Existence of nontrivial solutions for semilinear problems with strictly differentiable nonlinearity

# EXISTENCE OF NONTRIVIAL SOLUTIONS FOR SEMILINEAR PROBLEMS WITH STRICTLY DIFFERENTIABLE NONLINEARITY

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The existence of a nontrivial solution for semilinear elliptic problems with strictly differentiable nonlinearity is proved. A result of homological linking under nonstandard geometrical assumption is also shown. Techniques of Morse theory are employed.



## 1. Introduction

Since the paper of Amann and Zehnder [1], the existence of nontrivial solutions $u$ for semilinear elliptic problems of the form

$$-\Delta u = g(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

with $g(0) = 0$, has been the object of several studies, in which topological and variational methods are successfully applied. We refer the reader to [2, 3, 8, 10]. In particular, since the combination of linking theorems and Morse theory has turned out to be very fruitful, it is customary to impose conditions on $g$ that guarantee that the associated functional $f : H_0^1(\Omega) \to \mathbb{R}$, given by

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \int_\Omega G(u)dx, \qquad G(s) = \int_0^s g(t)dt, \tag{1.2}$$

is of class $C^2$.

In a recent paper [12], Perera and Schechter have proved a result of Amann-Zehnder type under assumptions that imply $f$ to be only of class $C^1$. More precisely, about the regularity of $g$, they assume that $g$ is continuous, there exist in $\mathbb{R}$ the limits

$$\lim_{s\to-\infty} \frac{g(s)}{s}, \qquad \lim_{s\to+\infty} \frac{g(s)}{s}, \qquad \lim_{s\to0} \frac{g(s)}{s} \tag{1.3}$$

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and that

$$\frac{g(s)}{s} \text{ is Lipschitz continuous in a neighbourhood of } 0. \tag{1.4}$$

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One could observe that hypothesis (1.4) allows $f$ not to be of class $C^2$, but it does not include every $g$ satisfying the usual assumption that $g$ is of class $C^1$ and $g'$ is bounded. In particular, condition (1.4) is not stable if we add to $g$ a term of the form

$$\frac{|s|^{3/2}}{1+s^2}. \tag{1.5}$$

The first purpose of this paper is to extend the result of [12] in such a way that also the classical smooth case is included. Our result is the following.

Theorem 1.1. *Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $g(0) = 0$ and*

(a) *there exists $C \geq 0$ such that*

$$|g(s)| \leq C(1+|s|); \tag{1.6}$$

(b) *there exists $\alpha \in \mathbb{R}$ such that*

$$\lim_{s\to\pm\infty} \frac{g(s)}{s} = \alpha. \tag{1.7}$$

*If we denote by $(\lambda_m)$ the sequence of the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition, let us assume that $\alpha \neq \lambda_m$ for any $m \in \mathbb{N}$. Moreover, let us suppose that $g$ is strictly differentiable at 0 (see Definition 3.1 below) and that there exists $m \in \mathbb{N}$ with either $g'(0) < \lambda_m < \alpha$ or $g'(0) > \lambda_m > \alpha$.*

*Then (1.1) admits a nontrivial solution.*

Theorem 1.1 is in fact a particular case of a more general result, which will be presented in Section 2.

*Remark 1.2.* If, as in [12], we have $g(s) = s\gamma(s)$, with $\gamma$ Lipschitz continuous in a neighbourhood of 0, then it is easy to see that $g$ is strictly differentiable at 0.

A second purpose of the paper is to improve the saddle theorem proved in [11, Theorem 1.4], also mentioned in [12], in which the functional is of class $C^2$, but nonstandard geometrical assumptions are considered. We will prove the following.

Theorem 1.3. *Let $H$ be a Hilbert space such that $H = H_- \oplus H_+$ with $\dim H_- < \infty$ and $H_+$ closed in $H$. Let $f : H \to \mathbb{R}$ be a functional of class $C^2$ and assume that*

$$c_0 = \inf_{H_+} f > -\infty, \qquad c_1 = \sup_{H_-} f < +\infty, \tag{1.8}$$

*$f$ satisfies $(PS)_c$ for every $c \in [c_0, c_1]$, $f''(u)$ is a Fredholm operator at every critical point $u$ in $f^{-1}([c_0, c_1])$.*

*Then there exists a critical point $u$ of $f$ with $c_0 \leq f(u) \leq c_1$ and $m(f,u) \leq \dim H_- \leq m^*(f,u)$.*

In [11] it is only shown that there exist critical points $\underline{u}$, $\overline{u}$ with $c_0 \leq f(\overline{u}) \leq f(\underline{u}) \leq c_1$ and $m(f,\underline{u}) \leq \dim H_- \leq m^*(f,\overline{u})$, but one cannot say if there exists a critical point $u = \underline{u} = \overline{u}$, as in the case with standard geometrical assumptions (see [8]), or not. Our improvement is related to the fact that, according to Proposition 4.3 below, also under the nonstandard geometrical assumptions of Theorem 1.3, it is possible to recognize a homological linking structure.

The paper is organized as follows: in Section 2 we state the result of existence of nontrivial solutions; Sections 3 and 4 are devoted to prove some auxiliary results, while in Section 5 we prove the main theorems.

## 2. Existence of a nontrivial solution

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying

($g_0$) $g(x,0) = 0$ for a.e. $x \in \Omega$;

($g_1$) there exists $C \geq 0$ such that $|g(x,s)| \leq C(1+|s|)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$;

($g_2$) for a.e. $x \in \Omega$, the function $\{s \mapsto g(x,s)\}$ is strictly differentiable at 0 (see Definition 3.1 below) with $D_s g(\cdot,0) \in L^\infty(\Omega)$;

($g_3$) there exist $\hat{C} \geq 0$ and $\delta > 0$ such that, for a.e. $x \in \Omega$, we have

$$\forall s,t \in ]-\delta,\delta[ : \ \big|g(x,s) - g(x,t)\big| \leq \hat{C}|s-t|. \tag{2.1}$$

We changed "(g0)" to "($g_0$)." Please check similar cases throughout.

If we set $G(x,s) = \int_0^s g(x,t)dt$, it is well known that the functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \int_\Omega G(x,u)dx \tag{2.2}$$

is of class $C^1$.

We denote by $m(f,0)$ the supremum of the dimensions of the linear subspaces of $H_0^1(\Omega)$ where the quadratic form

$$Q(u) = \int_\Omega |Du|^2 dx - \int_\Omega D_s g(x,0)u^2 dx \tag{2.3}$$

is negative definite, and by $m^*(f,0)$ the supremum of the dimensions of the linear subspaces of $H_0^1(\Omega)$ where $Q$ is negative semidefinite. We call $m(f,0)$ (resp., $m^*(f,0)$) *the strict* (resp., *large*) *Morse index of $f$ at* 0.

Theorem 2.1. *Assume that $H_0^1(\Omega) = X_- \oplus X_+$ with $\dim X_- < \infty$ and $X_+$ closed in $H_0^1(\Omega)$. Suppose also that*

$$c_0 = \inf_{X_+} f > -\infty, \qquad c_1 = \sup_{X_-} f < +\infty, \tag{2.4}$$

*and that $f$ satisfies $(PS)_c$ for every $c \in [c_0,c_1]$,*

*If it is $\dim X_- \notin [m(f,0), m^*(f,0)]$, then the problem*

$$-\Delta u = g(x,u) \quad in\ \Omega, \qquad u = 0 \quad on\ \partial\Omega, \tag{2.5}$$

*admits a nontrivial solution u.*

*Remark 2.2.* Under the assumption of Theorem 1.1, it is well known that $f$ satisfies $(PS)_c$ for any $c \in \mathbb{R}$ and the geometrical assumptions of Theorem 2.1. Since it is clear that also $(g_0)$–$(g_3)$ are satisfied, Theorem 1.1 is a consequence of Theorem 2.1.

## 3. Computations of critical groups

*Definition 3.1.* Let $\Phi$ be a map from an open subset $U$ of a normed space $X$ to a normed space $Y$ and let $u \in U$. We say that $\Phi$ is *strictly differentiable* at $u$ (*strongly differentiable* in the sense of [6]), if there exists a continuous linear map $L : X \to Y$ such that

$$\lim_{\substack{(w_1,w_2)\to(u,u)\\ w_1\neq w_2}} \frac{\Phi(w_1) - \Phi(w_2) - L(w_1 - w_2)}{\|w_1 - w_2\|} = 0. \tag{3.1}$$

Of course, in such a case $\Phi$ is Fréchet differentiable at $u$ and $L = \Phi'(u)$.

*Definition 3.2.* Let $\mathbb{K}$ be a field, $X$ be a metric space and $f : X \to \mathbb{R}$ be a continuous function. For $u \in X$ and $c = f(u)$, let us set

$$\forall q \in \mathbb{Z} : C_q(f,u) = H_q(f^c, f^c \setminus \{u\}), \tag{3.2}$$

where $f^c = \{v \in X : f(v) \le c\}$ and $H_q(A,B)$ denotes the $q$th singular homology group of the pair $(A,B)$, with coefficients in $\mathbb{K}$ (see, e.g., [14]). The vector space $C_q(f,u)$ is called *the qth critical group* of $f$ at $u$. Because of the excision property, we may replace $f$ by $f|_U$ for any neighborhood $U$ of $u$ in $X$.

*Definition 3.3.* Let $X$ be a Banach space, $U$ an open subset of $X$ and $f : U \to \mathbb{R}$ be a function of class $C^1$. Let $C$ be a closed subset of $X$ with $C \subseteq U$. We say that $f$ satisfies *the Palais-Smale condition* ($(PS)$, for short) *on* $C$, if every sequence $(u_h)$ in $C$ with $f(u_h)$ bounded and $f'(u_h) \to 0$ admits a convergent subsequence. In the case $C = A = X$, we simply say that $f$ satisfies $(PS)$.

Let $c \in \mathbb{R}$. We say that $f$ satisfies *the Palais-Smale condition at level c* ($(PS)_c$, for short), if every sequence $(u_h)$ in $U$ with $f(u_h) \to c$ and $f'(u_h) \to 0$ admits a convergent subsequence.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ $(n \ge 3)$, $1 \le p < (n+2)/(n-2)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying

$(g_1')$ there exists $C \ge 0$ such that $|g(x,s)| \le C(1 + |s|^p)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

Let $u_0 \in H_0^1(\Omega)$ be an isolated weak solution of the semilinear problem

$$-\Delta u = g(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.3}$$

By regularity theory, we automatically have $u_0 \in L^\infty(\Omega)$. Moreover, let us assume that:

$(g_2')$ for a.e. $x \in \Omega$, the function $\{s \mapsto g(x,s)\}$ is strictly differentiable at $u_0(x)$ and $D_s g(\cdot,u_0) \in L^\infty(\Omega)$;

($g_3'$) there exist $\widehat{C} \geq 0$ and $\delta > 0$ such that for a.e. $x \in \Omega$

$$\forall s,t \in ]-\delta,\delta[: \left| g(x,u_0(x)+s) - g(x,u_0(x)+t) \right| \leq \widehat{C}|s-t|. \tag{3.4}$$

Let $f : H_0^1(\Omega) \to \mathbb{R}$ be the functional

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \int_\Omega G(x,u)dx, \tag{3.5}$$

where $G(x,s) = \int_0^s g(x,t)dt$, and let $Q : H_0^1(\Omega) \to \mathbb{R}$ be the quadratic form

$$Q(u) = \int_\Omega |Du|^2 dx - \int_\Omega D_s g(x,u_0)u^2 dx. \tag{3.6}$$

Finally, let $m(f,u_0)$ and $m^*(f,u_0)$ be defined as in Section 2.

Theorem 3.4. *We have that* $C_q(f,u_0) = \{0\}$ *for every* $q \leq m(f,u_0) - 1$ *and every* $q \geq m^*(f,u_0) + 1$.

The proof will be given at the end of the section.

As a first step, we approximate the functional $f$ with suitable functionals $f_\lambda$ of class $C^1$ with $f_\lambda'$ strictly differentiable at $u_0$ and such that the critical groups of $f_\lambda$ at $u_0$ are independent of $\lambda$.

Let us denote by $\|\cdot\|_q$ the norm of $L^q(\Omega)$ and by $\|\cdot\|_{1,2}$ the norm of $H_0^1(\Omega)$.

*Remark 3.5.* Up to substitute $g$ with $\widetilde{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{g}(x,s) = g(x,u_0(x)+s) - g(x,u_0(x)), \tag{3.7}$$

we may assume that $u_0 = 0$ and that $g(x,0) = 0$.

Lemma 3.6. *There exists a constant* $\overline{C} > 0$ *such that, for a.e.* $x \in \Omega$ *and for any* $s \in \mathbb{R}$, *we have*

$$\left| g(x,s) \right| \leq \overline{C}\left(1 + |s|^{p-1}\right)|s|. \tag{3.8}$$

*Proof.* If $0 < |s| < \delta$, then by ($g_3'$) it is

$$\left| \frac{g(x,s)}{s} \right| \leq \widehat{C}. \tag{3.9}$$

Otherwise, if $|s| \geq \delta$, then it is

$$\left| \frac{g(x,s)}{s} \right| \leq \frac{C(1+|s|^p)}{|s|} \leq \frac{C}{\delta} + C|s|^{p-1}. \tag{3.10}$$

Hence the assertion follows. □

Now let $\delta > 0$ be as in $(g_3')$ and $\vartheta \in C_c^\infty(\mathbb{R})$ such that $0 \le \vartheta \le 1$, $\operatorname{supt}(\vartheta) \subseteq ]-\delta,\delta[$ and

$$\begin{aligned} \vartheta(s) = 1 \quad & \text{if } s \in \left[-\frac{\delta}{4}, \frac{\delta}{4}\right], \\ 0 \le \vartheta \le \frac{1}{2} \quad & \text{if } s \in [-\delta,\delta] \setminus \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]. \end{aligned} \tag{3.11}$$

For every $\lambda \in [0,1]$ let us define $g_\lambda(x,s) = g(x,\vartheta(\lambda s)s)$ and let $f_\lambda : H_0^1(\Omega) \to \mathbb{R}$ be the functional

$$f_\lambda(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \int_\Omega G_\lambda(x,u)dx, \tag{3.12}$$

where $G_\lambda(x,s) = \int_0^s g_\lambda(x,t)dt$. It is clear that:

(a) for every $\lambda > 0$ and for a.e. $x \in \Omega$, the function $\{s \mapsto g_\lambda(x,s)\}$ is Lipschitz continuous uniformly with respect to $x$;

(b) for every $\lambda$ and for a.e. $x \in \Omega$, the function $\{s \mapsto g_\lambda(x,s)\}$ is strictly differentiable at 0 with $D_s g_\lambda(x,0) = D_s g(x,0)$;

(c) for a.e. $x \in \Omega$, the functions $\{(\lambda,s) \mapsto g_\lambda(x,s)\}$ and $\{(\lambda,s) \mapsto G_\lambda(x,s)\}$ are continuous;

(d) there exists $\overline{C} \ge 0$ such that $|g_\lambda(x,s)| \le \overline{C}(1+|s|^p)$, $|G_\lambda(x,s)| \le \overline{C}(1+|s|^{p+1})$.

Theorem 3.7. *The following facts hold:*

(i) *for every $\lambda \in [0,1]$, the functional $f_\lambda$ is of class $C^1$;*

(ii) *there exists an open bounded neighbourhood $U$ of 0 in $H_0^1(\Omega)$ such that, for every $\lambda \in [0,1]$, 0 is the only critical point of $f_\lambda$ in $\overline{U}$;*

(iii) *for every $\lambda \in ]0,1]$, $f_\lambda'$ is strictly differentiable at 0 with $\langle f_\lambda''(0)v,v\rangle = Q(v)$.*

*Proof.* It is readily seen that assertion (i) holds.

Let us consider assertion (ii). By contradiction, let us assume that there exist $(\lambda_h)$ in $[0,1]$ and $(u_h)$ in $H_0^1(\Omega)$ with $u_h \neq 0$ and $u_h \to 0$ strongly in $H_0^1(\Omega)$ such that $f_{\lambda_h}'(u_h) = 0$. Up to a subsequence, $\lambda_h \to \lambda$ in $[0,1]$. Since $u_h$ is a critical point of $f_{\lambda_h}$, we have that $u_h$ is a weak solution of

$$-\Delta u = g_{\lambda_h}(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.13}$$

Let

$$a_h = \begin{cases} \dfrac{g_{\lambda_h}(x,u_h)}{u_h} & \text{where } u_h \neq 0, \\ 0 & \text{where } u_h = 0. \end{cases} \tag{3.14}$$

By Lemma 3.6 it is

$$|a_h| \le \left|\frac{g_{\lambda_h}(x,u_h)}{u_h}\right| = \left|\frac{g(x,\vartheta(\lambda_h u_h)u_h)}{u_h}\right| \le \overline{C}\Big(1+|\vartheta(\lambda_h u_h)u_h|^{p-1}\Big) \le \overline{C}\Big(1+|u_h|^{p-1}\Big). \tag{3.15}$$

Since $u_h$ is bounded in $L^{2n/(n-2)}(\Omega)$, then $a_h$ belongs to $L^q(\Omega)$ with $q > n/2$ and

$$\|a_h\|_q \le C'\left(1 + \|u_h\|_{2n/(n-2)}^{p-1}\right) \le M. \tag{3.16}$$

Hence $u_h$ is a weak solution of the linear problem

$$-\Delta u = a_h u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.17}$$

By [7, Theorem 3.13.1] $u_h \in L^\infty(\Omega)$ and there exists $C > 0$ such that $\|u_h\|_\infty \le C\|Du_h\|_2$. Hence $u_h \to 0$ in $L^\infty(\Omega)$. Since $\vartheta = 1$ on $[-\delta/4, \delta/4]$, for $h$ sufficiently large we have that $u_h$ is a weak solution of (3.3). It follows that 0 is not an isolated solution of (3.3): a contradiction.

Finally, let us consider assertion (iii). Let $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the continuous linear operator such that

$$\langle Lv, w\rangle = \langle Lw, v\rangle, \qquad \langle Lv, v\rangle = Q(v). \tag{3.18}$$

Let $(u_h)$, $(v_h)$, $(w_h)$ in $H_0^1(\Omega)$ be such that $u_h \to 0$, $w_h \to 0$ in $H_0^1(\Omega)$ and $\|v_h\|_{1,2} \le 1$. Up to a subsequence, $w_h \to 0$ and $u_h \to 0$ a.e. in $\Omega$. We have that

$$\begin{aligned}
&\left| \langle f_\lambda'(w_h), v_h\rangle - \langle f_\lambda'(u_h), v_h\rangle - \langle L(w_h - u_h), v_h\rangle \right| \\
&\quad = \left| \int_{\{x\in\Omega : w_h(x)\neq u_h(x)\}} \left[\frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0)\right](w_h - u_h) v_h \, dx \right| \\
&\quad \le C\left(\int_{\{x\in\Omega : w_h(x)\neq u_h(x)\}} \left|\frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0)\right|^{n/2} dx\right)^{2/n} \\
&\qquad \times \|w_h - u_h\|_{1,2} \|v_h\|_{1,2}.
\end{aligned} \tag{3.19}$$

Then it is

$$\begin{aligned}
&\frac{\left| \langle f_\lambda'(w_h), v_h\rangle - \langle f_\lambda'(u_h), v_h\rangle - \langle L(w_h - u_h), v_h\rangle \right|}{\|w_h - u_h\|_{1,2}} \\
&\quad \le C\left(\int_{\{x\in\Omega : w_h(x)\neq u_h(x)\}} \left|\frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0)\right|^{n/2} dx\right)^{2/n} \|v_h\|_{1,2} \\
&\quad \le C\left(\int_\Omega \left|\frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0)\right|^{n/2} \chi_{\{x\in\Omega : w_h(x)\neq u_h(x)\}} \, dx\right)^{2/n}.
\end{aligned} \tag{3.20}$$

By (a) and (b) we can apply Lebesgue's theorem, obtaining

$$\left(\int_\Omega \left|\frac{g_\lambda(x, w_h) - g_\lambda(x, u_h)}{w_h - u_h} - D_s g(x, 0)\right|^{n/2} \chi_{\{x\in\Omega : w_h(x)\neq u_h(x)\}} \, dx\right)^{2/n} \longrightarrow 0. \tag{3.21}$$

Therefore

$$\lim_{h\to+\infty} \frac{\langle f_\lambda'(w_h), v_h\rangle - \langle f_\lambda'(u_h), v_h\rangle - \langle L(w_h - u_h), v_h\rangle}{\|w_h - u_h\|_{1,2}} = 0 \tag{3.22}$$

and assertion (iii) follows. $\square$

THEOREM 3.8. *The critical groups $C_q(f_\lambda,0)$ are independent of $\lambda$. In particular*

$$\forall q \in \mathbb{Z} : C_q(f,0) \approx C_q(f_1,0). \tag{3.23}$$

*Proof.* Let $U$ be an open bounded neighbourhood of 0 in $H_0^1(\Omega)$ as in assertion (ii) of Theorem 3.7. We claim that if $\lambda_h \to \lambda$ in $[0,1]$, then $\| f_{\lambda_h|\overline{U}} - f_{\lambda|\overline{U}}\|_{1,\infty} \to 0$. Let $(u_h)$ be a sequence in $\overline{U}$. Up to a subsequence, $u_h \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_h \to u$ a.e in $\Omega$. It is

$$\begin{aligned} f_{\lambda_h}(u_h) - f_\lambda(u_h) &= \int_\Omega [G_{\lambda_h}(x,u_h) - G_\lambda(x,u_h)]dx \\ &= \int_\Omega [G_{\lambda_h}(x,u_h) - G_\lambda(x,u)]dx + \int_\Omega [G_\lambda(x,u) - G_\lambda(x,u_h)]dx. \end{aligned} \tag{3.24}$$

By (c), (d) and Lebesgue's theorem we deduce that

$$\int_\Omega [G_{\lambda_h}(x,u_h) - G_\lambda(x,u)]dx \longrightarrow 0. \tag{3.25}$$

Therefore $f_{\lambda_h} \to f_\lambda$ uniformly on $\overline{U}$.

Now, let $v_h \in H_0^1(\Omega)$ with $\|v_h\|_{1,2} \le 1$. Up to a subsequence $v_h \rightharpoonup v$ in $H_0^1(\Omega)$, $v_h \rightharpoonup v$ in $L^{2n/(n-2)}(\Omega)$ and $v_h \to v$ a.e. in $\Omega$. It is

$$\begin{aligned} &|\langle f'_{\lambda_h}(u_h), v_h\rangle - \langle f'_\lambda(u_h), v_h\rangle| \\ &\quad = \left| \int_\Omega [g_{\lambda_h}(x,u_h) - g_\lambda(x,u_h)]v_h dx \right| \\ &\quad = \left| \int_\Omega [g(x,\vartheta(\lambda_h u_h)u_h) - g(x,\vartheta(\lambda u_h)u_h)]v_h dx \right| \\ &\quad \le C\left( \int_\Omega |g(x,\vartheta(\lambda_h u_h)u_h) - g(x,\vartheta(\lambda u_h)u_h)|^{2n/(n+2)}dx \right)^{(n+2)/2n} ||v_h||_{1,2}. \end{aligned} \tag{3.26}$$

As before we have that

$$\int_\Omega |g_{\lambda_h}(x,u_h) - g_\lambda(x,u_h)|^{2n/(n+2)}dx \longrightarrow 0. \tag{3.27}$$

It follows that $f'_{\lambda_h} \to f'_\lambda$ uniformly on $\overline{U}$. Finally, since $U$ is bounded and $g$ has subcritical growth, we have that for every $\lambda \in [0,1]$ $f_\lambda$ satisfies $(PS)$ in $\overline{U}$. By [5, Theorem 5.2] the assertion follows. $\square$

In the second part of this section we deduce from [6] a generalization of the classical Shifting theorem (see [3, Theorem I.5.4], [10, Theorem 8.4]).

Let $H$ be a Hilbert space, $U$ be an open subset of $H$, $u_0 \in U$ and $f : U \to \mathbb{R}$ be a function of class $C^1$ such that $f'$ is strictly differentiable at $u_0$ and $f''(u_0)$ is a Fredholm operator. In particular, $f'$ is Lipschitz continuous in a neighbourhood of $u_0$. Let $L : H \to H$

be the linear operator defined by

$$\forall v, w \in H : \langle Lv, w\rangle = \langle f''(u_0)v, w\rangle, \tag{3.28}$$

let $V_0 = \ker L$ and let $P_{V_0}$ be the orthogonal projection on $V_0$. We also denote by $m(f,u_0)$ (resp., $m^*(f,u_0)$) the strict (resp., large) Morse index of $f$ at $u_0$.

Theorem 3.9. *Let $u_0$ be an isolated critical point of $f$. Then there exist a neighbourhood $\widehat{U}$ of $P_{V_0}u_0$ in $V_0$ and a function $\widehat{f} : \widehat{U} \to \mathbb{R}$ of class $C^1$ with locally Lipschitz gradient such that $P_{V_0}u_0$ is an isolated critical point of $\widehat{f}$ and*

$$\forall q \in \mathbb{Z} : C_q(f,u_0) \approx \begin{cases} C_{q-m(f,u_0)}(\widehat{f}, P_{V_0}u_0) & \text{if } m(f,u_0) < \infty, \\ \{0\} & \text{if } m(f,u_0) = \infty, \end{cases} \tag{3.29}$$

$$\begin{gathered} \forall q \le m(f,u_0) - 1 : C_q(f,u_0) = \{0\}, \\ \forall q \ge m^*(f,u_0) + 1 : C_q(f,u_0) = \{0\}. \end{gathered} \tag{3.30}$$

*Proof.* Without loss of generality, we may assume that $u_0 = 0$. From [6, Theorem 1.2] we also see that the generalized Morse lemma holds also in this setting. Arguing as in the proof of [10, Theorem 8.4], we find that (3.29) holds. Actually, in our case $f$ is of class $C^{2-0}$ instead of $C^2$, but the proof of [10, Theorem 8.4] remains valid also in this case.

On the other hand, also the proof of [10, Theorem 8.5] can be easily adapted from the $C^2$ to the $C^{2-0}$ case. Therefore we have that $C_q(\widehat{f}, P_{V_0}u_0) = \{0\}$ if $q \ge \dim V_0 + 1$. Since $m^*(f,u_0) = m(f,u_0) + \dim V_0$, the other assertions follow from (3.29). □

Finally, let us prove Theorem 3.4.

*Proof.* By Remark 3.5 we may assume that $u_0 = 0$. Let $f_\lambda : H_0^1(\Omega) \to \mathbb{R}$ be as in (3.12). By Theorem 3.7 we have that $f_1$ is of class $C^1$ with $f_1'$ strictly differentiable at 0 and 0 is an isolated critical point of $f_1$. Moreover, $f_1''(0)$ is a Fredholm operator. By Theorem 3.8 it is

$$\forall q \in \mathbb{Z} : C_q(f,0) \approx C_q(f_1,0). \tag{3.31}$$

On the other hand, since $Q(u) = \langle f_1''(0)u, u\rangle$, we have that $m(f,0) = m(f_1,0)$ and $m^*(f,0) = m^*(f_1,0)$. From Theorem 3.9 the assertion follows. □

## 4. Homological linking

Throughout this section, $X$ will denote a Banach space, $\mathrm{B}_r(u)$ the open ball of center $u \in X$ and radius $r$ and $f : X \to \mathbb{R}$ a function of class $C^1$. We set $K = \{u \in X : f'(u) = 0\}$ and, for every $c \in \mathbb{R}$,

$$K_c = \{u \in X : f'(u) = 0,\ f(u) = c\}. \tag{4.1}$$

We also denote by $H_*$ singular homology.

First of all, let us recall from [4] an extension of the homological linking of [3].

*Definition 4.1.* Let $D$, $S$, $A$ be three subsets of $X$, $m \in \mathbb{N}$ and $\mathbb{K}$ a field. We say that $(D,S)$ *links $A$ homologically in dimension $m$ (over $\mathbb{K}$)*, if $S \subseteq D$, $S \cap A = \varnothing$ and there exists $z \in H_m(X,S;\mathbb{K})$ belonging to the image of $i_* : H_m(D,S;\mathbb{K}) \to H_m(X,S;\mathbb{K})$ but not of $j_* : H_m(X \setminus A,S;\mathbb{K}) \to H_m(X,S;\mathbb{K})$, where $i : (D,S) \to (X,S)$ and $j : (X \setminus A,S) \to (X,S)$ are the inclusion maps.

It is clear that, if $(D,S)$ links $A$ homologically, then $D \cap A \neq \varnothing$.

Theorem 4.2. *Let $D$, $S$, $A$ be three subsets of $X$ such that $(D,S)$ links $A$ homologically in dimension $m$ and let $z \in H_m(X,S;\mathbb{K})$ be as in Definition 4.1. Assume that*

$$\inf_A f > -\infty, \quad \sup_D f < +\infty, \quad \forall u \in S : f(u) < \inf_A f \tag{4.2}$$

*and define*

$$\begin{aligned} c = \inf \{ b \in \mathbb{R} : S \subseteq f^b & \text{and } z \text{ belongs to the image of the} \\ & \text{homomorphism induced by inclusion } H_m(f^b,S;\mathbb{K}) \longrightarrow H_m(X,S;\mathbb{K}) \}. \end{aligned} \tag{4.3}$$

*Suppose that $f$ satisfies $(PS)$ and that each element of $K_c$ is isolated in $K$.*

*Then $\inf_A f \leq c \leq \sup_D f$ and there exists $u \in K_c$ with $C_m(f,u) \neq \{0\}$.*

To prove our main results we need the following.

Proposition 4.3. *Let $X = X_- \oplus X_+$, with $\dim X_- < \infty$ and $X_+$ closed in $X$. Assume that*

$$c_0 = \inf_{X_+} f > -\infty, \qquad c_1 = \sup_{X_-} f < +\infty \tag{4.4}$$

*and that $f$ satisfies $(PS)_c$ for every $c \in [c_0,c_1]$.*

*Then there exists a compact pair $(D,S)$ in $X$ such that*

$$\max_D f \leq c_1, \quad \forall u \in S : f(u) < c_0 \tag{4.5}$$

*and such that $(D,S)$ links $X_+$ homologically in dimension $\dim X_-$ over all $\mathbb{K}$.*

*Proof.* Since $f$ satisfies $(PS)_c$ for every $c \in [c_0,c_1]$, there exists $r > 0$ such that $K \cap f^{-1}([c_0, c_1]) \subseteq (\mathrm{B}_r(0) \cap X_-) \oplus X_+$. Moreover, there exist $\delta, \sigma > 0$ such that

$$\left.\begin{array}{c} \|P_{X_-}u\| \geq r, \\ c_0 - \delta \leq f(u) \leq c_1 + \delta \end{array}\right\} \Longrightarrow \|f'(u)\| > \sigma, \tag{4.6}$$

where $P_{X_-}$ denotes the projection on $X_-$ induced by the decomposition $X = X_- \oplus X_+$. Let $c > 0$ be such that $\|P_{X_-}u\| \leq c\|u\|$ for any $u \in X$ and let

$$\begin{gathered} R = c\frac{c_1 - c_0 + \delta}{\sigma} + r + \delta, \qquad \rho_1 = 1, \qquad \rho_2 = R - r - \delta, \\ C = X \setminus [(\mathrm{B}_{r+\rho_1+\rho_2}(0) \cap X_-) \oplus X_+]. \end{gathered} \tag{4.7}$$

By [5, Theorem 2.1] applied to the function $f_{|\{u\in X: f(u)\geq c_0-\delta\}}$, there exist a continuous function

$$\tau : \overline{\mathrm{B}_{\rho_1}(C)} \cap \{u \in X : c_0 - \delta \leq f(u) < c_1 + \delta\} \longrightarrow [0, +\infty) \tag{4.8}$$

and a continuous map

$$\eta : \left(\overline{\mathrm{B}_{\rho_1}(C)} \cap \{u \in X : c_0 - \delta \leq f(u) < c_1 + \delta\}\right) \times [0,1] \longrightarrow \{u \in X : f(u) \geq c_0 - \delta\} \tag{4.9}$$

such that

(a) $\tau(u) = 0 \Leftrightarrow f(u) = c_0 - \delta$;
(b) $\|\eta(u,t) - u\| \leq \tau(u)t$;
(c) $f(\eta(u,t)) \leq f(u) - \sigma\tau(u)t$;
(d) $f(\eta(u,1)) = c_0 - \delta$.

Let $\vartheta_1 : \mathbb{R} \to [0,1]$ be a continuous function such that

$$\vartheta_1(s) = 1 \quad \text{if } s \leq c_1, \qquad \vartheta_1(s) = 0 \quad \text{if } s \geq c_1 + \delta/2, \tag{4.10}$$

and let $\vartheta_2 : X \to [0,1]$ be a continuous function such that

$$\vartheta_2(u) = 1 \quad \text{if } \|u\| \geq R, \qquad \vartheta_2(u) = 0 \quad \text{if } \|u\| \leq R - \delta. \tag{4.11}$$

Let $\mathcal{H} : X \times [0,1] \to X$ be the deformation defined by

$$\mathcal{H}(u,t) = \begin{cases} \eta(u, \vartheta_1(f(u))\vartheta_2(P_{X_-}u)t) & \text{if } u \in \overline{\mathrm{B}_{\rho_1}(C)},\ c_0 - \delta \leq f(u) \leq c_1 + \delta, \\ u & \text{if } f(u) \leq c_0 - \delta, \\ u & \text{if } f(u) \geq c_1 + \delta/2, \\ u & \text{if } ||P_{X_-}u|| \leq R - \delta. \end{cases} \tag{4.12}$$

If $u \in X_-$, we have that

$$||P_{X_-}\mathcal{H}(u,t) - u|| \leq c\|\mathcal{H}(u,t) - u|| \leq c\frac{f(u) - f(\mathcal{H}(u,t))}{\sigma} \leq c\frac{c_1 - c_0 + \delta}{\sigma} < R - r. \tag{4.13}$$

It follows

$$\begin{gathered} ||P_{X_-}u|| \leq r \Longrightarrow \mathcal{H}(u,t) = u, \\ \begin{matrix} u \in X_-, \\ \|u\| \geq R \end{matrix} \Longrightarrow \begin{matrix} f(\mathcal{H}(u,1)) < c_0, \\ ||P_{X_-}(\mathcal{H}(u,t))|| \geq r, \quad \forall t \in [0,1]. \end{matrix} \end{gathered} \tag{4.14}$$

It is clear that $(X, (X_- \setminus \mathrm{B}_r(0)) \oplus X_+)$ links $X_+$ homologically in dimension $\dim X_-$ and that the inclusion map

$$i : \left(\overline{\mathrm{B}_R(0)} \cap X_-, \partial\mathrm{B}_R(0) \cap X_-\right) \longrightarrow (X, (X_- \setminus \mathrm{B}_r(0)) \oplus X_+) \tag{4.15}$$

induces an isomorphism in homology. Let $m = \dim X_-$ and

$$B = \overline{B_R(0)} \cap X_-, \qquad E = \partial B_R(0) \cap X_-, \qquad F = (X_- \setminus B_r(0)) \oplus X_+. \tag{4.16}$$

Consider now the commutative diagram

$$\begin{array}{ccccc} H_m(B,E) & \longrightarrow & H_m(X,E) & \longleftarrow & H_m(X \setminus X_+, E) \\ \downarrow & & \downarrow & & \downarrow \\ H_m(X,F) & \xrightarrow{Id} & H_m(X,F) & \longleftarrow & H_m(X \setminus X_+, F) \end{array} \tag{4.17}$$

where horizontal rows are induced by the inclusions and the vertical rows are isomorphisms. We have that there exists $z \in H_m(X,E)$ belonging to the image of $H_m(B,E) \to H_m(X,E)$ such that $i_*(z) \in H_m(X,F)$, but not to the image of $H_m(X \setminus X_+, F) \to H_m(X,F)$. Let us consider the compact sets $D = \mathcal{H}(B,1)$ and $S = \mathcal{H}(E,1)$. We have that

$$\max_D f \le c_1, \qquad \max_S f < c_0, \qquad S \subseteq F. \tag{4.18}$$

Consider now the commutative diagram

$$\begin{array}{ccccc} H_m(B,E) & \longrightarrow & H_m(X,E) & & \\ \downarrow{\scriptstyle \mathcal{H}_*(\cdot,1)} & & \downarrow{\scriptstyle \mathcal{H}_*(\cdot,1)} & & \\ H_m(D,S) & \longrightarrow & H_m(X,S) & \longleftarrow & H_m(X \setminus X_+, S) \\ \downarrow & & \downarrow & & \downarrow \\ H_m(X,F) & \xrightarrow{Id} & H_m(X,F) & \longleftarrow & H_m(X \setminus X_+, F) \end{array} \tag{4.19}$$

Since $\mathcal{H}(\cdot,1) : (X,E) \to (X,F)$ is homotopically equivalent to the identity map, then $(D,S)$ links $X_+$ homologically in dimension $m = \dim X_-$ and the assertions follows. $\square$

## 5. Proof of the main results

*proof of Theorem 2.1.* By contradiction, let us assume that 0 is the unique solution of (2.5). Since $m = \dim X_- \notin [m(f,0), m^*(f,0)]$, by Theorem 3.4 it is $C_m(f,0) = \{0\}$. By Proposition 4.3 there exists a compact pair $(D,S)$ in $H_0^1(\Omega)$ such that

$$\forall u \in S : f(u) < \inf_{X_+} f \tag{5.1}$$

and $(D,S)$ links $X_+$ homologically in dimension $m$ over all $\mathbb{K}$. By Theorem 4.2 there exists a critical point $u \in H_0^1(\Omega)$ of $f$ such that $C_m(f,u) \neq \{0\}$. Hence $u \neq 0$ and $u$ is a weak solution of (2.5): a contradiction. $\square$

*proof of Theorem 1.3.* Let $(D,S)$ be as in Proposition 4.3. By [13, Proposition 3.9 and Remark] there exists $\delta > 0$ such that $f$ satisfies $(PS)_c$ for every $c \in [c_0 - \delta, c_1 + \delta]$ and

*Please specify to which remark you are referring.*

$f''(u)$ is a Fredholm operator at every critical point $u$ in $f^{-1}([c_0-\delta,c_1+\delta])$. Let us argue by contradiction and set

$$\begin{aligned} K_1 &= \{u \in H : c_0-\delta \le f(u) \le c_1+\delta,\ f'(u)=0,\ m^*(f,u) < \dim H_-\}, \\ K_2 &= \{u \in H : c_0-\delta \le f(u) \le c_1+\delta,\ f'(u)=0,\ m(f,u) > \dim H_-\}. \end{aligned} \tag{5.2}$$

Then $K_1, K_2$ are two disjoint compact sets whose union is the critical set of $f$ in $f^{-1}([c_0-\delta,c_1+\delta])$. By Marino-Prodi perturbation lemma [9, Teorema 2.2], there exists a functional $\hat{f} : H \to \mathbb{R}$ of class $C^2$ such that

$$\inf_{H_+} \hat{f} > c_0 - \delta/2, \qquad \sup_{H_-} \hat{f} < c_1 + \delta/2, \qquad \max_S \hat{f} < \inf_{H_+} \hat{f}, \tag{5.3}$$

$\hat{f}$ satisfies $(PS)_c$ for every $c \in [c_0-\delta/2, c_1+\delta/2]$, $\hat{f}$ has only non-degenerate critical points $u$ in $\hat{f}^{-1}([c_0-\delta/2,c_1+\delta/2])$, with either $m(\hat{f},u) < \dim H_-$ or $m^*(\hat{f},u) > \dim H_-$. If we apply Theorem 4.2 to $\hat{f}$, we find a critical point $u$ of $\hat{f}$ with $c_0-\delta/2 \le \hat{f}(u) \le c_1+\delta/2$ and $C_m(\hat{f},u) \ne \{0\}$, where $m = \dim H_-$. By the Morse lemma, we have $m(\hat{f},u) = m$ and a contradiction follows. $\square$

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