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ROBUST *A POSTERIORI* ERROR ESTIMATES FOR FINITE ELEMENT DISCRETIZATIONS OF THE HEAT EQUATION WITH DISCONTINUOUS COEFFICIENTS *

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Abstract. In this work we derive *a posteriori* error estimates based on equations residuals for the heat equation with discontinuous diffusivity coefficients. The estimates are based on a fully discrete scheme based on conforming finite elements in each time slab and on the A-stable θ-scheme with $1/2 \leq \theta \leq 1$. Following remarks of [Picasso, *Comput. Methods Appl. Mech. Engrg.* **167** (1998) 223–237; Verfürth, *Calcolo* **40** (2003) 195–212] it is easy to identify a time-discretization error-estimator and a space-discretization error-estimator. In this work we introduce a similar splitting for the data-approximation error in time and in space. Assuming the *quasi-monotonicity* condition [Dryja *et al., Numer. Math.* **72** (1996) 313–348; Petzoldt, *Adv. Comput. Math.* **16** (2002) 47–75] we have upper and lower bounds whose ratio is independent of any meshsize, timestep, problem parameter and its jumps.

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1. Introduction

In many practical applications a heat conduction problem involving a non homogeneous medium, made for example by different materials, has to be solved. To deal with these situations, we propose robust *a posteriori* error estimates for the heat equation with discontinuous, piecewise constant coefficients based on a discretization by conforming finite elements and the classical A-stable θ-scheme with $1/2 \leq \theta \leq 1$.

Since the pioneering work of Babuška and Rheinboldt [1] *a posteriori* error estimates and adaptive algorithms have become an important field for scientific computing [2, 7, 10, 12, 16, 18] and many works were devoted to parabolic problems [3, 9, 14, 17].

The estimator here derived is based on equation residuals as in [3, 14, 17]. In [14] residual based error estimators bound the error measured in the norm $\int_{t^{[n-1]}}^{t^{[n]}} \| \nabla . \|_0^2 \mathrm{d}t$ from above and from below; the implicit Euler scheme with linear finite elements is considered and only refinement is allowed; further, a condition between the meshsize and the timestep-length has to be satisfied. In [17] residual based error estimators bounding the error containing also the term $\int_{t^{[n-1]}}^{t^{[n]}} \| \frac{\partial .}{\partial t} \|_{-1}^2 \mathrm{d}t$ are presented for constant diffusivity coefficients. The proof of

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the upper bound is performed deriving an upper bound of the error estimator in terms of a "time functional residual" and a "space functional residual". In [3] linear and semilinear heat equations are considered: first, a semidiscretization in time by implicit Euler scheme is introduced; second, a discretization in space by finite elements is performed. This approach is considered in order to uncouple as much as possible the time and the space errors. Continuously differentiable diffusivity coefficients are considered.

In our work we consider the case of discontinuous coefficients. We use a full discretization approach instead of considering a semidiscrete formulation as in [3]. Our estimates allow us to perform a control of the space-discretization used in each time slab and of the timestep-length. The ratio between the upper and the lower bounds for the error is independent on any meshsize, timestep, diffusivity parameter and its jumps across the domain. To ensure this robustness with respect to the parameters jumps the *quasi-monotonicity* condition [8,13] is assumed. In the proof of the lower bound we introduce an orthogonal space of edge bubble functions. In our estimates we consider the data-approximation error and we propose a splitting of this error in two terms: a data-approximation error in space and a data-approximation error in time; this splitting can be used in the adaptation of the mesh and in the choice of the timestep-length in each time slab.

In Section 4 we present some numerical results on uniform meshes and constant timestep-lengths to carefully analyze the behaviour of the effectivity indices and prove robustness of the estimates. These results also confirm that the terms forming the error estimator and the data-approximation error mainly depend either on the space discretization or on the time discretization. This splitting can be very useful in an adaptive algorithm to adapt mesh and timestep-length. A simple adaptive strategy and some preliminary numerical results are proposed in the Appendix.

2. The heat equation

2.1. The continuous problem

Let Ω be a polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and let $(0,\Xi)$ be the time interval of interest. For any $f\in \mathrm{L}^2(0,\Xi;\mathrm{L}^2(\Omega))$ and $u^{[0]}\in\mathrm{L}^2(\Omega)$, we want to find $u:\Omega\times(0,\Xi)\to\mathbb{R}$ such that

$$\begin{aligned} \frac{\partial u}{\partial t} - \nabla\cdot(\kappa\nabla u) &= f, && \text{in } \Omega\times(0,\Xi), && (1)\\ u(x,t) &= 0, && \text{on } \partial\Omega\times(0,\Xi), && (2)\\ u(x,0) &= u^{[0]}(x), && \text{in } \Omega. && (3)\end{aligned}$$

The diffusivity parameter $\kappa(x)$, $0<\kappa_{min}\le\kappa\le\kappa_{Max}<\infty$, is a function constant in time and piecewise constant on the polygonal subdomains Ω_d, $d=1,\dots,D$, with $\cup_{d=1}^{D}\overline{\Omega}_d=\overline{\Omega}$ and $\Omega_i\cap\Omega_j=\emptyset$, $\forall i\neq j$.

Setting $W=\left\{w\in\mathrm{L}^2(0,\Xi;\mathrm{H}^1_0(\Omega))\ :\ \frac{\partial w}{\partial t}\in\mathrm{L}^2(0,\Xi;\mathrm{H}^{-1}(\Omega))\right\}$ the variational formulation of the above problem is: *Find* $u\in W$ *such that* $u(.,0)=u^{[0]}$ *and*

$$\left\langle\frac{\partial u}{\partial t},v\right\rangle+(\kappa\nabla u,\nabla v)=(f,v),\quad\forall v\in\mathrm{H}^1_0(\Omega),\ \text{a.e. } in\ (0,\Xi). \tag{4}$$

Here $\langle\,.\,,\,.\,\rangle$ stands for the duality pairing between $\mathrm{H}^{-1}(\Omega)$ and $\mathrm{H}^1_0(\Omega)$,$(\,.\,,\,.\,)$ is the usual inner product in $\mathrm{L}^2(\Omega)$. If $u\in W$, then $u\in\mathcal{C}^0([0,\Xi];\mathrm{L}^2(\Omega))$ and the initial condition $u(.,0)=u^{[0]}$ is meaningful in $\mathrm{L}^2(\Omega)$.

2.2. The numerical discretization

Let us consider a partition of $(0,\Xi)$ into subintervals $\left(t^{[n-1]},t^{[n]}\right)$ of length $\Delta t^{[n]}=t^{[n]}-t^{[n-1]}$, with $0=t^{[0]}<t^{[1]}<\cdots<t^{[N]}=\Xi$; set $I^{[n]}=\left[t^{[n-1]},t^{[n]}\right]$. In each time-slab $\Omega\times I^{[n]}$, $n\ge1$, we consider a regular family of partitions $\mathcal{T}_h^{[n]}$ of $\overline{\Omega}$ into triangles $T\in\mathcal{T}_h^{[n]}$ which satisfy the usual conformity and minimal-angle conditions [5], we denote by $h_T^{[n]}$ the diameter of each element $T\in\mathcal{T}_h^{[n]}$ and by $h^{[n]}$ the maximum of $h_T^{[n]}$

over all the elements $T \in \mathcal{T}_h^{[n]}$. From now on the subscript h stands for $h^{[n]}$. We assume that each triangulation $\mathcal{T}_h^{[n]}$ induces triangulations $\mathcal{T}_{h,d}^{[n]}$ of the subdomains Ω_d, $d = 1,..,D$, such that $\mathcal{T}_h^{[n]} = \cup_{d=1}^{D} \mathcal{T}_{h,d}^{[n]}$. Let $\mathrm{V}_h^{[n]} = \left\{ v_h \in \mathrm{H}_0^1(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) : \ v_{|T} \in \mathbb{P}_k(T), \ \forall T \in \mathcal{T}_h^{[n]} \right\} \subset \mathrm{V} = \mathrm{H}_0^1(\Omega)$ be a family of conforming finite element spaces based on the partitions $\mathcal{T}_h^{[n]}$. We denote by $\mathbb{P}_k(T)$ the space of polynomials of degree $k \geq 1$ on the element $T \in \mathcal{T}_h^{[n]}$. Assuming $u^{[0]} \in \mathrm{L}^2(\Omega)$, we define $u_{h,\Delta t}^{[0]} = P_{h,k}^{[1]} u^{[0]}$ to be the $\mathrm{L}^2(\Omega)$-projection of $u^{[0]}$ on the finite element space $\mathrm{V}_h^{[1]}$ defined on $\mathcal{T}_h^{[1]}$. The subscript $h, \Delta t$ is used to refer to the full discretization in space and in time. Then, we introduce the discretization based on the classical θ-scheme for the time integration: *Find* $u_{h,\Delta t}^{[n]} \in \mathrm{V}_h^{[n]}$ *such that* $\forall v_h \in \mathrm{V}_h^{[n]}$, $n = 1, ..., N$

$$\begin{aligned}\left(\frac{u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]}}{t^{[n]} - t^{[n-1]}}, v_h \right) + \theta \left(\kappa \nabla u_{h,\Delta t}^{[n]}, \nabla v_h \right) + (1-\theta) \left(\kappa \nabla u_{h,\Delta t}^{[n-1]}, \nabla v_h \right) \\ = \theta \left(\Pi_T f^{[n]}, v_h \right) + (1-\theta) \left(\Pi_T f^{[n-1]}, v_h \right), \qquad \forall v_h \in \mathrm{V}_h^{[n]}. \end{aligned} \tag{5}$$

In the last scalar products of the previous equation we assume that $f \in \mathcal{C}^0([0,\Xi]; \mathrm{L}^2(\Omega))$ and we set $f^{[r]} = f(.,t^{[r]})$, $r \in \{n-1, n\}$. Moreover we introduce an arbitrary piecewise polynomial approximation $\Pi_T f$ of the data f. If the initial condition $u^{[0]}$ belongs to $\mathcal{C}^0(\overline{\Omega})$, instead of the projection operator $P_{h,k}^{[1]}$, we can use the interpolation operator $\pi_{h,k}^{[1]} : \mathcal{C}^0(\overline{\Omega}) \to \mathrm{V}_h^{[1]}$.

At last we define the continuous, piecewise affine in time approximation of the solution $u(.,t)$:

$$u_{h,\Delta t}(x,t) = \frac{t - t^{[n-1]}}{t^{[n]} - t^{[n-1]}} u_{h,\Delta t}^{[n]}(x) + \frac{t^{[n]} - t}{t^{[n]} - t^{[n-1]}} u_{h,\Delta t}^{[n-1]}(x), \qquad x \in \Omega, \qquad t \in I^{[n]}, \qquad n = 1, ..., N. \tag{6}$$

3. A residual-based *a posteriori* error estimator

In this section we derive a residual-based error estimator for our fully discretized model problem following the work in [14, 17]. In particular, we shall derive a global-in-space local-in-time upper and lower bounds. At first, we introduce some notation which will be used for the construction of the estimator.

3.1. Definitions and general results

For each time-slab $\Omega \times I^{[n]}$ we define a partitions $\mathcal{T}_h^{[n-1,n]}$ that is a common refinement of $\mathcal{T}_h^{[n-1]}$ and $\mathcal{T}_h^{[n]}$, satisfying conformity and minimal angle condition and a *transition condition* or *moderate coarsening condition* [17]: there exists a constants C_{tr} such that

$$\sup_{n=1,\ldots,N} \sup_{T \in \mathcal{T}_h^{[n]}} \sup_{T^* \in \mathcal{T}_h^{[n-1,n]} : T^* \subseteq T} \frac{h_T^{[n]}}{h_{T^*}^{[n-1,n]}} \leq C_{tr}, \tag{7}$$

being $h_{T^*}^{[n-1,n]}$ the diameter of the element $T^* \in \mathcal{T}_h^{[n-1,n]}$. For any $T \in \mathcal{T}_h^{[n-1,n]}$ we denote by $\mathcal{E}(T)$ the set of its edges; we denote by $\mathcal{E}_h^{[n-1,n]} = \cup_{T \in \mathcal{T}_h^{[n-1,n]}} \mathcal{E}(T)$ the set of all edges of the triangulation $\mathcal{T}_h^{[n-1,n]}$. Moreover, we split $\mathcal{E}_h^{[n-1,n]}$ into the form $\mathcal{E}_h^{[n-1,n]} = \mathcal{E}_{h,\Omega}^{[n-1,n]} \cup \mathcal{E}_{h,\partial\Omega}^{[n-1,n]}$ with $\mathcal{E}_{h,\Omega}^{[n-1,n]} = \left\{ E \in \mathcal{E}_h^{[n-1,n]} : E \not\subset \partial\Omega \right\}$, and $\mathcal{E}_{h,\partial\Omega}^{[n-1,n]} = \left\{ E \in \mathcal{E}_h^{[n-1,n]} : E \subset \partial\Omega \right\}$. Similarly, we define the corresponding sets $\mathcal{E}_h^{[n]}$, $\mathcal{E}_{h,\Omega}^{[n]}$ and $\mathcal{E}_{h,\partial\Omega}^{[n]}$ of edges E of $\mathcal{T}_h^{[n]}$. For any edge $E \in \mathcal{E}_h^{[n-1,n]}$ and we define:

$$\omega_E^{[n]} = \bigcup_{\{T' \in \mathcal{T}_h^{[n-1,n]} : \ E \in \mathcal{E}(T')\}} T'.$$

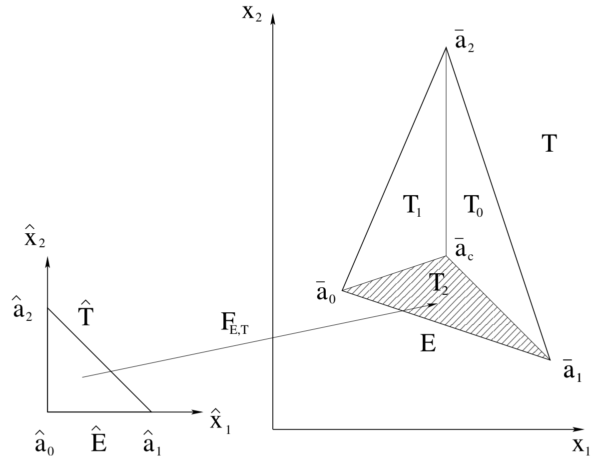

FIGURE 1. The mapping $F_{E,T} : \hat{T} \to T_2$.

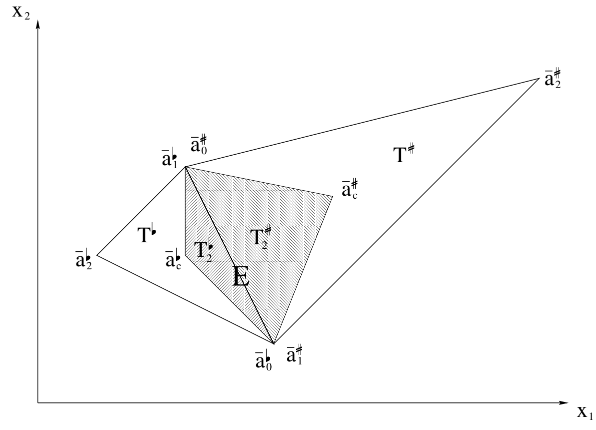

FIGURE 2. The support of the function $b_E^{[n]}$.

To any edge $E \in \mathcal{E}_{h,\Omega}^{[n-1,n]}$ we associate an orthogonal unit vector n_E and denote by $[\,.\,]_E$ the jump across E in the direction n_E. Let us denote by $\hat{T}$ the reference triangle and by $\hat{E}$ the reference edge as shown in Figure 1 on the left. Let λ_i, $i = 0, 1, 2$ be the barycentric coordinates on the reference triangle, then the *reference triangle bubble function* is $\hat{b}_{\hat{T}} = 27\lambda_0\lambda_1\lambda_2$, and the *reference edge bubble function* is $\hat{b}_{\hat{E}} = 4\hat{x}(1-\hat{x}-\hat{y})$ and $F_T^{[n]} : \hat{T} \to T$ is the affine mapping from the reference triangle to the triangle $T \in \mathcal{T}_h^{[n-1,n]}$ [5, 15]. For sake of simplicity, we will drop the superscript $[n]$ in the symbols of the mappings. For any $T \in \mathcal{T}_h^{[n-1,n]}$ we indicate with $b_T^{[n]}$ the *triangle bubble function* defined by $b_T^{[n]} = \hat{b}_{\hat{T}} \circ F_T^{-1}$. Note that this bubble function does not depend on time in each time-slab.

Given any $E \in \mathcal{E}_{h,\Omega}^{[n-1,n]}$, let $T^\sharp$ and $T^\flat$ the two triangles of $\mathcal{T}_h^{[n-1,n]}$ such that $\omega_E^{[n]} = T^\sharp \cup T^\flat$. Let us enumerate the vertices of $T^\sharp$ and $T^\flat$ counterclockwise in such a way that the vertices of E are numbered first. Let T be one of the triangles $T^\sharp$ and $T^\flat$, assume that E has vertices $\overline{a}_0$ and $\overline{a}_1$ and denote by $\overline{a}_c = (x_c, y_c)$ the barycentre of the triangle T; let us partition T into the triangles T_0, T_1, T_2 with T_2 having E as a side (see Fig. 1). Let $F_{E,T} : \hat{T} \to T_2$ be the invertible affine mapping that maps the reference triangle $\hat{T}$ onto the triangle T_2

$$F_{E,T}\,(\hat{x},\hat{y}) = \quad \overline{a}_0\,\lambda_0(\hat{x},\hat{y}) + \overline{a}_1\,\lambda_1(\hat{x},\hat{y}) + \overline{a}_c\,\lambda_2(\hat{x},\hat{y})\,, \quad \text{if } (\hat{x},\hat{y}) \in \hat{T}.$$

Then we define the *edge bubble function* $b_E^{[n]}$ by patching the two bubble functions:

$$b_{E,T^\sharp}^{[n]} = \hat{b}_{\hat{E}} \circ F_{E,T^\sharp}^{-1}, \qquad\qquad b_{E,T^\flat}^{[n]} = \hat{b}_{\hat{E}} \circ F_{E,T^\flat}^{-1},$$

each one being nonzero only on $T_2^\sharp$ and $T_2^\flat$, respectively. Finally, let us define the set $\hat{\omega}_E^{[n]} = T_2^\sharp \cup T_2^\flat$ (dashed area in Fig. 2). For the boundary edge E that belongs to the element T only, we naturally identify $b_E^{[n]}$ with $b_{E,T}^{[n]} = \hat{b}_{\hat{E}} \circ F_{E,T}^{-1}$.

With this definition of *edge bubble functions* we have a set of orthogonal functions, in the sense that the intersection of the supports of two different edge bubble functions is the empty set or a whole segment. This property is also true for the set of *triangle bubble functions*.

Moreover, for the reference edge $\hat{E}$ we define the extension operator $\hat{\mathcal{P}}_{\hat{E}} : \mathbb{P}_i(\hat{E}) \to \mathbb{P}_i(\hat{T})$ which extends a polynomial of degree i defined on the edge $\hat{E}$ to a polynomial of the same degree defined on $\hat{T}$ with constant values along lines orthogonal to the edge $\hat{E}$. Then, we define the extension operator $\mathcal{P}_E : \mathbb{P}_i(E) \to \mathbb{P}_i(\hat{\omega}_E^{[n]})$ which extends a polynomial of degree i defined on the edge E to a piecewise polynomial of the same degree defined on $\mathring{\omega}_E^{[n]}$ by patching the two operators:

$$\mathcal{P}_{E|_{T^\sharp}} = F_{E,T_2^\sharp} \circ \hat{\mathcal{P}}_{\hat{E}} \circ F^{-1}_{E,T_2^\sharp|_E}, \qquad\qquad \mathcal{P}_{E|_{T^\flat}} = F_{E,T_2^\flat} \circ \hat{\mathcal{P}}_{\hat{E}} \circ F^{-1}_{E,T_2^\flat|_E}.$$

The extension operator $\mathcal{P}_E$ is continuous, but not $\mathcal{C}^1$ in $\mathring{\omega}_E^{[n]}$. In the following we will need to collect all the triangles belonging to some set $\mathring{\omega}_E^{[n]}$, so let us define $\mathcal{T}^{[n-1,n]}_{h,\mathring{\omega}} = \left\{ T_2 \in \mathring{\omega}_E^{[n]} : E \in \mathcal{E}_h^{[n-1,n]} \right\}$.

The symbol $a \precsim b$ means that there exists a constant c independent of any meshsize, timestep, parameter and jump of parameters such that $a \le c\, b$. The symbol $a \asymp b$ means that $a \precsim b$ and $b \precsim a$.

For any time interval $I^{[n]}$, $T \in \mathcal{T}_h^{[n-1,n]}$ and $E \in \mathcal{E}_h^{[n-1,n]}$, the bubble functions $b_T^{[n]}$ and $b_E^{[n]}$ have the following properties: supp $b_T^{[n]} = T$, $0 \le b_T^{[n]} \le 1$, $\max_{(x,y)\in T} b_T^{[n]}(x,y) = 1$; supp $b_E^{[n]} = \mathring{\omega}_E^{[n]}$, $0 \le b_E^{[n]} \le 1$, $\max_{(x,y)\in E} b_E^{[n]}(x,y) = 1$; $\left\| b_T^{[n]} \right\|^2_{0,T} \asymp |\,T\,|$, $\left\| b_E^{[n]} \right\|^2_{0,T} \asymp |\,T\,|$, $\left\| b_E^{[n]} \right\|^2_{0,E} \asymp |\,E\,|$. Thanks to the regularity hypothesis on $\mathcal{T}_h^{[n-1,n]}$, there exist constants depending on the smallest angle in the triangulation, but not on the mesh size, such that for each $n = 1, ..., N$ we have: $|\,T\,| \asymp \left(h_T^{[n-1,n]} \right)^2$, $\forall T \in \mathcal{T}_h^{[n-1,n]}$, $h_T^{[n-1,n]} \asymp h_E^{[n-1,n]}$, $\forall E \in \mathcal{E}(T)$ and $|\,T\,| \asymp \left(h_E^{[n-1,n]} \right)^2$, $\forall T \in \omega_E^{[n]}$.

In the following κ_T will denote the constant value of κ in the triangle $T \in \mathcal{T}_h^{[n]}$, $\hat{\kappa}_{\omega_E^{[n]}}$ is the maximum of the values of κ_T over the two triangles $T \in \mathcal{T}_h^{[n]}$ sharing the edge E (we will use the same symbol to denote the maximum of κ_T over the two triangles $T \in \mathcal{T}_h^{[n-1,n]}$ sharing the edge $E \in \mathcal{E}_h^{[n-1,n]}$, it will be clear from the context which situation we are referring to). Moreover we shall use a modified *quasi-interpolation operator* $I_h \;:\; \mathrm{V} \to \mathrm{V}_h^{[n]}$ like the *quasi-interpolation operator* of Clément, [6]. The definition of this kind of interpolation operator requires the *quasi-monotonicity* hypothesis [8, 13] of $\kappa(x)$ with respect to any node $x_h^{[n]}$ of the triangulation $\mathcal{T}_h^{[n]}$. This hypothesis implies the existence of "robust" interpolation estimates [4, 8, 13]. For any subset $\omega \subseteq \overline{\Omega}$, let $\mathcal{N}_h^{[n]}(\omega)$ be the set of the vertices x of the triangulation $\mathcal{T}_h^{[n]}$ such that $x \in \omega$; let $\omega_{x_h^{[n]}}$ be the set of the triangles having $x_h^{[n]}$ as a vertex. Moreover, let $\hat{T}_{x_h^{[n]}}$ be a triangle from $\omega_{x_h^{[n]}}$ where the coefficient κ_T achieves its maximum in $\omega_{x_h^{[n]}}$. We recall the following definition of *quasi-monotonicity* for $\kappa(x)$ from [13], referring to this reference for more details.

Definition 3.1 (Quasi-monotonicity)**.** The distribution of coefficients κ_T, $T \in \omega_{x_h^{[n]}}$ is said to be *quasi-monotone with respect to the node* $x_h^{[n]} \in \mathcal{N}_h^{[n]}\left(\overline{\Omega}\right)$ if for each triangle $T \in \omega_{x_h^{[n]}}$ there exists a Lipschitz set $\tilde{\omega}_{T,x_h^{[n]}}$ containing triangles $T' \in \omega_{x_h^{[n]}}$ such that

- if $x_h^{[n]} \in \Omega$, then $T \cup \hat{T}_{x_h^{[n]}} \subseteq \tilde{\omega}_{T,x_h^{[n]}}$ and $\kappa_T \le \kappa_{T'}$, $\forall T' \in \tilde{\omega}_{T,x_h^{[n]}}$;
- if $x_h^{[n]} \in \partial\Omega$, then $T \subseteq \tilde{\omega}_{T,x_h^{[n]}}$, $\left| \partial\, \tilde{\omega}_{T,x_h^{[n]}} \cap \partial\Omega \right| > 0$ and $\kappa_T \le \kappa_{T'}$, $\forall T' \in \tilde{\omega}_{T,x_h^{[n]}}$.

Let the distribution of coefficients κ_T, $T \in \mathcal{T}_h^{[n]}$ be *quasi-monotone* with respect to every point $x_h^{[n]} \in \mathcal{N}_h^{[n]}\left(\overline{\Omega}\right)$. For a triangle $T \in \mathcal{T}_h^{[n]}$ and an edge $E \in \mathcal{E}_{h,\Omega}^{[n]}$, being $\mathcal{E}_{h,\Omega}^{[n]}$ the set of the edges of the triangulation $\mathcal{T}_h^{[n]}$, let us define two sets containing some neighboring triangles

$$\tilde{\omega}_T^{[n]} = \bigcup_{x_h^{[n]} \in \mathcal{N}_h^{[n]}(T)} \tilde{\omega}_{T,x_h^{[n]}}, \qquad \tilde{\omega}_E^{[n]} = \bigcup_{x_h^{[n]} \in \mathcal{N}_h^{[n]}(T_E)} \tilde{\omega}_{T,x_h^{[n]}},$$

where T_E is the triangle of the two triangles sharing E where κ_T achieves the maximum.

Definition 3.2 (Quasi-interpolation operator)**.** [4, 8, 13] Let the distribution of coefficients κ_T, $T \in \mathcal{T}_h^{[n]}$ be *quasi-monotone*, then we define the *quasi-interpolation operator* $I_h : V \to \mathrm{V}_h^{[n]}$ as

$$I_h v = \sum_{i=1}^{\dim \mathcal{N}_h^{[n]}(\Omega)} \lambda_i(x) p_{x_{h,i}^{[n]}}, \qquad p_{x_{h,i}^{[n]}} = \frac{1}{\hat{T}_{x_{h,i}^{[n]}}} \int_{\hat{T}_{x_{h,i}^{[n]}}} v \mathrm{d}\Omega, \quad \forall x_{h,i}^{[n]} \in \mathcal{N}_h^{[n]}(\Omega).$$

Let $p_x = 0$ for nodal points $x \in \partial\Omega$.

We recall from [13] the following results:

Lemma 3.3. *Let $T \in \mathcal{T}_h^{[n]}$ and $E \in \mathcal{E}_h^{[n]}$ be arbitrary. Let the* quasi-monotonicity *condition be satisfied with respect to any node $x_h^{[n]}$ of T and T_E. Then we have the following interpolation error estimates:*

$$\| v - I_h v \|_{0,T} \leq \tilde{C}l_R \frac{h_T^{[n]}}{\sqrt{\kappa_T}} \sum_{T' \in \tilde{\omega}_T^{[n]}} \| \sqrt{\kappa_{T'}} \nabla v \|_{0,T'}, \quad \forall v \in \mathrm{H}^1(\tilde{\omega}_T^{[n]}), \tag{8}$$

$$| v - I_h v |_{1,T} \leq \tilde{C}l_{R,1} \frac{1}{\sqrt{\kappa_T}} \sum_{T' \in \tilde{\omega}_T^{[n]}} \| \sqrt{\kappa_{T'}} \nabla v \|_{0,T'}, \quad \forall v \in \mathrm{H}^1(\tilde{\omega}_T^{[n]}), \tag{9}$$

$$\| v - I_h v \|_{0,E} \leq \tilde{C}l_E \frac{\sqrt{h_E^{[n]}}}{\sqrt{\hat{\kappa}_{\omega_E^{[n]}}}} \sum_{T' \in \tilde{\omega}_E^{[n]}} \| \sqrt{\kappa_{T'}} \nabla v \|_{0,T'}, \quad \forall v \in \mathrm{H}^1(\tilde{\omega}_E^{[n]}), \tag{10}$$

the constants $\tilde{C}l_R$, $\tilde{C}l_{R,1}$ and $\tilde{C}l_E$ depending only on the smallest angle in the triangulation.

For each triangle $T^* \in \mathcal{T}_h^{[n-1,n]}$ such that $T^* \subseteq T \in \mathcal{T}_h^{[n]}$ we define $\tilde{\omega}_{T^*}^{[n]} = \left\{ T' \in \mathcal{T}_h^{[n-1,n]} : T' \subseteq T'' \in \tilde{\omega}_T^{[n]} \subseteq \mathcal{T}_h^{[n]} \right\}$, *i.e.* the set of the triangles of $\mathcal{T}_h^{[n-1,n]}$ contained in, or equal to, a triangle $T'' \in \mathcal{T}_h^{[n]}$ belonging to $\tilde{\omega}_T^{[n]}$. Moreover if $E^* \in \mathcal{E}_h^{[n]}$ or if $E^* \subset E \in \mathcal{E}_h^{[n]}$, then we define $\tilde{\omega}_{E^*}^{[n]} = \left\{ T' \in \mathcal{T}_h^{[n-1,n]} : T' \subseteq T'' \in \tilde{\omega}_E^{[n]} \subseteq \mathcal{T}_h^{[n]} \right\}$, else if $E^* \in \mathcal{E}_h^{[n-1,n]} / \mathcal{E}_h^{[n]}$ and $E^* \not\subset E \in \mathcal{E}_h^{[n]}$ let $T \in \mathcal{T}_h^{[n]}$ be the triangle such that E^* is inside T, then $\tilde{\omega}_{E^*}^{[n]} = \left\{ T' \in \mathcal{T}_h^{[n-1,n]} : T' \subseteq T'' \in \tilde{\omega}_T^{[n]} \subseteq \mathcal{T}_h^{[n]} \right\}$, *i.e.* the sets of triangles of $\mathcal{T}_h^{[n-1,n]}$ contained in, or equal to, a triangle of $\tilde{\omega}_T^{[n]}$.

Lemma 3.4. *Let $T^* \in \mathcal{T}_h^{[n-1,n]}$ and $E^* \in \mathcal{E}_h^{[n-1,n]}$ be arbitrary. Then we have the following interpolation error estimates:*

$$\| v - I_h v \|_{0,T^*} \leq Cl_R \frac{h_{T^*}^{[n-1,n]}}{\sqrt{\kappa_{T^*}}} \sum_{T' \in \tilde{\omega}_{T^*}^{[n]}} \| \sqrt{\kappa_{T'}} \nabla v \|_{0,T^*} = Cl_R \frac{h_{T^*}^{[n-1,n]}}{\sqrt{\kappa_{T^*}}} \left\| \sqrt{\kappa} \nabla v \right\|_{0,\tilde{\omega}_{T^*}^{[n]}}, \forall v \in \mathrm{H}^1(\tilde{\omega}_{T^*}^{[n]}), \tag{11}$$

$$\| v - I_h v \|_{0,E^*} \leq Cl_E \frac{\sqrt{h_{E^*}^{[n-1,n]}}}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} \sum_{T' \in \tilde{\omega}_{E^*}^{[n]}} \| \sqrt{\kappa_{T'}} \nabla v \|_{0,T^*} = Cl_E \frac{\sqrt{h_{E^*}^{[n-1,n]}}}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} \left\| \sqrt{\kappa} \nabla v \right\|_{0,\tilde{\omega}_{E^*}^{[n]}}, \forall v \in \mathrm{H}^1(\tilde{\omega}_{E^*}^{[n]}), \tag{12}$$

the constants Cl_R, and Cl_E depending only on the smallest angle in the triangulation $\mathcal{T}_h^{[n]}$ and the constant C_{tr}.

Proof. Inequality (11) follows from (8) noting that $\| v - I_h v \|_{0,T^*} \leq \| v - I_h v \|_{0,T}$ where $T^* \in \mathcal{T}_h^{[n-1,n]} \subseteq T \in \mathcal{T}_h^{[n]}$, that $\kappa_T = \kappa_{T^*}$ and applying condition (7). If $E^* = E \in \mathcal{E}_h^{[n]}$ or $E^* \subset E \in \mathcal{E}_h^{[n]}$ inequality (12) comes from (10), (7) and considering that $\hat{\kappa}_{\omega_{E^*}^{[n]}} = \hat{\kappa}_{\omega_E^{[n]}}$. While if $E^* \in \mathcal{E}_h^{[n-1,n]} / \mathcal{E}_h^{[n]}$ and $E^* \not\subset E \in \mathcal{E}_h^{[n]}$ then E^* is inside $T \in \mathcal{T}_h^{[n]}$ and we apply standard trace inequality with (8) and (9) to one of the two triangles $T^* \in \mathcal{T}_h^{[n-1,n]}$ sharing E^*, with $\hat{\kappa}_{\omega_{E^*}^{[n]}} = \kappa_T$ and condition (7) to get (12). □

Let us consider the spaces $\mathrm{H}_0^1(\Omega)$ and $\mathrm{H}^{-1}(\Omega)$ respectively equipped with the norms:

$$\| v \|_{\kappa,1}^2 = \left\| \sqrt{\kappa}\, \nabla v \right\|_0^2 = \int_\Omega \kappa(x)\, \nabla v \cdot \nabla v \mathrm{d}\Omega, \qquad \| F \|_{\kappa,-1} = \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{\langle F\,,\, v\rangle}{\| v \|_{\kappa,1}}.$$

Let us define the error of our approximation $u_{h,\Delta t}$ in the interval $I^{[n]}$ as $e_{h,\Delta t} = u_{h,\Delta t} - u$ and define $\frac{\partial\, u_{h,\Delta t}}{\partial t} = \frac{u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]}}{t^{[n]} - t^{[n-1]}}$.

Definition 3.5. Let us define the residuals in the triangles $T^* \in \mathcal{T}_h^{[n-1,n]}$ and inter-element jumps on the edges $E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}$ of our approximation $u_{h,\Delta t}$

$$\begin{aligned}
R_{T^*}^{[n]} &= \left.\frac{\partial\, u_{h,\Delta t}}{\partial t} - \theta\, \kappa_{T^*} \triangle\, u_{h,\Delta t}^{[n]} - (1-\theta)\, \kappa_{T^*} \triangle\, u_{h,\Delta t}^{[n-1]} - \theta\, \Pi_T f^{[n]} - (1-\theta)\, \Pi_T f^{[n-1]}\right|_{T^*}, \\
R_{\Omega}^{[n]} &= \sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} R_{T^*}^{[n]}, \qquad J_{E^*}^{[n]} = \left[\!\!\left[\theta \kappa_{T'} \frac{\partial\, u_{h,\Delta t}^{[n]}}{\partial n_{E^*}} + (1-\theta)\, \kappa_{T'}\, \frac{\partial\, u_{h,\Delta t}^{[n-1]}}{\partial n_{E^*}} \right]\!\!\right]_{E^*}.
\end{aligned}$$

Definition 3.6. Let us define the following local-in-space local-in-time estimators

$$\begin{aligned}
\left(\eta_{R,T^*}^{[n]}\right)^2 &= \Delta t^{[n]} \left(\left(h_{T^*}^{[n-1,n]}\right)^2 \left\| \frac{1}{\sqrt{\kappa_{T^*}}}\, R_{T^*}^{[n]} \right\|_{0,T^*}^2 + \frac{1}{2} \sum_{E^* \in \mathcal{E}(T^*) \cap \mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}\, J_{E^*}^{[n]} \right\|_{0,E^*}^2 \right), \\
\left(\eta_{\nabla,T^*}^{[n]}\right)^2 &= \Delta t^{[n]} \left\| \sqrt{\kappa_{T^*}}\, \nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]}\right) \right\|_{0,T^*}^2.
\end{aligned}$$

Then, we define the following global-in-space and local-in-time estimators

$$\begin{aligned}
\left(\eta_R^{[n]}\right)^2 &= \sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} \left(\eta_{R,T^*}^{[n]}\right)^2, \qquad \left(\eta_\nabla^{[n]}\right)^2 = \sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} \left(\eta_{\nabla,T^*}^{[n]}\right)^2, \\
\left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2 &= \int_{t^{[n-1]}}^{t^{[n]}} \left\| \Pi_T f - \theta\, \Pi_T f^{[n]} - (1-\theta)\, \Pi_T f^{[n-1]} \right\|_{\kappa,-1}^2 \mathrm{d}t, \\
\left(\eta_{f,\Pi_T}^{[n]}\right)^2 &= \int_{t^{[n-1]}}^{t^{[n]}} \| f - \Pi_T f \|_{\kappa,-1}^2 \mathrm{d}t, \qquad \left(\eta_f^{[n]}\right)^2 = \left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2 + \left(\eta_{f,\Pi_T}^{[n]}\right)^2.
\end{aligned}$$

In the sequel we will derive upper and lower bounds for the error involving the following norm:

$$|\!|\!| e_{h,\Delta t} |\!|\!|_{\kappa,I^{[n]}} = \left(\int_{t^{[n-1]}}^{t^{[n]}} \left\| \frac{\partial\, e_{h,\Delta t}}{\partial t} \right\|_{\kappa,-1}^2 \mathrm{d}t + \int_{t^{[n-1]}}^{t^{[n]}} \| e_{h,\Delta t} \|_{\kappa,1}^2 \mathrm{d}t \right)^{\frac{1}{2}}.$$

Remark 3.7. Following considerations of [14,17], we can say that $\eta_R^{[n]}$ is a space error estimator related to the triangulation $\mathcal{T}_h^{[n]}$, whereas $\eta_\nabla^{[n]}$ gives information on the error due to time discretization.

Remark 3.8. The quantity $\eta_f^{[n]}$ is an estimator of the data approximation error and can be split into two terms: $\eta_{f,\Pi_T}^{[n]}$, that gives information essentially on the space data approximation error and $\eta_{f,\theta,\Delta t^{[n]}}^{[n]}$, that is a time data approximation error.

3.2. Upper bound

Theorem 3.9. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5)*, for each $n = 1, ..., N$, there exists a constant $\tilde{C}_{[n-1]}^{[n]}$ independent of any meshsize, timestep, problem-parameter and depending only on the smallest angle of the triangulation $\mathcal{T}_h^{[n]}$, on the constant C_{tr} and on the parameter θ, such that*

$$\left\| u_{h,\Delta t}^{[n]} - u^{[n]} \right\|_0^2 + \int_{t^{[n-1]}}^{t^{[n]}} \| \sqrt{\kappa}\, \nabla e_{h,\Delta t} \|_0^2 \mathrm{d}t \leq \left\| u_{h,\Delta t}^{[n-1]} - u^{[n-1]} \right\|_0^2 + \tilde{C}_{[n-1]}^{[n]} \left[\left(\eta_R^{[n]}\right)^2 + \left(\eta_\nabla^{[n]}\right)^2 + \left(\eta_f^{[n]}\right)^2 \right]. \tag{13}$$

Proof. Let us define

$$E_{h,\Delta t}^{[n]} = \int_{t^{[n-1]}}^{t^{[n]}} \left[\left\langle \frac{\partial\, e_{h,\Delta t}}{\partial t}, e_{h,\Delta t} \right\rangle + (\kappa\, \nabla\, e_{h,\Delta t}, \nabla\, e_{h,\Delta t}) \right] \mathrm{d}t \tag{14}$$

and

$$E_{h,\Delta t}^{[n]} = \frac{1}{2} \left\| u_{h,\Delta t}^{[n]} - u^{[n]} \right\|_0^2 - \frac{1}{2} \left\| u_{h,\Delta t}^{[n-1]} - u^{[n-1]} \right\|_0^2 + \int_{t^{[n-1]}}^{t^{[n]}} \| \sqrt{\kappa}\, \nabla\, e_{h,\Delta t} \|_0^2 \mathrm{d}t\,. \tag{15}$$

From the continuous variational formulation of problem (4) it immediately follows that

$$E_{h,\Delta t}^{[n]} = \int_{t^{[n-1]}}^{t^{[n]}} \left[\left(\frac{\partial\, u_{h,\Delta t}}{\partial t}, e_{h,\Delta t} \right) + (\kappa\, \nabla\, u_{h,\Delta t}, \nabla\, e_{h,\Delta t}) \right] \mathrm{d}t - \int_{t^{[n-1]}}^{t^{[n]}} \left[\left\langle \frac{\partial u}{\partial t}, e_{h,\Delta t} \right\rangle + (\kappa\, \nabla u, \nabla\, e_{h,\Delta t}) \right] \mathrm{d}t\,.$$

Recalling (5) with $I_h e_{h,\Delta t} \in \mathrm{V}_h^{[n]}$ as test function, we get

$$\begin{aligned} E_{h,\Delta t}^{[n]} = \int_{t^{[n-1]}}^{t^{[n]}} \Bigg[\left(\frac{\partial\, u_{h,\Delta t}}{\partial t}, e_{h,\Delta t} - I_h e_{h,\Delta t} \right) + \left(\theta\kappa\nabla\, u_{h,\Delta t}^{[n]} + (1-\theta)\kappa\nabla\, u_{h,\Delta t}^{[n-1]}, \nabla\left(e_{h,\Delta t} - I_h e_{h,\Delta t}\right) \right) \\ - \left(\theta\, \Pi_T f^{[n]} + (1-\theta)\, \Pi_T f^{[n-1]}, e_{h,\Delta t} - I_h e_{h,\Delta t} \right) \Bigg] \mathrm{d}t \\ + \int_{t^{[n-1]}}^{t^{[n]}} \theta \left(\kappa\, \nabla \left(u_{h,\Delta t} - u_{h,\Delta t}^{[n]} \right), \nabla\, e_{h,\Delta t} \right) \mathrm{d}t + \int_{t^{[n-1]}}^{t^{[n]}} (1-\theta) \left(\kappa\, \nabla \left(u_{h,\Delta t} - u_{h,\Delta t}^{[n-1]} \right), \nabla\, e_{h,\Delta t} \right) \mathrm{d}t \\ - \int_{t^{[n-1]}}^{t^{[n]}} (f - \Pi_T f, e_{h,\Delta t}) \mathrm{d}t - \int_{t^{[n-1]}}^{t^{[n]}} \left(\Pi_T f - \theta\, \Pi_T f^{[n]} - (1-\theta)\, \Pi_T f^{[n-1]}, e_{h,\Delta t} \right) \mathrm{d}t\,. \end{aligned}$$

Now we define $t^{[\theta,n]} = \theta\, t^{[n]} + (1-\theta)\, t^{[n-1]}$ and we note that

$$u_{h,\Delta t} - u_{h,\Delta t}^{[n]} = \frac{t - t^{[n]}}{t^{[n]} - t^{[n-1]}} \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right), \tag{16}$$

$$u_{h,\Delta t} - u_{h,\Delta t}^{[n-1]} = \frac{t - t^{[n-1]}}{t^{[n]} - t^{[n-1]}} \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right), \tag{17}$$

$$\theta \left(u_{h,\Delta t} - u_{h,\Delta t}^{[n]} \right) + (1-\theta) \left(u_{h,\Delta t} - u_{h,\Delta t}^{[n-1]} \right) = \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right). \tag{18}$$

After integration by parts of the term $\left(\theta\kappa\nabla\, u^{[n]}_{h,\Delta t} + (1-\theta)\kappa\nabla\, u^{[n-1]}_{h,\Delta t}, \nabla\left(e_{h,\Delta t} - I_h e_{h,\Delta t}\right)\right)$ we apply Cauchy-Schwarz's inequality, Poincaré-Friedrichs' inequality and the inequalities of Lemma 3.4 getting

$$
\begin{aligned}
E^{[n]}_{h,\Delta t} \lesssim & \int_{t^{[n-1]}}^{t^{[n]}} \sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\kappa_{T^*}}}\, R^{[n]}_{T^*} \right\|_{0,T^*} \|\sqrt{\kappa}\,\nabla\, e_{h,\Delta t}\|_{0,\tilde{\omega}^{[n]}_{T^*}} \mathrm{d}t \\
& + \int_{t^{[n-1]}}^{t^{[n]}} \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} \sqrt{h_{E^*}^{[n-1,n]}} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega^{[n]}_{E^*}}}}\, J^{[n]}_{E^*} \right\|_{0,E^*} \|\sqrt{\kappa}\,\nabla\, e_{h,\Delta t}\|_{0,\tilde{\omega}^{[n]}_{E^*}} \mathrm{d}t \\
& + \int_{t^{[n-1]}}^{t^{[n]}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} \left\| \sqrt{\kappa}\,\nabla\left(u^{[n]}_{h,\Delta t} - u^{[n-1]}_{h,\Delta t}\right)\right\|_0 \|\sqrt{\kappa}\,\nabla\, e_{h,\Delta t}\|_0 \mathrm{d}t \\
& + \int_{t^{[n-1]}}^{t^{[n]}} \| f - \Pi_T f\|_{\kappa,-1} \|\sqrt{\kappa}\,\nabla\, e_{h,\Delta t}\|_0 \mathrm{d}t + \int_{t^{[n-1]}}^{t^{[n]}} \left\| \Pi_T f - \theta\,\Pi_T f^{[n]} - (1-\theta)\,\Pi_T f^{[n-1]} \right\|_{\kappa,-1} \|\sqrt{\kappa}\,\nabla\, e_{h,\Delta t}\|_0 \mathrm{d}t\,.
\end{aligned}
$$

Applying Young's inequality with a suitable choice of constants, we conclude that there exists a constant $\tilde{\tilde{C}}^{[n]}_{[n-1]}$ such that

$$
\begin{aligned}
& \left\| u^{[n]}_{h,\Delta t} - u^{[n]} \right\|_0^2 + \int_{t^{[n-1]}}^{t^{[n]}} \|\sqrt{\kappa}\,\nabla\, e_{h,\Delta t}\|_0^2 \mathrm{d}t \le \left\| u^{[n-1]}_{h,\Delta t} - u^{[n-1]} \right\|_0^2 \\
& + \tilde{\tilde{C}}^{[n]}_{[n-1]} \Bigg[\Delta t^{[n]} \Bigg(\sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}}\, R^{[n]}_{T^*} \right\|_{0,T^*}^2 + \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega^{[n]}_{E^*}}}}\, J^{[n]}_{E^*} \right\|_{0,E^*}^2 \\
& \qquad + \sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} \left\| \sqrt{\kappa_{T^*}}\,\nabla\left(u^{[n]}_{h,\Delta t} - u^{[n-1]}_{h,\Delta t}\right)\right\|_{0,T^*}^2 \Bigg) \\
& + \int_{t^{[n-1]}}^{t^{[n]}} \| f - \Pi_T f\|_{\kappa,-1}^2 \mathrm{d}t + \int_{t^{[n-1]}}^{t^{[n]}} \left\| \Pi_T f - \theta\,\Pi_T f^{[n]} - (1-\theta)\,\Pi_T f^{[n-1]} \right\|_{\kappa,-1}^2 \mathrm{d}t \Bigg] \qquad (19)
\end{aligned}
$$

and we get the thesis. □

The result given by Theorem 3.9 is an upper bound of the error measured in the $\mathrm{L}^2(\Omega)$-norm at time $t^{[n]}$ and in the $\mathrm{L}^2(\,t^{[n-1]}, t^{[n]}; \mathrm{H}^1_0(\Omega)\,)$-norm. To have an upper bound in the same norm for which we can get a lower bound we also need an upper bound in the $\mathrm{L}^2(\,t^{[n-1]}, t^{[n]}; \mathrm{H}^{-1}(\Omega)\,)$-norm for $\partial\, e_{h,\Delta t}/\partial t$.

Lemma 3.10. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5) *for each* $n = 1, ..., N$ *and for each* $t \in \left(t^{[n-1]}, t^{[n]}\right)$, *we have*

$$
\begin{aligned}
\left\| \frac{\partial\, e_{h,\Delta t}}{\partial t} \right\|_{\kappa,-1} \le & \sqrt{\sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} \left(h_{T^*}^{[n-1,n]}\right)^2 \left\| \frac{1}{\sqrt{\kappa_{T^*}}}\, R^{[n]}_{T^*} \right\|_{0,T^*}^2} + \sqrt{\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega^{[n]}_{E^*}}}}\, J^{[n]}_{E^*} \right\|_{0,E^*}^2} \\
& + \| f - \Pi_T f\|_{\kappa,-1} + \left\| \Pi_T f - \theta\,\Pi_T f^{[n]} - (1-\theta)\,\Pi_T f^{[n-1]} \right\|_{\kappa,-1} \\
& + \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} \left\| \sqrt{\kappa}\,\nabla\left(u^{[n]}_{h,\Delta t} - u^{[n-1]}_{h,\Delta t}\right)\right\|_0 + \|\sqrt{\kappa}\,\nabla\, e_{h,\Delta t}\|_0\,. \qquad (20)
\end{aligned}
$$

Proof. Let us observe that

$$\left\langle \frac{\partial e_{h,\Delta t}}{\partial t}, v \right\rangle + (\kappa \nabla e_{h,\Delta t}, \nabla v) = \left(\frac{\partial u_{h,\Delta t}}{\partial t}, v \right) + (\kappa \nabla u_{h,\Delta t}, \nabla v) - (f, v)$$

$$= \left(\frac{\partial u_{h,\Delta t}}{\partial t}, v \right) + \left(\kappa \nabla \left(\theta\, u_{h,\Delta t}^{[n]} + (1-\theta)\, u_{h,\Delta t}^{[n-1]} \right), \nabla v \right) - \left(\theta\, \Pi_T f^{[n]} + (1-\theta)\, \Pi_T f^{[n-1]}, v \right)$$

$$+ \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} \left(\kappa \nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right), \nabla v \right) - (f - \Pi_T f, v) - \left(\Pi_T f - \theta\, \Pi_T f^{[n]} - (1-\theta)\, \Pi_T f^{[n-1]}, v \right).$$

Moreover, using (5) with $v_h = I_h v$, $v \in \mathrm{H}_0^1(\Omega)$, we have

$$\left\| \frac{\partial e_{h,\Delta t}}{\partial t} \right\|_{\kappa,-1} = \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{\left\langle \dfrac{\partial e_{h,\Delta t}}{\partial t}, v \right\rangle}{\|v\|_{\kappa,1}}$$

$$\leq \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} \sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} \left(R_{T^*}^{[n]}, v - I_h v \right)_{T^*} + \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \left(J_{E^*}^{[n]}, v - I_h v \right)_{E^*}$$

$$+ \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} \left(\kappa \nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right), \nabla v \right) + \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} (f - \Pi_T f, v)$$

$$+ \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} \left(\Pi_T f - \theta\, \Pi_T f^{[n]} - (1-\theta)\, \Pi_T f^{[n-1]}, v \right) + \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} (\kappa \nabla e_{h,\Delta t}, \nabla v).$$

Applying Hölder inequalities and Lemma 3.4 we get

$$\left\| \frac{\partial e_{h,\Delta t}}{\partial t} \right\|_{\kappa,-1} \lesssim \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} \sqrt{\sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} \left(h_{T^*}^{[n-1,n]} \right)^2 \left\| \frac{1}{\sqrt{\kappa_{T^*}}} R_{T^*}^{[n]} \right\|_{0,T^*}^2} \sqrt{\sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} \| \sqrt{\kappa}\, \nabla v \|_{0,\tilde{\omega}_{T^*}^{[n]}}^2}$$

$$+ \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} \sqrt{\sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} J_{E^*}^{[n]} \right\|_{0,E^*}^2} \sqrt{\sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \| \sqrt{\kappa}\, \nabla v \|_{0,\tilde{\omega}_{E^*}^{[n]}}^2}$$

$$+ \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} \left(\kappa \nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right), \nabla v \right) + \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} (f - \Pi_T f, v)$$

$$+ \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} \left(\Pi_T f - \theta\, \Pi_T f^{[n]} - (1-\theta)\, \Pi_T f^{[n-1]}, v \right) + \sup_{v \in \mathrm{H}_0^1(\Omega)} \frac{1}{\|v\|_{\kappa,1}} (\kappa \nabla e_{h,\Delta t}, \nabla v)$$

and then we easily get the thesis. □

Theorem 3.11. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5), *for each* $n = 1, ..., N$, *we have*

$$\int_{t^{[n-1]}}^{t^{[n]}} \left\| \frac{\partial e_{h,\Delta t}}{\partial t} \right\|_{\kappa,-1}^2 \mathrm{d}t \leq 6 \left[\left(\eta_R^{[n]} \right)^2 + \left(\eta_f^{[n]} \right)^2 + \left(\theta^2 - \theta + \frac{1}{3} \right) \left(\eta_\nabla^{[n]} \right)^2 + \int_{t^{[n-1]}}^{t^{[n]}} \| \sqrt{\kappa}\, \nabla e_{h,\Delta t} \|_0^2 \mathrm{d}t \right]. \tag{21}$$

Proof. Applying Young inequality to the square of (20) and integrating in time on the n-th time interval, the thesis readily follows. □

Corollary 3.12. *Under the hypotheses of Theorems* 3.9 *and* 3.11 *there exist constants* $C^{\uparrow,[n]}_{[n-1]}$, *independent of any meshsize, timestep, problem-parameter, but depending on the smallest angle of triangulations* $\mathcal{T}_h^{[n]}$ *and on the constant* C_{tr} *such that the following inequality holds true for* $n = 1, ..., N$

$$7\left\| u^{[n]}_{h,\Delta t} - u^{[n]} \right\|_0^2 + \| e_{h,\Delta t} \|^2_{\kappa,I^{[n]}} \leq 7\left\| u^{[n-1]}_{h,\Delta t} - u^{[n-1]} \right\|_0^2 + C^{\uparrow,[n]}_{[n-1]} \left(\left(\eta_R^{[n]}\right)^2 + \left(\eta_\nabla^{[n]}\right)^2 + \left(\eta_f^{[n]}\right)^2 \right). \tag{22}$$

Proof. Multiplying by 7 inequality (13) and summing to it (21) we get (22). □

Theorem 3.13. *Under the hypotheses of Theorems* 3.9 *and* 3.11 *there exists a constant* $C^{\uparrow,[N]}_{[0]} = \max_{n=1,\dots,N} C^{\uparrow,[n]}_{[n-1]}$, *independent of any meshsize, timestep, problem-parameter, but depending on the smallest angle of triangulations* $\mathcal{T}_h^{[n]}$ *and on the constant* C_{tr} *such that the following inequality holds true*

$$7\left\| u^{[m]}_{h,\Delta t} - u^{[m]} \right\|_0^2 + \| e_{h,\Delta t} \|^2_{\kappa,(0,t^{[m]})} \leq 7\left\| u^{[0]}_{h,\Delta t} - u^{[0]} \right\|_0^2 + C^{\uparrow,[N]}_{[0]} \sum_{n=1}^{m} \left(\left(\eta_R^{[n]}\right)^2 + \left(\eta_\nabla^{[n]}\right)^2 + \left(\eta_f^{[n]}\right)^2 \right). \tag{23}$$

3.3. Lower bound

We prove that the terms $\eta_R^{[n]}$ and $\eta_\nabla^{[n]}$ bound from below the error of the discretized problem with respect to the exact solution of the continuous variational formulation. We consider separately the contribution of the equation residual, the inter-element jumps and $\eta_\nabla^{[n]}$. We remark that the lower bound will not be local-in-space as in the elliptic case; instead, it will be global-in-space, but local-in-time.

3.3.1. *Equation residual*

Here we show how the residual of the equation can bound the error from below on the time interval $\left(t^{[n-1]}, t^{[n]}\right)$. For any triangle $T^* \in \mathcal{T}_h^{[n-1,n]}$, let us define the following functions in $\overline{\Omega}$

$$\begin{aligned} w^{[n]}_{R,T^*}(x) &= \begin{cases} h_{T^*}^{[n-1,n]^2} \dfrac{1}{\sqrt{\kappa_{T^*}}} R^{[n]}_{T^*}\, b^{[n]}_{T^*}(x), & \text{if } x \in T^*, \\ 0, & \text{if } x \notin T^*, \end{cases} \\ w^{[n]}_{R,\Omega} &= \sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} w^{[n]}_{R,T^*}\,. \end{aligned}$$

Lemma 3.14. *There exist constants* C_R *and* C_R^* *independent of any meshsize, timestep and problem-parameter such that*

$$h_{T^*}^{[n-1,n]^2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}} R^{[n]}_{T^*} \right\|^2_{0,T^*} \leq C_R \left(\frac{1}{\sqrt{\kappa_{T^*}}} R^{[n]}_{T^*}, w^{[n]}_{R,T^*} \right)_{T^*}, \tag{24}$$

$$\left\| w^{[n]}_{R,T^*} \right\|_{0,T^*} \leq h_{T^*}^{[n-1,n]^2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}} R^{[n]}_{T^*} \right\|_{0,T^*}, \tag{25}$$

$$\left\| \nabla w^{[n]}_{R,T^*} \right\|_{0,T^*} \leq C_R^* \frac{1}{h_{T^*}^{[n-1,n]}} \left\| w^{[n]}_{R,T^*} \right\|_{0,T^*}. \tag{26}$$

Proof. These results come exploiting the properties of bubble functions and the finite dimensionality of the residual function [16]. □

Proposition 3.15. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5), *on each time interval* $\left(t^{[n-1]}, t^{[n]}\right)$, $\forall \alpha \geq 0$, *we have*

$$\sqrt{\sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]^2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}} R_{T^*}^{[n]} \right\|_{0,T^*}^2 \Delta t^{[n]}} \leq C_R C_R^* \left[2 \frac{\alpha+1}{\sqrt{2\alpha+1}} \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2 \right)^{\frac{1}{2}} + \left| \theta - \frac{\alpha+1}{\alpha+2} \right| \eta_\nabla^{[n]} \right]. \qquad (27)$$

Proof. In the following we introduce an arbitrary function of time $b^{[n]}(t) \geq 0$, $\forall t \in I^{[n]}$. We start by subtracting to $\left(R_\Omega^{[n]} / \sqrt{\kappa}, w_{R,\Omega}^{[n]}\right)$ the continuous variational formulation (4) with $w_{R,\Omega}^{[n]} / \sqrt{\kappa}$ as test function and integrating on the time interval $I^{[n]}$ the result against $b^{[n]}$, then we apply (18). We have

$$\begin{aligned}
\int_{t^{[n-1]}}^{t^{[n]}} \sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} \left(\frac{1}{\sqrt{\kappa_{T^*}}} R_{T^*}^{[n]}, w_{R,T^*}^{[n]} \right)_{T^*} b^{[n]} \mathrm{d}t &= \int_{t^{[n-1]}}^{t^{[n]}} \left\langle \frac{1}{\sqrt{\kappa}} \frac{\partial e_{h,\Delta t}}{\partial t}, w_{R,\Omega}^{[n]} \right\rangle b^{[n]} \mathrm{d}t \\
&- \left(\sqrt{\kappa} \nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right), \nabla w_{R,\Omega}^{[n]} \right) \int_{t^{[n-1]}}^{t^{[n]}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} b^{[n]} \mathrm{d}t + \int_{t^{[n-1]}}^{t^{[n]}} \left(\sqrt{\kappa} \nabla e_{h,\Delta t}, \nabla w_{R,\Omega}^{[n]} \right) b^{[n]} \mathrm{d}t \\
&+ \int_{t^{[n-1]}}^{t^{[n]}} \left(\frac{1}{\sqrt{\kappa}} (f - \Pi_T f), w_{R,\Omega}^{[n]} \right) b^{[n]} \mathrm{d}t + \int_{t^{[n-1]}}^{t^{[n]}} \left(\frac{1}{\sqrt{\kappa}} \left(\Pi_T f - \theta \Pi_T f^{[n]} - (1-\theta) \Pi_T f^{[n-1]} \right), w_{R,\Omega}^{[n]} \right) b^{[n]} \mathrm{d}t .
\end{aligned}$$

Now we apply Cauchy-Schwarz's and Hölder inequality. Moreover we observe that

$$\left(\frac{1}{\sqrt{\kappa}} g, w_{R,\Omega}^{[n]} \right) = \left(g, \frac{1}{\sqrt{\kappa}} w_{R,\Omega}^{[n]} \right) \leq \| g \|_{\kappa,-1} \left\| \sqrt{\kappa} \nabla \left(\frac{1}{\sqrt{\kappa}} w_{R,\Omega}^{[n]} \right) \right\|_0 = \| g \|_{\kappa,-1} \left\| \nabla w_{R,\Omega}^{[n]} \right\|_0 .$$

Hence

$$\begin{aligned}
\int_{t^{[n-1]}}^{t^{[n]}} \sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} \left(\frac{1}{\sqrt{\kappa_{T^*}}} R_{T^*}^{[n]}, w_{R,T^*}^{[n]} \right)_{T^*} b^{[n]} \mathrm{d}t &\leq \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} \left\| \frac{\partial e_{h,\Delta t}}{\partial t} \right\|_{\kappa,-1}^2 \mathrm{d}t} \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} \left\| \nabla w_{R,\Omega}^{[n]} \right\|_0^2 b^{[n]^2} \mathrm{d}t} \\
&+ \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} \| \sqrt{\kappa} \nabla e_{h,\Delta t} \|_0^2 \mathrm{d}t} \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} \left\| \nabla w_{R,\Omega}^{[n]} \right\|_0^2 b^{[n]^2} \mathrm{d}t} \\
&+ \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} \| f - \Pi_T f \|_{\kappa,-1}^2 \mathrm{d}t} \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} \left\| \nabla w_{R,\Omega}^{[n]} \right\|_0^2 b^{[n]^2} \mathrm{d}t} \\
&+ \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} \left\| \Pi_T f - \theta \Pi_T f^{[n]} - (1-\theta) \Pi_T f^{[n-1]} \right\|_{\kappa,-1}^2 \mathrm{d}t} \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} \left\| \nabla w_{R,\Omega}^{[n]} \right\|_0^2 b^{[n]^2} \mathrm{d}t} \\
&+ \left\| \sqrt{\kappa} \nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right) \right\|_0 \left\| \nabla w_{R,\Omega}^{[n]} \right\|_0 \left| \int_{t^{[n-1]}}^{t^{[n]}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} b^{[n]} \mathrm{d}t \right| \\
&\leq 2 \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_{f,\Pi_T}^{[n]} \right)^2 + \left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]} \right)^2 \right)^{\frac{1}{2}} \left\| \nabla w_{R,\Omega}^{[n]} \right\|_0 \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} b^{[n]^2} \mathrm{d}t} \\
&+ \left\| \sqrt{\kappa} \nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right) \right\|_0 \left\| \nabla w_{R,\Omega}^{[n]} \right\|_0 \left| \int_{t^{[n-1]}}^{t^{[n]}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} b^{[n]} \mathrm{d}t \right| .
\end{aligned}$$

After, inequalities (24)–(26) give

$$\sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]^2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}}\, R_{T^*}^{[n]} \right\|_{0,T^*}^2 \int_{t^{[n-1]}}^{t^{[n]}} b^{[n]}\mathrm{d}t$$
$$\leq C_R C_R^* \left[2\left(\| e_{h,\Delta t} \|_{\kappa,I^{[n]}}^2 + \left(\eta_{f,\Pi_T}^{[n]}\right)^2 + \left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2 \right)^{\frac{1}{2}} \sqrt{\int_{t^{[n-1]}}^{t^{[n]}} b^{[n]^2}\mathrm{d}t} \right.$$
$$\left. + \left\| \sqrt{\kappa}\,\nabla\left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]}\right) \right\|_0 \left| \int_{t^{[n-1]}}^{t^{[n]}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}}\, b^{[n]}\mathrm{d}t \right| \right] \sqrt{\sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]^2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}}\, R_{T^*}^{[n]} \right\|_{0,T^*}^2}. \quad (28)$$

Now, let us choose the function $b^{[n]}$ as [17]

$$b^{[n]} = (\alpha+1)\left(\frac{t - t^{[n-1]}}{t^{[n]} - t^{[n-1]}}\right)^{\alpha}, \qquad \alpha \geq 0 \quad (29)$$

and compute the following integrals to be inserted into the previous inequality:

$$\int_{t^{[n-1]}}^{t^{[n]}} b^{[n]}\mathrm{d}t = \Delta t^{[n]}, \quad \left| \int_{t^{[n-1]}}^{t^{[n]}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}}\, b^{[n]}\mathrm{d}t \right| = \left| \theta - \frac{\alpha+1}{\alpha+2} \right| \Delta t^{[n]}, \quad \int_{t^{[n-1]}}^{t^{[n]}} b^{[n]^2}\mathrm{d}t = \frac{(\alpha+1)^2}{2\alpha+1}\, \Delta t^{[n]}.$$

From inequality (28) we obtain the following relation

$$\sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]^2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}}\, R_{T^*}^{[n]} \right\|_{0,T^*}^2 \Delta t^{[n]} \leq C_R C_R^* \left[2\frac{\alpha+1}{\sqrt{2\alpha+1}} \left(\| e_{h,\Delta t} \|_{\kappa,I^{[n]}}^2 + \left(\eta_{f,\Pi_T}^{[n]}\right)^2 + \left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2 \right)^{\frac{1}{2}} \right.$$
$$\left. + \sqrt{\Delta t^{[n]}} \left| \theta - \frac{\alpha+1}{\alpha+2} \right| \left\| \sqrt{\kappa}\,\nabla\left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]}\right) \right\|_0 \right] \sqrt{\Delta t^{[n]} \sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]^2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}}\, R_{T^*}^{[n]} \right\|_{0,T^*}^2}$$

from which inequality (27) follows. □

3.3.2. *Inter-element jumps*

Now we consider the edges $E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}$ and we show how the jumps $J_{E^*}^{[n]}$ can bound the error from below. Let us define

$$w_{J,E^*}^{[n]}(x) = \begin{cases} h_{E^*}^{[n-1,n]}\, \mathcal{P}_E\left(\dfrac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}\, J_{E^*}^{[n]} \right) b_{E^*}^{[n]}(x), & \text{if } x \in \mathring{\omega}_{E^*}^{[n]}, \\ 0, & \text{if } x \notin \mathring{\omega}_{E^*}^{[n]}. \end{cases}$$

We remark that $w_{J,E^*}^{[n]}$ vanishes on the edges of the triangles $T^* \in \mathcal{T}_{h,\mathring{\omega}}^{[n-1,n]}$ inside the triangles $T^* \in \mathcal{T}_h^{[n-1,n]}$.

Remark 3.16. Thanks to the orthogonality of our system of edge-bubble functions we have

$$\left\| \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} \nabla\, w_{J,E^*}^{[n]} \right\|_0^2 = \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} \left\| \nabla\, w_{J,E^*}^{[n]} \right\|_0^2.$$

Lemma 3.17. *There exist constants C_E and C_E^* independent of any meshsize, timestep and problem-parameter such that*

$$h_{E^*}^{[n-1,n]}\left\|\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}J_{E^*}^{[n]}\right\|_{0,E^*}^2 \leq C_E\left(\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}J_{E^*}^{[n]},w_{J,E^*}^{[n]}\right)_{E^*}, \tag{30}$$

$$\left\|w_{J,E^*}^{[n]}\right\|_{0,\hat{\omega}_{E^*}^{[n]}} \leq \sqrt{h_{E^*}^{[n-1,n]}}\,h_{E^*}^{[n-1,n]}\left\|\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}J_{E^*}^{[n]}\right\|_{0,E^*}, \tag{31}$$

$$\left\|\nabla w_{J,E^*}^{[n]}\right\|_{0,\hat{\omega}_{E^*}^{[n]}} \leq C_E^*\frac{1}{h_{E^*}^{[n-1,n]}}\left\|w_{J,E^*}^{[n]}\right\|_{0,\hat{\omega}_{E^*}^{[n]}}. \tag{32}$$

Proof. The previous results are derived exploiting the properties of bubble functions and inverse inequalities for the jump functions. □

Proposition 3.18. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5), *on each time interval $\left(t^{[n-1]},t^{[n]}\right)$ we have*

$$\sqrt{\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}}h_{E^*}^{[n-1,n]}\left\|\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}J_{E^*}^{[n]}\right\|_{0,E^*}^2\Delta t^{[n]}} \leq C_E\left(C_E^*+C_RC_R^*\right)\left[2\frac{\alpha+1}{\sqrt{2\alpha+1}}\left(\|\!|\!| e_{h,\Delta t}|\!|\!|_{\kappa,I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2\right)^{\frac{1}{2}}+\left|\theta-\frac{\alpha+1}{\alpha+2}\right|\eta_\nabla^{[n]}\right]. \tag{33}$$

Proof. We start by integrating $\left(J_{E^*}^{[n]}/\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}},w_{J,E^*}^{[n]}\right)$ against the arbitrary bubble function $b^{[n]}(t)$, we introduce the continuous variational formulation (4) and we apply Poincaré-Friedrichs' inequality obtaining

$$\begin{aligned}
&\int_{t^{[n-1]}}^{t^{[n]}}\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}}\left(\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}J_{E^*}^{[n]},w_{J,E^*}^{[n]}\right)_{E^*}b^{[n]}\mathrm{d}t = \\
&\qquad\int_{t^{[n-1]}}^{t^{[n]}}\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}}\sum_{T'\in\hat{\omega}_{E^*}^{[n]}}\int_{T'}\nabla\cdot\left[\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}\left(\theta\kappa\nabla u_{h,\Delta t}^{[n]}+(1-\theta)\kappa\nabla u_{h,\Delta t}^{[n-1]}\right)w_{J,E^*}^{[n]}\right]\mathrm{d}\Omega\ b^{[n]}\mathrm{d}t \\
&=\int_{t^{[n-1]}}^{t^{[n]}}\left\langle\frac{\partial e_{h,\Delta t}}{\partial t},\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}}\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}w_{J,E^*}^{[n]}\right\rangle b^{[n]}\mathrm{d}t+\int_{t^{[n-1]}}^{t^{[n]}}\left(\nabla e_{h,\Delta t}^{[n]},\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}}\frac{\kappa}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}\nabla w_{J,E^*}^{[n]}\right)b^{[n]}\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{t^{[n-1]}}^{t^{[n]}} \left(f - \Pi_T f, \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} w_{J,E^*}^{[n]} \right) b^{[n]} \mathrm{d}t \\
&+ \int_{t^{[n-1]}}^{t^{[n]}} \left(\Pi_T f - \theta f^{[n]} - (1-\theta) f^{[n-1]}, \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} w_{J,E^*}^{[n]} \right) b^{[n]} \mathrm{d}t \\
&- \int_{t^{[n-1]}}^{t^{[n]}} \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \sum_{T' \in \hat{\omega}_{E^*}^{[n]}} \left(\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} R_{T^*}^{[n]}, w_{J,E^*}^{[n]} \right)_{T'} b^{[n]} \mathrm{d}t \\
&- \int_{t^{[n-1]}}^{t^{[n]}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} \left(\nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right), \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \frac{\kappa}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} \nabla w_{J,E^*}^{[n]} \right) b^{[n]} \mathrm{d}t \,.
\end{aligned}
$$

Then we apply Cauchy-Schwarz's inequality to get

$$
\begin{aligned}
&\int_{t^{[n-1]}}^{t^{[n]}} \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \left(\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} J_{E^*}^{[n]}, w_{J,E^*}^{[n]} \right) b^{[n]} \mathrm{d}t \\
&\quad \le \int_{t^{[n-1]}}^{t^{[n]}} \left\| \frac{\partial e_{h,\Delta t}}{\partial t} \right\|_{\kappa,-1} \left\| \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \frac{\sqrt{\kappa}}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} \nabla w_{J,E^*}^{[n]} \right\|_0 b^{[n]} \mathrm{d}t \\
&\quad + \int_{t^{[n-1]}}^{t^{[n]}} \left\| \sqrt{\kappa} \nabla e_{h,\Delta t} \right\|_0 \left\| \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \frac{\sqrt{\kappa}}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} \nabla w_{J,E^*}^{[n]} \right\|_0 b^{[n]} \mathrm{d}t \\
&\quad + \int_{t^{[n-1]}}^{t^{[n]}} \left\| f - \Pi_T f \right\|_{\kappa,-1} \left\| \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \frac{\sqrt{\kappa}}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} \nabla w_{J,E^*}^{[n]} \right\|_0 b^{[n]} \mathrm{d}t \\
&\quad + \int_{t^{[n-1]}}^{t^{[n]}} \left\| \Pi_T f - \theta \Pi_T f^{[n]} - (1-\theta) \Pi_T f^{[n-1]} \right\|_{\kappa,-1} \left\| \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \frac{\sqrt{\kappa}}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} \nabla w_{J,E^*}^{[n]} \right\|_0 b^{[n]} \mathrm{d}t \\
&\quad + \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \sum_{T' \in \hat{\omega}_{E^*}^{[n]}} \left\| \frac{1}{\sqrt{\kappa_{T'}}} R_{T'}^{[n]} \right\|_{0,T'} \left\| \frac{\sqrt{\kappa}}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} w_{J,E^*}^{[n]} \right\|_{0,T'} \int_{t^{[n-1]}}^{t^{[n]}} b^{[n]} \mathrm{d}t \\
&\quad + \left\| \sqrt{\kappa} \nabla \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]} \right) \right\|_0 \left\| \sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} \frac{\sqrt{\kappa}}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} \nabla w_{J,E^*}^{[n]} \right\|_0 \\
&\qquad \times \left| \int_{t^{[n-1]}}^{t^{[n]}} \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} b^{[n]} \mathrm{d}t \right| .
\end{aligned}
$$

Applying Hölder's inequality, inequalities (31), (32), the definition of $b^{[n]}$ given by (29) and orthogonality of the edge-bubble functions we get

$$\int_{t^{[n-1]}}^{t^{[n]}} \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} \left(\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} J_{E^*}^{[n]}, w_{J,E^*}^{[n]}\right) b^{[n]}\mathrm{d}t \leq C_E^* \left[2\frac{\alpha+1}{\sqrt{2\alpha+1}}\left(\| e_{h,\Delta t}\|_{\kappa,I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2\right)^{\frac{1}{2}}\right.$$
$$\left. + \left|\theta - \frac{\alpha+1}{\alpha+2}\right| \eta_\nabla^{[n]}\right] \sqrt{\Delta t^{[n]} \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} J_{E^*}^{[n]}\right\|_{0,E^*}^2}$$
$$+ \Delta t^{[n]} \sqrt{\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} \sum_{T'\in\hat{\omega}_{E^*}^{[n]}} h_{E^*}^{[n-1,n]2} \left\|\frac{1}{\sqrt{\kappa_{T'}}} R_{T'}^{[n]}\right\|_{0,T'}^2} \sqrt{\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} J_{E^*}^{[n]}\right\|_{0,E^*}^2}.$$

Inequality (30) give us the following relation

$$\sqrt{\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} J_{E^*}^{[n]}\right\|_{0,E^*}^2 \Delta t^{[n]}} \leq C_E C_E^* \left[2\frac{\alpha+1}{\sqrt{2\alpha+1}}\left(\| e_{h,\Delta t}\|_{\kappa,I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2\right)^{\frac{1}{2}}\right.$$
$$\left. + \left|\theta - \frac{\alpha+1}{\alpha+2}\right| \eta_\nabla^{[n]}\right] + C_E \sqrt{\sum_{T'\in\mathcal{T}_{h,\hat{\omega}}^{[n-1,n]}} h_{T'}^{[n-1,n]2} \left\|\frac{1}{\sqrt{\kappa_{T'}}} R_{T'}^{[n]}\right\|_{0,T'}^2 \Delta t^{[n]}}$$

and by (27) we conclude with (33). □

3.3.3. *Time discretization estimator*

Now we show how the norm $\left\| \sqrt{\kappa}\,\nabla\left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]}\right)\right\|_0$ can bound the error from below. Let us define

$$w_{\nabla,\Omega}^{[n]} = \frac{t - t^{[\theta,n]}}{t^{[n]} - t^{[n-1]}} \left(u_{h,\Delta t}^{[n]} - u_{h,\Delta t}^{[n-1]}\right).$$

Proposition 3.19. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5)*, on each time interval* $\left(t^{[n-1]}, t^{[n]}\right)$ *the following inequality*

$$\eta_\nabla^{[n]} \leq 2\sqrt{3}\, Cl_E \sqrt{\Delta t^{[n]} \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} J_{E^*}^{[n]}\right\|_{0,E^*}^2}$$
$$+ 2\sqrt{3}\, Cl_R \sqrt{\Delta t^{[n]} \sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]2} \left\|\frac{1}{\sqrt{\kappa_{T^*}}} R_{T^*}^{[n]}\right\|_{0,T^*}^2} + 4\sqrt{3}\left(\| e_{h,\Delta t}\|_{\kappa,I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2\right)^{\frac{1}{2}} \quad (34)$$

holds true.

Proof. In the proof we use the following relations

$$\begin{aligned}
\int_{t^{[n-1]}}^{t^{[n]}} \frac{t-t^{[\theta,n]}}{t^{[n]}-t^{[n-1]}} \left(\kappa\,\nabla\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right),\nabla\, w_{\nabla,\Omega}^{[n]}\right)\mathrm{d}t &= \left(\theta^2-\theta+\frac{1}{3}\right)\Delta t^{[n]}\left\|\sqrt{\kappa}\,\nabla\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right)\right\|_0^2,\\
\frac{t-t^{[\theta,n]}}{t^{[n]}-t^{[n-1]}}\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right) &= u_{h,\Delta t}-\theta\, u_{h,\Delta t}^{[n]}-(1-\theta)\,u_{h,\Delta t}^{[n-1]},\\
\int_{t^{[n-1]}}^{t^{[n]}}\left\langle\nabla\cdot(\kappa\,\nabla u)\,,w_{\nabla,\Omega}^{[n]}\right\rangle\mathrm{d}t &= -\int_{t^{[n-1]}}^{t^{[n]}}\left(\kappa\,\nabla u,\nabla\, w_{\nabla,\Omega}^{[n]}\right)\mathrm{d}t\,.
\end{aligned}$$

From the first of the previous relations we get

$$\begin{aligned}
\left(\theta^2-\theta+\frac{1}{3}\right)\Delta t^{[n]}\left\|\sqrt{\kappa}\,\nabla\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right)\right\|_0^2 &= \int_{t^{[n-1]}}^{t^{[n]}} \frac{t-t^{[\theta,n]}}{t^{[n]}-t^{[n-1]}} \left(\kappa\,\nabla\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right),\nabla\, w_{\nabla,\Omega}^{[n]}\right)\mathrm{d}t\\
&= \int_{t^{[n-1]}}^{t^{[n]}} \frac{t-t^{[\theta,n]}}{t^{[n]}-t^{[n-1]}} \sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} \left(\gamma_{\partial_{T^*}}^{[n]}\left(\hat{n}\cdot\left(\kappa\,\nabla\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right)\right)\right),\gamma_{\partial_{T^*}}^{[n]}\left(w_{\nabla,\Omega}^{[n]}\right)\right)_{\partial_{T^*}}\mathrm{d}t\\
&\quad -\int_{t^{[n-1]}}^{t^{[n]}} \frac{t-t^{[\theta,n]}}{t^{[n]}-t^{[n-1]}} \sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} \left(\nabla\cdot\left(\kappa\,\nabla\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right)\right),w_{\nabla,\Omega}^{[n]}\right)_{T^*}\mathrm{d}t\\
&= -\int_{t^{[n-1]}}^{t^{[n]}} \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} \left(J_{E^*}^{[n]},\gamma_{E^*}^{[n]}\left(w_{\nabla,\Omega}^{[n]}\right)\right)_{E^*}\mathrm{d}t - \int_{t^{[n-1]}}^{t^{[n]}} \sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} \left(R_{T^*}^{[n]},w_{\nabla,\Omega}^{[n]}\right)_{T^*}\mathrm{d}t\\
&\quad + \int_{t^{[n-1]}}^{t^{[n]}} \left\langle\frac{\partial\, e_{h,\Delta t}}{\partial t},w_{\nabla,\Omega}^{[n]}\right\rangle\mathrm{d}t - \int_{t^{[n-1]}}^{t^{[n]}} \left(\kappa\,\nabla\, e_{h,\Delta t},\nabla\, w_{\nabla,\Omega}^{[n]}\right)\mathrm{d}t\\
&\quad + \int_{t^{[n-1]}}^{t^{[n]}} \left(f-\Pi_{\mathcal{T}} f,w_{\nabla,\Omega}^{[n]}\right)\mathrm{d}t + \int_{t^{[n-1]}}^{t^{[n]}} \left(\Pi_{\mathcal{T}} f-\theta\,\Pi_{\mathcal{T}} f^{[n]}-(1-\theta)\,\Pi_{\mathcal{T}} f^{[n-1]},w_{\nabla,\Omega}^{[n]}\right)\mathrm{d}t\,.
\end{aligned}$$

Now we observe that the discrete equation (5) can be written as

$$\sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} \left(R_{T^*}^{[n]},v_h\right)_{T^*} + \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} \left(J_{E^*}^{[n]},\gamma_{E^*}^{[n]}(v_h)\right)_{E^*} = 0,$$

then we insert this equation with the test function $v_h = I_h(w_{\nabla,\Omega}^{[n]})\in \mathrm{V}_h^{[n]}$ in the previous one, obtaining

$$\begin{aligned}
\left(\theta^2-\theta+\frac{1}{3}\right)\Delta t^{[n]}\left\|\sqrt{\kappa}\,\nabla\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right)\right\|_0^2 &\le \int_{t^{[n-1]}}^{t^{[n]}} \sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} \left\| J_{E^*}^{[n]}\right\|_{0,E^*}\left\| w_{\nabla,\Omega}^{[n]}-I_h(w_{\nabla,\Omega}^{[n]})\right\|_{0,E^*}\mathrm{d}t\\
+\int_{t^{[n-1]}}^{t^{[n]}} \sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} \left\| R_{T^*}^{[n]}\right\|_{0,T^*}\left\| w_{\nabla,\Omega}^{[n]}-I_h(w_{\nabla,\Omega}^{[n]})\right\|_{0,T^*}\mathrm{d}t &+ \int_{t^{[n-1]}}^{t^{[n]}} \left\|\frac{\partial\, e_{h,\Delta t}}{\partial t}\right\|_{\kappa,-1}\left\|\sqrt{\kappa}\,\nabla\, w_{\nabla,\Omega}^{[n]}\right\|_0\mathrm{d}t\\
+\int_{t^{[n-1]}}^{t^{[n]}} \left\|\sqrt{\kappa}\,\nabla\, e_{h,\Delta t}\right\|_0\left\|\sqrt{\kappa}\,\nabla\, w_{\nabla,\Omega}^{[n]}\right\|_0\mathrm{d}t &+ \int_{t^{[n-1]}}^{t^{[n]}} \left\| f-\Pi_{\mathcal{T}} f\right\|_{\kappa,-1}\left\|\sqrt{\kappa}\,\nabla\, w_{\nabla,\Omega}^{[n]}\right\|_0\mathrm{d}t\\
&+ \int_{t^{[n-1]}}^{t^{[n]}} \left\|\Pi_{\mathcal{T}} f-\theta\,\Pi_{\mathcal{T}} f^{[n]}-(1-\theta)\,\Pi_{\mathcal{T}} f^{[n-1]}\right\|_{\kappa,-1}\left\|\sqrt{\kappa}\,\nabla\, w_{\nabla,\Omega}^{[n]}\right\|_0\mathrm{d}t\,.
\end{aligned}$$

Now we apply *quasi-interpolation inequalities* (11), (12) and Hölder inequality to get

$$\left(\theta^2-\theta+\frac{1}{3}\right)\Delta t^{[n]}\left\|\sqrt{\kappa}\,\nabla\left(u_{h,\Delta t}^{[n]}-u_{h,\Delta t}^{[n-1]}\right)\right\|_0^2 \leq \left[Cl_E\sqrt{\Delta t^{[n]}\sum_{E^*\in\mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]}\left\|\frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}}J_{E^*}^{[n]}\right\|_{0,E^*}^2}\right.$$

$$\left.+Cl_R\sqrt{\Delta t^{[n]}\sum_{T^*\in\mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]2}\left\|\frac{1}{\sqrt{\kappa_{T^*}}}R_{T^*}^{[n]}\right\|_0^2}+2\left(|\!|\!|\, e_{h,\Delta t}\,|\!|\!|_{\kappa,I^{[n]}}^2+\left(\eta_f^{[n]}\right)^2\right)^{\frac{1}{2}}\right]$$

$$\times\sqrt{\int_{t^{[n-1]}}^{t^{[n]}}\sum_{T^*\in\mathcal{T}_h^{[n-1,n]}}\left\|\sqrt{\kappa_{T^*}}\,\nabla\, w_{\nabla,\Omega}^{[n]}\right\|_{0,T^*}^2\mathrm{d}t}$$

from which we derive (34) observing that $\forall\theta\in\mathbb{R}$, $\theta^2-\theta+\frac{1}{3}\geq\frac{1}{12}$. □

3.3.4. *Final lower bounds*

Now we want to collect the results of Propositions 3.15, 3.18 and 3.19 to get a lower bound for the error $\sqrt{|\!|\!|\, e_{h,\Delta t}\,|\!|\!|_{\kappa,I^{[n]}}^2+\left(\eta_f^{[n]}\right)^2}$ in term of $\eta_\nabla^{[n]}$. Then we use this result to derive a lower bound for the same quantity with respect to $\eta_R^{[n]}$.

Theorem 3.20. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5)*, on each time interval* $\left(t^{[n-1]},t^{[n]}\right)$ *there exists a constant* $\tilde{c}_{[n-1]}^{[n]}$ *independent of any meshsize, timestep and problem-parameter, but depending on the quality of the mesh* $\mathcal{T}_h^{[n]}$ *and on* C_{tr} *such that*

$$\eta_\nabla^{[n]}\leq\tilde{c}_{[n-1]}^{[n]}\left(|\!|\!|\, e_{h,\Delta t}\,|\!|\!|_{\kappa,I^{[n]}}^2+\left(\eta_f^{[n]}\right)^2\right)^{\frac{1}{2}}. \tag{35}$$

Proof. We use (27), (33) and (34) to get

$$\eta_\nabla^{[n]}\leq 2\sqrt{3}\left[\left[Cl_EC_E\left(C_E^*+C_RC_R^*\right)+Cl_RC_RC_R^*\right]2\frac{\alpha+1}{\sqrt{2\alpha+1}}+2\right]\left(|\!|\!|\, e_{h,\Delta t}\,|\!|\!|_{\kappa,I^{[n]}}^2+\left(\eta_f^{[n]}\right)^2\right)^{\frac{1}{2}}$$

$$+2\sqrt{3}\left(Cl_EC_E\left(C_E^*+C_RC_R^*\right)+Cl_RC_RC_R^*\right)\left|\theta-\frac{\alpha+1}{\alpha+2}\right|\eta_\nabla^{[n]}.$$

We define $C=2\sqrt{3}\max\left\{Cl_EC_E\left(C_E^*+C_RC_R^*\right)+Cl_RC_RC_R^*,1\right\}$ and we play with the parameter α to get

$$C\left|\theta-\frac{\alpha+1}{\alpha+2}\right|\leq\frac{1}{2}\cdot \tag{36}$$

By simple algebraic manipulations it is straightforward to see that exist values of $\alpha\geq 0$ satisfying (36). Being (36) satisfied we conclude that

$$\eta_\nabla^{[n]}\leq 4\left[C\frac{\alpha+1}{\sqrt{2\alpha+1}}+2\sqrt{3}\right]\left(|\!|\!|\, e_{h,\Delta t}\,|\!|\!|_{\kappa,I^{[n]}}^2+\left(\eta_f^{[n]}\right)^2\right)^{\frac{1}{2}}. \tag{37}$$

□

Now we collect the previous results to get a lower bound for the error in term of the residual $\eta_R^{[n]}$.

Theorem 3.21. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5)*, on each time interval* $\left(t^{[n-1]}, t^{[n]}\right)$ *there exists a constant* $\tilde{\tilde{c}}_{[n-1]}^{[n]}$ *independent of any meshsize, timestep and problem-parameter, but depending on the quality of the mesh* $\mathcal{T}_h^{[n]}$ *and on* C_{tr} *such that*

$$\eta_R^{[n]} \leq \tilde{\tilde{c}}_{[n-1]}^{[n]} \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2 \right)^{\frac{1}{2}}. \tag{38}$$

Proof. If $\theta = 1/2$, $\alpha = 0$ and the thesis comes immediately from (27) and (33), else we consider the square of inequalities (27), (33) and (37), Young's inequality to get

$$\sum_{T^* \in \mathcal{T}_h^{[n-1,n]}} h_{T^*}^{[n-1,n]2} \left\| \frac{1}{\sqrt{\kappa_{T^*}}} R_{T^*}^{[n]} \right\|_{0,T^*}^2 \Delta t^{[n]} \leq 2C_R^2 \left(C_R^*\right)^2 \left[4 \frac{(\alpha+1)^2}{2\alpha+1} \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2 \right) + \frac{1}{4C^2} \left(\eta_\nabla^{[n]}\right)^2 \right],$$

$$\sum_{E^* \in \mathcal{E}_{h,\Omega}^{[n-1,n]}} h_{E^*}^{[n-1,n]} \left\| \frac{1}{\sqrt{\hat{\kappa}_{\omega_{E^*}^{[n]}}}} J_{E^*}^{[n]} \right\|_{0,E^*}^2 \Delta t^{[n]} \leq 2C_E^2 (C_E^* + C_R C_R^*)^2 \left[4 \frac{(\alpha+1)^2}{2\alpha+1} \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2 \right) + \frac{1}{4C^2} \left(\eta_\nabla^{[n]}\right)^2 \right],$$

$$\left(\eta_\nabla^{[n]}\right)^2 \leq 16 \left(C \frac{\alpha+1}{\sqrt{2\alpha+1}} + 2\sqrt{3} \right)^2 \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2 \right).$$

We define $D^2 = 2C_R^2 \left(C_R^*\right)^2 + 2C_E^2 \left(C_E^* + C_R C_R^*\right)^2$ and we get

$$\left(\eta_R^{[n]}\right)^2 \leq 4D^2 \left(\frac{(\alpha+1)^2}{2\alpha+1} + \left(\frac{\alpha+1}{\sqrt{2\alpha+1}} + \frac{2\sqrt{3}}{C} \right)^2 \right) \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2 \right).$$

□

We finally state the lower bound collecting Theorems 3.20 and 3.21.

Theorem 3.22. *Under the assumptions on the continuous problem* (4) *and on the discrete formulation* (5)*, there exists a constant* $C_{\downarrow,[n-1]}^{[n]}$ *independent of any meshsize, timestep, problem-parameter, but depending on the parameter* θ*, on the smallest angle of the triangulation* $\mathcal{T}_h^{[n]}$ *and on* C_{tr} *such that the following inequalities hold true*

$$\left(\eta_R^{[n]}\right)^2 + \left(\eta_\nabla^{[n]}\right)^2 \leq C_{\downarrow,[n-1]}^{[n]} \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2 \right), \tag{39}$$

$$\sum_{n=1}^{m} \left(\left(\eta_R^{[n]}\right)^2 + \left(\eta_\nabla^{[n]}\right)^2 \right) \leq C_{\downarrow,[0]}^{[N]} \sum_{n=1}^{m} \left(\| e_{h,\Delta t} \|_{\kappa, I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2 \right), \qquad m = 1, ..., N, \tag{40}$$

where $C_{\downarrow,[0]}^{[N]} = \max_{n=1,\dots,N} C_{\downarrow,[n-1]}^{[n]}$.

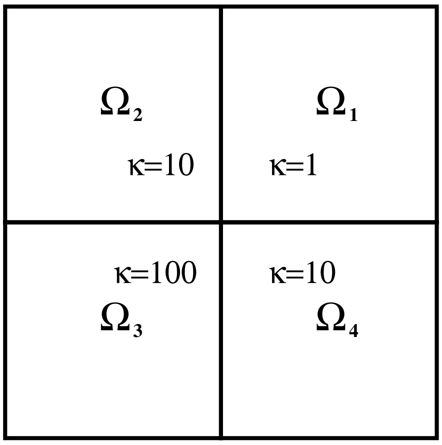

FIGURE 3. Test Problem 1: domain.

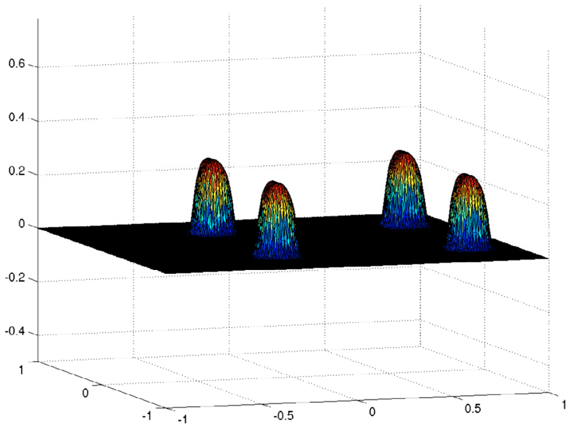

FIGURE 4. Test Problem 1: solution, $t = 0$.

4. Numerical results on uniform grids and constant timestep-length

In this section we want to compare the true error and the error estimator in the norms we have defined in the previous sections for some test problem in order to get some indication about the values of the constants and the sharpness of the estimates with respect to the meshsize and the timestep-length. For this reason we consider different uniform meshes and different constant timestep-lengths. We consider some challenging test problem that are designed to be good test problems for an adaptive algorithm, so their solutions are obtained by the sum of different functions each one describing a structure evolving with a different velocity. These solutions display sharp internal layers that make the solution hard to describe on uniform grids. One of our target is to see how we can detect each one of the possible errors (poor time discretization or poor mesh) by the values of the different components of the error estimator. The domains of the test problems we consider are shown in Figures 3 and 13 and a frame of their solutions in Figures 4 and 14, respectively. All the $\| v \|_{\kappa,-1}$-norms are approximated by the $\| r \|_{\kappa,1}$-norms of the solution of the problem $\nabla \cdot (\kappa \nabla r) = v$ with homogeneous Dirichlet boundary conditions. The norms $\int_{t^{[n-1]}}^{t^{[n]}} \| v \|_{\kappa,-1}^2 \mathrm{d}t$ are then computed by a Gaussian quadrature formula with three nodes in time.

4.1. **Test case 1**

The problem (1)–(3) is solved in the domain $\Omega = (-1,1) \times (-1,1)$ and in the time interval $(0,\Xi) = (0,0.5)$ with $\kappa = 1$ in $(0,1) \times (0,1)$, with $\kappa = 10$ in $(-1,0) \times (0,1)$ and in $(0,1) \times (-1,0)$, and with $\kappa = 100$ in $(-1,0) \times (-1,0)$ (Fig. 3). The initial condition $u^{[0]}(x,y)$ and the forcing function $f(x,y,t)$ are defined such that the solution is

$$u(x,y,t) = \begin{cases} U(x,y,t,1), & \text{if } x > 0,\ y > 0, \\ U(x+1,y,t,10), & \text{if } x < 0,\ y > 0, \\ U(x+1,y+1,t,100), & \text{if } x < 0,\ y < 0, \\ U(x,y+1,t,10), & \text{if } x > 0,\ y < 0, \\ 0, & \text{if } x = 0 \text{ or } y = 0, \end{cases}$$

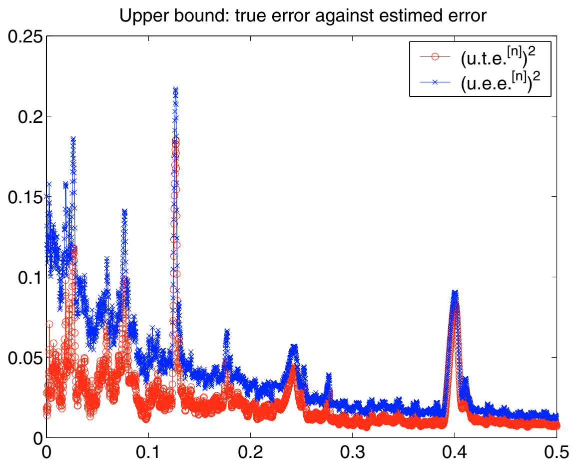

FIGURE 5. Test Problem 1: Comparison between the true error and the estimated error of the upper bound.

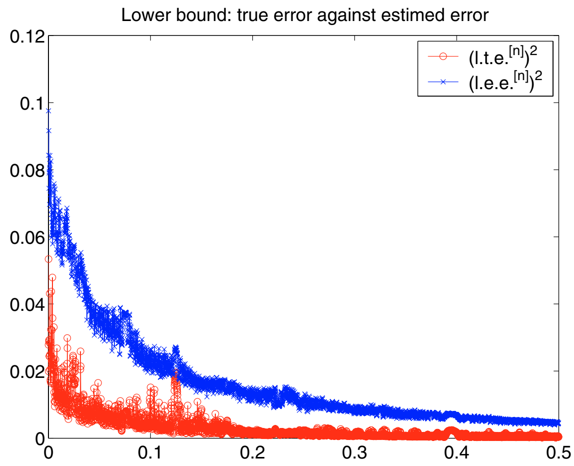

FIGURE 6. Test Problem 1: Comparison between the true error and the estimated error of the lower bound.

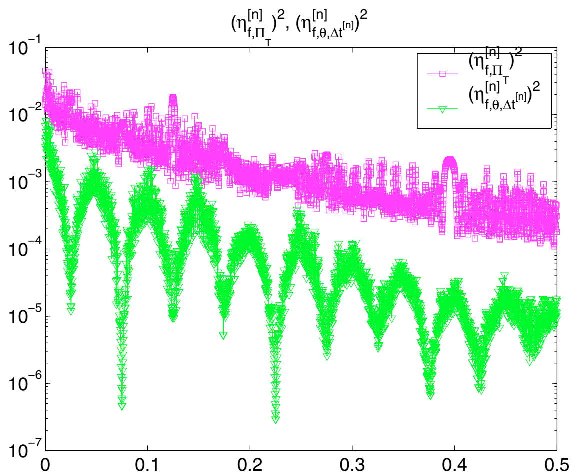

FIGURE 7. Test Problem 1: Data approximation errors.

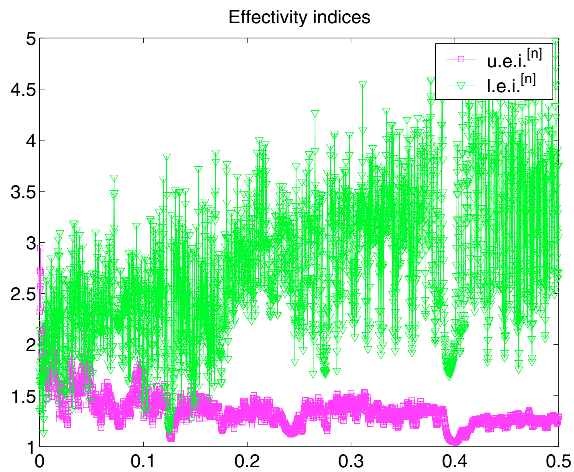

FIGURE 8. Test Problem 1: Upper and lower effectivity indices.

where

$$U(x,y,t,\kappa) = 500\,x^2\,(1-x)^2\,y^2\,(1-y)^2$$
$$\mathrm{e}^{-\mathrm{e}^{\left(R_3 - R_4\,\sqrt{\kappa t}\right)}\left(\left(x-\frac{1}{2}-\frac{1}{4}\,\cos\left(2\,\pi\,\left(1+\sin\left(2\,\pi\,\sqrt{\kappa t}\right)\right)\right)\right)^2+\left(y-\frac{1}{2}-\frac{1}{4}\,\sin\left(2\,\pi\,\left(1+\sin\left(2\,\pi\,\sqrt{\kappa t}\right)\right)\right)\right)^2\right)^4}$$
$$\left(1-\mathrm{e}^{-R_5\,\left(\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\right)x(1-x)\,y(1-y)}\right)\frac{1}{1+\ln\left(1+\sqrt{\kappa t}\right)}.$$

This solution is made by four Gaussian peaks running and smearing with different velocities; the velocity of each peak is proportional to the value of κ in the corresponding portion of the domain (Fig. 4). The parameters R_i are: $R_3 = 18$, $R_4 = 1$, $R_5 = 100$. In Figures 5–8 we plot some results concerning a constant uniform discretization with a number of nodes $N_{node} = 2113$ and $N = 3200$ uniform timesteps. In Figure 5 we compare the behaviour

Table 1. Test Problem 1: $N_{node} = 2113$, $N = 3200$.

n	$\left\| u_{h,\Delta t}^{[n]} - u^{[n]} \right\|_2^2$	$\left(\varepsilon_{\kappa,-1}^{[n]}\right)^2$	$\left(\varepsilon_{\kappa,1}^{[n]}\right)^2$	$\left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2$	$\left(\eta_{f,\Pi_T}^{[n]}\right)^2$	$\left(\eta_{\nabla}^{[n]}\right)^2$	$\left(\eta_R^{[n]}\right)^2$
0	1.39E-04	0.00	0.00	0.00	0.00	0.00	0.00
1	3.73E-03	1.50E-03	2.73E-04	6.85E-03	4.48E-02	2.66E-02	7.09E-02
800	1.59E-02	1.24E-03	1.96E-04	1.08E-05	1.81E-02	1.37E-05	2.62E-02
1600	2.88E-03	8.90E-05	1.05E-04	2.15E-04	1.77E-03	5.28E-05	9.49E-03
2400	1.33E-03	5.87E-05	8.90E-05	1.26E-06	6.12E-04	2.26E-06	6.97E-03
3200	1.00E-03	6.28E-05	6.25E-05	1.69E-05	1.80E-04	1.58E-05	4.53E-03

Table 2. Test Problem 1: $N_{node} = 2113$, $N = 3200$.

	$(e.i.)_{max}$	$(e.i.)_{min}$	$(e.i.)_{mean}$	$(e.i.)_{rms}$
upper	2.9447	1.0443	1.3576	1.7263E-1
lower	4.9689	1.1380	2.7774	6.8730E-1

of the true error for which we derive an upper bound $\left(u.t.e.^{[n]}\right)^2 = 7\left\| u_{h,\Delta t}^{[n]} - u^{[n]} \right\|_0^2 + \left\| e_{h,\Delta t}^{[n]} \right\|_{\kappa,I^{[n]}}^2$ and the upper error estimator $\left(u.e.e.^{[n]}\right)^2 = 7\left\| u_{h,\Delta t}^{[n-1]} - u^{[n-1]} \right\|_0^2 + \left(\eta_R^{[n]}\right)^2 + \left(\eta_{\nabla}^{[n]}\right)^2 + \left(\eta_f^{[n]}\right)^2$. We can observe a perfect agreement between the true error and the error estimator. In Figure 6 we compare the behaviour of the true error for which we derive the lower bound $\left(l.t.e.^{[n]}\right)^2 = \left\| e_{h,\Delta t}^{[n]} \right\|_{\kappa,I^{[n]}}^2 + \left(\eta_f^{[n]}\right)^2$ and the lower error estimator $\left(l.e.e.^{[n]}\right)^2 = \left(\eta_R^{[n]}\right)^2 + \left(\eta_{\nabla}^{[n]}\right)^2$, again the agreement is good. In Figure 7 we plot the two data approximation errors and in Figure 8 we plot, for each timestep, the effectivity indices $u.e.i.^{[n]} = u.e.e.^{[n]} \,/\, u.t.e.^{[n]}$ and $l.e.i^{[n]} = l.e.e.^{[n]} \,/\, l.t.e.^{[n]}$. We remark that the values of this effectivity indices are not far from one.

In Table 1, we report some values of the terms of the local-in-time global-in-space true error and error estimator at $t = 0$, at the end of the first time-step and at the times $t = 0.125, 0.25, 0.375, 0.5$ for $N_{node} = 2113$ and $N = 3200$ timesteps. We define $\left(\varepsilon_{\kappa,-1}^{[n]}\right)^2 = \int_{t^{[n-1]}}^{t^{[n]}} \left\| \frac{\partial e_{h,\Delta t}^{[n]}}{\partial t} \right\|_{\kappa,-1}^2 \mathrm{d}t$ and $\left(\varepsilon_{\kappa,1}^{[n]}\right)^2 = \int_{t^{[n-1]}}^{t^{[n]}} \left\| e_{h,\Delta t}^{[n]} \right\|_{\kappa,1}^2 \mathrm{d}t$. In Table 2, we report the maximal and minimal values of the effectivity indices for the upper and the lower estimate. In these table we report also the mean values and the root mean square of these values. In Table 3, we report some values of the terms of the global-in-time and global-in-space true error and error estimator at the times $t = 0.125, 0.25, 0.375, 0.5$. In Figures 9 and 10 we plot the quantities η_{∇}^2 and $\eta_{f,\theta,\Delta t^{[n]}}^2$ with respect to the number of timesteps N for two different constant uniform grids. We can clearly see that these quantities are strongly dependent on the timestep-length. In Figures 11 and 12 we plot the quantities η_R^2 and η_{f,Π_T}^2 with respect to the number of timesteps N for the same grids. We can clearly see that these quantities are essentially independent of the timestep-length, but they are dependent on the mesh-size. These behaviours justify the Remarks 3.7 and 3.8.

TABLE 3. Test Problem 1: $N_{node} = 2113$, $N = 3200$.

m	$\sum_{n=1}^{m}\left(\varepsilon_{\kappa,-1}^{[n]}\right)^2$	$\sum_{n=1}^{m}\left(\varepsilon_{\kappa,1}^{[n]}\right)^2$	$\sum_{n=1}^{m}\left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2$	$\sum_{n=1}^{m}\left(\eta_{f,\Pi_T}^{[n]}\right)^2$	$\sum_{n=1}^{m}\left(\eta_{\nabla}^{[n]}\right)^2$	$\sum_{n=1}^{m}\left(\eta_{R}^{[n]}\right)^2$
800	5.40E-01	1.74E-01	5.09E-01	5.60	6.44E-01	2.95E+01
1600	7.19E-01	2.83E-01	6.22E-01	7.26	7.40E-01	4.11E+01
2400	8.03E-01	3.62E-01	6.51E-01	7.90	7.72E-01	4.78E+01
3200	8.68E-01	4.26E-01	6.59E-01	8.34	7.84E-01	5.22E+01

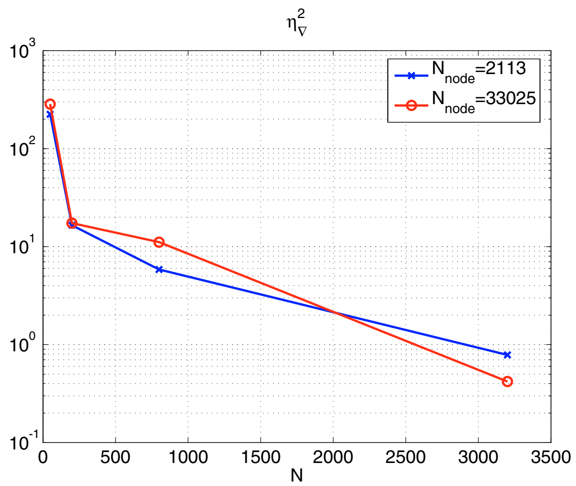

FIGURE 9. Test Problem 1: η_{∇}^2 for different timesteps.

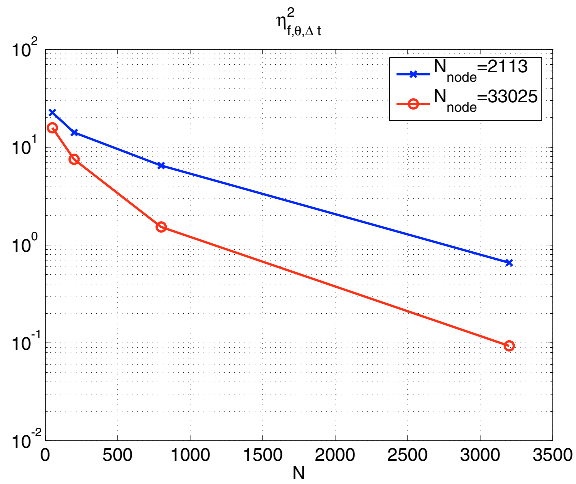

FIGURE 10. Test Problem 1: $\eta_{f,\theta,\Delta t^{[n]}}^2$ for different timesteps.

4.2. **Test case 2**

The problem (1)–(3) is solved in the domain $\Omega = (-1,1)\times(0,1)$ and in the time interval $(0,\Xi) = (0,1)$ with $\kappa = 1$ in $(0,1)\times(0,1)$ and with $\kappa = 100$ in $(-1,0)\times(0,1)$ (Fig. 13). The initial condition $u^{[0]}(x,y)$ and the forcing function $f(x,y,t)$ are defined such that the solution is

$$u(x,y,t) = \begin{cases} u_1(x,y,t), & \text{if } x \geq 0,\ y \geq 0,\\ u_2(x,y,t), & \text{if } x < 0,\ y \geq 0,\end{cases}$$

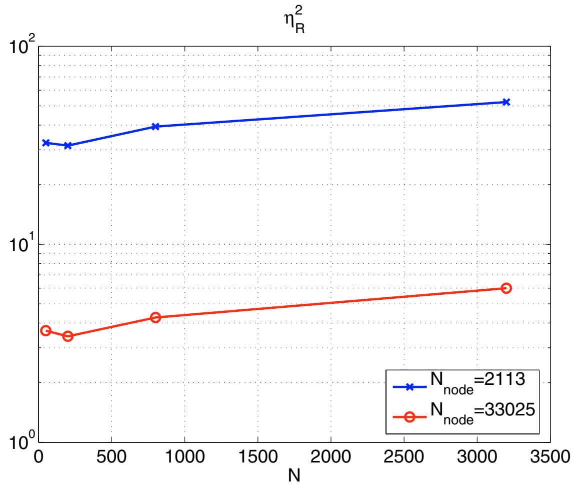

FIGURE 11. Test Problem 1: η_R^2 for different timesteps.

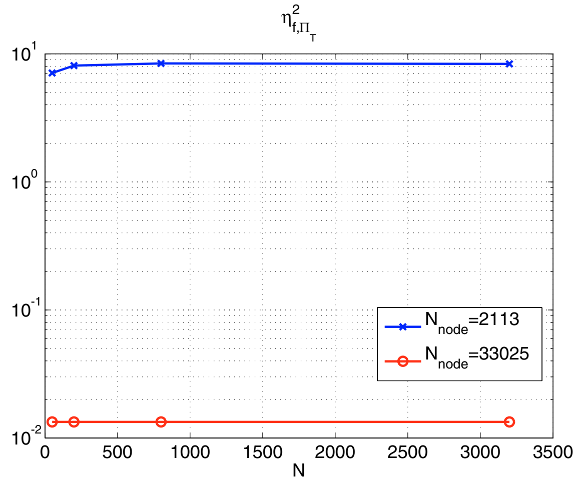

FIGURE 12. Test Problem 1: η^2_{f,Π_T} for different timesteps.

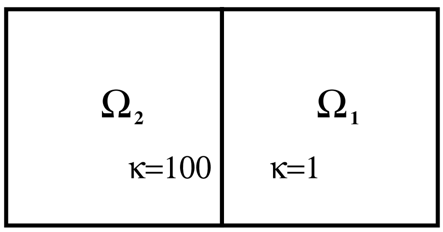

FIGURE 13. Test Problem 2: domain.

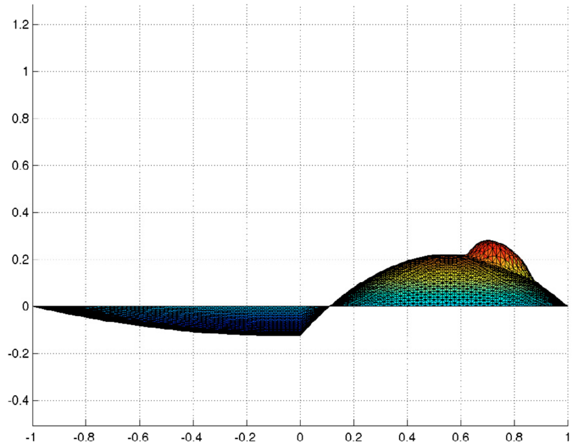

FIGURE 14. Test Problem 2: solution, $t = 0.74$.

with

$$u_1(x,u,t) = 500x^2(1-x)^2 y^2\,(1-y)^2 \mathrm{e}^{-\mathrm{e}^{R_3-\kappa t}\left(\left(x-\frac{1}{2}-\frac{1}{4}\cos(2\pi(1+\sin(2\pi\kappa t)))\right)^2+\left(y-\frac{1}{2}-\frac{1}{4}\sin(2\pi(1+\sin(2\pi\kappa t)))\right)^2\right)^4}$$
$$\times \frac{1-\mathrm{e}^{-R_4\left(\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\right)x(1-x)\,y(1-y)}}{1+\ln(1+\kappa t)} + \left(\left(-\frac{R_5}{R_1}-\sin(2\pi t)\right)x^2+\frac{R_5 x}{R_1}+\sin(2\pi t)\right)y\,(1-y)$$

and

$$u_2(x,y,t) = \left(\left(\frac{R_5}{R_2}-\sin(2\pi t)\right)x^2+\frac{R_5 x}{R_2}+\sin(2\pi t)\right)y\,(1-y)\,.$$

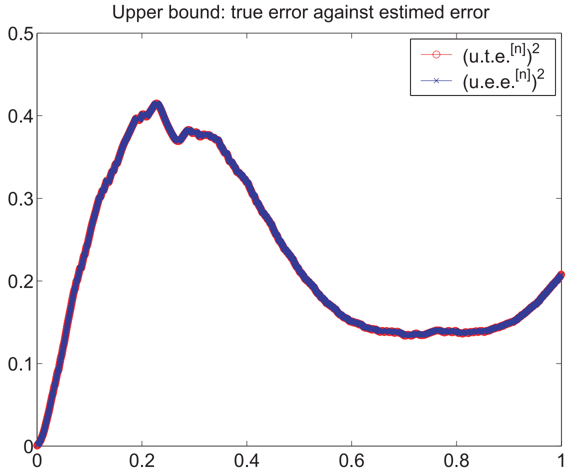

FIGURE 15. Test Problem 2: Comparison between the true error and the estimated error of the upper bound.

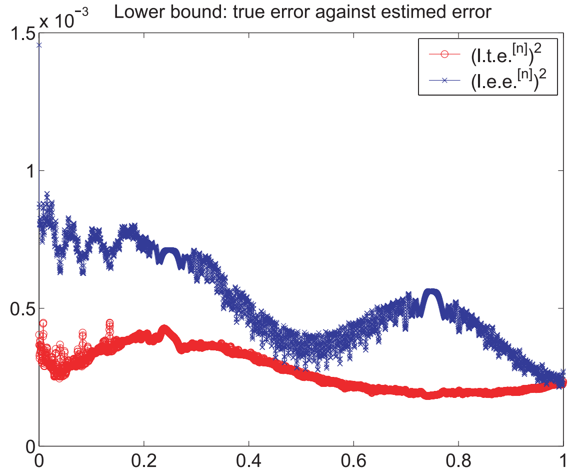

FIGURE 16. Test Problem 2: Comparison between the true error and the estimated error of the lower bound.

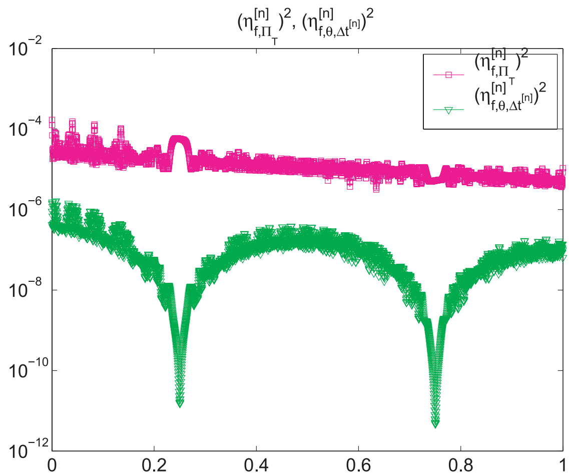

FIGURE 17. Test Problem 2: Data approximation errors.

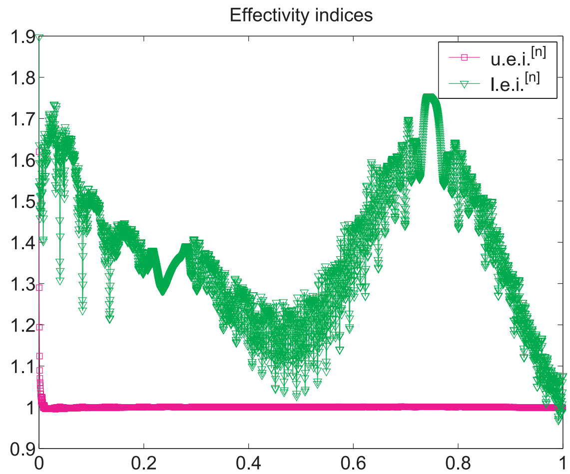

FIGURE 18. Test Problem 2: Upper and lower effectivity indices.

The normal derivatives $\partial u/\partial x$ of this function on the edge $x = 0,\ 0 \le y \le 1$ are discontinuous with continuous $\kappa\,\partial u/\partial x$. Moreover the function u in $x > 0,\ y > 0$ includes a Gaussian peak moving at a velocity proportional to κ (Fig. 14). The parameters R_i are: $R_1 = 1$, $R_2 = 100$, $R_3 = 18$, $R_4 = 100$, $R_5 = 10$.

In Figures 15–18 we plot some results concerning a uniform discretization with a number of nodes $N_{node} = 2145$ and $N = 3200$ uniform timesteps. In Figure 15 we compare the behaviour of the error for which we derive an upper bound and the upper error estimator. We can observe a perfect agreement between the error and the error estimator. In Figure 16 we compare the behaviour of the error for which we derive the lower bound and the lower error estimator, again the agreement is good. In Figure 17 we plot the two data approximation errors and in Figure 18 we plot the effectivity index for each timestep for the upper and lower error estimate. Again the values of this effectivity indices are close to one.

TABLE 4. Test Problem 2: $N_{node} = 2145$, $N = 3200$.

n	$\left\| u^{[n]}_{h,\Delta t} - u^{[n]} \right\|^2_2$	$\left(\varepsilon^{[n]}_{\kappa,-1}\right)^2$	$\left(\varepsilon^{[n]}_{\kappa,1}\right)^2$	$\left(\eta^{[n]}_{f,\theta,\Delta t^{[n]}}\right)^2$	$\left(\eta^{[n]}_{f,\Pi_T}\right)^2$	$\left(\eta^{[n]}_{\nabla}\right)^2$	$\left(\eta^{[n]}_{R}\right)^2$
0	1.55E-05	0.00	0.00	0.00	0.00	0.00	0.00
1	6.08E-05	1.71E-04	6.40E-05	4.14E-07	1.69E-04	4.64E-05	1.41E-03
800	5.52E-02	4.50E-06	3.47E-04	1.87E-11	5.69E-05	6.26E-10	7.12E-04
1600	3.04E-02	9.60E-06	2.36E-04	1.72E-07	7.94E-06	1.61E-07	3.97E-04
2400	1.96E-02	3.80E-07	1.78E-04	6.24E-12	5.21E-06	1.20E-10	5.62E-04
3200	2.97E-02	1.06E-05	2.13E-04	6.37E-08	1.07E-05	1.12E-07	2.72E-04

TABLE 5. Test Problem 2: $N_{node} = 2145$, $N = 3200$.

	$(e.i.)_{max}$	$(e.i.)_{min}$	$(e.i.)_{mean}$	$(e.i.)_{rms}$
upper	1.6196	0.9956	1.0010	1.3114E-2
lower	1.8976	0.9685	1.3756	1.7288E-1

TABLE 6. Test Problem 2: $N_{node} = 2145$, $N = 3200$.

m	$\sum_{n=1}^{m}\left(\varepsilon^{[n]}_{\kappa,-1}\right)^2$	$\sum_{n=1}^{m}\left(\varepsilon^{[n]}_{\kappa,1}\right)^2$	$\sum_{n=1}^{m}\left(\eta^{[n]}_{f,\theta,\Delta t^{[n]}}\right)^2$	$\sum_{n=1}^{m}\left(\eta^{[n]}_{f,\Pi_T}\right)^2$	$\sum_{n=1}^{m}\left(\eta^{[n]}_{\nabla}\right)^2$	$\sum_{n=1}^{m}\left(\eta^{[n]}_{R}\right)^2$
800	2.80E-02	2.30E-01	1.94E-04	2.30E-02	2.84E-04	5.92E-01
1600	3.23E-02	4.79E-01	2.79E-04	3.62E-02	4.03E-04	1.02
2400	3.57E-02	6.40E-01	3.49E-04	4.38E-02	4.98E-04	1.37
3200	3.84E-02	7.93E-01	3.93E-04	4.89E-02	5.62E-04	1.67

In Table 4, we report some values of the terms of the local-in-time true error and error estimator at the time $t = 0$, at the end of the first time-step and at the times $t = 0.25, 0.5, 0.75, 1$, for $N_{node} = 2145$ and $N = 3200$. In Table 5, we report the maximal and minimal values of the effectivity indices for the upper and the lower estimate with their mean values and the root mean square. In Table 6, we report some values of the terms of the global-in-time true error and error estimator at the times $t = 0.25, 0.5, 0.75, 1$.

In Figures 19 and 20 we plot the quantities η^2_{∇} and $\eta^2_{f,\theta,\Delta t^{[n]}}$ with respect to the number of timesteps N for two different grids. In Figures 21 and 22 we plot the quantities η^2_R and η^2_{f,Π_T} with respect to the number of timesteps N for the same grids. Again, these behaviours agree with Remarks 3.7 and 3.8.

5. CONCLUSIONS

From our numerical experiments we conclude that both the upper and the lower bounds are sharp and robust. Moreover, for the test problems here proposed, hard to solve with a uniform grid and with a constant

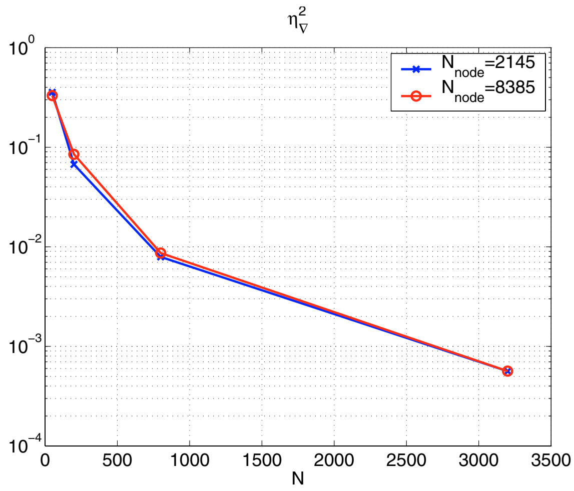

FIGURE 19. Test Problem 2: η_∇^2 for different timesteps.

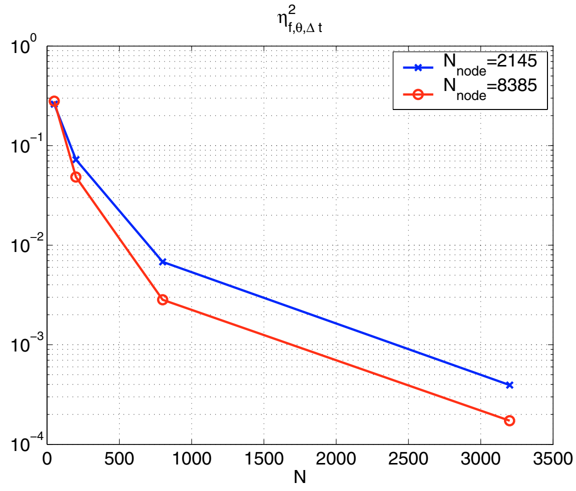

FIGURE 20. Test Problem 2: $\eta^2_{f,\theta,\Delta t^{[n]}}$ for different timesteps.

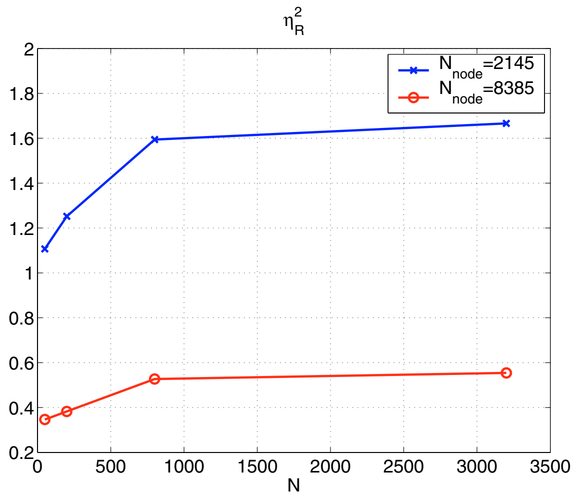

FIGURE 21. Test Problem 2: η_R^2 for different timesteps.

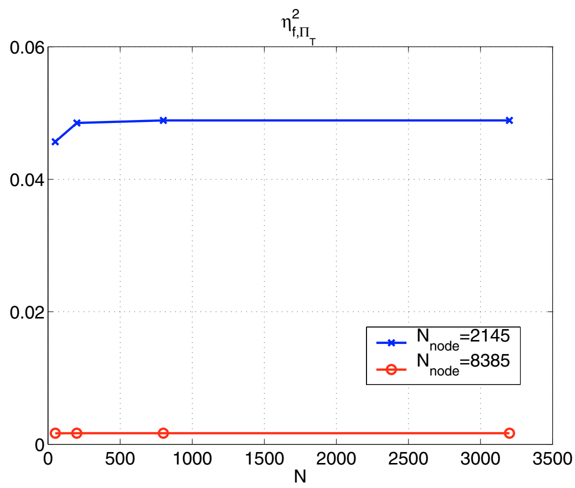

FIGURE 22. Test Problem 2: η^2_{f,Π_T} for different timesteps.

timestep-length, the components of the error estimator $\eta_\nabla^{[n]}$ and $\eta_R^{[n]}$ are good estimators respectively for the quality of the time-discretization and the space-discretization. Comparing these quantities for the same test problem on the same grid for different timesteps we find a stronger reduction of $\left(\eta_\nabla^{[n]}\right)^2$ and $\sum_{n=1}^m \left(\eta_\nabla^{[n]}\right)^2$ with respect to the changes of $\left(\eta_R^{[n]}\right)^2$ and $\sum_{n=1}^m \left(\eta_R^{[n]}\right)^2$ and comparing the results obtained with the same number of timesteps on different grids we find a stronger reduction of $\left(\eta_R^{[n]}\right)^2$ and $\sum_{n=1}^m \left(\eta_R^{[n]}\right)^2$ with respect to the changes of $\left(\eta_\nabla^{[n]}\right)^2$ and $\sum_{n=1}^m \left(\eta_\nabla^{[n]}\right)^2$. These remarks justify the idea to use $\eta_\nabla^{[n]}$ to adapt the timestep-length and $\eta_R^{[n]}$ to adapt the grid in each time-slab in an adaptive algorithm.

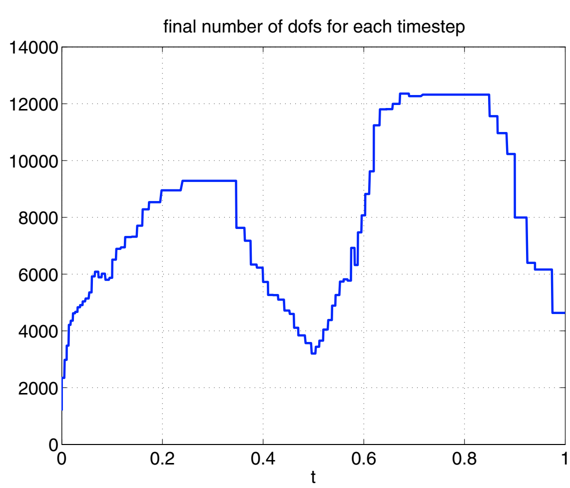

FIGURE 23. Test Problem 2: number of dofs for each timestep, $\mathrm{TOL}_\Omega = 0.4$, $\mathrm{TOL}_{I^{[n]}} = 0.2$.

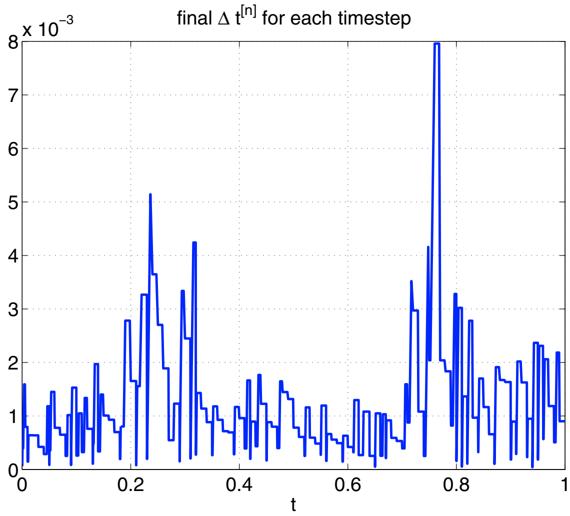

FIGURE 24. Test Problem 2: timestep-length for each timestep, $\mathrm{TOL}_\Omega = 0.4$, $\mathrm{TOL}_{I^{[n]}} = 0.2$.

Moreover, the behaviour showed in Figures 10, 12, 20 and 22 confirm the splitting of the data-approximation errors suggested in Remark 3.8.

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APPENDIX

A simple adaptive algorithm

In this section we present a simple possible use of the estimates presented in the previous sections to adapt the grid and the timestep-length for each timestep. The presented estimates can be applied in many different adaptive algorithms, possibly more complex and efficient than the one presented here. Our approach is based on equidistribution and on the splitting between space and time error estimators and data-approximation errors described in Remarks 3.7 and 3.8. First, let us define the following space error estimator on the triangles $T \in \mathcal{T}_h^{[n]}$:

$$\left(\eta_{R,T}^{[n]}\right)^2 = \sum_{\left\{T^* \in \mathcal{T}_h^{[n-1,n]} : T^* \subseteq T\right\}} \left(\eta_{R,T^*}^{[n]}\right)^2$$

and let us denote by N_T the number of elements in $\mathcal{T}_h^{[n]}$. We choose a space tolerance TOL_Ω and a time tolerance $\mathrm{TOL}_{I^{[n]}}$ for each timestep.

We mark for refining or coarsening each triangle $T \in \mathcal{T}_h^{[n]}$ according the following rules:

- `if`

$$\frac{(1+\alpha)^2\,\mathrm{TOL}_\Omega^2}{N_T} \, |\!|\!| u_{h,\Delta t} |\!|\!|_{\kappa, I^{[n]}}^2 < \left(\eta_{R,T}^{[n]}\right)^2 + \frac{\left(\eta_{f,\Pi_T}^{[n]}\right)^2}{N_T}$$

mark for refinement,

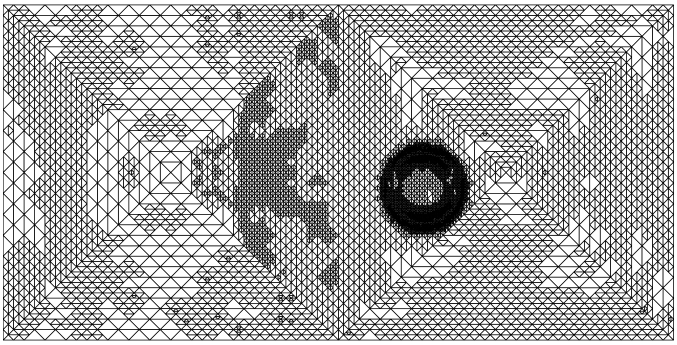

FIGURE 25. Test Problem 2: adapted mest at time $t = 0.1$, $\mathrm{TOL}_\Omega = 0.4$, $\mathrm{TOL}_{I^{[n]}} = 0.2$.

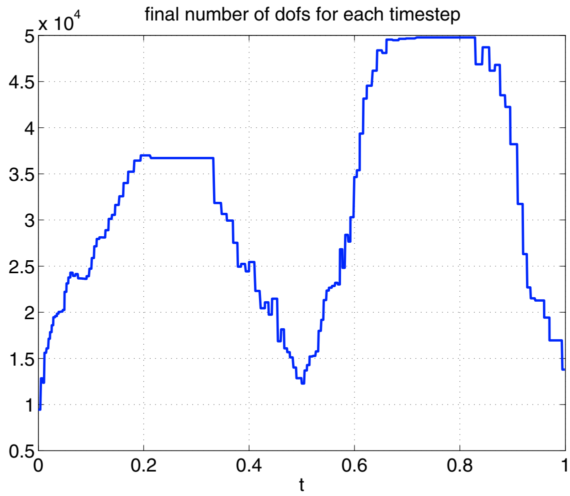

FIGURE 26. Test Problem 2: number of dofs for each timestep, $\mathrm{TOL}_\Omega = 0.2$, $\mathrm{TOL}_{I^{[n]}} = 0.2$.

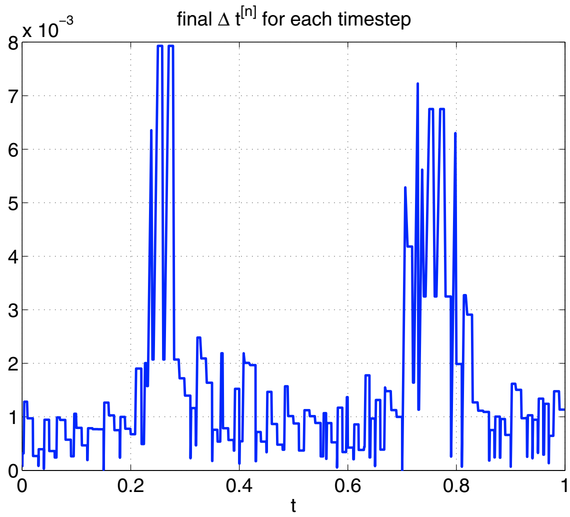

FIGURE 27. Test Problem 2: timestep-length for each timestep, $\mathrm{TOL}_\Omega = 0.2$, $\mathrm{TOL}_{I^{[n]}} = 0.2$.

- `else if`

$$\left(\eta_{R,T}^{[n]}\right)^2 < \frac{(1-\alpha)^2\,\mathrm{TOL}_\Omega^2}{N_T}\,\|\!|\, u_{h,\Delta t} \,\|\!|_{\kappa,I^{[n]}}^2$$

mark for coarsening.

We enlarge or shorten the timestep-length according to the following rules:

- `if`

$$(1+\alpha)^2\,\mathrm{TOL}_{I^{[n]}}^2\,\|\!|\, u_{h,\Delta t} \,\|\!|_{\kappa,I^{[n]}}^2 < \left(\eta_{\nabla}^{[n]}\right)^2 + \left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2$$

`then` $\Delta t^{[n]} := \dfrac{\Delta t^{[n]}}{\rho}$,

- `else if`

$$\left(\eta_{\nabla}^{[n]}\right)^2 < (1-\alpha)^2\,\mathrm{TOL}_{I^{[n]}}^2\,\|\!|\, u_{h,\Delta t} \,\|\!|_{\kappa,I^{[n]}}^2$$

`then` $\Delta t^{[n]} := \dfrac{\Delta t^{[n]}}{\rho}$, where

$$\rho = \min\left\{ \frac{\left(\eta_{\nabla}^{[n]}\right)^2 + \left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2}{\mathrm{TOL}_{I^{[n]}}^2 \, \| u_{h,\Delta t} \|_{\kappa,I^{[n]}}^2}, 2 \right\}.$$

We repeat the same timestep performing the mesh changes required by the previous marking strategy if $(1+\alpha)^2 \, \mathrm{TOL}_{\Omega}^2 \, \| u_{h,\Delta t} \|_{\kappa,I^{[n]}}^2 < \left(\eta_R^{[n]}\right)^2 + \left(\eta_{f,\Pi_T}^{[n]}\right)^2$ or if $\left(\eta_R^{[n]}\right)^2 < (1-\alpha)^2 \, \mathrm{TOL}_{\Omega}^2 \, \| u_{h,\Delta t} \|_{\kappa,I^{[n]}}^2$ at most ten times for each timestep. Moreover, we repeat the timestep also if $(1+\alpha)^2 \, \mathrm{TOL}_{I^{[n]}}^2 \, \| u_{h,\Delta t} \|_{\kappa,I^{[n]}}^2 < \left(\eta_{\nabla}^{[n]}\right)^2 + \left(\eta_{f,\theta,\Delta t^{[n]}}^{[n]}\right)^2$. If none of the previous situations occurs, we keep the same mesh and we possibly enlarge the timestep-length for the next timestep. Leaving the last mesh of the previous timestep unchanged we never need to project the solution at the end of the previous timestep on a different mesh for the current timestep so we never have any transmission error to consider.

Numerical experiments with the adaptive algorithm

In this subsection we present some numerical results obtained by a simplified version of our adaptive algorithm applied to the test case 2. In Figures 23 and 24 we report the final number of degrees of freedom and the timestep-length accepted for each timestep for the following choices of the tolerances: $\mathrm{TOL}_{\Omega} = 0.4$, $\mathrm{TOL}_{I^{[n]}} = 0.2$, $\alpha = 0.5$. In these computations we neglect the contribution of the data-approximation errors considering only the space error estimator $\eta_R^{[n]}$ and the data error estimator $\eta_{\nabla}^{[n]}$. This implementation is based on the library LibMesh [11], in these computations we allow the presence of hanging nodes and we do not coarsen the elements of the starting mesh, that is a uniform mesh with 1073 nodes. We point out that, due to the implementation of LibMesh, not all the elements marked for coarsening are actually coarsened, since some additional conditions have to be satisfied to have LibMesh coarsening such elements [11]. In Figure 25 we report the mesh obtained at time $t = 0.1$. In Figures 26 and 27 we report the same quantities of Figures 23 and 24 for the choices $\mathrm{TOL}_{\Omega} = 0.2$, $\mathrm{TOL}_{I^{[n]}} = 0.2$, $\alpha = 0.5$.

From Figures 23 and 26 we see that the adaptive algorithm recognize that the two big structures in the two subdomains are vanishing at times $t = 0$, $t = 0.5$ and $t = 1.0$. This means that the number of degrees of freedom required by the solution is only related to the small Gaussian peak moving in the second subdomain. On the other hand, the timestep-lengths plotted in Figures 24 and 27 reflect the fact that the Gaussian peak in the second subdomain is moving with a varying velocity and stops at times $t = 0.25$ and $t = 0.75$, allowing the use of larger timestep-lengths.

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