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Dynamical Superalgebra and Supersymmetry for a Many-Fermion System

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A linearization scheme is proposed for the general Hamiltonian of an interacting fermion system, consisting of a mean-field approximation in which pairing and umklapp play dominant roles. A variety of models emerge, characterized by a hierarchy of spectrum-generating algebras and superalgebras, of which some are supersymmetric. The superconducting phase transition is shown to be related to supersymmetry breaking.

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It has been recently emphasized that interacting many-fermion systems can be handled by a method which is similar to that introduced by Solomon.[1] The strategy adopted in this method amounts essentially to linearizing the Hamiltonian by a generalized Hartree-Fock approximation, which leads to a conventional pair-reduced mean-field model. This linearization results in an effective Hamiltonian which is an element of a Lie algebra $\mathcal{A}$, referred to as the spectrum-generating algebra; the rotation which effects the diagonalization of the Hamiltonian, leading to energy spectrum, is an automorphism of $\mathcal{A}$. The use of dynamical algebras permits a unified treatment of systems capable of simultaneously exhibiting more than one condensed phase.[2]

In this paper we show how, starting from a general many-fermion interacting Hamiltonian, the reduction process can be extended to include more terms than in the standard case, and there results a hierarchy of spectrum-generating algebras and superalgebras. The appearance of the latter is important not only for the diagonalization of the Hamiltonian; they also allow us to write the Hamiltonian in supersymmetric form.[3]

The general Hamiltonian H for a system of fermions interacting via a two-body potential V is given by

$$H=\sum_i \epsilon_i a_i a_i + \tfrac{1}{2} \sum_{i,j,l,k} \langle i,j\,|\,V\,|\,k,l\rangle a_i^\dagger a_j^\dagger a_l a_k. \qquad (1)$$

The index i on the annihilation operator a_i for a fermion includes both momentum and spin indices $i=(\mathbf{i},\sigma)$, and the operators satisfy $\{a_i,a_j^\dagger\}=\delta_{ij}$, $\{a_i,a_j\}=0$.

Methods of solution usually involve the reduction of the four-fermion term in the interaction to quadratic terms, a process which we refer to as linearization. The resulting approximate (mean field) Hamiltonian, being quadratic in $a_i,a_j^\dagger$, is then diagonalizable. One method of linearization is the following: We consider the interaction term in (1) as being the product of two operators A and B, and write the identity

$$AB=(A-\langle A\rangle)(B-\langle B\rangle)+\langle A\rangle B+A\langle B\rangle-\langle A\rangle\langle B\rangle. \qquad (2)$$

We may interpret $\langle A\rangle$ and $\langle B\rangle$ as being the respective expectations of the operators A and B in some state. If we only consider states in our model for which the first term on the right-hand side of (2) is "small" in some sense, then we may approximate

$$AB \sim \langle A\rangle B + A\langle B\rangle - \langle A\rangle\langle B\rangle. \tag{3}$$

However, this approximation is consistent only in the following circumstances: (i) A,B commute; then BA leads to the same linear approximation, with $\langle A\rangle,\langle B\rangle$ ordinary c numbers. (ii) A,B anticommute; then $BA=-AB$ implies that $\langle A\rangle,\langle B\rangle$ must be thought of as anticommuting variables, which anticommute with the operators A,B as well. However, in case (ii) it is in fact sufficient, for our purposes here, to demand that these variables anticommute with the operators, since the term $\langle A\rangle\langle B\rangle$ plays no role in the calculations, giving simply an additive constant in the Hamiltonian. For instance, if we consider the average values $\langle a_{k_1}\cdots a_{k_m}\rangle$ of a product of an odd number m of fermion operators a_k, this condition is implemented by imposing

$$\langle a_{k_1}\cdots a_{k_m}\rangle = \eta\alpha_{k_1,\ldots,k_m}, \tag{4}$$

where $\alpha_{k_1,\ldots,k_m}$ is a c number, and η a variable satisfying

$$\{\eta, a_k\}=0 \quad \text{for all } k, \quad \eta^2 = -\text{identity}. \tag{5}$$

We note from (5) that η is not a Grassmann variable.

In the conventional linearization referred to above, for example in the BCS model of superconductivity, where A and B are commuting fermion pairs, then case (i) is operative. We shall also consider the case where A and B are single or triple fermion operators, and then we shall have case (ii).

In case (i) reduction of the Hamiltonian (1) leads to a bilinear form which is an element of a Lie algebra, the spectrum-generating algebra of the model. In the BCS case, for example, on which the additional constraint of pairing in equal and opposite momentum and spin states is imposed, this leads essentially to su(2).[4] In case (ii), we are lead naturally to a superalgebra.

We first consider a model for an interacting electron gas in a crystal lattice, and retain only the following terms in the interaction of (1), which can be treated in the framework of case (i): (a)

$$\tfrac{1}{2}\textstyle\sum_{i,j}\langle i,-i\,|\,V\,|\,j,-j\rangle a_i^\dagger a_{-i}^\dagger a_{-j}a_j,$$

the Cooper-pairing terms, responsible for superconductivity, and (b)

$$\tfrac{1}{2}\textstyle\sum_{i,j}'\langle i,j\,|\,V\,|-j,-i\rangle a_i^\dagger a_j^\dagger a_{-i}a_{-j},$$

which refer to an umklapp process,[5] permitted if momentum is conserved modulo a vector of the reciprocal lattice L of the crystal; thus $2(\mathbf{i}+\mathbf{j})\in L$ (the prime on the above summation indicates this limitation).

With this approximation, the Hamiltonian (1) reduces to $H^{(1)}$, a direct sum of commuting single-k Hamiltonians, $H^{(1)}=\sum_k H_k^{(1)}$;

$$H_k^{(1)} = \epsilon_k(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + (\Delta_k a_k^\dagger a_{-k}^\dagger + u_k a_k^\dagger a_{-k} + \text{H.c.}) \tag{6}$$

after we use the linearization technique of case (i), taking $A=B^\dagger$ as $a_i^\dagger a_{-i}^\dagger$ and $a_i^\dagger a_{-i}$, successively [we have fixed $k=(\mathbf{k},\uparrow)$, $-k=(-\mathbf{k},\downarrow)$]. The Δ_k term is the conventional superconductivity one

$$\Delta_k = \tfrac{1}{2}\textstyle\sum_j \langle k,-k\,|\,V\,|\,j,-j\rangle\langle a_j a_{-j}\rangle,$$

while u_k arises from the type of umklapp process considered above,

$$u_k = \tfrac{1}{2}\textstyle\sum_j \langle k,j\,|\,V\,|-j,-k\rangle\langle a_j^\dagger a_{-j}\rangle.$$

If we disregard the umklapp term ($u_k=0$), then we recover the BCS reduced model, for which the spectrum-generating algebra is su(2), generated by

$$J_+ = J_-^\dagger = a_k^\dagger a_{-k}^\dagger, \quad J_z = \tfrac{1}{2}(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} - 1). \tag{7}$$

The additional operators in (6) generate a commuting su(2),

$$J'_+ = J'^\dagger_- = a_k^\dagger a_{-k}, \quad J'_z = \tfrac{1}{2}(a_k^\dagger a_k - a_{-k}^\dagger a_{-k}). \tag{8}$$

The spectrum-generating algebra of this model is therefore su(2)$\oplus$(2), and the spectrum is immediately obtainable since

$$H_k^{(1)} \to (\epsilon_k^2 + |\Delta_k|^2)^{1/2}(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} - 1) + |u_k|(a_k^\dagger a_k - a_{-k}^\dagger a_{-k}) \tag{9}$$

under a rotation in su(2)$\oplus$su(2) space. The interest in $H_k^{(1)}$ as given by Eq. (6) is that it may be regarded as the Hamiltonian of a model exhibiting spontaneously broken supersymmetry.

The principal model that we wish to consider in this Letter, whose dynamical algebra will be a superalgebra, includes additional terms coming from the following umklapp processes: (c)

$$\tfrac{1}{2}\textstyle\sum_{i,k}\langle i,-i\,|\,V\,|\,k,i\rangle a_i^\dagger a_{-i}^\dagger a_i a_k \quad (\mathbf{i}+\mathbf{k}\in L),$$

(d)

$$\tfrac{1}{2}\textstyle\sum_{i,k}\langle i,-i\,|\,V\,|\,k,-i\rangle a_i^\dagger a_{-i}^\dagger a_{-i} a_k \quad (\mathbf{i}-\mathbf{k}\in L),$$

and their Hermitean conjugates. We linearize these terms in the framework of case (ii), so that, for example, $a_i^\dagger a_{-i}^\dagger a_i a_k \sim \langle a_i^\dagger a_{-i}^\dagger a_i \rangle a_k + a_i^\dagger a_{-i}^\dagger a_i \langle a_k \rangle$, where, following the prescription discussed before, we have dropped the constant $\langle a_i^\dagger a_{-i}^\dagger a_i \rangle \langle a_k \rangle$, and the expectations $\langle\, \rangle$ are given by the expressions (4). Expectations such as $\langle a_k \rangle$ must be evaluated self-consistently, and will be nonzero in "supercoherent" states like $\exp\{\eta a_k^\dagger\} | 0 \rangle$.

By this method we obtain a new reduced Hamiltonian, $H^{(2)} = \sum_k H_k^{(2)}$, where the operators in $H_k^{(2)}$ are, in addition to those in $H_k^{(1)}$ (with $a_+ \equiv a_k$, $a_- \equiv a_{-k}$, $n_\pm \equiv a_\pm^\dagger a_\pm$),

$$\{a_+(1-n_-),\ a_-(1-n_+),\ a_+^\dagger(1-n_-),\ a_-^\dagger(1-n_+),\ n_+ a_-,\ n_- a_+,\ a_-^\dagger n_+,\ a_+^\dagger n_-\} = \{g_i \,|\, i=1,\ldots,8\}. \tag{10}$$

Together with the six operators of (7),(8),

$$\{\tfrac{1}{2}(n_+ + n_- - 1),\ a_+^\dagger a_-^\dagger,\ a_- a_+,\ \tfrac{1}{2}(n_+ - n_-), a_+^\dagger a_-,\ a_-^\dagger a_+\}, \tag{11}$$

these eight operators generate the fourteen-dimensional superalgebra $A(1,1) = \mathrm{su}(2|2)/\mathscr{C}I$, where $\mathscr{C}$ is the center.[6] The reduced Hamiltonian $H_k^{(2)}$ is a linear combination of elements of a representation of the compact real form of this superalgebra, with coefficients in the extended field $\mathbb{R}[\eta]$ ($\mathbb{R}[\eta] = \{a + \eta b \,|\, a,b \in \mathbb{R}\}$). The coefficients of the bosonic terms (11) are in the even part of $\mathbb{R}[\eta]$ (isomorphic with $\mathbb{R}$), while those of the fermionic terms (10) are in the odd part ($\eta\mathbb{R}$):

$$H_k^{(2)} = H_k^{(1)} + \eta \sum_{g_i \in \mathscr{F}(\mathrm{su}(2|2))} v_{g_i}(k) g_i, \tag{12}$$

where $\mathscr{F}(\mathrm{su}(2|2))$ denotes the fermionic sector of su(2|2) defined in (10), and if g_i is one of the first four elements in the set, e.g., $g_1 = a_+(1-n_-)$,

$$\eta v_{g_1}(k) = \textstyle\sum_{q \in L} \langle k+q, -(k+q) | V | k, -(k+q) \rangle \langle a_{k+q}^\dagger a_{-(k+q)}^\dagger a_{-(k+q)} \rangle, \tag{13}$$

whereas if g_i is one of the four remaining elements, e.g., $g_5 = n_+ a_-$,

$$\eta v_{g_5}(k) = \textstyle\sum_{g \in L} \langle k+q, k | V | -k \rangle \langle a_{k+q}^\dagger \rangle + \eta v_{g_1}(k). \tag{14}$$

The existence of a dynamical superalgebra permits us to derive in standard fashion both the spectrum of $H_k^{(2)}$ and the self-consistency equations from which the coefficients $v_{g_i}(k)$ can be determined. Here we report the results for a simplified version of the model, in which it is assumed that $v_{g_i} = v_i$ and $v_{g_{i+4}} = v_1 + v_2$, $i = 1, \ldots, 4$.

The eigenvalues λ_i of $H_k^{(2)}$ are the roots of the quartic equation

$$\Delta_k XY - Z\{(2\epsilon_k - \lambda)X + 2\Delta_k (v_1 + v_2)^2\} = 0, \tag{15}$$

where $X = X(\lambda) = \Delta_k(\epsilon_k - u_k - \lambda)$, $Y = Y(\lambda) = \Delta_k(\epsilon_k + u_k - \lambda)$, and $Z = Z(\lambda) = 2v_1^2 - \lambda(\epsilon_k + u_k - \lambda)$. The corresponding eigenvectors can be conveniently written as linear combinations of vectors $\{|n_+ n_-\rangle\}$, where n_+, n_- are occupation numbers of + and − particles, respectively, taking values in the set $\{0,1\}$, with coefficients in $\mathbb{R}[\eta]$:

$$|\lambda_i\rangle = N_i(|00\rangle - (Z_i/Y_i)|11\rangle + (\eta\Delta_k/X_iY_i)\{[v_1X_i - (v_1+v_2)Z_i]|10\rangle + [v_1X_i + (v_1+v_2)Z_i]|01\rangle\}), \quad i = 1, \ldots, 4, \tag{16}$$

where, for example, $X_i = X(\lambda_i)$ and N_i is the normalization factor. Letting v_1, v_2 go to zero, one recovers the eigenvalues and eigenvectors of $H_k^{(1)}$.

The self-consistency equations for v_1 and v_2, with use of the Gibbs ensemble for $H_k^{(2)}$, are

$$v_1 \sum_{i=1}^4 \exp\left(-\frac{\lambda_i}{k_\mathrm{B}T}\right) = \sum_{q \in L} \langle k+q, -(k+q) | V | k, -(k+q) \rangle \times \sum_{i=1}^4 N_i^2 \exp(-\lambda_i/k_\mathrm{B}T)(\Delta_k/X_1Y_i^2)[-v_1X_i(Y_i - Z_i) + (v_1+v_2)Z_i(Y_i+Z_i)], \tag{17}$$

$$v_2 \sum_{i=1}^4 \exp\left(-\frac{\lambda_i}{k_\mathrm{B}T}\right) = \sum_{q \in L} \langle k+q, k | V | -k, k \rangle \sum_{i-1}^4 N_i^2 \exp\left(-\frac{\lambda_i}{k_\mathrm{B}T}\right) \frac{\Delta_k Z_i}{X_i Y_i^2}[v_1X_i + (v_1+v_2)Z_i]. \tag{18}$$

Note that these equations no longer involve the anticommuting variable η. A detailed analysis of the solutions of (17), (18), and their physical meaning will be given elsewhere. The v_i play the role of fermionic order parameters; nonzero solutions indicate the existence of the corresponding fermionic phase, and a consequent lowering of the free energy below that given by the BCS model.

We note in passing that the fourteen operators of (10),(11) generate the fifteen-dimensional Lie algebra su(4) under *commutation*; the additional element generated being the operator $n_+ n_-$. However, since the coefficients of the fer-

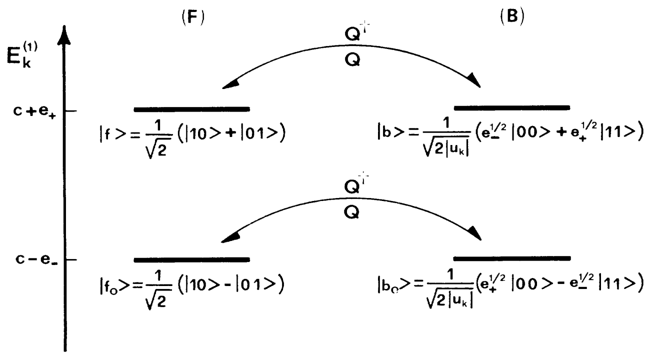

FIG. 1. The energy spectrum of $\tilde{H}_k^{(1)}$; $c=|\gamma|^2+|\delta|^2$, $e_\pm=|u_k|\pm(\epsilon_+\epsilon_-)^{1/2}$.

mionic terms in $H_k^{(2)}$ anticommute with the operators, it would not appear natural to consider su(4) as the dynamical Lie algebra of this model. In addition the su(2)$\oplus$su(2) Lie algebra of the Hamiltonian $H_k^{(1)}$ lies in the even component of the su(2|2) superalgebra; we now show how $H_k^{(1)}$ can be considered as a supersymmetric Hamiltonian in terms of charge operators $Q,Q^\dagger$ which lie in $\mathcal{F}$(su(2|2)).

In order to get the most general element of su(2)$\oplus$(2), we modify $H_\kappa^{(1)}$ of Eq. (6) to

$$\tilde{H}_k^{(1)}=\epsilon_+n_++\epsilon_-n_-+(\Delta_k a_+^\dagger a_-^\dagger+u_k a_+^\dagger a_-+\text{H.c.}) \tag{19}$$

by including a J_3' term $\epsilon'(n_+-n_-)$ (with $\epsilon_\pm=\epsilon_k\pm\epsilon'$; physically this would correspond to the inclusion of an external magnetic field). Now define the charge Q,

$$Q=\alpha a_+n_-+\beta a_-n_++\gamma a_+^\dagger(1-n_-)+\delta a_-^\dagger(1-n_+), \tag{20}$$

which is an element of $\mathcal{F}$(su(2|2)), and whose square is zero. Then, provided $|\alpha|^2-|\gamma|^2=\epsilon_-$, $|\beta|^2-|\delta|^2=\epsilon_+$, $\beta^*\gamma-\alpha^*\delta=\Delta_k$, and $\delta^*\gamma-\alpha^*\beta=u_k$, we may express the Hamiltonian (19), up to an additive constant $c=|\gamma|^2+|\delta|^2$, as

$$\tilde{H}_k^{(1)}=\{Q,Q^\dagger\},\quad\text{with } [\tilde{H}_k^{(1)},Q]=0. \tag{21}$$

In this case the potentials must satisfy $|u_k|^2=|\Delta_k|^2+\epsilon_+\epsilon_-$, and then we have a spontaneously broken supersymmetric model; $H_k^{(1)}$ satisfies (21), but $Q^\dagger|f_0\rangle\neq0$ and $Q|b_0\rangle\neq0$ [unless $c=|u_k|-(\epsilon_+\epsilon_-)^{1/2}$], where $\{|f_0\rangle,|b_0\rangle\}$ is the degenerate ground state of the model. In Fig. 1 we exhibit the spectrum of this system and give the states in the basis $\{|n_+n_1\rangle\}$. If we choose $\gamma=\delta=0$, so that $\Delta_k=0$ and $u_k=\exp(i\phi_k)(\epsilon_+\epsilon_-)^{1/2}$, then we obtain the following Hamiltonian:

$$H_k^{(0)}=\epsilon_+n_-+\epsilon_-n_++u_k a_+^\dagger a_-+u_k^* a_-^\dagger a_+. \tag{22}$$

This Hamiltonian describes a system with unbroken supersymmetry, because it is of the form $H_k^{(0)}=\{Q_0,Q_0^\dagger\}$, with $Q_0=\sqrt{\epsilon_-}a_+n_--\exp(-i\phi_k)\sqrt{\epsilon_+}a_-n_+$, $Q_0^2=0$, such that $[H_k^{(0)},Q_0]=0$ and both Q_0 and $O_0^\dagger$ annihilate the (degenerate) ground state $\{(1/\sqrt{2})(|10\rangle-|01\rangle, |00\rangle\}$.

Identifying Δ_k in (19) as the superconducting order parameter, the Hamiltonian $H^{(0)}$ describes a system above the critical temperature for pairing, where $\Delta_k=0$ and $|u_k|\neq0$. In this sense the superconducting transition may be considered as a spontaneous breaking of supersymmetry for this model. We note in passing that the spectrum-generating algebra of $H_\kappa^{(0)}$ is the algebra u(2) generated by

$$\{\tfrac{1}{2}(n_+n_--1),\tfrac{1}{2}(n_+-n_-),a_+^\dagger a_-,a_-^\dagger a_+\}.$$

The appropriate superalgebra for discussion of this model is that obtained by our adding to these bosonic elements the fermionic elements $\{a_+n_-,a_-n_+,n_-a_+^\dagger,n_+a_-^\dagger\}$ of Q_0 and $Q_0^\dagger$. The resulting superalgebra is su(2|1).[6] In Fig. 2 we give a diagram of the connections between the various Lie algebras and superalgebras of the models discussed here.

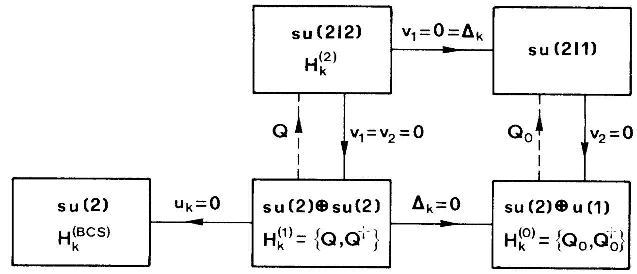

FIG. 2. Lie algebra-superalgebra chain.

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