *for all* x_j*,* $j = 1, \ldots, i-1$,

then $\mathbf{X} \leq_{st} \mathbf{Y}$*.*

Conditions for a random vector $\mathbf{X}$ having an Archimedean copula (or survival copula) to be CIS may be found in Müller and Scarsini (2005).

Finally, we remark that random variables having log-convex densities play a crucial role in the next section. Log-convexity and log-concavity are popular concepts both in reliability and in economics (see for example Shaked and Shanthikumar, 1987, or An, 1998). Moreover, most of the Archimedean copulas considered in the applications, like the Clayton, the Gumbel-Barnett or the Ali-Mikhail-Haq copulas, are such that their corresponding functions W, inverses of their generators, are survival functions of variables having log-convex densities, thus satisfy the assumptions of the main results in Section 2.

Some conventions that are used in this paper are the following. By “increasing” and “decreasing”, we mean “nondecreasing” and “nonincreasing”, respectively. The relation $=_{st}$ stands for equality in law. For any random vector $\mathbf{X}$, or random variable, we denote by $[\mathbf{X}|A]$ a random vector, or random variable, whose distribution is the conditional distribution of $\mathbf{X}$ given A.

In the next section we will state and prove the main results, while Section 3 deals with examples of applications.

2 Results

Let $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ and $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$ be two bivariate lifetimes described by two different time-transformed exponential models, i.e, let $\mathbf{X}$ and $\mathbf{Y}$ be two bivariate random vectors with survival functions $\overline{F}_{\mathbf{X}}(t, s) = W_{\mathbf{X}}(R_{\mathbf{X}}(t) + R_{\mathbf{X}}(s))$ and $\overline{F}_{\mathbf{Y}}(t, s) = W_{\mathbf{Y}}(R_{\mathbf{Y}}(t) + R_{\mathbf{Y}}(s))$, respectively, for two suitable univariate survival functions $W_{\mathbf{X}}$ and $W_{\mathbf{Y}}$ and for two suitable continuous and strictly increasing functions $R_{\mathbf{X}}, R_{\mathbf{Y}} : [0, +\infty) \to [0, +\infty)$ such that $R_{\mathbf{X}}(0) = R_{\mathbf{Y}}(0) = 0$ and

$$\lim_{t\to\infty} R_{\mathbf{X}}(t) = \lim_{t\to\infty} R_{\mathbf{Y}}(t) = +\infty.$$

Note that, in this case, $\bar{G}_{\mathbf{X}}(t) = W_{\mathbf{X}}(R_{\mathbf{X}}(t))$ and $\bar{G}_{\mathbf{Y}}(t) = W_{\mathbf{Y}}(R_{\mathbf{Y}}(t))$ are the univariate marginal survival functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively, i.e., the survival functions of X_i and Y_i, respectively.

Let us denote with X^* and Y^* the univariate lifetimes whose survival functions are $W_{\mathbf{X}}$ and $W_{\mathbf{Y}}$, respectively.

A first immediate sufficient condition one can prove to get stochastic comparisons between $\mathbf{X}$ and $\mathbf{Y}$ is the following.

Theorem 2.1. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. *If: (i)* $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$ *and (ii)* $X^* \leq_{st} Y^*$, *then* $\mathbf{X} \leq_{uo} \mathbf{Y}$.

Proof. It is enough to observe that

$$\overline{F}_{\mathbf{X}}(t, s) = W_{\mathbf{X}}(R(t) + R(s)) \leq W_{\mathbf{Y}}(R(t) + R(s)) = \overline{F}_{\mathbf{Y}}(t, s)$$

for all $s, t \geq 0$, where the inequality follows from $X^* \leq_{st} Y^*$. □

An immediate question one can consider is if, under the same assumptions, it is possible to get stronger comparisons between $\mathbf{X}$ and $\mathbf{Y}$. Actually, the answer to this question is negative, as shown in Counterexample 2.1 below. However, under some additional assumptions it is possible to get $\mathbf{X} \leq_{st} \mathbf{Y}$, as shown in the following statement.

For it, let us denote with $w_{\mathbf{X}}$ and $w_{\mathbf{Y}}$ the density functions of the random variables X^* and Y^*, i.e., let $w_{\mathbf{X}}(x) = -W'_{\mathbf{X}}(x)$ and $w_{\mathbf{Y}}(x) = -W'_{\mathbf{Y}}(x)$.

Theorem 2.2. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$, *and assume that the derivatives* $W'_{\mathbf{X}}$ *and* $W'_{\mathbf{Y}}$ *exist. If: (i)* $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$, *(ii)* $X^* \leq_{lr} Y^*$ *and (iii)* X^*, *or* Y^*, *has log-convex density, then* $\mathbf{X} \leq_{st} \mathbf{Y}$.

Proof. Let us suppose that X^* has density function $w_{\mathbf{X}}$ that is log-convex. Then by Proposition 1 in Averous and Dortet-Bernadet(2004), it follows that $\mathbf{X}$ is CIS. Thus, by Lemma 1.1 in order to prove that $\mathbf{X} \leq_{st} \mathbf{Y}$ it is sufficient to verify that: (a) $X_1 \leq_{st} Y_1$, and (b) $[X_2|X_1 = u] \leq_{st} [Y_2|Y_2 = u]$ for all $u \in \mathbb{R}^+$.

By assumption (ii) it holds $X^* \leq_{st} Y^*$, and therefore, clearly, $W_{\mathbf{X}}(R(t)) \leq W_{\mathbf{Y}}(R(t))$ for all $t \geq 0$, i.e., $X_1 \leq_{st} Y_1$.

With straightforward computations it is easy to verify that

$$\overline{F}_{[X_2|X_1=u]}(t) = \frac{w_{\mathbf{X}}(R(t) + R(u))}{w_{\mathbf{X}}(R(u))}$$

for all $u, t \geq 0$ (and similarly for the survival function of $[Y_2|Y_1 = u]$).

Now observe that $[X_2|X_1 = u] \leq_{st} [Y_2|Y_2 = u]$ for all u if, and only if,

$$\overline{F}_{[X_2|X_1=u]}(t) \leq \overline{F}_{[Y_2|Y_1=u]}(t) \ \forall t \geq 0$$

i.e., if, and only if,

$$\frac{w_{\mathbf{X}}(R(t) + R(u))}{w_{\mathbf{X}}(R(u))} \leq \frac{w_{\mathbf{Y}}(R(t) + R(u))}{w_{\mathbf{Y}}(R(u))} \ \forall t \geq 0.$$

This inequality is clearly verified if $\frac{w_{\mathbf{Y}}(x)}{w_{\mathbf{X}}(x)}$ is an increasing function in x, that is equivalent to $X^* \leq_{lr} Y^*$, which is satisfied again by assumption (ii). □

Note that the assumption (ii) in the above theorem can not be replaced by a weaker one, like a comparison between X^* and Y^* in the usual stochastic or the hazard rate order, as shown in the following counterexample.

Counterexample 2.1. *Let* $\mathbf{X} \sim TTE(W_{\mathbf{X}}, R)$ *and* $\mathbf{Y} \sim TTE(W_{\mathbf{Y}}, R)$, *with*

$$W_{\mathbf{X}}(x) = \exp\{1 - e^x\}, \ W_{\mathbf{Y}}(x) = \frac{1}{1 + x},$$

(inverses of the generators of the Gumbel-Barnett copula and of the Clayton copula, respectively) and

$$R(t) = \frac{e^t - 1}{2}.$$

It can be easily verified with straightforward calculations that Y^* *has log-convex density and that inequality* $X^* \leq_{hr} Y^*$ *holds, while* $X^* \leq_{lr} Y^*$ *does not hold. It can also be observed that it holds* $\mathbf{X} \leq_{uo} \mathbf{Y}$, *but not* $\mathbf{X} \leq_{lo} \mathbf{Y}$. *In fact, for example, taking* $t_0 = 0.1$ *and* $s_0 = 0.2$, *it is* $F_{\mathbf{X}}(t_0, s_0) \leq F_{\mathbf{Y}}(t_0, s_0)$ *(as one can verify with a direct computation). Thus, it can not be* $\mathbf{X} \leq_{st} \mathbf{Y}$. *Moreover, since the hazard rate order implies the usual stochastic order, this example also shows that the usual univariate stochastic order between* X^* *and* Y^* *does not imply the usual multivariate stochastic order between the corresponding bivariate lifetimes.*

The previous statement can be generalized to the case where $R_{\mathbf{X}}$ and $R_{\mathbf{Y}}$ are different functions.

Corollary 2.1. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$, *and assume that the derivatives* $W'_{\mathbf{X}}$ *and* $W'_{\mathbf{Y}}$ *exist. If: (i)* $R_{\mathbf{X}}(t) \geq R_{\mathbf{Y}}(t)$ *for all* $t \geq 0$, *(ii)* $X^* \leq_{lr} Y^*$ *and (iii)* X^*, *or* Y^*, *has log-convex density, then* $\mathbf{X} \leq_{st} \mathbf{Y}$.

Proof. Let $\mathbf{Z}$ be a bivariate random vector with survival function $\overline{F}_{\mathbf{Z}}(t,s) = W_{\mathbf{X}}(R_{\mathbf{Y}}(t) + R_{\mathbf{Y}}(s))$. By Theorem 2.2, it follows that $\mathbf{Z} \leq_{st} \mathbf{Y}$.

On the other hand, $\mathbf{X}$ and $\mathbf{Z}$ have the same copula, and marginals ordered in the usual stochastic order. Thus it follows $\mathbf{X} \leq_{st} \mathbf{Z}$ (see Müller and Scarsini, 2001).

Combining $\mathbf{X} \leq_{st} \mathbf{Z}$ and $\mathbf{Z} \leq_{st} \mathbf{Y}$, it follows $\mathbf{X} \leq_{st} \mathbf{Y}$. □

Under a quite stronger assumption on the functions $R_{\mathbf{X}}$ and $R_{\mathbf{Y}}$ it is possible to obtain the comparison between the residual lifetimes of $\mathbf{X}$ and $\mathbf{Y}$ at any time $t \geq 0$.

Corollary 2.2. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$, *and assume that the derivatives* $W'_{\mathbf{X}}, W'_{\mathbf{Y}}, R'_{\mathbf{X}}$ *and* $R'_{\mathbf{Y}}$ *exist. If: (i)* $R'_{\mathbf{X}}(t) \geq R'_{\mathbf{Y}}(t)$ *for all* $t \geq 0$, *(ii)* $X^* \leq_{lr} Y^*$, *(iii)* X^* *has log-convex density, (iv)* Y^* *has log-concave density, then* $\mathbf{X}_t \leq_{st} \mathbf{Y}_t$ *for all* $t \geq 0$.

Proof. As pointed out in the previous section, for any fixed t it holds $\mathbf{X}_t \sim TTE(W_{\mathbf{X}_t}, R_{\mathbf{X}_t})$ and $\mathbf{Y}_t \sim TTE(W_{\mathbf{Y}_t}, R_{\mathbf{Y}_t})$, where

$$R_{\mathbf{X}_t}(x) = R_{\mathbf{X}}(t+x) - R_{\mathbf{X}}(t), \ x \geq 0,$$

and $W_{\mathbf{X}_t}$ is the survival function of a variable $X^*_{\tilde{t}_X} = [X^* - \tilde{t}_X \mid X^* > \tilde{t}_X]$ where $\tilde{t}_X = 2R_{\mathbf{X}}(t)$, being

$$W_{\mathbf{X}_t}(x) = \frac{W_{\mathbf{X}}(2R_{\mathbf{X}}(t) + x)}{W_{\mathbf{X}}(2R_{\mathbf{X}}(t))} = \frac{W_{\mathbf{X}}(\tilde{t}_X + x)}{W_{\mathbf{X}}(\tilde{t}_X)}$$

(and similarly for $R_{\mathbf{Y}_t}$ and $W_{\mathbf{Y}_t}$).

Now observe that, since $X^* \leq_{lr} Y^*$, by Theorem 1.C.6 in Shaked and Shanthikumar (2007) it follows that $X^*_{\tilde{t}_X} \leq_{lr} Y^*_{\tilde{t}_X}$. On the other hand, since X^* has log-convex density then also $X^*_{\tilde{t}_X}$ has log-convex density (since clearly log-convexity of $w_{\mathbf{X}}(x)$ implies log-convexity of $w_{\mathbf{X}_t}(x)$). Moreover, since Y^* has log-concave density, it holds $Y^*_{\tilde{t}_X} \leq_{lr} Y^*_{\tilde{t}_Y}$ (being $\tilde{t}_X = 2R_{\mathbf{X}}(t) \geq 2R_{\mathbf{Y}}(t) = \tilde{t}_Y$). Thus $X^*_{\tilde{t}_X} \leq_{lr} Y^*_{\tilde{t}_Y}$. Finally, since $R'_{\mathbf{X}}(t) \geq R'_{\mathbf{Y}}(t)$ for all $t \geq 0$, then

$$R_{\mathbf{X}}(t+x) - R_{\mathbf{X}}(t) \geq R_{\mathbf{Y}}(t+x) - R_{\mathbf{Y}}(t)$$

for all $t, x \geq 0$, i.e., $R_{\mathbf{X}_t}(x) \geq R_{\mathbf{Y}_t}(x)$ for all $x \geq 0$. Thus the assertion follows by application of Corollary 2.1. □

In a similar manner, it is possible to get conditions for negative bivariate aging of the bivariate lifetime $\mathbf{X}$, in the sense described in the following statement.

Corollary 2.3. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$*, where* $W_{\mathbf{X}}$ *and* $R_{\mathbf{X}}$ *are such that both* $W'_{\mathbf{X}}$ *and* $R'_{\mathbf{X}}$ *exist. Suppose that* X^* *has log-convex density and* $R_{\mathbf{X}}$ *is concave. Then* $\mathbf{X}_t \leq_{st} \mathbf{X}_{t+s}$ *for all* $t, s > 0$.

Proof. Fix $t, s > 0$, and denote $\widetilde{t+s} = 2R_{\mathbf{X}}(t+s)$ and $\tilde{t} = 2R_{\mathbf{X}}(t)$. Since X^* has log-convex density, it follows that $X^*_{\widetilde{t+s}} \geq_{lr} X^*_{\tilde{t}}$, where $X^*_{\widetilde{t+s}}$ has survival function

$$W_{\mathbf{X}_{t+s}}(x) = \frac{W_{\mathbf{X}}(2R(t+s)+x)}{W_{\mathbf{X}}(2R(t+s))} = \frac{W_{\mathbf{X}}(\widetilde{t+s}+x)}{W_{\mathbf{X}}(\widetilde{t+s})},\ x \geq 0,$$

while $X^*_{\tilde{t}}$ has survival function $W_{\mathbf{X}_t}$ defined as in the previous proof. Moreover, since X^* has log-convex density, then also $X^*_{\tilde{t}}$ has log-convex density.

On the other hand, if $R_{\mathbf{X}}$ is a concave function, then $R_{\mathbf{X}_{t+s}}(x) = R(t+s+x) - R(t+s) \leq R(t+x) - R(t) = R_{\mathbf{X}_t}(x)$ for all $x \geq 0$. Thus, by Corollary 2.1, the assertion follows. □

Note that the statement of Corollary 2.3 generalizes Theorem 3.2 in Mulero and Pellerey (2010), where conditions for bivariate aging are considered. It is also strictly related to Theorem 4.3 in the same paper, where a comparison between $\mathbf{X}_{t+s}$ and $\mathbf{X}_t$ in the weaker $\leq_{lo}$ order is obtained under weaker conditions on $W_{\mathbf{X}}$.

In the following results is considered the case where X^* and Y^* are comparable in a stochastic sense that is weaker than $\leq_{st}$. Even in this case it is possible to get comparisons between $\mathbf{X}$ and $\mathbf{Y}$, but, obviously, in a stochastic sense that is weaker than $\leq_{uo}$.

Theorem 2.3. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$*. Let* $g_1(x)$ *and* $g_2(x)$ *be two nonnegative and increasing functions. Let* g_1*, or* g_2*, be convex [concave]. If: (i)* $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$ *is a concave [convex] function, and (ii)* $X^* \leq_{icx}$ $[\leq_{icv}$ $]$ Y^**, then* $\mathbf{E}[g_1(X_1)g_2(X_2)] \leq \mathbf{E}[g_1(Y_1)g_2(Y_2)]$*, provided the expectations exist.*

Proof. We give here the proof of the statement without the bracket, the other being similar. Assume that $g_2(x)$ is a nonnegative increasing convex function. Let $\mathbf{g}(\mathbf{X}) = (g_1(X_1), g_2(X_2))$ be a random vector with survival function $\bar{F}_{\mathbf{g}(\mathbf{X})}(t, s)$. Then, for all $t \geq g_1(0)$ and $s \geq g_2(0)$,

$$\begin{aligned} \bar{F}_{\mathbf{g}(\mathbf{X})}(t,s) &= P[g_1(X_1) > t, g_2(X_2) > s] \\ &= P[X_1 > g_1^{-1}(t), X_2 > g_2^{-1}(s)] \\ &= \bar{F}(g_1^{-1}(t), g_2^{-1}(s)) \\ &= W(R(g_1^{-1}(t)) + R(g_2^{-1}(s))), \end{aligned}$$

where $g_i^{-1}(t) = \sup\{x : g_i(x) \leq t\}$, $i = 1, 2$.

Now, let $h_1(u) = g_1(R^{-1}(u))$ and $h_2(v) = g_2(R^{-1}(v))$, where $u, v \geq 0$. It is easy to see that, by the assumptions, h_1 is an increasing function and h_2 is an increasing and convex function. Moreover, with straightforward calculations it is easy to verify that

$$\mathbf{E}[g_1(X_1)g_2(X_2)] \quad = \quad \int_0^\infty \left(\int_0^\infty W_{\mathbf{X}}(u+v)dh_2(v) \right) dh_1(u),$$

and that

$$
\begin{aligned}
\int_0^\infty W_{\mathbf{X}}(u+v)dh_2(v) &= \int_u^\infty \left(\int_v^\infty d(1-W_{\mathbf{X}}(z)) \right) dh_2(v) \\
&= \int_u^\infty \left(\int_u^z dh_2(v) \right) d(1-W_{\mathbf{X}}(z)) \\
&= \int_u^\infty [h_2(z)-h_2(u)] d(1-W_{\mathbf{X}}(z)) \\
&= \int_0^\infty H_u(z) d(1-W_{\mathbf{X}}(z)) = \mathbf{E}[H_u(X^*)],
\end{aligned}
$$

where $H_u(z) = [h_2(z) - h_2(u)] \cdot \mathbf{1}_{[u,\infty)}(z)$. Since h_2 is non-negative, increasing and convex, it follows that also H_u is increasing and convex, whatever $u \geq 0$ is. Thus, for all $u \geq 0$,

$$
\int_0^\infty W_{\mathbf{X}}(u+v)dh_2(v) - \int_0^\infty W_{\mathbf{Y}}(u+v)dh_2(v) = \mathbf{E}[H_u(X^*)] - \mathbf{E}[H_u(Y^*)] \leq 0,
$$

where the inequality follows from assumption (ii). It follows

$$
\begin{aligned}
\mathbf{E}[g_1(X_1)g_2(X_2)] \quad &- \quad \mathbf{E}[g_1(Y_1)g_2(Y_2)] \\
&= \int_0^\infty \left(\int_0^\infty W_{\mathbf{X}}(u+v)dh_2(v) - \int_0^\infty W_{\mathbf{Y}}(u+v)dh_2(v) \right) dh_1(u) \leq 0,
\end{aligned}
$$

being h_1 increasing. □

Using the same arguments as in the proof of the previous result, a similar statement can be proved. Here the functions g_1 and g_2 are assumed to be decreasing instead of increasing.

Theorem 2.4. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. *Let* $g_1(x)$ *and* $g_2(x)$ *be two nonnegative and decreasing functions where* $g_1(x)$ *or* $g_2(x)$ *is convex [concave]. If: (ii)* $R_{\mathbf{X}} = R_{\mathbf{Y}} \equiv R$ *is a convex [concave] function, and (ii)* $X^* \geq_{icv}$ $[\geq_{icx}]$ Y^*, *then* $\mathbf{E}[g_1(X_1)g_2(X_2)] \geq \mathbf{E}[g_1(Y_1)g_2(Y_2)]$, *provided the expectations exist.*

As immediate consequences of previous theorems one gets the following statements.

Corollary 2.4. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_{\mathbf{X}}, R_{\mathbf{X}})$ *and* $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_{\mathbf{Y}}, R_{\mathbf{Y}})$. *If: (i)* $R_{\mathbf{X}}(t) \geq R_{\mathbf{Y}}(t)$ *for all* $t \geq 0$ *and* $R_{\mathbf{Y}}$, *or* $R_{\mathbf{X}}$, *is a concave [convex] function, and (ii)* $X^* \leq_{icx}$ $[\leq_{icv}]$ Y^*, *then* $\mathbf{X} \leq_{uo-cx}$ $[\leq_{lo-cv}]$ $\mathbf{Y}$.

Proof. The proof is for the case that $R_{\mathbf{Y}}$ is concave (the other case is similar). Let $\mathbf{Z} = (Z_1, Z_2)$ be a bivariate random vector with survival function $\overline{F}_{\mathbf{Z}}(t,s) = W_{\mathbf{X}}(R_{\mathbf{Y}}(t) + R_{\mathbf{Y}}(s))$. Since $\mathbf{X}$ and $\mathbf{Z}$ have the same copula and marginals ordered in the usual stochastic order, by Theorem 4.1 in Müller and Scarsini (2001) it follows that $\mathbf{X} \leq_{st} \mathbf{Z}$, which in turns implies $\mathbf{X} \leq_{uo-cx} \mathbf{Z}$ and $\mathbf{X} \leq_{lo-cv} \mathbf{Z}$. Moreover, from Theorem 2.3 easy follows that

$$
\mathbf{E}[g_1(Z_1)g_2(Z_2)] \leq \mathbf{E}[g_1(Y_1)g_2(Y_2)]
$$

for all univariate nonnegative increasing convex [concave] functions g_1 and g_2, i.e., that $\mathbf{Z} \le_{uo-cx}$ $[\le_{lo-cv}]$ $\mathbf{Y}$ holds. Combining the two stochastic inequalities the assertion is obtained. □

Corollary 2.5. *Let* $\mathbf{X} = (X_1, X_2) \sim TTE(W_\mathbf{X}, R_\mathbf{X})$ *and* $\mathbf{Y} = (Y_1, Y_2) \sim TTE(W_\mathbf{Y}, R_\mathbf{Y})$. *If: (i)* $R_\mathbf{X}(t) \ge R_\mathbf{Y}(t)$ *for all* $t \ge 0$ *and* $R_\mathbf{Y}$, *or* $R_\mathbf{X}$, *is a concave [convex] function, and (ii)* $X^* \le_{icx}$ $[\le_{icv}]$ Y^*, *then* $\mathbf{E}[X_1X_2] \le \mathbf{E}[Y_1Y_2]$.

Proof. By taking $g_1(x)$ and $g_2(x)$ both the identity function, the result follows immediately from Corollary 2.4. □

Note that all the results stated in this section can be easily generalized to the case where $\mathbf{X}$ and $\mathbf{Y}$ are vectors of non exchangeable variables, as well as to the case of conditioned residual lifetimes of the kind $\mathbf{X}_{(t_1,t_2)} = [(X_1 - t_1, X_2 - t_2)|\ X_1 > t_1, X_2 > t_2]$ instead of $\mathbf{X}_t = \mathbf{X}_{(t,t)} = [(X_1 - t, X_2 - t)|\ X_1 > t, X_2 > t]$.

3 Examples of applications

Some possible applications of the results presented in Section 2 follow immediately from the properties of the usual stochastic order. For example, from Theorem 2.2 it follows that, if its assumptions are satisfied, then $h(\mathbf{X}) \le_{st} h(\mathbf{Y})$ for every increasing function h. Thus, in particular, the sums $X_1 + X_2$ and $Y_1 + Y_2$, or the maximum and the minimum of $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$, are ordered in usual stochastic order. Also, for example, an order between the Values-at-Risk, of any order $\alpha \in (0, 1)$, for two risk positions $aX_1 + bX_2$ and $aY_1 + bY_2$, $a, b \in \mathbb{R}$, follows from Theorem 2.2 (see Embrechts et. al., 2003, for applications in risk management of comparisons between Values-at-Risk of this kind).

Some other simple examples are illustrated in this section.

3.1 Bounds for expected values

Bounds for expected values, based on comparisons with respect to the independent case, can be provided making use of the results described in the previous section. Usefulness of these bounds is of course due to the fact that, in general, expectations are easier to compute under independence.

Let for example $\mathbf{X} \sim TTE(W_\mathbf{X}, R_\mathbf{X})$ be such that $X^* \le_{lr} Z_\lambda$, where Z_λ is exponentially distributed with mean $1/\lambda$, i.e., let $w_\mathbf{X}(x) \le w_\mathbf{X}(0)\exp(-\lambda x)$ for all $x \ge 0$. Since Z_λ has log-convex density, then for every increasing function h one has $\mathbf{E}[h(\mathbf{X})] \le \mathbf{E}[h(\mathbf{Z})]$, where $\mathbf{Z} = (Z_1, Z_2)$ has independent components having survival functions $\overline{G}_\mathbf{Z}(t) = \exp(-\lambda R_\mathbf{X}(t))$. Note that under the same assumptions we also have

$$\overline{F}_{X_1+X_2}(t) \le \int_0^s \overline{G}_\mathbf{Z}(t-s)dG_\mathbf{Z}(s), \text{ for all } t \ge 0,$$

since $\mathbf{X} \le_{st} \mathbf{Z}$.

Assume now that X^* possesses the HNBUE (*Harmonically New Better than Used in Expectation*) property (see Klefsjö, 1983, or Pellerey, 2000, for properties and applications of this aging notion and its dual notion HNWUE). Then $X^* \le_{icx} Z_\lambda$, where Z_λ is exponentially distributed with mean $\mathbf{E}[X^*] = 1/\lambda$. Moreover, let $R_\mathbf{X}$ be a concave function. Then for every pair of functions g_1 and g_2 that are increasing and convex we have $\mathbf{E}[g_1(X_1)g_2(X_2)] \le \mathbf{E}[g_1(Z_1)]\mathbf{E}[g_2(Z_2)]$, where $\mathbf{Z} = (Z_1, Z_2)$ is defined as before (with $\lambda = 1/\mathbf{E}[X^*]$).

3.2 Frailty models

In the frailty approach, $W_\mathbf{X}$ is a Laplace transform. Thus, its corresponding density $w_\mathbf{X}$ is always log-convex (see, e.g., An, 1998) and therefore assumption (iii) of Theorem 2.2 is always satisfied. Let now $\mathbf{X}$ and $\mathbf{Y}$ be two vectors defined as in Section 1, mixtures of conditionally independent variables with respect to two environmental random parameters $\Theta_\mathbf{X}$ and $\Theta_\mathbf{Y}$, respectively, i.e., let

$$\overline{F}_\mathbf{X}(t,s) = \mathbf{E}[\overline{H}(t)^{\Theta_\mathbf{X}}\overline{H}(s)^{\Theta_\mathbf{X}}] \quad \text{and} \quad \overline{F}_\mathbf{Y}(t,s) = \mathbf{E}[\overline{H}(t)^{\Theta_\mathbf{Y}}\overline{H}(s)^{\Theta_\mathbf{Y}}]$$

for some survival function $\overline{H}$. By Theorem 2.2 and Theorem 1 in Bartoszewicz and Skolimowska (2006), it follows that a sufficient condition for $\mathbf{X} \ge_{st} \mathbf{Y}$ is the inequality $\Theta_\mathbf{X} \le_{lr} \Theta_\mathbf{Y}$.

Moreover, let $\Theta_\mathbf{Y} = \theta$ a.s., so that $\mathbf{Y} = (Y_1, Y_2)$ has independent components, with survival function $\overline{G}_\mathbf{Y}(t) = \overline{H}(t)^\theta$. If $\overline{H}$ is DFR, so that $R_\mathbf{X}$ is concave, from Theorem 2.3 it follows $\mathbf{E}[g_1(X_1)g_2(X_2)] \ge \mathbf{E}[g_1(Y_1)]\mathbf{E}[g_2(Y_2)]$ for all increasing and convex functions g_1 and g_2, being X^* with log-convex density and therefore also HNWUE.

3.3 Portfolio optimization

In actuarial and financial literature it is a common assumption that utility functions are increasing and concave. In particular, exponential utilities are often considered in portfolio theory (see, e.g., Kaas et al., 2001). Thus, let us consider the case of an exponential utility u defined as $u(t) = c(1-e^{-\alpha t})$, with $c, \alpha > 0$. Let $\mathbf{X} \sim TTE(W_\mathbf{X}, R_\mathbf{X})$ and $\mathbf{Y} \sim TTE(W_\mathbf{Y}, R_\mathbf{Y})$ be two different pairs of assets. Assume that $R_\mathbf{X} = R_\mathbf{Y} = R$, where R is convex and consider the two portfolios $S_\mathbf{X} = X_1 + X_2$ and $S_\mathbf{Y} = Y_1 + Y_2$. By Theorem 2.4 it follows that if $X^* \ge_{icv} Y^*$ then $\mathbf{E}[u(S_\mathbf{X})] \le \mathbf{E}[u(S_\mathbf{Y})]$. In fact:

$$\begin{aligned}\mathbf{E}[u(S_\mathbf{X})] &= c(1 - \mathbf{E}[\exp(-\alpha X_1)\exp(-\alpha X_2)]) \\ &\le c(1 - \mathbf{E}[\exp(-\alpha Y_1)\exp(-\alpha Y_2)]) = \mathbf{E}[u(S_\mathbf{Y})]\end{aligned}$$

being $g(t) = \exp(-\alpha t)$ decreasing and convex.

Let now $\mathbf{Z} = (Z_1, Z_2) \sim TTE(W_\mathbf{Z}, R_\mathbf{Z})$, where $W_\mathbf{Z}(x) = \exp(-x/\mathbf{E}[X^*])$, so that Z_1 and Z_2 are independent, and let $R_\mathbf{X} = R_\mathbf{Z} = R$ be concave. Assume that X^* is HNBUE, which can be rewritten as $X^* \geq_{icv} Z_\lambda$ where Z_λ is exponentially distributed with mean $\mathbf{E}[X^*]$. Then one can get the following upper bound for the expected utility $\mathbf{E}[u(S_\mathbf{X})]$:

$$\mathbf{E}[u(S_\mathbf{X})] \leq \mathbf{E}[u(S_\mathbf{Z})] = c[1 - (\mathbf{E}[e^{-\alpha Z_1}])^2]$$

where Z_1 has survival function $\overline{G}_\mathbf{Z}(t) = \exp(-R(t)/\mathbf{E}[X^*])$.

3.4 Stochastic ordering of mixtures

Consider a family $\{\mathbf{X}_\theta, \theta \in \mathcal{T} \subseteq \mathbb{R}\}$ of bivariate lifetimes where $\mathbf{X}_\theta \sim TTE(W_\theta, R)$ with

$$W_\theta(x) = [W(x)]^\theta, \ \forall x \in \mathbb{R}^+, \theta \in \mathcal{T}, \tag{3.1}$$

for some suitable convex survival function W. Nelsen (1997) call the families of Archimedean copulas of this kind as α *families*, and this is the case, for example, of Clayton copulas (where $W(x) = (1+x)^{-1}$ and $\mathcal{T} = (0, +\infty)$) and Gumbel-Barnett copulas (where $W(x) = \exp(1 - e^x)$ and $\mathcal{T} = (0, +\infty)$).

Let now W be such that the corresponding variable X^* has log-convex density (like for the two examples above), then X^*_θ also has log-convex density. It is easy to verify that $X^*_{\theta_1} \leq_{lr} X^*_{\theta_2}$ whenever $\theta_1 \geq \theta_2$ (here $X^*_{\theta_i}$ has survival function $W_{\theta_i}(x) = [W(x)]^{\theta_i}$, $i = 1, 2$). It immediately follows, by Theorem 2.2, that in this case $\mathbf{X}_\theta$ is stochastically decreasing in θ, which means that $\mathbf{X}_{\theta_1} \leq_{st} \mathbf{X}_{\theta_2}$ whenever $\theta_1 \geq \theta_2$.

Assume now that the parameter θ describes some environmental factor, related to the degree of dependence between the components of $\mathbf{X}_\theta$, and consider two different random environmental parameters Θ_1 and Θ_2, assuming values in $\mathcal{T}$. Thus, consider the two vectors $\mathbf{X}_{\Theta_1}$ and $\mathbf{X}_{\Theta_2}$ defined as mixtures of an α family $\{\mathbf{X}_\theta, \theta \in \mathcal{T} \subseteq \mathbb{R}\}$, defined as above, and the random parameters Θ_1 and Θ_2. The following holds.

Proposition 3.1. *Let $\{\mathbf{X}_\theta, \theta \in \mathcal{T} \subseteq \mathbb{R}\}$ be an α family having the survival function W as inverse of the basic generator of the copula, and let $-W$ have log-convex derivative. Then $\Theta_1 \geq_{st} \Theta_2$ implies $\mathbf{X}_{\Theta_1} \leq_{st} \mathbf{X}_{\Theta_2}$.*

Proof. As pointed out before, from the assumptions and Theorem 2.2 it follows that $\mathbf{X}_\theta$ is stochastically decreasing in θ. Therefore, for all increasing functions ϕ it holds $\mathbf{E}[\phi(\mathbf{X}_{\theta_1})] \geq \mathbf{E}[\phi(\mathbf{X}_{\theta_2})]$ whenever $\theta_1 \leq \theta_2$, provided the expectations exist. Thus $\Psi(\theta) = \mathbf{E}[\phi(\mathbf{X}_\theta)]$ is a decreasing function in θ. On the other hand, from the condition $\Theta_1 \geq_{st} \Theta_2$, it follows that $\mathbf{E}[h(\Theta_1)] \leq \mathbf{E}[h(\Theta_2)]$ for all decreasing functions h, provided the expectations exist. In particular, since $\Psi(\theta)$ is a decreasing function in θ, then

$$\mathbf{E}[\phi(\mathbf{X}_{\Theta_1})] = \mathbf{E}[\Psi(\Theta_1)] \leq \mathbf{E}[\Psi(\Theta_2)] = \mathbf{E}[\phi(\mathbf{X}_{\Theta_2})]$$

for all increasing functions ϕ, and this yields the stated result. □

As already stated, Clayton and Gumbel-Barnett copulas satisfy the assumptions of Proposition 3.1. Furthermore, it is possible to prove also the above result dealing with the Frank family of copulas which is not an α family.

Proposition 3.2. *Let* $\mathbf{X}_\theta \sim TTE(W_\theta, R)$ *with* $W_\theta(x) = -\frac{1}{\theta}\log[e^{-x}(e^{-\theta}-1)+1]$ *(inverse of the generator of the Frank copula) and let* Θ_1 *and* Θ_2 *be two nonnegative random variables. If* $\Theta_1 \geq_{st} \Theta_2$*, then* $\mathbf{X}_{\Theta_1} \leq_{st} \mathbf{X}_{\Theta_2}$.

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