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Vector functions for singular fields on curved triangular elements, truly defined in the parent space.

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I. Introduction.

This paper presents singular curl- and divergence-conforming functions of interpolative kind on curved triangular elements, directly defined in their parent triangle of area coordinates ($\xi_1, \xi_2\,; \xi_3 = 1 - \xi_1 - \xi_2$), without introducing any intermediate (polar) reference frame as previously done in other works. Curl-conforming functions are useful in the FEM solution of the transverse vector Helmholtz equation, whereas divergence-conforming functions are used in the Moments Method solution of surface integral equations. Singular vector functions for hierarchical families are easily extracted from those given here. Our functions incorporate the edge condition and are able to approximate the unknown field components in the neighborhood of the edge of a wedge for any order of the singularity coefficient ν, that is supposed given and known *a priori*. The wedge can be penetrable in the curl-conforming case, while it is supposed metallic in the divergence conforming case. For metal wedges of aperture angle α, one has $\nu = \pi/(2\pi - \alpha)$. The curl (divergence) conforming singular functions provided here are compatible with standard zero-order vector functions [1] in adjacent elements and guarantee tangential (normal) continuity along the edges of the elements allowing for the discontinuity of normal (tangential) components, adequate modelling of the curl (divergence), and removal of spurious modes (solutions). These functions, unless ν equals one, are of non-integer polynomial order and singular. Therefore it does not make too much sense to look for any completeness

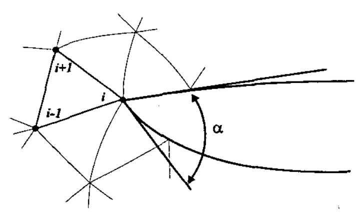

Fig. 1. Cross-sectional view of the region around a sharp, but curved edge of aperture angle α, meshed with curved triangles. All the singular curl-conforming elements have their i-th node (local numbering scheme) on the edge tip and their two triangle edges $i \pm 1$ departing from it.

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property while deriving them; as opposite to the case of standard higher order vector functions [1]. Nevertheless, one can in general derive singular vector functions of either substitutive or non-substitutive kind. The non-substitutive basis functions reduce to the standard (regular) basis in the limit for singularity coefficient $\nu = 1$, whereas the substitutive functions differ from the regular ones at $\nu = 1$. As a matter of fact, curl-conforming functions of substitutive kind are required to discretize the transverse magnetic field $\boldsymbol{H}_t$ in the neighborhood of the edge of a metal wedge, because the curl of $\boldsymbol{H}_t$ is related to the longitudinal electric field component that must vanish at a metallic edge. Conversely, the curl-conforming functions required to discretize a transverse electric field $\boldsymbol{E}_t$, or $\boldsymbol{H}_t$ in presence of a non metallic wedge, could be non-substitutive, since they ought to model also a constant curl component on the triangular element.

II. Curl conforming functions.

Curl conforming singular vector functions have been derived in [2], [3] by mapping the triangular element from real space to a unit square. These elements require to use a local polar coordinate system (ρ, ϕ) to be defined and cannot be distorted to match wedges of curved cross-section such as that shown in Fig. 1. Singular elements were derived in [3] by invoking a quasi-static field behavior in the vicinity of the edge, thereby constructing each singular function as the gradient of a linear combination of nodal-based scalar functions. Conversely, in [2], the curl of the singular component is not zero on the whole singular element but it rather goes to zero at the edge of the wedge as ρ^ν. New curl-conforming, singular triangular elements able to deal with curved edges are reported in Table I . The substitutive functions yield $\boldsymbol{\Omega}_j^s = \boldsymbol{\Omega}_j - \xi_i \boldsymbol{\Omega}_i$ at $\nu = 1$, where one has to substitute them with the regular zero order functions $\boldsymbol{\Omega}_j$ (also given in Table I). To obtain a non substitutive basis one may add to each of the substitutive functions of Table I the first order edgeless function $\xi_i \boldsymbol{\Omega}_i$, or $\nu\, \xi_i \boldsymbol{\Omega}_i$, with zero tangent component on each of the triangle edges. The substitutive functions have equal curls, that vanish with the correct power ν at the edge $\xi_i = 1$. The substitutive and non-substitutive functions of Table I have identical tangent component at the triangle edges, explicitly provided in the Table. They all model the singularity of the transverse field at the edge $(\xi_i = 1)$ via their $\nu(1 - \xi_i)^{\nu - 1}$ behavior of the tangent component along the triangle edges $(i \pm 1)$. Notice also that all the singular functions of Table I yield the curl-free vector $\boldsymbol{\Omega}_{i+1}^s(\boldsymbol{r}) - \boldsymbol{\Omega}_{i-1}^s(\boldsymbol{r}) = \nu\,(1 - \xi_i)^{\nu - 1}\, \nabla \xi_i$. A curl-free combination $\boldsymbol{H}_t = a\boldsymbol{\Omega}_i^s + b\boldsymbol{\Omega}_{i+1}^s + c\boldsymbol{\Omega}_{i-1}^s$ is obtained by setting $(a + b + c) = 0$ for the substitutive functions, and $a = 0$, $c = -b$ for the non-substitutive functions of Table I. Thus,

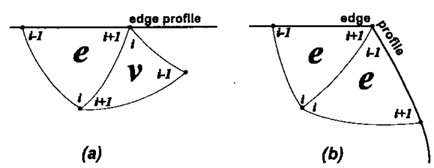

Fig. 2. a) Edge (e) and vertex (v) singularity triangles with local, node numbering scheme. b) Although two edge singularity triangles can have an edge in common, the basis functions cannot model a corner singularity.

these functions are capable of modelling the null space of the curl operator, and therefore are able to eliminate spurious numerical solutions.

III. Divergence conforming functions.

Substitutive bases incorporating the singular behavior of the current density near the edge of a wedge have been derived in [4], by starting from the divergence of the basis functions and enforcing the correct behavior for the charge density. In the following we adopt the same notation used in [4] and consider two types of singularity triangles: the edge (e) and the vertex (v) singularity triangle (Fig. 2). All the basis functions of Table II are identical to those of [4], with the exception of the edge singular triangle function $\mathbf{\Lambda}_i^e(\boldsymbol{r})$, since that in [4] does not have a constant normal component on the i-th edge (in [4]: $-\boldsymbol{\nabla}\xi_i \cdot \mathbf{\Lambda}_i^e|_{\xi i=0} = \xi_{i+1}^\nu + \xi_{i-1}^\nu$). Our edge singularity triangle functions can model the singular behavior of the current and charge density at the i-th edge. In fact, any linear combination of these functions, $\boldsymbol{J}(\boldsymbol{r}) = a\,\mathbf{\Lambda}_i^e(\boldsymbol{r}) + b\,\mathbf{\Lambda}_{i+1}^e(\boldsymbol{r}) + c\,\mathbf{\Lambda}_{i-1}^e(\boldsymbol{r})$, and its divergence, can be written as:

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{r}) &= a\,\boldsymbol{J}^0(\boldsymbol{r}) + (b+c)\,\boldsymbol{J}^{\nu-1}(\boldsymbol{r}) + (b-c)\,\boldsymbol{J}^{\text{null}}(\boldsymbol{r}) & (1)\\ \boldsymbol{\nabla}\cdot\boldsymbol{J}(\boldsymbol{r}) &= a\,\boldsymbol{\nabla}\cdot\boldsymbol{J}^0(\boldsymbol{r}) + (b+c)\,\boldsymbol{\nabla}\cdot\boldsymbol{J}^{\nu-1}(\boldsymbol{r}) = \frac{2}{\mathcal{J}}a + (b+c)\,\frac{1+\nu}{\mathcal{J}}\,\xi_i^{\nu-1} & (2)\end{aligned}$$

with:

$$\left\{\begin{aligned} \boldsymbol{J}^0(\boldsymbol{r}) &= \mathbf{\Lambda}_i^e(\boldsymbol{r})\\ \boldsymbol{J}^{\nu-1}(\boldsymbol{r}) &= \left[\mathbf{\Lambda}_{i+1}^e(\boldsymbol{r}) + \mathbf{\Lambda}_{i-1}^e(\boldsymbol{r})\right]/2\\ \boldsymbol{J}^{\text{null}}(\boldsymbol{r}) &= \left[\mathbf{\Lambda}_{i+1}^e(\boldsymbol{r}) - \mathbf{\Lambda}_{i-1}^e(\boldsymbol{r})\right]/2 = \frac{\boldsymbol{\ell}_i}{2\mathcal{J}}\,\xi_i^{\nu-1}\end{aligned}\right. \tag{3}$$

and where the divergence-free vector $\boldsymbol{J}^{\text{null}}(\boldsymbol{r})$ models the singular behavior of the divergence-free current component parallel to the i-th edge (i.e. parallel to $\boldsymbol{\ell}_i$), whereas the correct singular behavior of the charge density at $\xi_i = 0$ is modelled by the divergence of the field $\boldsymbol{J}^{\nu-1}(\boldsymbol{r})$. Spurious numerical solutions are suppressed since these functions can model the null space of the divergence operator, a divergence-free field being obtained for $a = 0, c = -b$. Notice also that the first two terms on the right-hand side of (1) model, at edge $\xi_i = 0$, the current component in the direction normal to the edge (i.e. parallel to $\boldsymbol{\nabla}\xi_i$) as the sum of a constant plus an independent term vanishing with the correct power (ν), since one has:

$$\left\{\begin{aligned} \boldsymbol{J}^0(\boldsymbol{r})\cdot\boldsymbol{\nabla}\xi_i &= \mathbf{\Lambda}_i^e(\boldsymbol{r})\cdot\boldsymbol{\nabla}\xi_i = -\frac{1-\xi_i}{\mathcal{J}} = -\frac{1}{\mathcal{J}} \text{ at } \xi_i = 0\\ \boldsymbol{J}^{\nu-1}(\boldsymbol{r})\cdot\boldsymbol{\nabla}\xi_i &= \mathbf{\Lambda}_{i\pm1}^e(\boldsymbol{r})\cdot\boldsymbol{\nabla}\xi_i = \frac{1}{\mathcal{J}}\,\xi_i^\nu\\ \boldsymbol{J}^{\text{null}}(\boldsymbol{r})\cdot\boldsymbol{\nabla}\xi_i &= 0\end{aligned}\right. \tag{4}$$

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TABLE I

LOWEST-ORDER CURL-CONFORMING SINGULAR BASIS FUNCTIONS ON TRIANGLES, WITH SINGULARITY ON NODE i, AT $\xi_i = 1$. SUBSCRIPTS ARE COUNTED MODULO 3, FOR $i = 1, 3$.

Basis Functions	Surface Curl
Substitutive $\boldsymbol{\Omega}_i^s(\boldsymbol{r}) = \frac{1}{2}(1-\xi_i)^{\nu+2}\,\boldsymbol{\nabla}\left(\frac{\xi_{i-1}-\xi_{i+1}}{1-\xi_i}\right) = (1-\xi_i)^{\nu}\,\boldsymbol{\Omega}_i(\boldsymbol{r})$ $\boldsymbol{\Omega}_{i\pm1}^s(\boldsymbol{r}) = \boldsymbol{\Omega}_i^s(\boldsymbol{r}) \mp \boldsymbol{\nabla}\left[(1-\xi_i)^{\nu-1}\,\xi_{i\mp1}\right] = (1-\xi_i)^{\nu}\,\boldsymbol{\Omega}_i(\boldsymbol{r}) \mp (1-\xi_i)^{\nu-1}\left[\boldsymbol{\nabla}\xi_{i\mp1} + (1-\nu)\,\frac{\xi_{i\mp1}}{1-\xi_i}\boldsymbol{\nabla}\xi_i\right]$	$\boldsymbol{\nabla}\times\boldsymbol{\Omega}_j^s(\boldsymbol{r}) = \frac{2+\nu}{\mathcal{J}}\,(1-\xi_i)^{\nu}\,\hat{\boldsymbol{n}}$ for $j = i, i\pm1$
Non Substitutive $\boldsymbol{\Omega}_i^s(\boldsymbol{r}) = \left[(1-\xi_i)^{\nu} + \nu\xi_i\right]\boldsymbol{\Omega}_i(\boldsymbol{r})$ $\boldsymbol{\Omega}_{i\pm1}^s(\boldsymbol{r}) = \boldsymbol{\Omega}_i(\boldsymbol{r}) \mp (1-\xi_i)^{\nu-1}\left[\boldsymbol{\nabla}\xi_{i\mp1} + (1-\nu)\,\frac{\xi_{i\mp1}}{1-\xi_i}\boldsymbol{\nabla}\xi_i\right]$	$\boldsymbol{\nabla}\times\boldsymbol{\Omega}_i^s(\boldsymbol{r}) = \frac{2+\nu}{\mathcal{J}}\,(1-\xi_i)^{\nu}\,\hat{\boldsymbol{n}} - \frac{\nu}{\mathcal{J}}\,(1-3\xi_i)\,\hat{\boldsymbol{n}}$ $\boldsymbol{\nabla}\times\boldsymbol{\Omega}_{i\pm1}^s(\boldsymbol{r}) = \frac{2}{\mathcal{J}}\,\hat{\boldsymbol{n}}$
Non Substitutive at $\nu = 1 \implies$ Regular $\boldsymbol{\Omega}_j(\boldsymbol{r}) = \xi_{j+1}\boldsymbol{\nabla}\xi_{j-1} - \xi_{j-1}\boldsymbol{\nabla}\xi_{j+1}$ for $j = i, i\pm1$	$\boldsymbol{\nabla}\times\boldsymbol{\Omega}_j(\boldsymbol{r}) = \frac{2}{\mathcal{J}}\,\hat{\boldsymbol{n}}$ for $j = i, i\pm1$
Tangent component at the edges: $\boldsymbol{\Omega}_k^s(\boldsymbol{r})\cdot\boldsymbol{\ell}_j\big\vert_{\xi_j=0} = \begin{cases} 0, & k\neq j \\ 1, & k=j=i \\ \nu\,(1-\xi_i)^{\nu-1}, & k=j=i\pm1 \end{cases}$	

TABLE II

NON SUBSTITUTIVE, LOWEST-ORDER DIVERGENCE-CONFORMING BASIS FUNCTIONS ON TRIANGLES, WITH SINGULARITY ON EDGE i ($\xi_i = 0$) WHEN EDGE SINGULAR, AND ON NODE i ($\xi_i = 1$) WHEN VERTEX SINGULAR.

Basis Functions	Surface Divergence
Edge Singular $\boldsymbol{\Lambda}_{i\pm1}^e(\boldsymbol{r}) = \xi_i^{\nu-1}\boldsymbol{\Lambda}_{i\pm1}$ $\boldsymbol{\Lambda}_i^e(\boldsymbol{r}) = \boldsymbol{\Lambda}_i$	$\boldsymbol{\nabla}\cdot\boldsymbol{\Lambda}_{i\pm1}^e(\boldsymbol{r}) = \frac{1+\nu}{\mathcal{J}}\,\xi_i^{\nu-1}$ $\boldsymbol{\nabla}\cdot\boldsymbol{\Lambda}_i^e(\boldsymbol{r}) = \frac{2}{\mathcal{J}}$
Vertex Singular $\boldsymbol{\Lambda}_{i\pm1}^v(\boldsymbol{r}) = (1-\xi_i)^{\nu-1}\times \times\left[\boldsymbol{\Lambda}_{i\pm1} + \frac{(\nu-1)\,\xi_i}{\nu\,(1-\xi_i)}\,\boldsymbol{\Lambda}_i\right]$ $\boldsymbol{\Lambda}_i^v(\boldsymbol{r}) = (1-\xi_i)^{\nu-1}\,\boldsymbol{\Lambda}_i$	$\boldsymbol{\nabla}\cdot\boldsymbol{\Lambda}_{i\pm1}^v(\boldsymbol{r}) = \frac{1+\nu}{\mathcal{J}}\,\frac{(1-\xi_i)^{\nu-1}}{\nu}$ $\boldsymbol{\nabla}\cdot\boldsymbol{\Lambda}_i^v(\boldsymbol{r}) = \frac{1+\nu}{\mathcal{J}}\,(1-\xi_i)^{\nu-1}$
Edge/Vertex Sing. at $\nu = 1 \implies$ Regular $\boldsymbol{\Lambda}_j(\boldsymbol{r}) = \frac{1}{\mathcal{J}}\,(\xi_{j+1}\boldsymbol{\ell}_{j-1} - \xi_{j-1}\boldsymbol{\ell}_{j+1})$ for $j = 1, 3$	$\boldsymbol{\nabla}\cdot\boldsymbol{\Lambda}_j(\boldsymbol{r}) = \frac{2}{\mathcal{J}}$ for $j = 1, 3$
Subscripts are counted modulo 3, for $i = 1, 3$	