The exceptional set for the distribution of primes between consecutive powers

THE EXCEPTIONAL SET FOR THE DISTRIBUTION OF PRIMES BETWEEN CONSECUTIVE POWERS

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Abstract

A well known conjecture about the distribution of primes asserts that between two consecutive squares there is always at least one prime number. The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis. This paper is concerned with the distribution of prime numbers between two consecutive powers of integers, as a natural generalization of the afore-mentioned conjecture.

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[1]

1 Introduction

A well known conjecture about the distribution of primes asserts that all intervals of type $[n^2,(n+1)^2]$ contain at least one prime.

The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis. To get a conditional proof of the conjecture we need to assume a stronger hypothesis about the behavior of Selberg's integral in short intervals, see Bazzanella [2].

This paper is concerned with the distribution of prime numbers between two consecutive powers of integers, as a natural generalization of the above conjecture.

The well known result of Huxley about the distribution of prime in short intervals, see [11], implies that, for $n \to \infty$, all intervals $[n^\alpha,(n+1)^\alpha]$ contain the expected numbers of prime for $\alpha > \frac{12}{5}$ and "almost all" contain the expected numbers of prime for $\alpha > \frac{6}{5}$. By "almost all" we mean that the number of integers $X \le n \le 2X$ for which the interval $[n^\alpha,(n+1)^\alpha]$ does not contain the expected number of primes is $o(X)$. Our first result is an unconditional estimate for the measure of the exceptional set for the distribution of primes between two consecutive power of integers.

Theorem 1 *Let $\varepsilon > 0$. Then for every $[n^\alpha,(n+1)^\alpha] \subset [N,2N]$ with $O(N^{\eta(\alpha)+\varepsilon})$ exceptions we have the expected number of primes , where*

$$\eta(\alpha) = \begin{cases} \dfrac{8}{5\alpha} - \dfrac{1}{2} & \dfrac{6}{5} < \alpha \le \dfrac{6}{5} + c \\[2ex] \dfrac{5}{2\alpha} - 1 & \dfrac{27}{16} < \alpha \le \dfrac{53}{26} \\[2ex] \dfrac{72 - 9\alpha - 8\alpha^2}{3\alpha(\alpha+12)} & \dfrac{53}{26} \le \alpha < \dfrac{12}{5} \end{cases}$$

and c positive suitable constant.

For the sake of simplicity we consider the function $c(\alpha)$ only for the extremes and more interesting values of the parameter α. Arguing in the same way we can obtain the explicit values of the function $c(\alpha)$ for every α.
A corollary of this theorem is Theorem 1 of Bazzanella [2], which states that for every $[n^2,(n+1)^2] \subset [N,2N]$ with $O(N^{1/4+\varepsilon})$ exceptions we have the expected number of primes.

Assuming some hypotheses we can have strong result about the regularity of distribution of primes and then a better estimate for the number of exceptions for the distribution of primes between two consecutive powers of integers. First we assume the Lindelöf hypothesis, which states that, for every $\eta > 0$, the Riemann Zeta-function satisfies

$$\zeta(\sigma + it) \ll t^{\eta} \quad (\sigma \geq \frac{1}{2}, t \geq 2).$$

Assuming this hypothesis we can prove that all intervals $[n^{\alpha}, (n+1)^{\alpha}]$ contain the expected numbers of prime for $\alpha > 2$ and "almost all" contain the expected numbers of prime for $\alpha > 1$, see Bazzanella [1]. Our second result is a conditional estimate for the measure of the exceptional set for the distribution of primes between two consecutive powers of integers under the assumption of the Lindelöf hypothesis.

Theorem 2 *Assume the Lindelöf hypothesis, let* $\varepsilon > 0$ *and* $1 < \alpha \leq 2$. *Then for every* $[n^{\alpha}, (n+1)^{\alpha}] \subset [N, 2N]$ *with* $O(N^{2/\alpha - 1 + \varepsilon})$ *exceptions we have the expected number of primes.*

A corollary of this theorem is Theorem 2 of Bazzanella [1], which states that almost all intervals $[n^{\alpha}, (n+1)^{\alpha}]$ contain the expected number of primes for $\alpha > 1$.

Assuming The Riemann Hypothesis we can set our last result.

Theorem 3 *Assume the Riemann Hypothesis, let* $1 < \alpha \leq 2$ *and* $g(x) \to \infty$ *arbitrarily slowly.*

Then for every $[n^{\alpha}, (n+1)^{\alpha}] \subset [N, 2N]$ *with* $O(N^{2/\alpha - 1} \log^2 N \, g(N))$ *exceptions we have the expected number of primes.*

A corollary of this theorem is Theorem 2 of Bazzanella [2], which states that for every $[n^2, (n+1)^2] \subset [N, 2N]$ with $O(g(N) \log^2 N)$ exceptions we have the expected number of primes.

2 The necessary lemmas

First lemma is a result about the structure of the exceptional set for the asymptotic formula

$$\psi(x + h(x)) - \psi(x) \sim h(x) \text{ as } x \to \infty. \tag{1}$$

Let X be a large positive number, $\delta > 0$ and let $|\ \ |$ denote the modulus of a complex number or the Lebesgue measure of a set. Let $h(x)$ be an increasing function such that $x^{\varepsilon} \le h(x) \le x$ for some $\varepsilon > 0$,

$$\Delta(x,h) = \psi(x + h(x)) - \psi(x) - h(x)$$

and

$$E_{\delta}(X,h) = \{X \le x \le 2X : |\Delta(x,h)| \ge \delta h(x)\}.$$

It is clear that (1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_{\delta}(X,h) = \emptyset$ for $X \ge X_0(\delta)$. Hence for small $\delta > 0$, X tending to ∞ and $h(x)$ suitably small with respect to x, the set $E_{\delta}(X,h)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_{\delta}(X,h) \subset E_{\delta'}(X,h) \quad \text{if} \quad 0 < \delta' < \delta.$$

We will consider increasing functions $h(x)$ of the form $h(x) = x^{\theta+\varepsilon(x)}$, with some $0 < \theta < 1$ and a function $\varepsilon(x)$ such that $|\varepsilon(x)|$ is decreasing,

$$\varepsilon(x) = o(1) \quad \text{and} \quad \varepsilon(x+y) = \varepsilon(x) + O\left(\frac{|y|}{x}\right).$$

A function satisfying these requirements will be called of *type* θ.

First lemma provides the basic structure of the exceptional set $E_{\delta}(X,h)$.

Lemma 1 *Let $0 < \theta < 1$, $h(x)$ be of type θ, X be sufficiently large depending on the function $h(x)$ and $0 < \delta' < \delta$ with $\delta - \delta' \ge \exp(-\sqrt{\log X})$. If $x_0 \in E_{\delta}(X,h)$ then $E_{\delta'}(X,h)$ contains the interval $[x_0 - ch(X), x_0 + ch(X)] \cap [X, 2X]$, where $c = (\delta - \delta')\theta/5$. In particular, if $E_{\delta}(X,h) \ne \emptyset$ then*

$$|E_{\delta'}(X,h)| \gg_{\theta} (\delta - \delta')h(X).$$

Lemma 1 is part (i) of Theorem 1 of Bazzanella and Perelli, see [3], and essentially says that if we have a single exception in $E_{\delta}(X,h)$, with a fixed δ, then we necessarily have an interval of exceptions in $E_{\delta'}(X,h)$, with δ' little smaller than δ. The interesting consequence of this lemma is that we can use an average estimate to prove the non-existence of the exceptions.

Remainder lemmas provide conditional and unconditional bounds for the Selberg's integral and the fourth-power integral linked to the distribution of primes in short intervals.

Lemma 2 *Let $\varepsilon > 0$ then exists $c > 0$ such that*

$$\int_X^{2X} |\psi(x+\delta x) - \psi(x) - \delta x + B(x,\theta)|^2 dx \ll X^{(11+14\theta)/10+\varepsilon}$$

with

$$\frac{1}{6} < \theta < \frac{1}{6} + c,$$

$$B(x,\theta) \ll \frac{X^\theta}{\log X},$$

uniformly for $X^{\theta-1} \ll \delta \ll X^{\theta-1}$.

Proof.

In order to prove the lemma we use the classical explicit formula, see ch. 17 of Davenport [5], to write

$$\psi(x+\delta x) - \psi(x) - \delta x = \sum_{|\gamma|\le T} x^\varrho c_\varrho(\delta) + O\left(\frac{X \log^2 X}{T}\right), \tag{2}$$

uniformly for $X \le x \le 2X$, where $10 \le T \le X$, $\varrho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$,

$$c_\varrho(\delta) = \frac{(1+\delta)^\varrho - 1}{\varrho} \qquad \text{and} \qquad c_\varrho(\delta) \ll \min(X^{\theta-1}, \frac{1}{|\gamma|}). \tag{3}$$

Choose

$$T = X^{1-\theta} \log^3 X, \tag{4}$$

We use Theorem 3 of Halász-Turán [6], which asserts that there exists a constant $c_1 > 0$ such that

$$N(\sigma, T) \ll T^{(1-\sigma)^{3/2} \log^3 \frac{1}{1-\sigma}} \tag{5}$$

for $1 - c_1 \le \sigma \le 1$. From (3) - (5) and Vinogradov's zero-free region, see ch. 6 of Titchmarsh [14], by a standard argument we see that there exists a constant $c_2 > 0$ such that

$$\sum_{\substack{|\gamma|\le T \\ 1-c_2\le\beta\le 1}} x^\rho c_\rho(\delta) \ll X^{\theta-1} \log^2 X \max_{1-c_2\le\sigma\le 1} X^\sigma N(\sigma, T) \ll \frac{X^\theta}{\log X} \tag{6}$$

uniformly for $X \le x \le 2X$.

Again by a standard argument, from (3), (4) and Ingham-Huxley density estimate, we obtain

$$\int_X^{2X} \Big| \sum_{\substack{|\gamma|\le T \\ 1-c_2\le\beta\le 1}} x^\rho c_\rho(\delta)\Big|^2 dx \ll X^{2\theta-1+\varepsilon} \max_{1/2\le\sigma\le 1-c_2} X^{2\sigma} N(\sigma,T) \ll X^{(11+14\theta)/10+\varepsilon},$$

for $\theta > 1/6$ and sufficiently small. This completes the proof of the Lemma 2.

Lemma 3 *Let $\varepsilon > 0$ then*

$$\int_X^{2X} |\psi(x+\Delta x) - \psi(x) - \Delta x + B(x,\theta)|^4 dx \ll X^{\mu(\theta)+\varepsilon}$$

where

$$\mu(\theta) = \begin{cases} X^{(3+5\theta)/2} & \frac{11}{27} < \theta \le \frac{27}{53} \\ X^{(55+60\theta^2-108\theta)/(39-36\theta)} & \frac{27}{53} \le \theta < \frac{7}{12}. \end{cases}$$

and

$$B(x,\theta) \ll \frac{X^\theta}{\log X}$$

uniformly for $X^{\theta-1} \ll \Delta \ll X^{\theta-1}$.

Proof.

We proceed along similar lines of the Lemma 2, using fourth power moments instead of mean square estimates. Using The Ingham-Huxley density estimate we can prove that for every $\eta > 0$ we have

$$\sum_{|\gamma|\le T,\, \beta\notin I} x^\rho c_\rho(\Delta) \ll X^{\theta-1}\log^2 X \max_{1/2\le\sigma\le 1,\, \sigma\notin I} X^\sigma N(\sigma,T) \ll \frac{X^\theta}{\log X} \tag{7}$$

with

$$I = [a,b],$$

$$a = \max\left\{\frac{1}{2}, 3\theta - 1 - \eta\right\}$$

and

$$b = \begin{cases} \dfrac{(10-9\theta)}{7} + \eta & \dfrac{11}{27} < \theta \le \dfrac{27}{53} \\ \dfrac{4}{3} - \theta + \eta & \dfrac{27}{53} \le \theta < \dfrac{7}{12} \end{cases},$$

uniformly for $X \le x \le 2X$ and $11/27 < \theta < 7/12$.

We bound the contribution of the zeros such that $\beta \in I$ by a fourth power moment and we get

$$\int_X^{2X} |\sum_{|\gamma|\le T,\, \beta\in I} x^\rho c_\rho(\xi)|^4 dx \ll X^{4\theta-3+\varepsilon} \max_{\sigma\in I} X^{4\sigma} N^*(\sigma, T). \tag{8}$$

From Theorem 2 of Heath-Brown [8] we have

$$N^*(\sigma, T) \ll \begin{cases} T^{(10-11\sigma)/(2-\sigma)} \log^k T & \text{if } \frac{1}{2} \le \sigma \le \frac{2}{3} \\ T^{(18-19\sigma)/(4-2\sigma)} \log^k T & \text{if } \frac{2}{3} \le \sigma \le \frac{3}{4} \\ T^{12(1-\sigma)/(4\sigma-1)} \log^k T & \text{if } \frac{3}{4} \le \sigma \le 1, \end{cases} \tag{9}$$

where k is an absolute constant. Hence from (8) and (9) we obtain

$$\int_X^{2X} |\sum_{|\gamma|\le T,\, \beta\in I} x^\rho c_\rho(\xi)|^4 dx \ll X^{\mu(\theta)+\varepsilon},$$

with

$$\mu(\theta) = \begin{cases} X^{(3+5\theta)/2} & \dfrac{11}{27} < \theta \le \dfrac{27}{53} \\ X^{(55+60\theta^2-108\theta)/(39-36\theta)} & \dfrac{27}{53} \le \theta < \dfrac{7}{12}, \end{cases}$$

and then Lemma 3 follows.

Lemma 4 *Assume the Lindelöf hypothesis and let $\varepsilon > 0$. Then there exists a function $E(y, T)$ such that*

$$\int_X^{2X} \left|\psi\left(y + \frac{y}{T}\right) - \psi(y) - \frac{y}{T} + B(y,T)\right|^4 dy \ll X^{4+\varepsilon} T^{-3}$$

and

$$B(y,T) \ll \frac{y}{T \ln y}$$

uniformly for $X \geq 2, 1 \leq T \leq X$ *and* $X \leq y \leq 2X$.

Lemma 4 is Lemma B of Yu, see [15].

Lemma 5 *Assume the Riemann hypothesis, then for* $x \geq 4$

$$\int_X^{2X} |\psi(x+\theta x) - \psi(x) - \theta x|^2 dx \ll \theta X^2 \left(\log \frac{2}{\theta}\right)^2,$$

uniformly in $0 < \theta \leq 1$.

This last lemma is Lemma 5 of Saffari and Vaughan, see [13].

3 Proof of the Theorems

We define $H = (n+1)^\alpha - n^\alpha$ and

$$A_\delta(N,\alpha) = \{N^{1/\alpha} \leq n \leq (2N)^{1/\alpha} : |\psi((n+1)^\alpha) - \psi(n^\alpha) - H| \geq \delta H\}.$$

This set contains the exceptions, if any, to the expected asymptotic formula for the number of primes in intervals of the type $[n^\alpha, (n+1)^\alpha]$ in $[N, 2N]$.

The main step of the proofs is to connect the exceptional set $A_\delta(N,\alpha)$ with the exceptional set for the distribution of primes in short intervals and show that

$$|A_\delta(N,\alpha)| \ll \frac{E_{\delta/2}(N,h)}{N^{1-1/\alpha}} + 1 \tag{10}$$

for every $\delta > 0$, $\alpha > 1$ and $h(x) = (x^{1/\alpha}+1)^\alpha - x$.

In order to prove (10) we let $n \in A_\delta(N,\alpha)$ and $x = n^\alpha \in [N, 2N]$. From the definition of the set $A_\delta(N,\alpha)$ we get

$$|\psi((n+1)^\alpha) - \psi(n^\alpha) - H| \geq \delta H,$$

and then

$$|\psi(x+h(x)) - \psi(x) - h(x)| \geq \delta h(x),$$

which implies $x \in E_\delta(N,h)$. Using Lemma 1, with $\delta' = \delta/2$, we obtain that there exists an effective constant c such that

$$[x, x + ch(x)] \cap [N, 2N] \subset E_{\delta/2}(N,h).$$

Let $m \in A_\delta(N,\alpha)$, $m > n$. As before we can define $y = m^\alpha \in [N, 2N]$ such that

$$[y, y + ch(y)] \cap [N, 2N] \subset E_{\delta/2}(N,h).$$

Choosing $c < 1$ we find

$$y - x = m^\alpha - n^\alpha \geq (n+1)^\alpha - n^\alpha > ch(x),$$

and then

$$[x, x + ch(x)] \cap [y, y + ch(y)] = \emptyset.$$

Hence (10) is proved, since for every $n \in A_\delta(N,\alpha)$ and $x = n^\alpha$, with at most one exception, we have

$$[x, x + ch(x)] \subset [N, 2N].$$

Now we can conclude the proof of the theorems providing a suitable bound for the measure of the exceptional set $E_{\delta/2}(N,h)$. Let $g(x)$ a function such that $g(x) \to \infty$ arbitrarily slowly, for $x \to \infty$. Then we subdivide $[N, 2N]$ into $\ll g(N)$ intervals of type $I_j = [N_j, N_j + Y]$ with

$$N \leq N_j < 2N \quad \text{and} \quad Y \ll \frac{N}{g(N)}.$$

For every $x \in E_{\delta/2}(N,h)$ we have

$$|\psi(x + h(x)) - \psi(x) - h(x)| \gg N^{1-1/\alpha}$$

and then

$$|E_{\delta/2}(N,h)| N^{2-2/\alpha} \ll \int\limits_{E_{\delta/2}(N,h)} |\psi(x + h(x)) - \psi(x) - h(x)|^2 dx$$

$$\ll \sum_j \int\limits_{E^j_{\delta/2}(N,h)} |\psi(x + h(x)) - \psi(x) - h(x)|^2 dx \ ,$$

where $E^j_{\delta/2}(N,h) = E_{\delta/2}(N,h) \cap [N_j, N_j + Y]$. Choosing $T_j = N_j^{1/\alpha}/\alpha$ we can deduce

$$|E_{\delta/2}(N,h)| N^{2-2/\alpha} \ll \sum_j \int\limits_{E^j_{\delta/2}(N,h)} \left| \psi(x + \frac{x}{T_j}) - \psi(x) - \frac{x}{T_j} \right|^2 dx$$

$$\leq \sum_j \int_N^{2N} \left| \psi(x + \frac{x}{T_j}) - \psi(x) - \frac{x}{T_j} \right|^2 dx.$$

Hence

$$|E_{\delta/2}(N,h)| \ll N^{-2+2/\alpha} \sum_j \int_N^{2N} \left| \psi(x + \frac{x}{T_j}) - \psi(x) - \frac{x}{T_j} \right|^2 dx. \tag{11}$$

Follow the same line, but choosing the function $g(x) \ll N^\varepsilon$, for every $\varepsilon > 0$, we can also obtain

$$|E_{\delta/2}(N,h)| \ll N^{-2+2/\alpha} \sum_j \int_N^{2N} \left| \psi(x + \frac{x}{T_j}) - \psi(x) - \frac{x}{T_j} + B(x, T_j) \right|^2 dx, \tag{12}$$

and

$$|E_{\delta/2}(N,h)| \ll N^{-4+4/\alpha} \sum_j \int_N^{2N} \left| \psi(x + \frac{x}{T_j}) - \psi(x) - \frac{x}{T_j} + B(x, T_j) \right|^4 dx, \tag{13}$$

for every function $B(x,T)$ such that

$$B(x,T) \ll \frac{x}{T \log N}$$

uniformly for $1 \leq T \leq N$ and $N \leq x \leq 2N$.

Lemma 2 and (12) imply that

$$|E_{\delta/2}(N,h)| \ll N^{1/2+3/(5\alpha)+\varepsilon}. \tag{14}$$

Lemma 3 and (13) lead to

$$|E_{\delta/2}(N,h)| \ll N^{\tau(\alpha)+\varepsilon}, \tag{15}$$

with

$$\tau(\alpha)=\begin{cases}\dfrac{3}{2\alpha} & \dfrac{27}{16}\le\alpha\le\dfrac{53}{26}\\[2ex] \dfrac{36+24\alpha-5\alpha^2}{3\alpha(\alpha+12)} & \dfrac{53}{26}\le\alpha<\dfrac{12}{5}\end{cases}$$

Theorem 1 is consequence of (10), (14) and (15).

We now assume the Lindelöf hypothesis. Lemma 4 and (13) yields

$$|E_{\delta/2}(N,h)|\ll N^{-4+4/\alpha}\sum_j N^{4+\varepsilon}T_j^{-3}\ll N^{1/\alpha+2\varepsilon}. \tag{16}$$

From (10) and (16) we can conclude

$$|A_\delta(N,\alpha)|\ll\frac{E_{\delta/2}(N,h)}{N^{1-1/\alpha}}+1\ll\frac{N^{1/\alpha+2\varepsilon}}{N^{1-1/\alpha}}+1\ll N^{2/\alpha-1+2\varepsilon},$$

for every $1<\alpha\le 2$ and $\delta>0$. This completes the proof of Theorem 2.

Lastly we assume the Riemann hypothesis. Lemma 5 and (11) lead to

$$|E_{\delta/2}(N,h)|\ll N^{-2+2/\alpha}\sum_j T_j^{-1}N^2\log^2(2T_j)\ll N^{1/\alpha}\log^2 N\ g(N). \tag{17}$$

From (10) and (17) we can conclude

$$|A_\delta(N,\alpha)|\ll\frac{E_{\delta/2}(N,h)}{N^{1-1/\alpha}}+1\ll\frac{N^{1/\alpha}\log^2 N\ g(N)}{N^{1-1/\alpha}}+1\ll N^{2/\alpha-1}\log^2 N\ g(N),$$

for every $1<\alpha\le 2$ and $\delta>0$. Thus Theorem 3 is proved.

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