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# A geometric proof of the Karpelevich-Mostow's Theorem

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## ABSTRACT

In this paper we give a geometric proof of the Karpelevich's theorem that asserts that a semisimple Lie subgroup of isometries, of a symmetric space of non compact type, has a totally geodesic orbit. In fact, this equivalent to a well-known result of Mostow about existence of compatible Cartan decompositions.

## 1. Introduction.

In this paper we address the problem of giving a geometric proof of the following theorem of Karpelevich.

**THEOREM 1.1.** (*Karpelevich [7]*) *Let  $M$  be a Riemannian symmetric space of non positive curvature without flat factor. Then any connected and semisimple subgroup  $G \subset \text{Iso}(M)$  has a totally geodesic orbit  $G.p \subset M$ .*

It is well-known that Karpelevich's theorem is equivalent to the following algebraic theorem.

**THEOREM 1.2.** (*Mostow [8, Theorem 6]*) *Let  $\mathfrak{g}'$  be a real semisimple Lie algebra of non compact type and let  $\mathfrak{g} \subset \mathfrak{g}'$  be a semisimple Lie subalgebra. Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition for  $\mathfrak{g}$ . Then there exists a Cartan decomposition  $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$  for  $\mathfrak{g}'$  such that  $\mathfrak{k} \subset \mathfrak{k}'$  and  $\mathfrak{p} \subset \mathfrak{p}'$ .*

The proof of the above theorems is very algebraic in nature and uses delicate arguments related to automorphisms of semisimple Lie algebras.

For the real hyperbolic spaces, i.e. when  $\mathfrak{g}' = \mathfrak{so}(n, 1)$ , there are two geometric proofs of Karpelevich's theorem [4], [2]. The proof in [4] is based on the study of minimal orbits of isometries subgroups, i.e. orbits with zero mean curvature. The approach in [2] is based on hyperbolic dynamics. It is interesting to note that both proofs are strongly based on the fact that the boundary at infinity of real hyperbolic spaces has a simple structure.

The only non-trivial algebraic tool that we will use is the existence of a Cartan decomposition of a non compact semisimple Lie algebra. But this can also be proved geometrically as was explained by S.K. Donaldson in [5].

Here is a brief explanation of our proof of Theorem 1.1. We first show that a simple subgroup  $G \subset \text{Iso}(M)$  has a minimal orbit  $G.p \subset M$ . Then, by using a standard totally geodesic embedding  $M \hookrightarrow \mathcal{P}$ , where  $\mathcal{P} = SL(n, \mathbb{R})/SO(n)$ , we will show that  $G.p$  is, actually, a totally geodesic submanifold of  $M$ .

## 2. Preliminaries.

The results in this section are well known and are included to orient the non-specialist reader.

The equivalence between Theorems 1.1 and 1.2 is a consequence of the following Elie Cartan's famous and remarkable theorem.

**THEOREM 2.1.** *(Elie Cartan) Let  $M$  be a Riemannian symmetric space of non positive curvature without flat factor. Then the Lie group  $\text{Iso}(M)$  is semisimple of non compact type. Conversely, if  $\mathfrak{g}$  is a semisimple Lie algebra of non compact type then there exist a Riemannian symmetric space  $M$  of non positive curvature without flat factor such that  $\mathfrak{g}$  is the Lie algebra of  $\text{Iso}(M)$ .*

The difficult part of the proof of the above theorem is the second part. Namely, the construction of the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , where  $\mathfrak{k}$  is maximal compact subalgebra of  $\mathfrak{g}$  and the Killing form  $B$  of  $\mathfrak{g}$  is positive definite on  $\mathfrak{p}$ . The standard and well-known proof of the existence of a Cartan decomposition is long and via the classification theory of complex semisimple Lie algebras, i.e. the existence of a real compact form (see e.g. [6]). There is also a direct and geometric proof of the existence of a Cartan decomposition [5].

On the other hand, when  $\mathfrak{g} = \text{Lie}(\text{Iso}(M))$ , where  $M$  is a Riemannian symmetric space of non positive curvature without flat factor, a Cartan decomposition of  $\mathfrak{g}$  can be constructed geometrically. Namely,  $\mathfrak{g} = \text{Lie}(\text{Iso}(M)) = \mathfrak{k} \oplus \mathfrak{p}$  where  $\mathfrak{k}$  is the Lie algebra of the isotropy group  $K_p \subset \text{Iso}(M)$  and  $\mathfrak{p} := \{X \in \text{Lie}(\text{Iso}(M)) : (\nabla X)_p = 0\}$ .

It is well-known that the Riemannian symmetric spaces  $\mathcal{P} = SL(n, \mathbb{R})/SO(n)$  are the *universal* Riemannian symmetric space of non positive curvature. Namely, any Riemannian symmetric space of non compact type  $M = G/K$  can be totally geodesically embedded in some  $\mathcal{P}$  (up to rescaling the metric in the irreducible De Rham factors). A proof of this fact follows from the following well-known result (c.f. Theorem 1 in [5]).

**PROPOSITION 2.2.** *Let  $\mathfrak{g} \subset \mathfrak{sl}(n, \mathbb{R})$  be a semisimple Lie subalgebra and let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition. Then there exists a Cartan decomposition  $\mathfrak{sl}(n, \mathbb{R}) = \mathcal{A} \oplus \mathcal{S}$  such that  $\mathfrak{k} \subset \mathcal{A}$  and  $\mathfrak{p} \subset \mathcal{S}$ . Thus, if  $G \subset SL(n)$  is semisimple,  $G$  has a totally geodesic orbit in  $\mathcal{P} = SL(n)/SO(n)$ . Indeed, any Riemannian symmetric space of non positive curvature  $M$ , without flat factor, can be totally geodesically embedded in some  $\mathcal{P} = SL(n)/SO(n)$ .*

*Proof.* Notice that any Cartan decomposition of  $\mathfrak{sl}(n, \mathbb{R})$  is given by the anti-symmetric  $\mathcal{A}$  and symmetric matrices  $\mathcal{S}$  w.r.t. a positive definite inner product on  $\mathbb{R}^n$ . Since  $g^* := \mathfrak{k} \oplus i\mathfrak{p}$  is a compact Lie subalgebra of  $\mathfrak{sl}(n, \mathbb{C})$ , there exists a positive definite Hermitian form  $(\cdot | \cdot)$  of  $\mathbb{C}^n$

invariant by  $g^*$ . By defining  $\langle , \rangle := \text{Real}(\ | )$  it follows that  $\mathfrak{k} \subset \mathcal{A}$  and  $\mathfrak{p} \subset \mathcal{S}$ .  $\square$

Let  $S_\infty(M)$  be the sphere or boundary at infinity of  $M$ , i.e.  $S_\infty(M)$  is the set of equivalence classes of asymptotic geodesics rays (see [5] or [3, Chapter II.8] for details).

Here is another corollary of the existence of the totally geodesic embedding  $M \hookrightarrow \mathcal{P}$ .

**COROLLARY 2.3.** *Let  $M$  be a Riemannian symmetric space of non positive curvature without flat factor. Then a connected and semisimple Lie subgroup  $G \subset \text{Iso}(M)$  of non compact type has no fixed points in  $S_\infty(M)$ .*

**This corollary is false see Corrigendum in <http://arxiv.org/abs/1104.0892>**

We include the following proposition.

**PROPOSITION 2.4.** *Let  $M$  be a Riemannian symmetric space of non positive curvature without flat factor. Let  $S = \mathbb{R}^N \times M$  be a symmetric space of non positive curvature with flat factor  $\mathbb{R}^N$ . If  $G \subset \text{Iso}(S)$  is a connected non compact simple Lie group then  $G \subset \text{Iso}(M)$ .*

*Proof.* Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . Then the projection  $\pi : \mathfrak{g} \mapsto \text{Lie}(\text{Iso}(\mathbb{R}^N))$  is injective or trivial i.e.  $\pi \equiv 0$ . If  $\pi$  is injective then a further composition with the projection to  $\mathfrak{so}(N)$  gives that  $\mathfrak{g}$  must carry a bi-invariant metric. So,  $\mathfrak{g}$  can not be simple and non compact.  $\square$

Let  $G$  be a semisimple Lie group and let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition. A subspace  $T \subset \mathfrak{p}$  is called a *Lie triple system* if  $[T, [T, T]] \subset T$ . It is well-known that there is a 1-1 correspondence between Lie triple systems  $T$  of  $\mathfrak{p}$  and totally geodesic submanifolds through the base point  $[K] \in G/K$  (see [6]).

### 3. Minimal and totally geodesic orbits.

We will need the following proposition (see Lemma 3.1. in [4] or Proposition 5.5. in [1]).

**PROPOSITION 3.1.** *Let  $M$  be a Riemannian symmetric space of non positive curvature without flat factor and let  $G \subset \text{Iso}(M)$  be a connected group of isometries. Assume that  $G$  has a totally geodesic orbit  $G.p$ . Then any other minimal orbit  $G.q$  is also a totally geodesic submanifold of  $M$ . Moreover, if  $G$  is semisimple then the union of totally geodesic  $G$ -orbits  $T_G$  is a totally geodesic submanifold of  $M$  which is a Riemannian product  $T_G = (G.p) \times A$  where  $A$  is a totally geodesic submanifold of  $M$ .*

*Proof.* Let  $G.p$  be the totally geodesic orbit and let  $G.q \neq \{q\}$  be another orbit. Let  $\gamma$  be a geodesic in  $M$  that minimizes the distance between  $q$  and  $G.p$  (such geodesic do exists since totally geodesic submanifolds of  $M$  are closed and embedded). Eventually by changing the base point  $p$  by another in the orbit we may assume that  $\gamma(0) = p$  and  $\gamma(1) = q$ . A simple computation using the Killing equation shows that  $\dot{\gamma}(t)$  is perpendicular to  $T_{\gamma(t)}(G.\gamma(t))$ , for all  $t$ .

Let  $X$  be a Killing field in the Lie algebra of  $G$  such that  $X.q \neq 0$  and let  $\phi_s^X$  be the one-parameter group of isometries generated by  $X$ . Define  $h : I \times \mathbb{R} \rightarrow M$  by  $h_s(t) := \phi_s^X.\gamma(t)$ . Note that  $X.h_s(t) = \frac{\partial h}{\partial s}$  and that, for a fixed  $s$ ,  $h_s(t)$  is a geodesic.

Let  $A_{\dot{\gamma}(t)}$  be the shape operator, in the direction of  $\dot{\gamma}(t)$  of the orbit  $G \cdot \gamma(t)$ . Define  $f(t) := -\langle A_{\dot{\gamma}(t)}(X \cdot \gamma(t)), X \cdot \gamma(t) \rangle = \langle \frac{D}{ds} \frac{\partial h}{\partial t}, X \cdot h_s(t) \rangle |_{s=0}$ . Now a computation as in Lemma 3.1. in [4] or Proposition 5.5. in [1] implies that  $\frac{d}{dt} f(t) \geq 0$ . Since  $f(0) = 0$ , due to the fact that  $G \cdot p$  is totally geodesic, we obtain that  $f(1) = -\langle A_{\dot{\gamma}(1)}(X \cdot q), X \cdot q \rangle \geq 0$ . Hence  $A_{\dot{\gamma}(1)}$  is negative semidefinite. Since  $G \cdot q$  is minimal,  $\text{trace}(A_{\dot{\gamma}(1)}) = 0$ , we get that  $f(t) \equiv 0$ . Thus,  $\langle R(\dot{\gamma}(t), X \cdot \gamma(t))\dot{\gamma}(t), X \cdot \gamma(t) \rangle \equiv 0$  and  $\nabla_{\dot{\gamma}(t)}(X \cdot \gamma(t)) \equiv 0$ . Notice that the tangent spaces  $T_{\gamma(t)}G \cdot \gamma(t)$  are parallel along  $\gamma(t)$  in  $M$ . So the normal spaces  $\nu_{\gamma(t)}G \cdot \gamma(t)$  are also parallel along  $\gamma(t)$  in  $M$ . Since  $M$  is a symmetric space of non positive curvature the condition  $\langle R(\dot{\gamma}(t), X \cdot \gamma(t))\dot{\gamma}(t), X \cdot \gamma(t) \rangle \equiv 0$  implies  $R(\dot{\gamma}(t), X \cdot \gamma(t))(\cdot) \equiv 0$ . Let  $\eta(t) \in \nu_{\gamma(t)}G \cdot \gamma(t)$  be a parallel vector along  $\gamma(t)$  and let  $X, Y$  two Killing vector fields in the Lie algebra of  $G$ . Then  $\frac{d}{dt} \langle \nabla_X Y, \eta(t) \rangle = \langle \nabla_{\dot{\gamma}(t)} \nabla_X Y, \eta(t) \rangle = \langle \nabla_X \nabla_{\dot{\gamma}(t)} Y, \eta(t) \rangle + \langle R(\dot{\gamma}(t), X \cdot \gamma(t))(Y), \eta(t) \rangle \equiv 0$ . Since  $\langle \nabla_X Y, \eta(0) \rangle = 0$  we get that the  $G$ -orbits  $G \cdot \gamma(t)$  are totally geodesic submanifolds of  $M$ . This show the first part.

For the second part let  $K' := \text{Iso}(M)_p$  be the isotropy subgroup at  $p \in M$  and let  $\mathfrak{k}'$  its Lie algebra. Let  $\mathfrak{p}' \subset \text{Lie}(\text{Iso}(M))$  be such that  $X \in \mathfrak{p}'$  iff  $(\nabla X)_p = 0$ . Thus,  $\text{Lie}(\text{Iso}(M)) = \mathfrak{k}' \oplus \mathfrak{p}'$  is a Cartan decomposition of  $\text{Lie}(\text{Iso}(M))$ . Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition of the Lie algebra  $\mathfrak{g} = \text{Lie}(G)$ . Since  $G \cdot p$  is totally geodesic in  $M$  we get that  $\mathfrak{k} \subset \mathfrak{k}'$  and  $\mathfrak{p} \subset \mathfrak{p}'$ . Let  $\alpha := \{Y \in \mathfrak{p}' : Y \perp \mathfrak{p} \text{ and } [Y, \mathfrak{p}] = 0\}$  which is a Lie triple system of  $\mathfrak{p}'$ . Moreover,  $\mathfrak{n} := \mathfrak{p} \oplus \alpha$  is also a Lie triple system of  $\mathfrak{p}'$ . So,  $N := \exp_p(\mathfrak{n}) = \exp_p(\mathfrak{p}) \times \exp_p(\alpha)$  is a  $G$ -invariant totally geodesic submanifold of  $M$ . Notice that (by construction)  $N \subset T_G$ .

Let  $G \cdot q$  any other totally geodesic  $G$ -orbit. From the computation in the first part we get  $R(\dot{\gamma}(t), X \cdot \gamma(t))(\cdot) \equiv 0$  which implies  $\gamma'(0) \in \alpha$ . This shows  $T_G \subset N$ . Then  $N = T_G = (G \cdot p) \times A$  where  $A := \exp_p(\alpha)$  is a totally geodesic submanifold of  $M$  associated to the Lie triple system  $\alpha$ .  $\square$

#### 4. Karpelevich's Theorem for $G$ a simple Lie group.

Here is the first step to prove Theorem 1.1.

**THEOREM 4.1.** *Let  $M$  be a Riemannian symmetric space of non positive curvature without flat factor. Then any connected, simple and non compact Lie subgroup  $G \subset \text{Iso}(M)$  has a minimal orbit  $G \cdot p \subset M$ .*

*Proof.* Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be a Cartan decomposition of the Lie algebra  $\mathfrak{g} := \text{Lie}(G)$  and let  $K \subset G$  be the maximal compact subgroup associated to  $\mathfrak{k}$ . Let  $\Sigma$  be the set of fixed points of  $K$ . Notice that  $\Sigma \neq \emptyset$  by Cartan's fixed point theorem. Since  $G$  is simple all  $G$ -orbits  $G \cdot x$  through points in  $x \in \Sigma$  are homothetic i.e. the Riemannian metric induced on  $G \cdot x$  and  $G \cdot y$  differ from a constant multiple for  $x, y \in \Sigma$ . Let  $x_0 \in \Sigma$  be a point in  $\Sigma$  and let  $g_0$  be the Riemannian metric on  $G \cdot x_0 = G/K$  induced by the Riemannian metric  $g = \langle \cdot, \cdot \rangle$  of  $M$ . So if  $y \in \Sigma$  the Riemannian metric  $g_y$  on  $G \cdot y$  is given by  $g = \lambda(y) \cdot g_0$ . Notice that if  $X \in \mathfrak{p}$  is unitary at  $x_0$  (i.e.  $g_0(X(x_0), X(x_0)) = 1$ ) then  $\lambda(y) = g(X(y), X(y)) = \|X(y)\|^2$ . We claim that  $\lambda(y)$  has a minimum in  $\Sigma$ . Indeed, if  $y_n \rightarrow \infty \in S_\infty(\Sigma) \subset S_\infty(M)$  ( $y_n \in \Sigma$ ) and  $\lambda(y_n) \leq \text{const}$  then the monoparametric Lie group  $\psi_t^X \subset G$  associated to any unitary  $X \in \mathfrak{p}$  at  $x_0 \in \Sigma$  must fix  $\infty \in S_\infty(\Sigma) \subset S_\infty(M)$ . Thus, since  $X \in \mathfrak{p}$  is arbitrary and  $\mathfrak{p}$  generate  $\mathfrak{g}$  we get that  $\infty \in S_\infty(\Sigma) \subset S_\infty(M)$  is a fixed point of  $G$ . This contradicts Corollary 2.3. So there exist  $y_0 \in \Sigma$  such that  $\lambda$  has a minimum. Notice that the volume element  $\text{Vol}_y$  of an orbit  $G \cdot y$  is given by  $\lambda^{\frac{n}{2}} \text{Vol}_{x_0}$ , where  $n = \dim(G/K)$ . Now a simple computation shows that the mean curvature vector of  $G \cdot y_0$  vanish and we are done.  $\square$

Now we are ready to prove Karpelevich's Theorem 1.1 for  $G$  a simple non compact Lie subgroup of  $Iso(M)$ .

**THEOREM 4.2.** *Let  $M$  be a Riemannian symmetric space of non positive curvature. Then any connected, simple and non compact Lie subgroup  $G \subset Iso(M)$  has a totally geodesic orbit  $G.p \subset M$ .*

*Proof.* According to Proposition 2.4 we can assume that  $M$  has no flat factor. Let  $i : (M, g) \hookrightarrow (\mathcal{P}, h)$  be a totally geodesic embedding as in Proposition 2.2. Notice that the pull-back metric  $i^*h$  can eventually differ (up to constant factors) from  $g$  on each irreducible De Rham factor of  $M$ . Anyway, totally geodesic submanifolds of  $(M, g)$  and  $(M, i^*h)$  are the same since totally geodesic submanifolds are defined in terms of the same Levi-Civita connection  $\nabla^g = \nabla^{i^*h}$ . Notice that  $G$  also acts by isometries on  $(M, i^*h)$ . Indeed,  $G$  can be also regarded as a subgroup of  $Iso(\mathcal{P})$ . Now Proposition 2.2 implies that  $G$  has a totally geodesic orbit  $G.p$  in  $\mathcal{P}$ . The above proposition shows that  $G$  has a minimal orbit  $G.y_0$  in  $(M, i^*h)$ . Since the embedding  $M \hookrightarrow \mathcal{P}$  is totally geodesic we get that the  $G$ -orbit  $G.y_0$  is also a minimal submanifold of  $\mathcal{P}$ . Then Proposition 3.1 implies that  $G.y_0$  is a totally geodesic submanifold of  $\mathcal{P}$ . Thus,  $G.y_0$  is a totally geodesic submanifold of  $(M, i^*h)$  and so  $G.y_0$  is also a totally geodesic submanifold of  $(M, g)$ .  $\square$

## 5. Karpelevich's Theorem.

Let  $G \subset Iso(M)$  be a semisimple, connected Lie group. Then the Lie algebra  $\mathfrak{g} = Lie(G) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  is a sum of a simple Lie algebra  $\mathfrak{g}_1$  and a semisimple Lie algebra  $\mathfrak{g}_2$ . Due to Cartan's fixed point theorem we can assume that each simple factor of  $\mathfrak{g}$  is non compact. We are going to make induction on the number of simple factors of the semisimple Lie algebra  $\mathfrak{g}$ . Let  $G_1$  (resp.  $G_2$ ) be the simple Lie group associated to  $\mathfrak{g}_1$  (resp. the semisimple Lie subgroup associated to  $\mathfrak{g}_2$ ). Let  $T_{G_1} \subset M$  be the union of the totally geodesic orbits of the simple subgroup  $G_1$  acting on  $M$ . Notice that Theorem 4.2 implies that  $T_{G_1} \neq \emptyset$  and Proposition 3.1 implies that  $T_{G_1} = (G_1 \cdot p) \times A$  is a totally geodesic submanifold of  $M$ , where  $G_1 \cdot p$  is a totally geodesic  $G_1$ -orbit. Notice that  $G_2$  acts on  $T_{G_1} = (G_1 \cdot p) \times A$ . Then  $\mathfrak{g}_2$  (or eventually a quotient  $\mathfrak{g}_2 / \sim$  of it) acts on  $A$ . Since  $A$  is symmetric space of non positive curvature we get (by induction) that the semisimple subgroup  $G_2$  (or eventually a quotient  $G_2 / \sim$  of it) has a totally geodesic orbit  $S \subset A$ . Then  $(G_1 \cdot p) \times S$  is a totally geodesic orbit of  $G$  and this finish our proof of Karpelevich's Theorem 1.1.

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