Transitions between stable states in the dynamics of soil development.

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Transition between stable states in the dynamics of soil development

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Abstract.
The dynamics of soil development and erosion is studied through a stochastic soil mass balance in which the soil production by bedrock weathering is expressed as a state-dependent deterministic function, while the erosion by landslides is modeled as a marked Poisson process. For a range of values of the parameters the dynamics is bistable and the noise drives the transitions from a state to another through a potential barrier. The rates of such a transition are analytically estimated and their dependence on the system parameters is briefly discussed.

The analysis of the dynamics of soil development in a hillslope is crucially important to the understanding and modeling of the process of landscape evolution. This dynamics has been traditionally studied [*Kirkby*, 1971; *Ahnert*, 1988; *Dietrich et al*, 1995; *Roering et al.*, 1999] through a soil mass balance equation in which the variability in time of the soil thickness, h, was expressed as the difference between the rates of the process of soil production (at the interface between regolith and bedrock) and erosion (at the ground surface). The former is usually given by a deterministic function, of the soil depth [*Carson and Kirkby*, 1972; *Heimsath et al.*, 1997]. This approach has lead to important contributions to the modelling of the spatial [*Dietrich et al.*, 1995; *Roering et al.*, 1999] and temporal [*Carson and Kirkby*, 1972] variability of soil depth. When the landscape is subject to shallow landslides soil erosion is a discontinuous process, frequently controlled by the hydroclimatic (random) forcing [*Iida*, 1999]. A stochastic model of soil development has been recently suggested [*D'Odorico*, 2000] through a stochastic soil mass balance in which soil production is given by a deterministic function, $l(h)$, and erosion is modeled as a stochastic process in time, consisting of a sequence of marked Poissonian occurrences [e.g., *Wu and Swanston*, 1980], representing serious and abrupt erosive events (e.g., landslides). The deterministic function, $l(h)$, accounts also for those erosive processes (e.g., rainsplash, rainwash, etc.) which are always (gradually) active in time (at least if observed at the timescales of landscape evolution) and which are therefore suitable for a deterministic modeling. Thus the soil mass balance finally reads as

$$\frac{dh}{dt} = l(h) + f(t) \qquad (1)$$

where $l(h)$ (Fig. 1a) can be expressed [*D'Odorico*, 2000] as $l(h) = \frac{\rho_r}{\rho_s} P_0 \left(\frac{h}{d} + b\right)\left(1 - \frac{h}{d}\right)^2$ (for $0 \le h \le d$) and $f(t)$ is a marked Poisson process $f(t) = \sum_i \gamma_i \delta(t - \tau_i)$. In the above ρ_r and ρ_s are the bulk densities of the parent rock and the soil, respectively; P_0 measures the rate of bedrock weathering; d is the maximum thickness of the regolith above which the rates of soil production are negligible; b is a dimensionless parameter controlling the shape of $l(h)$; $\delta()$ is the Dirac-delta function; $t = \tau_i - \tau_{i-1}$ is an exponentially distributed random variable

$$\psi(t) = \lambda e^{-\lambda t} \qquad (2)$$

and γ_i is a random variable ($\gamma_i < 0$) with exponential distribution, $\Theta(\gamma_i) = -\frac{1}{\gamma} Exp\left(-\frac{\gamma_i}{\gamma}\right)$. To maintain the bound at $h = 0$, $\Theta(\gamma_i)$ is more properly represented as a state-dependent distribution

$$\Theta(\gamma) = \begin{cases} -\frac{1}{\gamma} e^{-\frac{\gamma_i}{\gamma}} & -h < \gamma_i \le 0 \\ \delta(\gamma_i - h) e^{\frac{h}{\gamma}} & \gamma_i \le -h. \end{cases} \qquad (3)$$

The rationale behind the above soil production function (Fig. 1a) comes from the observation that above a certain value of the soil thickness, say d, the rates of bedrock weathering are negligible [e.g., *Ahnert*, 1988; *Dietrich et al*, 1995; *Heimsath et al.*, 1997]. At the same time the biogenic mechanical weathering by tree roots is weaker in shallow soils, where significant vegetation seldom grows [*Carson and Kirkby*, 1972]. The dynamics discussed in this paper is bistable only when the soil production function has the characteristics described above (i.e. a maximum for a positive value of h). Equations (1-3) represent a stochastic differential problem, that has been integrated [*Rodriguez-Iturbe et al.*, 1999; *D'Odorico*, 2000], providing the steady-state probability distribution of soil thickness, h. It was observed that there are (likely) ranges of values of the pa-

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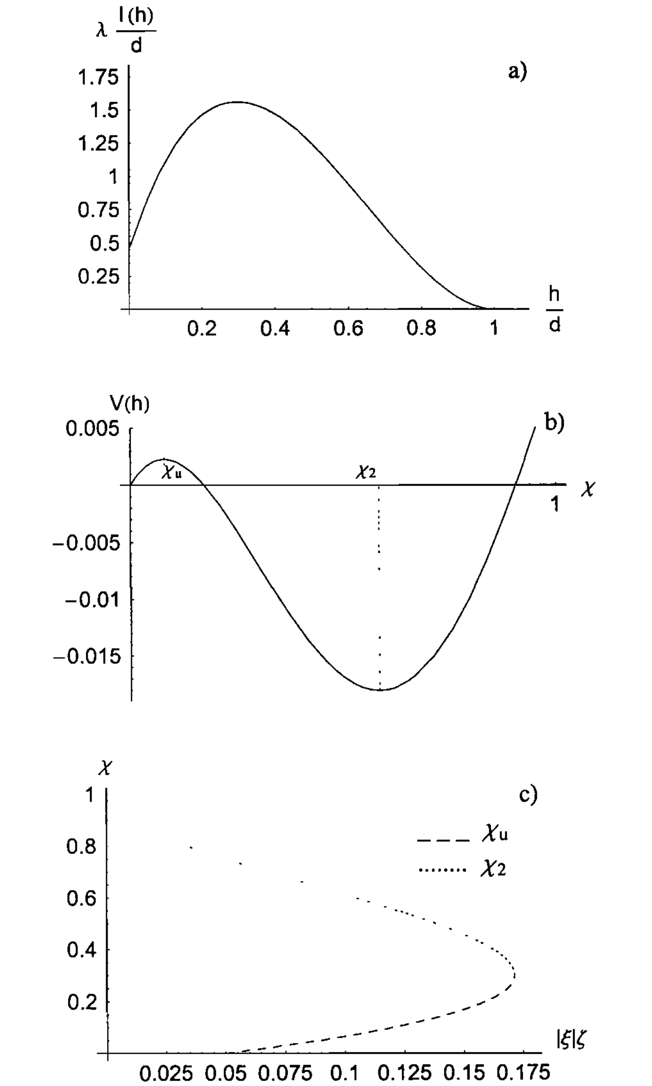

Figure 1. a) Example of a possible soil production function: $L(\frac{h}{d}) = \frac{\lambda l(h)}{d}$ ($\zeta = 0.1$; $b = 0.05$). b) Non-dimensional potential function ($b = 0.05$; $\zeta = 0.1$; $\xi = -1$); c) States of equilibrium - χ_2 (stable) and χ_u (unstable) - as a function of $|\xi\zeta|$, for $b = 0.05$.

rameters in which such a distribution may be bimodal with the implication that the system has two preferential states corresponding to *weathering-limited* and *transport-limited* dynamics [*Gilbert*, 1877] and that, because of the random forcing, the system may fluctuate between these two regimes, without ever (asymptotically) converging to any steady-state condition.

In some bistable systems the dynamics can be studied in terms of a potential function [*Porrà and Masoliver*, 1993]: the existence of two preferential states means that the potential has two minima (stable states) or *potential wells*, separated by a maximum (unstable state) or *potential barrier*. In the absence of the external forcing the system would tend to either one of the potential wells, depending on the initial condition. The noise induces the transition of the dynamics between the two preferential states through the potential barrier (noise-induced transitions). The properties of such a dynamics are generally characterized through the rates of these transitions [*Porrà and Masoliver*, 1993].

In this paper we study the rates of transition from a stable state to another as well as their dependence on the rates of the processes of soil production and removal. Normalizing h by d in equation (1) we have

$$\frac{d\chi}{dt} = \lambda L(\chi) + F(t) \qquad (4)$$

where $\chi = \frac{h}{d}$, $F(t) = \sum_i \xi_i \, \delta(t - \tau_i)$, $L(\chi) = \frac{1}{\zeta}(\chi + b)(1 - \chi)^2$; with $\xi = \frac{\gamma}{d}$, $\xi_i = \frac{\gamma_i}{d}$ and $\zeta = \frac{\rho_s}{\rho_r}\frac{\lambda d}{P_0}$. ξ (ξ_i) represents the ratio between the average (actual) landslide depth and the maximum thickness of the regolith; $|\xi\zeta|$ measures the ratio between rates of erosion due to landsliding, and soil production due to bedrock weathering.

A potential function can be defined for h [*D'Odorico*, 2000] as (see equation (1))

$$v(h) = -\int_h (l(h) + \langle f(t)\rangle) \; dh = -\int_h (l(h) + \gamma\lambda) dh \quad (5)$$

in a way that $-v'(h)$ gives the average rate of change of the soil thickness (i.e. $\langle\frac{dh}{dt}\rangle$); in dimensionless terms, equation (5) becomes

$$V(\chi) \propto -\int_h \left[\left(\frac{h}{d} + b\right)\left(1 - \frac{h}{d}\right)^2 + \xi\zeta\right] dh =$$
$$\left(b - \frac{1}{2}\right)\chi^2 - \frac{1}{3}(b-2)\chi^3 - \frac{\chi^4}{4} - \chi(b+\xi\zeta) \qquad (6)$$

Fig. 1b shows a plot of (6) for an assigned set of parameters: the potential has a maximum at χ_u and two minima at $\chi_1 = 0$ and χ_2. Values of χ_u and χ_2 are shown in Fig. 1c as functions of ξ and ζ. Notice how for $|\xi\zeta| > r \approx 0.172$, $V(h)$ has only one minimum (at $\chi_1 = 0$), corresponding to a weathering-limited dynamics with completely denuded ground.

The rates of transition from χ_1 to χ_2 (*forward transition*; Fig. 2a) can be studied following the approach by [*Masoliver*, 1987; *Porrà and Masoliver*, 1993], leading to the equation

$$\lambda T_f = \frac{1}{\xi}\int_{\chi_2}^{\chi_1} \frac{d\chi}{L(\chi)} e^{M(\chi)} \int_{-\infty}^{\chi} d\chi' \; e^{-M(\chi')} \qquad (7)$$

where T_f is the average time needed for a forward transition (i.e. $\chi_1 \to \chi_2$), and $M(\chi) = \frac{\chi}{\xi} + \int_\chi \frac{d\chi'}{L(\chi')} = \frac{\chi}{\xi} + \zeta\left[\frac{1}{(1+b)(1-\chi)} + \frac{1}{(1+b)^2} Log\left(\frac{b+\chi}{1-\chi}\right)\right]$.

The average time of *backward transition* (i.e. from χ_2 to χ_1; Fig. 2b) can be estimated in a similar way [*Masoliver*, 1987; *Porrà and Masoliver*, 1993] leading to the equation

$$\lambda T_b = \frac{1}{\xi}\int_{\chi_1}^{\chi_2} \frac{d\chi}{L(\chi)} \; e^{M(\chi)} \int_{1}^{\chi} d\chi' \; e^{-M(\chi')}$$
$$+ \left(1 + \frac{1}{\xi}\int_{1}^{\chi_1} d\chi \; e^{-[M(\chi) - M(\chi_1)]}\right). \qquad (8)$$

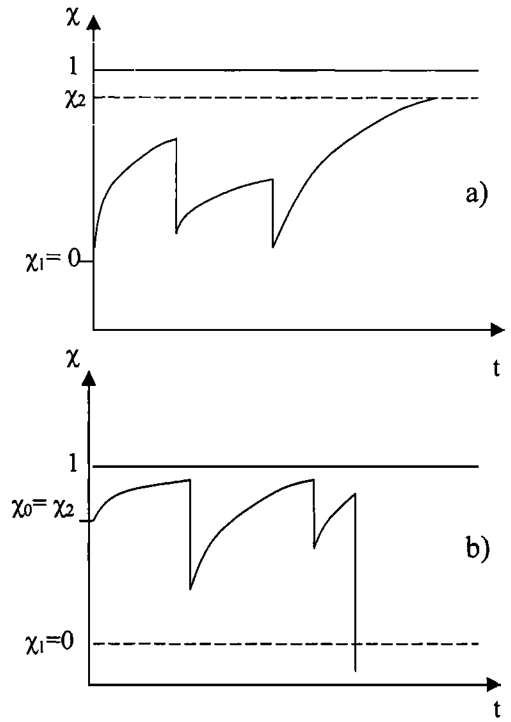

Figure 2. Forward (a) and backward (b) transitions between the two stable states, χ_1 and χ_2: The trajectories between the two stable states consist of temporal intervals of soil development interrupted by instantaneous erosive events.

The above analytical solutions have been derived without accounting for the bound that the dynamics has at $h = 0$ (i.e., $\chi = 0$). Nevertheless it is straightforward to observe that the existence of such a bound (corresponding to a condition of completely denuded hillslope) may affect only the rates of forward transition from χ_1 to χ_2 and not vice versa. In fact, the backward transitions (Fig. 2b) occur only as random jumps (erosion), here modeled by the Poisson process. The bound at $h = 0$ (completely denuded ground) would be crossed at the same time as the level $h_1 = d\,\chi_1$ (in fact $\chi_1 = 0$), without affecting the estimation of the average time needed to reach χ_1 starting from χ_2. Nevertheless, the existence of the bound at $h = 0$ affects in general the rates of forward transition from χ_1 to χ_2. To account for this, a "modified" soil production function, $L'(\chi)$ can be used

$$L'(\chi) = \begin{cases} L(\chi) & \chi \geq 0 \\ c\chi + b & \chi < 0. \end{cases} \qquad (9)$$

If $|c|$ is large enough, $\chi = 0$ becomes a reflecting barrier since a (fictitious) instantaneous (i.e. $c \to -\infty$) soil production would keep χ in the positive domain almost always with the exception of a finite number of instants. For negative values of χ the function $M(\chi)$ becomes $M(\chi) = \frac{\chi}{\xi} + \frac{Log(\chi)}{c}$; for $c \to -\infty$ $M(\chi) \to \frac{\chi}{\xi}$ and equation (7) reads as

$$\lambda T_f = \frac{1}{\xi}\int_{\chi_2}^{\chi_1} \frac{d\chi}{L(\chi)} e^{M(\chi)} \left[\int_0^{\chi} d\chi'\, e^{-M(\chi')} - \xi\right]. \qquad (10)$$

Fig. 3a shows the average time, T_f, needed for a forward transition (equation (10)) between the stable states of the system; λT_f is an increasing function of ζ in a wide range of values of ζ ($0 < \zeta < \zeta^+$): for given λ and ξ, high rates of soil production, P_0, lead to faster (forward) transitions to the transport-limited regime. At the same time, given P_0, despite the increase of λT_f, with increasing values of λ, T_f is a decreasing function of λ. This fact can be better understood by recalling that the values of χ_2 and χ_u are functions of λ. For given P_0 and ξ, larger values of λ correspond to lower values of χ_2 and shorter trajectories from χ_1 to χ_2 (Fig. 1c).

λT_f has a maximum (marked by "+" in Fig. 3a) for $\zeta = \zeta^+$ and, then, for $\zeta > \zeta^+$ it decreases with increasing values of ζ. In fact, also the height of the potential barriers for both forward and backward tran-

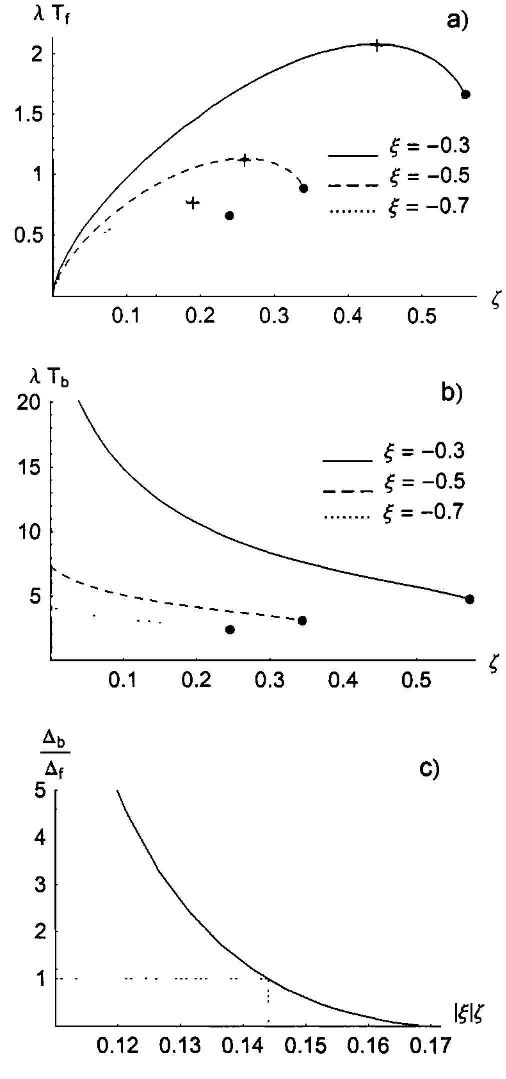

Figure 3. Average (normalized) times, λT, of forward (a) and backward (b) transitions as a function of the system parameters. The black points correspond to values of ζ above which $|\xi\zeta| > r$ and the system is no more bistable (see Fig. 1c); (c) Ratio between the heights of the potential barrier in the backward and forward transitions.

sitions ($\Delta_f = V(\chi_u) - V(\chi_1)$ and $\Delta_b = V(\chi_u) - V(\chi_2)$, respectively) are functions of $|\xi\zeta|$ (Fig. 3c): increasing values of $|\xi\zeta|$ correspond to decreasing values of Δ_f with respect to Δ_b, a condition that favors the forward transitions and explains the decrease of T_f for $\zeta > \zeta^+$; in fact, for $\zeta = \zeta^+$, $|\xi\zeta| \approx 0.144$ and the barrier is the same in both directions ($\Delta_f = \Delta_b$).

For a given ζ, low values of ξ (i.e. low rates of erosion) should favor the forward transitions (see Fig. 3c), being $\Delta_f << \Delta_b$. However, lower values of ξ correspond to long times of forward transition because of the larger distance between $\chi_1 = 0$ and χ_2 (Fig. 1c). Vice versa, high rates of soil erosion (i.e. high λ and $|\gamma|$) shorten the distance between the two stable states, leading to faster transitions in both directions.

The average time of backward transition is a decreasing function of both ξ and ζ (Fig. 3b). For given rates of soil erosion (i.e. λ and $|\gamma|$) slow processes of soil production (i.e. small P_0's) correspond to shorter times of backward transition and vice versa. Similarly, for a given P_0, high rates of soil erosion (either λ or $|\gamma|$) correspond to lower values of λT_b (and to even lower T_b's). This is due to the interplay between losses (erosion) and yields (production) in the soil mass balance (as it is explained, for example, by the dependence of the height of the potential barrier on the by-product $|\xi\zeta|$; see Fig. 3c) but also on the shifting of the point of stable equilibrium along the χ axis for different values of ξ and ζ (Fig. 1c).

For a given value of $|\xi\zeta|$ (i.e. for given Δ_b/Δ_f) and $|\chi_2 - \chi_1|$ high values of ξ (i.e. low values of ζ) correspond to shorter times of forward and backward transition. In fact, a backward transition occurs when, because of a strong erosion (high ξ), the system overcomes the potential barrier and falls in the potential well of χ_1. This leads the system to the complete denudation because of both the erosion and the lower rates of production (i.e. $L(\chi)$). At the same time a forward transition can be slowed down more by strong than by frequent events, and this can be explained again on the basis of the state-dependence of the rates of soil production through the function $L(\chi)$.

Conclusions

A model of soil development is here studied through a stochastic soil mass balance in which the soil production is expressed by a state-dependent deterministic function, while the soil erosion by landslides is represented by a marked Poisson process. The system is shown to be bistable in a wide range of values of the parameters and the two preferential states correspond to transport limited and weathering-limited dynamics. The average times of transition between the two stable states are analytically estimated and their dependence on the dynamical parameters is briefly discussed. The rates of backward transition are higher either with more frequent or with more intense erosive events. The existence of a bound for the process of soil erosion (the amount of removable soil can not exceed that of the soil effectively present) as well as the dependence of the distance between the stable states on the parameters of the system make the overall dynamics extremely non-linear and strongly affect the rates of transition between such states. This explains the possibility of having shorter forward transitions even with higher rates of erosion.

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