Strictly stable boundary treatment with wavelets: application to tansient solution of distributed electrical networks

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Implementazione stabile di condizioni al contorno con ondine: applicazione alla soluzione transitoria di reti elettriche distribuite

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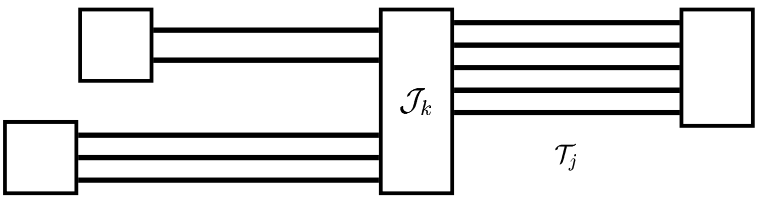
Transient simulation of distributed electrical networks has become a critical issue in the design of modern electronic systems. Indeed, due to the continuous increase of the clock speed in digital circuits, signal distortion, crosstalk, and spurious electromagnetic coupling must be carefully assessed because they could seriously affect the overall performance of the system. This paper investigates a discretization scheme for the hyperbolic partial differential equations governing the electrical behavior of the network. Specifically, we study a systematic procedure for the implementation of the boundary conditions that proves to be strictly stable for any type of passive network. This procedure is applied to wavelet-based discretizations to perform adaptive transient simulations.



We focus herewith with general structures of the type shown above. A set of electrical interconnects (or *tubes*) $\{\mathcal{T}_j,\, j = 1, \ldots, M\}$, each formed by $P^j + 1$ conductors, provides the link between a set of junctions $\{\mathcal{J}_k,\, k = 1, \ldots, T\}$. The evolution of the electrical behavior of each tube $\{\mathcal{T}_j\}$ can be modeled through the following transmission-line equations [1],

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{a}^j(z,t)\\ \mathbf{b}^j(z,t)\end{bmatrix} + \begin{bmatrix}\mathbf{c}^j & \mathbf{0}\\ \mathbf{0} & -\mathbf{c}^j\end{bmatrix}\frac{\partial}{\partial z}\begin{bmatrix}\mathbf{a}^j(z,t)\\ \mathbf{b}^j(z,t)\end{bmatrix} + \mathbf{R}^j\begin{bmatrix}\mathbf{a}^j(z,t)\\ \mathbf{b}^j(z,t)\end{bmatrix} = 0, \qquad (1)$$

here expressed in terms of forward and backward propagating wave vectors $\mathbf{a}^j(z,t)$ and $\mathbf{b}^j(z,t)$ of size P^j, which are related to voltages and currents along the tube conductors. The longitudinal coordinate z is assumed to be normalized to the unit interval. The diagonal matrix $\mathbf{c}^j$ includes the possibly different and strictly positive propagation wave velocities, while the matrix $\mathbf{R}^j$ is positive semidefinite and represents distributed power losses along the conductors.

Boundary conditions for the above set of equations are provided by the junctions equations. On order to treat the cases of interest in the applications we deal with the most general type of nonlinear and dynamic termination, expressed for the k-th junction as

$$\left\{ \begin{array}{lcl} \mathbf{x}'_k(t) &=& \mathcal{F}_k(\mathbf{x}_k(t), \mathbf{A}_k(t), \mathbf{s}(t)), \\ \mathbf{B}_k(t) &=& \mathcal{G}_k(\mathbf{x}_k(t), \mathbf{A}_k(t), \mathbf{s}(t)), \end{array} \right. \tag{2}$$

where $\mathbf{x}_k(t)$ is a state-variable vector, $\mathbf{s}(t)$ is a vector collecting the independent sources, and $\mathbf{A}_k(t)$, $\mathbf{B}_k(t)$ are vectors collecting, respectively, all the incident and reclected waves by the junction (note that thay can come from different tubes). Obviously, there is a one-to-one mapping between the sets $\{\mathbf{a}^j, \mathbf{b}^j, \forall j\}$ and $\{\mathbf{A}_k(t), \mathbf{B}_k(t), \forall k\}$. The prime $'$ indicates time differentiation.

The discretization of (1) is performed through suitable spatial finite difference approximations of order τ,

$$\mathbf{u}_z = \mathbf{P}^{-1}\mathbf{Q}\,\mathbf{u} + \mathbf{e}, \qquad |\mathbf{e}| = O(h^\tau),$$

where $\mathbf{u}$ is the array collecting the $N+1$ nodal values of any function $u(z,t)$ defined on $[0,1]$, and $\mathbf{u}_z$ is the array collecting the nodal values of its first derivative. The operators $\mathbf{P}$ and $\mathbf{Q}$ are constrained to satisfy certain symmetry conditions, which are essential to prove strict stability of the boundary treatment [2, 3]. Specifically,

$$\mathbf{P} = \mathbf{P}^T, \qquad \mathbf{P} > 0, \qquad \mathbf{u}^T(\mathbf{Q} + \mathbf{Q}^T)\mathbf{u} = u_0 + u_N.$$

Note that the above difference approximation includes the boundary nodes of each tube, so that the boundary conditions are still to be enforced.

The proposed boundary treatment is based on a weak enforcement of the junctions equations. All the nodal variables, including edges, are retained in the discrete system, thus allowing a small approximation error at the boundaries. This error is added in a stable way to the dynamics of the discrete equations [3] through a Simultaneous Approximation Term (SAT).

This procedure leads to a straightforward implementation and can be proved to be strictly stable for any passive termination. We get

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}\overline{\mathbf{a}}^j(t)\\ \overline{\mathbf{b}}^j(t)\end{bmatrix} &= -\left(\begin{bmatrix}\mathbf{c}^j & \mathbf{0}\\ \mathbf{0} & -\mathbf{c}^j\end{bmatrix}\otimes[\mathbf{P}^j]^{-1}\mathbf{Q}^j\right)\begin{bmatrix}\overline{\mathbf{a}}^j(t)\\ \overline{\mathbf{b}}^j(t)\end{bmatrix}\\ &\quad -(\mathbf{R}^j\otimes I)\begin{bmatrix}\overline{\mathbf{a}}^j(t)\\ \overline{\mathbf{b}}^j(t)\end{bmatrix}-\begin{bmatrix}\mathbf{c}^j\,\Xi_0^j(t)\otimes\mathbf{p}_0^j\\ \mathbf{c}^j\,\Xi_N^j(t)\otimes\mathbf{p}_N^j\end{bmatrix},\end{aligned}$$

where $\overline{\mathbf{a}}^j(t)$, $\overline{\mathbf{b}}^j(t)$ collect the nodal values of the wave vectors for the j-th tube, $\otimes$ indicates the Kronecker matrix product, $\mathbf{p}_0^j$, $\mathbf{p}_N^j$ are the first and last column of $[\mathbf{P}^j]^{-1}$, and $\Xi_0^j(t)$, $\Xi_N^j(t)$ are the approximation errors in the reflected waves at the left and right termination of the tube. More precisely,

$$\Xi_0^j(t)=\mathbf{a}_0^j(t)-\widehat{\mathbf{a}}_0^j(t),\qquad \Xi_N^j(t)=\mathbf{b}_N^j(t)-\widehat{\mathbf{b}}_N^j(t), \tag{3}$$

where $\mathbf{a}_0^j(t)$, $\mathbf{b}_N^j(t)$ are the values of the waves launched at the boundaries into the j-th tube, and $\widehat{\mathbf{a}}_0^j(t)$, $\widehat{\mathbf{b}}_N^j(t)$ are the output waves obtained by the dynamics of the junctions equations

$$\begin{cases}\mathbf{x}_k'(t) &= \mathcal{F}_k(\mathbf{x}_k(t),\mathbf{A}_k(t),\mathbf{s}(t)),\\ \widehat{\mathbf{B}}_k(t) &= \mathcal{G}_k(\mathbf{x}_k(t),\mathbf{A}_k(t),\mathbf{s}(t)).\end{cases}$$

Defining the continuous energy in the network as

$$\mathcal{E}(t)=\sum_{j=1}^{M}\sum_{i=1}^{P^j}\left\{\frac{||a_i^j(\cdot,t)||^2}{c_i^j}+\frac{||b_i^j(\cdot,t)||^2}{c_i^j}\right\}+\frac{1}{2}\sum_{k=1}^{T}||x_k(t)||^2,$$

and the discrete energy of the proposed scheme as

$$E(t)=\sum_{j=1}^{M}\left\{[\overline{\mathbf{a}}^j]^T([\mathbf{c}^j]^{-1}\otimes\mathbf{P}^j)\overline{\mathbf{a}}^j+[\overline{\mathbf{b}}^j]^T([\mathbf{c}^j]^{-1}\otimes\mathbf{P}^j)\overline{\mathbf{b}}^j\right\}+\frac{1}{2}\sum_{k=1}^{T}||x_k(t)||^2,$$

it is not difficult to prove that

$$\frac{d}{dt}\mathcal{E}(t)\le 0\ \Rightarrow\ \frac{d}{dt}E(t)\le 0.$$

In other words, the scheme is strictly stable whan applied to any distributed electrical network characterized by bounded energy. In addition, it can be shown that the scheme is uniformly of order τ, including the boundary nodes.

The above procedure has the remarkable advantage that high order is achieved with a straightforward implementation of the boundary conditions.

In turn, this allows switching from a standard nodal representation of the solution vectors into a hierarchical representation through wavelet bases defined on the unit interval. This can be accomplished without the need of complicated redefinitions of the boundary wavelets, and therefore, without having to apply different thresholding strategies for the implementation of adaptive algorithms. The wavelet coefficients related to the boundary wavelets can therefore be treated in the same way as the internal wavelet coefficients are. The scheme operating in the wavelet basis can be derived through application of the matrix operator $\mathbf{W}$ mapping the nodal into the wavelet coefficients, here denoted with a tilde. We get, for the j-th tube,

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \widetilde{\mathbf{a}}^j(t) \\ \widetilde{\mathbf{b}}^j(t) \end{bmatrix} &= -\left(\begin{bmatrix} \mathbf{c}^j & \mathbf{0} \\ \mathbf{0} & -\mathbf{c}^j \end{bmatrix} \otimes \mathbf{W}[\mathbf{P}^j]^{-1}\mathbf{Q}^j\mathbf{W}^{-1}\right)\begin{bmatrix} \widetilde{\mathbf{a}}^j(t) \\ \widetilde{\mathbf{b}}^j(t) \end{bmatrix} \\ &\quad - (\mathbf{R}^j \otimes I)\begin{bmatrix} \widetilde{\mathbf{a}}^j(t) \\ \widetilde{\mathbf{b}}^j(t) \end{bmatrix} - \begin{bmatrix} \mathbf{c}^j\, \boldsymbol{\Xi}_0^j(t) \otimes \mathbf{p}_0^j \\ \mathbf{c}^j\, \boldsymbol{\Xi}_N^j(t) \otimes \mathbf{p}_N^j \end{bmatrix}. \end{aligned}$$

The boundary approximation errors remains unchanged from Eq. (3), provided that the first and last rows of $\mathbf{W}^{-1}$ are used to relate the unknowns in the above system to the nodal values at the edges of the wave vectors. Obviously, the same considerations on strict stability and approximation error hold also in this wavelet-based implementation.

In conclusion, a discretization scheme based on the combination of high-order finite differencing, weak boundary treatment, and use of wavelet representations has been presented. The scheme is proved to be strictly stable when applied to the simulation of passive distributed electrical networks formed by multiconductor transmission-line segments and nonlinear dynamic junctions. These structures are found in any modern electronic device. Application of the proposed technique seems promising in the reduction of the computational burden of existing simulation tools based on low-order and non-adaptive discretizations.

References

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