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Perturbations of critical values in nonsmooth critical point theory

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1 Introduction

Let X be a metric space and $f : X \to \mathbf{R}$ be a continuous function. Recently, in [4, 7, 8, 9], a critical point theory has been elaborated for such a setting, which extends the classical case concerning smooth functionals on smooth Finsler manifolds.

A possible development consists in the study of stability under perturbation. More precisely, we can assume that $c \in \mathbf{R}$ is a critical value of f and ask whether any $g : X \to \mathbf{R}$ sufficiently close to f has a critical value near c. For functionals of class C^1, such a problem has been already treated in [12, 13]. In our setting, the question has been the object of [6, 10].

In the first two sections, we recall the main aspects of the abstract theory of [6, 10]. Let us mention that we are able to deal also with non-isolated critical values. In addition, we study here in some detail the stability of a critical value originated by a local minimum.

As it is shown in Theorem (3.1), the critical values, we are able to treat, are stable if the perturbed functional g is uniformly close to f. In section 4, we treat a class of functionals in the Sobolev space $H^1_0(\Omega)$, for which Γ-convergence is sufficient to get the same result.

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In the last section, we briefly outline a particular case which generalizes some results of [6, 10], concerning eigenvalue problems for variational inequalities.

2 Trivial pairs and essential values

Throughout this section X will denote a metric space endowed with the metric d and $f : X \to \mathbf{R}$ a continuous function. If $b \in \overline{\mathbf{R}} := \mathbf{R} \cup \{-\infty, +\infty\}$, let us set

$$f^b = \{u \in X : \, f(u) \le b\} .$$

We also denote by $\mathrm{B}_r(u)$ the open ball of centre u and radius r. More generally, if $Y \subseteq X$, $\mathrm{B}_r(Y)$ denotes the open r-neighbourhood of Y. For the topological notions involved this section, the reader is referred to [14].

Definition 2.1 *Let $a, b \in \overline{\mathbf{R}}$ with $a \le b$. The pair $\left(f^b, f^a\right)$ is said to be* trivial*, if for every neighbourhood $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b in $\overline{\mathbf{R}}$ there exists a continuous map $\mathcal{H} : f^{\beta'} \times [0,1] \to f^{\beta''}$ such that*

$$\mathcal{H}(x, 0) = x \qquad \forall x \in f^{\beta'} ,$$

$$\mathcal{H}\left(f^{\beta'} \times \{1\}\right) \subseteq f^{\alpha''} ,$$

$$\mathcal{H}\left(f^{\alpha'} \times [0,1]\right) \subseteq f^{\alpha''} .$$

Remark 2.2 *If $\alpha < \alpha'$ in the above definition, we can suppose, without loss of generality, that $\mathcal{H}(x,t) = x$ on $f^\alpha \times [0,1]$. Actually, it is sufficient to substitute $\mathcal{H}(x,t)$ with $\mathcal{H}(x, t\vartheta(x))$, where $\vartheta : f^{\beta'} \to [0,1]$ is a continuous function with $\vartheta(x) = 0$ for $f(x) \le \alpha$ and $\vartheta(x) = 1$ for $f(x) \ge \alpha'$.*

Theorem 2.3 *Let $a, c, d, b \in \overline{\mathbf{R}}$ with $a < c < d < b$. Let us assume that the pairs $\left(f^b, f^c\right)$ and $\left(f^d, f^a\right)$ are trivial.*

Then the pair $\left(f^b, f^a\right)$ is trivial.

Proof. Let $[\alpha', \alpha'']$ be a neighbourhood of a and $[\beta', \beta'']$ a neighbourhood of b. Without loss of generality, we can assume $\alpha'' < c$ and $\beta' > d$. Moreover, let $c < \gamma < d$. There exists a continuous map $\mathcal{H}_1 : f^{\beta'} \times [0,1] \to f^{\beta''}$ such that $\mathcal{H}_1(x,0) = x \;\; \forall x \in f^{\beta'}$, $\mathcal{H}_1\left(f^{\beta'} \times \{1\}\right) \subseteq f^\gamma$, $\mathcal{H}_1\left(f^{\alpha''} \times [0,1]\right) \subseteq f^\gamma$ and such that $\mathcal{H}_1(x,t) = x$ on $f^{\alpha'} \times [0,1]$. Moreover there exists a continuous map $\mathcal{H}_2 : f^\gamma \times [0,1] \to f^{\beta'}$ such that

$\mathcal{H}_2(x,0) = x \quad \forall x \in f^\gamma$, $\mathcal{H}_2(f^\gamma \times \{1\}) \subseteq f^{\alpha''}$, $\mathcal{H}_2\left(f^{\alpha'} \times [0,1]\right) \subseteq f^{\alpha''}$. If we define $\mathcal{H} : f^{\beta'} \times [0,1] \to f^{\beta''}$ by

$$\mathcal{H}(x,t) = \begin{cases} \mathcal{H}_1(x,2t) & 0 \le t \le \frac{1}{2} \\ \mathcal{H}_2\left(\mathcal{H}_1(x,1), 2t-1\right) & \frac{1}{2} \le t \le 1 \end{cases},$$

it turns out that $\mathcal{H}$ is a continuous map with the required properties. Therefore the assertion follows. ∎

Definition 2.4 *A real number c is said to be* an essential value *of f , if for every $\varepsilon > 0$ there exist $a, b \in]c-\varepsilon, c+\varepsilon[$ with $a < b$ such that the pair $\left(f^b, f^a\right)$ is not trivial.*

Remark 2.5 *The set of the essential values of f is closed in $\mathbf{R}$.*

Theorem 2.6 *Let $a, b \in \overline{\mathbf{R}}$ with $a < b$. Let us assume that f has no essential value in $]a, b[$.*

Then the pair $\left(f^b, f^a\right)$ is trivial.

Proof. Let $[\alpha', \alpha'']$ be a neighbourhood of a , $[\beta', \beta'']$ be a neighbourhood of b and let $a' \in]a, \alpha''[$ and $b' \in]\beta', b[$ with $a' < b'$. For every $c \in [a', b']$ there exists $\varepsilon > 0$ such that for every $\overline{a}, \overline{b} \in]c-\varepsilon, c+\varepsilon[$ with $\overline{a} < \overline{b}$ the pair $\left(f^{\overline{b}}, f^{\overline{a}}\right)$ is trivial. Since $[a', b']$ is compact, there exist $a' \le c_1 < \cdots < c_k \le b'$ and $\varepsilon_i > 0$ for $i = 1, \cdots, k$, such that

$$[a', b'] \subseteq \bigcup_{i=1}^{k}]c_i - \varepsilon_i, c_i + \varepsilon_i[$$

and such that for every $\overline{a}, \overline{b} \in]c_i - \varepsilon_i, c_i + \varepsilon_i[$ with $\overline{a} < \overline{b}$ the pair $\left(f^{\overline{b}}, f^{\overline{a}}\right)$ is trivial. Arguing by induction on k and taking into account Theorem (2.3), we deduce that the pair $\left(f^{b'}, f^{a'}\right)$ is trivial. Then there exists a continuous map $\mathcal{H} : f^{\beta'} \times [0,1] \to f^{\beta''}$ such that

$$\mathcal{H}(x,0) = x \qquad \forall x \in f^{\beta'} ,$$

$$\mathcal{H}\left(f^{\beta'} \times \{1\}\right) \subseteq f^{\alpha''} ,$$

$$\mathcal{H}\left(f^{\alpha'} \times [0,1]\right) \subseteq f^{\alpha''} .$$

It follows that the pair (f^b, f^a) is trivial. ∎

3 Properties of essential values

Let X denote again a metric space and $f : X \to \mathbf{R}$ a continuous function.

Theorem 3.1 *Let $c \in \mathbf{R}$ be an essential value of f.*

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous function $g : X \to \mathbf{R}$ with

$$\sup\{|g(x) - f(x)| : x \in X\} < \delta$$

admits an essential value in $]c - \varepsilon, c + \varepsilon[$.

Proof. By contradiction, assume there exist $\varepsilon > 0$ and a sequence of continuous functions $g_k : X \to \mathbf{R}$ such that

$$\sup\{|g_k(x) - f(x)| : x \in X\} < \frac{1}{k}$$

and such that g_k has no essential value in $]c - \varepsilon, c + \varepsilon[$.

Let $a, b \in]c - \varepsilon, c + \varepsilon[$ with $a < b$. Let us show that the pair $\left(f^b, f^a\right)$ is trivial. Let $[\alpha', \alpha'']$ be a neighbourhood of a and $[\beta', \beta'']$ a neighbourhood of b. Since the function g_k has no essential value in $]a, b[$, the pair $\left(g_k^b, g_k^a\right)$ is trivial, by Theorem (2.6). Moreover, if k is sufficiently large, we have $\alpha' + 1/k < a < \alpha'' - 1/k$ and $\beta' + 1/k < b < \beta'' - 1/k$. Then there exists a continuous map $\mathcal{H}_k : g_k^{\beta'+\frac{1}{k}} \times [0, 1] \to g_k^{\beta''-\frac{1}{k}}$ such that

$$\mathcal{H}_k(x, 0) = x \qquad \forall x \in g_k^{\beta'+\frac{1}{k}},$$

$$\mathcal{H}_k\left(g_k^{\beta'+\frac{1}{k}} \times \{1\}\right) \subseteq g_k^{\alpha''-\frac{1}{k}},$$

$$\mathcal{H}_k\left(g_k^{\alpha'+\frac{1}{k}} \times [0, 1]\right) \subseteq g_k^{\alpha''-\frac{1}{k}}.$$

Since $f^{\alpha'} \subseteq g_k^{\alpha'+\frac{1}{k}} \subseteq g_k^{\alpha''-\frac{1}{k}} \subseteq f^{\alpha''}$ and $f^{\beta'} \subseteq g_k^{\beta'+\frac{1}{k}} \subseteq g_k^{\beta''-\frac{1}{k}} \subseteq f^{\beta''}$, it follows that the pair (f^b, f^a) is trivial. Therefore, c is not an essential value of f: a contradiction. ∎

Now, let us recall a notion from [4, 7, 9].

Definition 3.2 *For every $u \in X$ let us denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : \mathrm{B}_\delta(u) \times [0, \delta] \to X$ with*

$$d(\mathcal{H}(v, t), v) \leq t,$$

$$f(\mathcal{H}(v, t)) \leq f(v) - \sigma t.$$

The extended real number $|df|(u)$ is called the weak slope *of f at u.*

It is readily seen that the function $|df| : X \to [0, +\infty]$ is lower semicontinuous.

Definition 3.3 *An element $u \in X$ is said to be* a critical point *of f, if $|df|(u) = 0$. A real number c is said to be* a critical value *of f, if there exists a critical point $u \in X$ of f such that $f(u) = c$. Otherwise c is said to be* a regular value *of f.*

Definition 3.4 *Let c be a real number. The function f is said to satisfy* the Palais - Smale condition at level c *($(PS)_c$ for short), if every sequence (u_h) in X with $|df|(u_h) \to 0$ and $f(u_h) \to c$ admits a subsequence (u_{h_k}) converging in X to some v (which is a critical point of f, by the lower semicontinuity of $|df|$).*

For every $c \in \mathbf{R}$ let us set

$$K_c = \{u \in X : \ f(u) = c, \ |df|(u) = 0\} .$$

Theorem 3.5 (Deformation Theorem) *Let $c \in \mathbf{R}$. Let us assume that X is complete and that f satisfies the Palais-Smale condition at level c.*

Then, for every $\overline{\varepsilon} > 0$, $\mathcal{O}$ neighbourhood of K_c (if $K_c = \emptyset$, we allow $\mathcal{O} = \emptyset$) and $\lambda > 0$, there exist $\varepsilon > 0$ and a continuous map $\eta : X \times [0, 1] \to X$ such that:

(a) $d(\eta(u, t), u) \leq \lambda t$;

(b) $f(\eta(u, t)) \leq f(u)$;

(c) $f(u) \notin]c - \overline{\varepsilon}, c + \overline{\varepsilon}[\implies \eta(u, t) = u$;

(d) $\eta(f^{c+\varepsilon} \setminus \mathcal{O}, 1) \subseteq f^{c-\varepsilon}$.

Proof. See [4, Theorem (2.14)]. ■

Theorem 3.6 (Noncritical Interval Theorem) *Let $a \in \mathbf{R}$ and $b \in \mathbf{R} \cup \{+\infty\}$ ($a < b$). Let us assume that X is complete, that f has no critical point u with $a \leq f(u) \leq b$ and that f satisfies $(PS)_c$ for every $c \in [a, b]$.*

Then there exists a continuous map $\eta : X \times [0, 1] \to X$ such that

(a) $\eta(u, 0) = u$;

(b) $f(\eta(u, t)) \leq f(u)$;

(c) $f(u) \leq a \implies \eta(u, t) = u$;

(d) $f(u) \leq b \implies f(\eta(u, 1)) \leq a$.

Proof. See [4, Theorem (2.15)]. ■

Theorem 3.7 *Let c be an essential value of f. Let us assume that X is complete and that $(PS)_c$ holds.*

Then c is a critical value of f.

Proof. By contradiction, let us assume that c is not a critical value of f. Since the function $|df|$ is lower semicontinuous and $(PS)_c$ holds, there exists $\varepsilon > 0$ such that

$$\inf \{|df|(x) : x \in X, c - \varepsilon < f(x) < c + \varepsilon\} > 0.$$

In particular, f has no critical value in $]c - \varepsilon, c + \varepsilon[$ and $(PS)_d$ holds whenever $c - \varepsilon < d < c + \varepsilon$. Let $a, b \in]c - \varepsilon, c + \varepsilon[$ with $a < b$. By the Noncritical Interval Theorem there exists a continuous map $\eta : X \times [0, 1] \to X$ such that

$$\eta(x, 0) = x,$$

$$f(\eta(x, t)) \leq f(x),$$

$$f(x) \leq b \Longrightarrow f(\eta(x, 1)) \leq a,$$

$$f(x) \leq a \Longrightarrow \eta(x, t) = x.$$

In particular the pair $\left(f^b, f^a\right)$ is trivial. Therefore, c is not an essential value of f: a contradiction. ■

Example 3.8 *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by*

$$f(x, y) = e^x - y^2.$$

Then 0 is an essential value of f, but not a critical value of f. On the other hand, $(PS)_0$ is not satisfied for f.

Let us show that the values arising from usual min–max procedures are all essential.

Theorem 3.9 *Let Γ be a non-empty family of closed non-empty subsets of X and let $d \in \mathbf{R} \cup \{-\infty\}$. Let us assume that for every $C \in \Gamma$ and for every deformation $\eta : X \times [0, 1] \to X$ with $\eta(x, t) = x$ on $f^d \times [0, 1]$, we have $\overline{\eta(C \times \{1\})} \in \Gamma$. Let us set*

$$c = \inf_{C \in \Gamma} \sup_{x \in C} f(x)$$

and let us suppose that $d < c < +\infty$.

Then c is an essential value of f.

Proof. By contradiction, let us assume that c is not an essential value of f. Let $d < a < c$ and $b > c$ be such that the pair $\left(f^b, f^a\right)$ is trivial. Let

$$d < \alpha' < a < \alpha'' < c < \gamma < \beta < b\,.$$

Then there exists a continuous map $\mathcal{H} : f^\beta \times [0,1] \to X$ such that

$$\mathcal{H}(x,0) = x \qquad \forall x \in f^\beta\,,$$

$$\mathcal{H}\left(f^\beta \times \{1\}\right) \subseteq f^{\alpha''}\,,$$

$$\mathcal{H}\left(f^{\alpha'} \times [0,1]\right) \subseteq f^{\alpha''}\,,$$

$$\mathcal{H}(x,t) = x \qquad \forall\,(x,t) \in f^d \times [0,1]\,.$$

Let $\vartheta : X \to [0,1]$ be a continuous function such that $\vartheta(x) = 1$ for $f(x) \leq \gamma$ and $\vartheta(x) = 0$ for $f(x) \geq \beta$. Let us define $\eta : X \times [0,1] \to X$ by

$$\eta(x,t) = \begin{cases} \mathcal{H}(x, \vartheta(x)t) & \text{if } f(x) \leq \beta \\ x & \text{if } f(x) \geq \beta \end{cases}.$$

It turns out that η is a deformation with $\eta(x,t) = x$ on $f^d \times [0,1]$. Let $C \in \Gamma$ be such that $C \subseteq f^\gamma$. Then $\overline{\eta(C \times \{1\})} \in \Gamma$ and $\overline{\eta(C \times \{1\})} \subseteq f^{\alpha''}$; this is absurd, as $\alpha'' < c$. ■

Corollary 3.10 *Let (D,S) be a pair of compact sets, let $\psi : S \to X$ be a continuous map and let*

$$\Phi \;=\; \{\varphi \in C(D;X) :\; \varphi_{|S} = \psi\}\,.$$

Let us assume that $\Phi \neq \emptyset$ and let us set

$$c = \inf_{\varphi \in \Phi} \max_{x \in \varphi(D)} f(x)\,.$$

If $c > \max\limits_{x \in \psi(S)} f(x)$, then c is an essential value of f.

Proof. Let us set

$$\Gamma \;=\; \{\varphi(D) :\; \varphi \in \Phi\}\,,$$

$$d \;=\; \max_{x \in \psi(S)} f(x)\,.$$

Then the assertion follows from the previous theorem. ■

Corollary 3.11 *Assume that X is non-empty and f is bounded from below.*

Then $\inf_X f$ *is an essential value of f.*

Proof. Let us set

$$\Gamma = \{\{x\} : x \in X\} ,$$

and $d = -\infty$. Then the assertion follows from Theorem (3.9). ∎

Now we want to study in more detail the case of a local minimum.

Example 3.12 *Let $X = \mathbf{R}$ and let $f : \mathbf{R} \to \mathbf{R}$ be defined by*

$$f(x) = \begin{cases} (x+1)^3 & \text{if } x < -1 \\ 0 & \text{if } -1 \le x \le 1 \\ (x-1)^3 & \text{if } x > 1 \end{cases} .$$

Then 0 is a local minimum of f, but $0 = f(0)$ is not an essential value of f. In fact

$$f_\varepsilon(x) = f(x) + \varepsilon \arctan x , \qquad \varepsilon > 0$$

has no critical value, even if f_ε satisfies $(PS)_c$ for any $c \in \mathbf{R}$ and $(f_\varepsilon - f)$ is uniformly small. From Theorems (3.1) and (3.7) it follows that 0 is not an essential value of f.

Now we study the situation for a strict local minimum u.

Definition 3.13 *We say that $u \in X$ is* a strict local minimum *for f, if there exists a neighbourhood U of u such that*

$$\forall v \in U \setminus \{u\} : f(v) > f(u) .$$

Example 3.14 *Let X be the Hilbert space l^2. For any integer $j \ge 1$, let $\varphi_j : \mathbf{R} \to \mathbf{R}$ be the continuous function defined by*

$$\varphi_j(s) = \begin{cases} -1 - s & \text{if } s < -1 \\ \frac{1}{j} s(s+1) & \text{if } -1 \le s \le 0 \\ s & \text{if } s > 0 \end{cases} .$$

It is readily seen that

$$\varphi_j(s) \ge -\frac{1}{4j} ,$$
$$|\varphi_j(s)| \le |s| .$$

Let $\Phi_j : \mathbf{R} \to \mathbf{R}$ be the primitive of φ_j such that $\Phi_j(0) = 0$ and let $f : X \to \mathbf{R}$ be defined by

$$f(u) = \sum_{j=1}^{\infty} \Phi_j \left(u^{(j)} \right) .$$

Then f is of class C^1 and has a strict local minimum at the origin. Define $f_h : X \to \mathbf{R}$ by

$$f_h(u) = f(u) + \frac{1}{h} \arctan\left(u^{(h)}\right) .$$

Then f_h is of class C^1 and uniformly close to f . Moreover it is

$$\forall u \in X : \, f_h'(u)e_h = \varphi_h\left(u^{(h)}\right) + \frac{1}{h}\frac{1}{1+\left(u^{(h)}\right)^2} \geq \frac{1}{4h} .$$

It follows that f_h satisfies $(PS)_c$ for any $c \in \mathbf{R}$ and has no critical value. From Theorems (3.1) *and* (3.7) *we deduce again that* $0 = f(0)$ *is not an essential value of f . Observe that f does not satisfy $(PS)_0$.*

In the next theorem, we give a positive result, when the minimum is strict in a stronger sense. In particular, the cases where X is finite dimensional or $(PS)_c$ holds for f are covered.

Theorem 3.15 *Let $u \in X$. Assume there exists a neighbourhood U of u such that*

$$\forall v \in U : \, f(v) \geq f(u) ,$$

$$\inf\{f(v) : \, v \in \partial U\} \, > \, f(u)$$

(the agree that $\inf \emptyset = +\infty$*).*

Then $f(u)$ is an essential value of f .

Proof. Let $c = f(u)$, $0 < \delta < \inf\{f(v) : v \in \partial U\} - f(u)$ and let $\varepsilon \in]0, \delta[$. Let $a \in]c-\varepsilon, c-\frac{\varepsilon}{2}[$ and $b \in]c+\frac{\varepsilon}{2}, c+\varepsilon[$. We claim that (f^b, f^a) is not trivial. By contradiction, let $\mathcal{H} : f^{c+\frac{\varepsilon}{2}} \times [0,1] \to f^{c+\varepsilon}$ be a deformation such that $\mathcal{H}\left(f^{c+\frac{\varepsilon}{2}} \times \{1\}\right) \subseteq f^{c-\frac{\varepsilon}{2}}$. We have $\mathcal{H}\left(f^{c+\frac{\varepsilon}{2}} \times [0,1]\right) \cap \partial U = \emptyset$, hence $\mathcal{H}\left(\{u\} \times [0,1]\right) \subseteq U$. This is absurd, as $f(\mathcal{H}(u,1)) \leq c - \frac{\varepsilon}{2}$. ■

Corollary 3.16 *Let X be locally compact and let $u \in X$ be a strict local minimum of f .*

Then $f(u)$ is an essential value of f .

Proof. It follows from the previous theorem. ■

Corollary 3.17 *Let u be a strict local minimum of f. Assume that X is complete and that the Palais-Smale condition is satisfied at level $f(u)$.*

Then $f(u)$ is an essential value of f.

Proof. Let $r > 0$ be such that

$$\forall v \in \overline{\mathrm{B}_{2r}(u)} \setminus \{u\} : \ f(v) > f(u).$$

By Theorem (3.15) it is sufficient to show that

$$\inf\{f(v) : \ v \in \partial \mathrm{B}_r(u)\} \ > \ f(u).$$

Let us set $c = f(u)$. By contradiction, let (v_h) be a sequence in $\partial \mathrm{B}_r(u)$ with $f(v_h) \to f(u)$. By the Deformation Theorem, there exist $\varepsilon > 0$ and a deformation $\eta : X \times [0,1] \to X$ such that

$$d(\eta(u,t), u) \leq rt,$$

$$\eta\left((f^{c+\varepsilon} \setminus \mathrm{B}_r(K_c)) \times \{1\}\right) \subseteq f^{c-\varepsilon}.$$

For h sufficiently large, it follows $\eta(v_h, 1) \in \overline{\mathrm{B}_{2r}(u)}$ and $f(\eta(v_h,1)) \leq c - \varepsilon$: a contradiction. ∎

4 Perturbations with variable domain

Definition 4.1 *Let X be a topological space and, for any $h \in \overline{\mathbf{N}} := \mathbf{N} \cup \{+\infty\}$, let $f_h : X \to \mathbf{R} \cup \{+\infty\}$ be a function. According to* [1, 5], *we write that*

$$f_\infty = \Gamma(X^-) \lim_h f_h,$$

if the following facts hold:

(*a*) *if (u_h) is a sequence in X convergent to u, we have*

$$f_\infty(u) \leq \liminf_h f_h(u_h);$$

(*b*) *for every $u \in X$ there exists a sequence (u_h) in X convergent to u such that*

$$f_\infty(u) = \lim_h f_h(u_h).$$

Definition 4.2 *Let X be a normed space and, for any $h \in \overline{\mathbf{N}}$, let $\mathbf{K}_h$ be a closed convex subset of X. According to* [11]*, we say that the sequence $(\mathbf{K}_h)$ is convergent to $\mathbf{K}_\infty$ in the sense of Mosco, if the following facts hold:*

(a) if $h_j \to +\infty$, $u_j \in \mathbf{K}_{h_j}$ and the sequence (u_j) is weakly convergent to u in X, then $u \in \mathbf{K}_\infty$;

(b) for every $u \in \mathbf{K}_\infty$ there exists a sequence (u_h) strongly convergent to u in X with $u_h \in \mathbf{K}_h$.

Now let Ω be a bounded open subset of $\mathbf{R}^n$ with $n \geq 3$. For every $h \in \overline{\mathbf{N}}$ let $f_h : H_0^1(\Omega) \to \mathbf{R} \cup \{+\infty\}$ be a functional and let us denote by

$$\mathcal{D}(f_h) = \{u \in H_0^1(\Omega) : \, f_h(u) < +\infty\}$$

the effective domain of f_h. In the following $\|\cdot\|_p$ will denote the norm in $L^p(\Omega)$ and $\|\cdot\|$ the norm in $H_0^1(\Omega)$. Let us assume that:

(i) for every $h \in \overline{\mathbf{N}}$ the functional $f_{h|\mathcal{D}(f_h)}$ is continuous with respect to the strong topology of $H_0^1(\Omega)$;

(ii) $f_\infty = \Gamma(w - H_0^1(\Omega)^-) \lim_h f_h$, where $w - H_0^1(\Omega)$ denotes the space $H_0^1(\Omega)$ endowed with the weak topology;

(iii) if (u_h) and (v_h) are weakly convergent to u in $H_0^1(\Omega)$ with $u_h, v_h \in \mathcal{D}(f_\infty)$ and

$$\limsup_h \left(\|v_h\| - \|u_h\|\right) \leq 0\,,$$

then

$$\limsup_h \left(f_\infty(v_h) - f_\infty(u_h)\right) \leq 0\,;$$

(iv) if (u_h) is weakly convergent to u in $H_0^1(\Omega)$ and $\lim_h f_h(u_h) = f_\infty(u) < +\infty$, then u_h is strongly convergent to u in $H_0^1(\Omega)$;

(v) if (u_h) is strongly convergent to u in $H_0^1(\Omega)$ with $u_h \in \mathcal{D}(f_h)$, then $f_h(u_h) \to f_\infty(u)$;

(vi) if we set $\mathbf{K}_h := \mathcal{D}(f_h)$ for every $h \in \overline{\mathbf{N}}$, $\mathbf{K}_h$ is a closed convex subset of $H_0^1(\Omega)$ with $0 \in \mathbf{K}_h$;

(vii) we have

$$\lim_{\|u\| \to \infty} f_\infty(u) = +\infty$$

and for every $R > 0$ and $b \in \mathbf{R}$ there exist $\tilde{h} \in \mathbf{N}$ and $R_1, R_2 > 0$ with $R < R_1 < R_2$ such that

$$\left[\left(\bigcup_{h \geq \tilde{h}} f_h^b\right) \cap \mathrm{B}_{R_2}(0)\right] \subseteq \mathrm{B}_{R_1}(0)\,.$$

First of all, let us investigate the stability of the assumptions $(i), \ldots, (vii)$.

Proposition 4.3 *Let us assume that* (f_h) *satisfies the hypotheses* $(i), \ldots, (vii)$ *and let* (g_h) *be a sequence of continuous functions from* $H_0^1(\Omega)$ *to* $\mathbf{R}$ *such that* $g_h \to 0$ *uniformly on bounded subsets of* $H_0^1(\Omega)$.

Then $(f_h + g_h)$ *satisfies the hypotheses* $(i), \ldots, (vii)$.

Proof. It is easy to see that the hypotheses $(i), \ldots, (vi)$ hold for $(f_h + g_h)$.

Let us prove that the hypothesis (vii) holds for $(f_h + g_h)$. Let $R > 0$ and $b \in \mathbf{R}$. Let $\overline{h}, R_1, R_2$ be related to f_h, R and $(b+1)$ as in the hypothesis (vii). Let $\tilde{h} > \overline{h}$ be such that $|g_h| < 1$ on $\mathrm{B}_{R_2}(0)$ for $h \geq \tilde{h}$. Then

$$\left[\left(\bigcup_{h \geq \tilde{h}} (f_h + g_h)^b\right) \cap \mathrm{B}_{R_2}(0)\right] \subseteq \mathrm{B}_{R_1}(0)$$

and (vii) follows. ∎

For $\rho > 0$, let us set

$$S_\rho = \left\{u \in H_0^1(\Omega) : \int_\Omega u^2\,dx = \rho^2\right\}.$$

In the following, the set $\mathbf{K}_h \cap S_\rho$ will be endowed with the H_0^1-metric.

For every $h \in \overline{\mathbf{N}}$ let us set $\tilde{f}_h := f_{h|\mathbf{K}_h \cap S_\rho}$. Evidently $\tilde{f}_h : \mathbf{K}_h \cap S_\rho \to \mathbf{R}$ is continuous. Our aim is to obtain a result like Theorem (3.1) in this setting. Observe, however, that $\tilde{f}_h$ is not uniformly close to $\tilde{f}_\infty$. Actually, even the domain of $\tilde{f}_h$ is variable.

Let us recall a definition from [3].

Definition 4.4 *Let* C *be a convex subset of a Banach space* X*, let* M *be a hypersurface in* X *of class* C^1*, let* $u \in C \cap M$ *and let* $\nu(u) \in X'$ *be a unit normal vector to* M *at* u*. The sets* C *and* M *are said to be* tangent at u*, if we have either*

$$\langle \nu(u), v - u \rangle \leq 0 \qquad \forall v \in C$$

or

$$\langle \nu(u), v - u \rangle \geq 0 \qquad \forall v \in C\,,$$

where $\langle \cdot, \cdot \rangle$ *is the pairing between* X' *and* X.

The sets C *and* M *are said to be* tangent*, if they are tangent at some point of* $C \cap M$.

Now we can state the main result of this section.

Theorem 4.5 *Let $c \in \mathbf{R}$ be an essential value of $\tilde{f}_\infty$. Let us assume that $\mathbf{K}_\infty$ and S_ρ are not tangent at any point of $\tilde{f}^c_\infty$.*

Then for every $\varepsilon > 0$ there exists $\overline{h} \in \mathbf{N}$ such that for every $h \geq \overline{h}$ the functional $\tilde{f}_h$ has an essential value in $]c - \varepsilon, c + \varepsilon[$.

The proof of this theorem will be given at the end of this section, after some auxiliary lemmas.

Lemma 4.6 *For every $u \in \mathbf{K}_\infty$ there exists a sequence (u_h) strongly convergent to u in $H^1_0(\Omega)$ with $u_h \in \mathbf{K}_h$.*

Proof. From the definition of Γ-convergence, it follows that there exists a sequence (u_h) weakly convergent to u in $H^1_0(\Omega)$ with $f_h(u_h)$ convergent to $f_\infty(u)$. From assumption (iv) we deduce that (u_h) is strongly convergent to u and the assertion follows. ■

Let us set

$$D = \Big\{ \, (h,u) \in \overline{\mathbf{N}} \times S_\rho : \ u \in \mathbf{K}_h \text{ and } \mathbf{K}_h \text{ and } S_\rho \text{ are not tangent at } u \Big\} \, .$$

In the following, D will be endowed with the topology induced by $\overline{\mathbf{N}} \times L^2(\Omega)$.

Theorem 4.7 *For every $\tilde{\varepsilon} > 0$ there exists a continuous map*

$$\eta : D \to H^1_0(\Omega)$$

such that for every $(h,u) \in D$ we have

$$\eta(h,u) \in \mathbf{K}_h \, ,$$

$$\int_\Omega u(\eta(h,u) - u) \, dx \ > \ 0 \, ,$$

$$\|\eta(h,u) - u\|_2 \ \leq \ \tilde{\varepsilon} \, ,$$

$$\|D\eta(h,u)\|_2 \ \leq \ \|Du\|_2 \, + \, \tilde{\varepsilon} \, ,$$

$$\|\eta(h,u)\| \ \leq \ \|u\| \, + \, \tilde{\varepsilon} \, .$$

Proof. It is sufficient to prove the assertion without the last inequality.

For every $(h,u) \in D$ let us denote by $\Sigma(h,u)$ the set of σ's in $]0, +\infty[$ such that there exists $u^+ \in \mathbf{K}_h$ with

$$\int_\Omega u(u^+ - u) dx \ > \ \sigma \, , \ \ \|u^+ - u\|_2 \ < \ \tilde{\varepsilon} \, , \ \ \|Du^+\|_2 \ < \ \|Du\|_2 \, + \, \tilde{\varepsilon} \, .$$

Because of the definition of D, for every $(h,u) \in D$ we can find $u^+ \in \mathbf{K}_h$ with $\int_\Omega u(u^+ - u)dx > 0$. By substituting u^+ with $(1-t)u + tu^+$ for some $t \in]0,1[$, we can also suppose that $\|u^+ - u\|_2 < \tilde{\varepsilon}$ and $\|Du^+\|_2 < \|Du\|_2 + \tilde{\varepsilon}$. Therefore $\Sigma(h,u)$ is a non-empty interval in $\mathbf{R}$.

Moreover, let us consider $\sigma \in \Sigma(\infty, u)$ and let us choose $u^+ \in \mathbf{K}_\infty$ according to the definition of $\Sigma(\infty,u)$. Let (u_h^+) be a sequence converging to u^+ in $H_0^1(\Omega)$ with $(u_h^+) \in \mathbf{K}_h$. Then it is readily seen that $\sigma \in \Sigma(h,v)$ for every (h,v) sufficiently close to (∞,u) in D.

Now it is easy to see that, for every $(h,u) \in D$ and for every $\sigma \in \Sigma(h,u)$, we have $\sigma \in \Sigma(k,v)$ whenever (k,v) is sufficiently close to (h,u) in D. Therefore there exists a continuous function $\sigma : D \to]0,+\infty[$ such that $\sigma(h,u) \in \Sigma(h,u)$.

For every $(h,u) \in D$ let us denote by $\mathcal{F}(h,u)$ the set of u^+'s in $\mathbf{K}_h$ such that

$$\int_\Omega u(u^+ - u)dx \geq \sigma(h,u), \ \|u^+ - u\|_2 \leq \tilde{\varepsilon}, \ \|Du^+\|_2 \leq \|Du\|_2 + \tilde{\varepsilon}.$$

Then $\mathcal{F}(h,u)$ is a non-empty closed convex subset of $H_0^1(\Omega)$.

Let $(\infty,u) \in D$, $u^+ \in \mathcal{F}(\infty,u)$ and $\varepsilon > 0$. Let $\hat{u}^+ \in K_\infty$ be related to $\sigma(\infty,u)$, as in the definition of $\Sigma(\infty,u)$. By substituting $\hat{u}^+$ with $(1-t)u^+ + t\hat{u}^+$ for some $t \in]0,1[$, we can suppose that $\|\hat{u}^+ - u^+\| < \frac{\varepsilon}{2}$. Let $(\hat{u}_h^+)$ be a sequence converging to $\hat{u}^+$ in $H_0^1(\Omega)$ with $\hat{u}_h^+ \in \mathbf{K}_h$. Then it is readily seen that $\|\hat{u}_h^+ - u^+\| < \varepsilon$ and $\hat{u}_h^+ \in \mathcal{F}(h,v)$ for every (h,v) sufficiently close to (∞,u) in D.

Now it is easy to see that the multifunction $\{(h,u) \longmapsto \mathcal{F}(h,u)\}$ is lower semi-continuous on D. By Michael Selection Theorem [2, Theorem (1.11.1)] there exists a continuous map $\eta : D \to H_0^1(\Omega)$ such that $\eta(h,u) \in \mathcal{F}(h,u)$ and the assertion follows. ■

Lemma 4.8 *Let $b \in \mathbf{R}$ and $\hat{\varepsilon} > 0$. Let us assume that $\mathbf{K}_\infty$ and S_ρ are not tangent at any point of $\tilde{f}_\infty^{b+\hat{\varepsilon}}$.*

Then there exists a function $\eta : D \to H_0^1(\Omega)$ as in Theorem (4.7) such that

$$\|\eta(\infty,u)\| < \|u\| + \hat{\varepsilon},$$

$$\tilde{f}_\infty(v) < \tilde{f}_\infty(u) + \hat{\varepsilon}$$

whenever $u \in \tilde{f}_\infty^{b+\hat{\varepsilon}}$, $t \in [0,1]$ and

$$v = \rho\frac{(1-t)u + t\eta(\infty,u)}{\|(1-t)u + t\eta(\infty,u)\|_2}.$$

Proof. By contradiction, let us assume that there exist $u_j \in \tilde{f}_\infty^{b+\hat{\varepsilon}}$, $t_j \in [0,1]$ and a sequence of continuous functions $\eta_j : D \to H_0^1(\Omega)$ such that

$$\|\eta_j(\infty, u_j) - u_j\|_2 \leq \frac{1}{j},$$

$$\|\eta_j(\infty, u_j)\| \leq \|u_j\| + \frac{1}{j}$$

and

$$\tilde{f}_\infty(v_j) \geq \tilde{f}_\infty(u_j) + \hat{\varepsilon}$$

with

$$v_j = \rho \frac{(1-t_j)u_j + t_j\eta_j(\infty, u_j)}{\|(1-t_j)u_j + t_j\eta_j(\infty, u_j)\|_2}.$$

Because of (vii), up to a subsequence, (u_j) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \mathbf{K}_\infty \cap S_\rho$. Hence we have that $\eta_j(\infty, u_j) \rightharpoonup u$ in $H_0^1(\Omega)$. It follows that $[(1-t_j)u_j + t_j\eta_j(\infty, u_j)] \rightharpoonup u$ in $H_0^1(\Omega)$, hence $v_j \rightharpoonup u$ in $H_0^1(\Omega)$. Moreover, from $\|(1-t_j)u_j + t_j\eta_j(\infty, u_j)\|_2 \geq \rho$ we deduce that $v_j \in \mathbf{K}_\infty \cap S_\rho$. Since

$$\limsup_j \left(\|v_j\| - \|u_j\|\right) \leq 0,$$

from assumption (iii) we deduce that

$$\limsup_j \left(\tilde{f}_\infty(v_j) - \tilde{f}_\infty(u_j)\right) \leq 0.$$

Therefore, for j sufficiently large, $\tilde{f}_\infty(v_j) \geq \tilde{f}_\infty(u_j) + \hat{\varepsilon}$ implies a contradiction and the assertion follows. ∎

For every $h \in \overline{\mathbf{N}}$ let us denote by $\pi_h : H_0^1(\Omega) \to \mathbf{K}_h$ the orthogonal projection in $H_0^1(\Omega)$ on the closed convex set $\mathbf{K}_h$.

Lemma 4.9 *Let $b \in \mathbf{R}$, $\hat{\varepsilon} > 0$ and $R > 0$ with $\tilde{f}_\infty^b \subseteq \mathrm{B}_R(0)$. Assume that $\mathbf{K}_\infty$ and S_ρ are not tangent at any point of $\tilde{f}_\infty^{b+\hat{\varepsilon}}$. Let $\eta : D \to H_0^1(\Omega)$ be a map as in the previous lemma. Moreover, if $u \in \tilde{f}_\infty^{b+\hat{\varepsilon}}$ and $\pi_h(\eta(\infty, u)) \neq 0$, let*

$$P_h(u) = \rho \frac{\pi_h(\eta(\infty, u))}{\|\pi_h(\eta(\infty, u))\|_2}.$$

Then there exists $\overline{h} \in \mathbf{N}$ such that the following facts hold:

(a) for every $h \geq \overline{h}$ the sets $\mathbf{K}_h$ and S_ρ are not tangent at any point of $\tilde{f}_h^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0)$;

(*b*) *for every* $h, k \in \overline{\mathbf{N}}$ *with* $h, k \geq \overline{h}$ *and* $u \in \tilde{f}_k^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0)$ *we have*

$$\|\pi_h(\eta(k,u))\|_2 > \rho\,,$$
$$\tilde{f}_h\left(\rho\frac{\pi_h(\eta(k,u))}{\|\pi_h(\eta(k,u))\|_2}\right) < \tilde{f}_k(u) + \hat{\varepsilon}\,;$$

(*c*) *for every* $h \geq \overline{h}$, $u \in \tilde{f}_\infty^b$ *and* $t \in [0,1]$ *we have*

$$\|P_h(u)\| < \|u\| + \hat{\varepsilon}\,,$$

$$\|(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))\|_2 > \rho\,,$$
$$\tilde{f}_\infty\left(\rho\frac{(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))}{\|(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))\|_2}\right) < \tilde{f}_\infty(u) + 2\hat{\varepsilon}\,.$$

Proof. Let us prove property (a). By contradiction, let us assume that there exist $h_k \to +\infty$ and $u_k \in \tilde{f}_{h_k}^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0)$ such that $\mathbf{K}_{h_k}$ and S_ρ are tangent at u_k. Since $0 \in \mathbf{K}_{h_k}$, we have

$$\int_\Omega u_k(v - u_k)\,dx \leq 0 \qquad \forall v \in \mathbf{K}_{h_k}\,.$$

Up to a subsequence, (u_k) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \tilde{f}_\infty^{b+\hat{\varepsilon}}$. Let $v \in \mathbf{K}_\infty$. Let (v_h) be weakly convergent to v in $H_0^1(\Omega)$ with $f_h(v_h) \to f_\infty(v)$. It follows that, eventually, $v_h \in \mathbf{K}_h$. Therefore, for k sufficiently large, we have

$$\int_\Omega u_k\,(v_{h_k} - u_k)\;dx \leq 0\,,$$

which implies

$$\int_\Omega u(v-u)\,dx \leq 0\;:$$

a contradiction, because $\mathbf{K}_\infty$ and S_ρ are not tangent at u.

Let us prove property (b). First of all, by contradiction, let us assume that there exist $h_j \to +\infty$, $k_j \to +\infty$ and $u_j \in \tilde{f}_{k_j}^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0)$ such that

$$\left\|\pi_{h_j}(\eta(k_j, u_j))\right\|_2 \leq \rho\,.$$

Up to a subsequence, (u_j) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \tilde{f}_\infty^{b+\hat{\varepsilon}}$. Consequently, $(\eta(k_j, u_j))$ is strongly convergent in $H_0^1(\Omega)$ to $\eta(\infty, u)$. Let (v_h) be weakly convergent to $\eta(\infty,u)$ in $H_0^1(\Omega)$ with $f_h(v_h) \to f_\infty(\eta(\infty,u))$. From assumption (iv) we deduce that (v_h) is strongly convergent to $\eta(\infty,u)$ in $H_0^1(\Omega)$. For j sufficiently large, we have that

$$\left\|\pi_{h_j}(\eta(k_j,u_j)) - \eta(k_j,u_j)\right\| \leq \|v_{h_j} - \eta(k_j,u_j)\|\,.$$

Therefore $\pi_{h_j}(\eta(k_j, u_j)) \to \eta(\infty, u)$ in $H_0^1(\Omega)$, which implies $\|\eta(\infty,u)\|_2 \le \rho$. This is absurd, as $\int_\Omega u(\eta(\infty,u)-u)dx > 0$.

Now, by contradiction, let us assume that there exist $h_j \to +\infty$, $k_j \to +\infty$ and $u_j \in \tilde{f}_{k_j}^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0)$ such that

$$\tilde{f}_{h_j}\left(\rho\frac{\pi_{h_j}(\eta(k_j,u_j))}{\|\pi_{h_j}(\eta(k_j,u_j))\|_2}\right) \ge \tilde{f}_{k_j}(u_j) + \hat{\varepsilon}.$$

Up to a subsequence, (u_j) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \tilde{f}_\infty^{b+\hat{\varepsilon}}$. As in the previous argument, it follows $\pi_{h_j}(\eta(k_j,u_j)) \to \eta(\infty,u)$ in $H_0^1(\Omega)$. Since

$$\rho\frac{\pi_{h_j}(\eta(k_j,u_j))}{\|\pi_{h_j}(\eta(k_j,u_j))\|_2} \in \mathbf{K}_{h_j} \cap S_\rho,$$

from assumption (v) we deduce that

$$\tilde{f}_{h_j}\left(\rho\frac{\pi_{h_j}(\eta(k_j,u_j))}{\|\pi_{h_j}(\eta(k_j,u_j))\|_2}\right) \to \tilde{f}_\infty\left(\rho\frac{(\eta(\infty,u))}{\|(\eta(\infty,u))\|_2}\right).$$

Combining this fact with $f_\infty(u) \le \liminf_j f_{kj}(u_j)$, by Lemma (4.8) we get a contradiction.

Let us prove property (c). Since $\|\eta(\infty,u)\| < \|u\| + \hat{\varepsilon}$ and $0 \in \mathbf{K}_h$, it is clear that $\|P_h(u)\| < \|u\| + \hat{\varepsilon}$. Now, by contradiction, let us assume that there exist $h_k \to +\infty$, $u_k \in \tilde{f}_\infty^b$ and $t_k \in [0,1]$ such that

$$\|(1-t_k)\eta(\infty, P_\infty(u_k)) + t_k\pi_\infty(\eta(h_k, P_{h_k}(u_k)))\|_2 \le \rho.$$

Up to a subsequence, (u_k) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \tilde{f}_\infty^b$. As in the proof of property (b), we have that $\pi_{h_k}(\eta(\infty,u_k)) \to \eta(\infty,u)$ in $H_0^1(\Omega)$. It follows $P_{h_k}(u_k) \to P_\infty(u)$ and $\eta(h_k, P_{h_k}(u_k)) \to \eta(\infty, P_\infty(u))$ in $H_0^1(\Omega)$. As in the proof of (b), we get a contradiction.

Finally, by contradiction, let us assume that there exist $h_k \to +\infty$, $u_k \in \tilde{f}_\infty^b$ and $t_k \in [0,1]$ such that

$$\tilde{f}_\infty\left(\rho\frac{(1-t_k)\eta(\infty,P_\infty(u_k)) + t_k\pi_\infty(\eta(h_k,P_{h_k}(u_k)))}{\|(1-t_k)\eta(\infty,P_\infty(u_k)) + t_k\pi_\infty(\eta(h_k,P_{h_k}(u_k)))\|_2}\right) \ge \tilde{f}_\infty(u_k) + 2\hat{\varepsilon}.$$

Up to a subsequence, (u_k) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \tilde{f}_\infty^b$. As in the previous argument, we have $(1-t_k)\eta(\infty,P_\infty(u_k)) + t_k\pi_\infty(\eta(h_k,P_{h_k}(u_k))) \to \eta(\infty,P_\infty(u))$ in $H_0^1(\Omega)$. Therefore by assumption (v) and Lemma (4.8) we get a contradiction. ∎

Lemma 4.10 *Let $R > 0$, $b \in \mathbf{R}$ and $\hat{\varepsilon} > 0$. Let us assume that $\tilde{f}_\infty^b \subseteq \mathrm{B}_R(0)$ and that $\mathbf{K}_\infty$ and S_ρ are not tangent at any point of $\tilde{f}_\infty^{b+\hat{\varepsilon}}$.*

Then there exists $\overline{h} \in \mathbf{N}$ and, for every $h \geq \overline{h}$, two continuous maps

$$P_h : \tilde{f}_\infty^b \to \mathbf{K}_h \cap S_\rho \cap \mathrm{B}_{R+\hat{\varepsilon}}(0)\,, \qquad Q_h : \tilde{f}_h^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0) \to \mathbf{K}_\infty \cap S_\rho$$

such that $\tilde{f}_h(P_h(u)) \leq \tilde{f}_\infty(u) + \hat{\varepsilon}$, $\tilde{f}_\infty(Q_h(v)) \leq \tilde{f}_h(v) + \hat{\varepsilon}$ for every $u \in \tilde{f}_\infty^b$, $v \in \tilde{f}_h^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0)$ and such that $Q_h \circ P_h : \tilde{f}_\infty^b \to \tilde{f}_\infty^{b+2\hat{\varepsilon}}$ is homotopic to the inclusion map $\tilde{f}_\infty^b \to \tilde{f}_\infty^{b+2\hat{\varepsilon}}$ by a homotopy $\mathcal{H} : \tilde{f}_\infty^b \times [0,1] \to \tilde{f}_\infty^{b+2\hat{\varepsilon}}$ such that

$$\forall (u,t) \in \tilde{f}_\infty^b \times [0,1] : \qquad \tilde{f}_\infty(\mathcal{H}(u,t)) \leq \tilde{f}_\infty(u) + 2\hat{\varepsilon}\,.$$

Proof. Let $\eta : D \to H_0^1(\Omega)$ be as in Lemma (4.8) and let $\overline{h} \in \mathbf{N}$ be as in Lemma (4.9). According to Lemma (4.9), for every $h \in \overline{\mathbf{N}}$ with $h \geq \overline{h}$ let us set

$$\forall u \in \tilde{f}_\infty^b : \qquad P_h(u) = \rho \frac{\pi_h(\eta(\infty,u))}{\|\pi_h(\eta(\infty,u))\|_2}\,,$$

$$\forall v \in \tilde{f}_h^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0) : \qquad Q_h(v) = \rho \frac{\pi_\infty(\eta(h,v))}{\|\pi_\infty(\eta(h,v))\|_2}\,.$$

By Lemma (4.9) it is readily seen that P_h and Q_h are well defined, continuous and satisfy $\tilde{f}_h(P_h(u)) \leq \tilde{f}_\infty(u) + \hat{\varepsilon}$, $\tilde{f}_\infty(Q_h(v)) \leq \tilde{f}_h(v) + \hat{\varepsilon}$ for every $u \in \tilde{f}_\infty^b$, $v \in \tilde{f}_h^{b+\hat{\varepsilon}} \cap \mathrm{B}_{R+\hat{\varepsilon}}(0)$. Now let us define $\mathcal{H}_0 : \tilde{f}_\infty^b \times [0,1] \to \tilde{f}_\infty^{b+\hat{\varepsilon}}$ by

$$\mathcal{H}_0(u,t) = \rho \frac{(1-t)u + t\eta(\infty,u)}{\|(1-t)u + t\eta(\infty,u)\|_2}\,.$$

Then $\mathcal{H}_0(u,0) = u$ and, by Lemma (4.8), we have $\tilde{f}_\infty(\mathcal{H}_0(u,t)) \leq \tilde{f}_\infty(u) + \hat{\varepsilon}$. Essentially in the same way, we can define $\mathcal{H}_1 : \tilde{f}_\infty^b \times [0,1] \to \tilde{f}_\infty^{b+2\hat{\varepsilon}}$ by

$$\mathcal{H}_1(u,t) = \rho \frac{(1-t)P_\infty(u) + t\eta(\infty, P_\infty(u))}{\|(1-t)P_\infty(u) + t\eta(\infty, P_\infty(u))\|_2}\,.$$

Thus, $\mathcal{H}_1(u,0) = \mathcal{H}_0(u,1)$ and $\tilde{f}_\infty(\mathcal{H}_1(u,t)) \leq \tilde{f}_\infty(u) + 2\hat{\varepsilon}$.

Finally, let us define $\mathcal{H}_2 : \tilde{f}_\infty^b \times [0,1] \to \tilde{f}_\infty^{b+2\hat{\varepsilon}}$ by

$$\mathcal{H}_2(u,t) = \rho \frac{(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))}{\|(1-t)\eta(\infty, P_\infty(u)) + t\pi_\infty(\eta(h, P_h(u)))\|_2}\,.$$

By Lemma (4.9), $\mathcal{H}_2$ is well defined, continuous, with $\tilde{f}_\infty(\mathcal{H}_2(u,t)) \leq \tilde{f}_\infty(u) + 2\hat{\varepsilon}$. Moreover, $\mathcal{H}_2(u,0) = \mathcal{H}_1(u,1)$ and $\mathcal{H}_2(u,1) = Q_h(P_h(u))$. The proof is complete. ∎

Proof of Theorem (4.5). Let $\tilde{\varepsilon} > 0$ be such that $\mathbf{K}_\infty$ and S_ρ are not tangent at any point of $\tilde{f}_\infty^{c+\tilde{\varepsilon}}$. Infact, by contradiction, let us assume that there exists $u_j \in \tilde{f}_\infty^{c+\frac{1}{j}}$ such that

$\mathbf{K}_\infty$ and S_ρ are tangent at u_j. Up to a subsequence, (u_j) is weakly convergent in $H_0^1(\Omega)$ to some $u \in \tilde{f}_\infty^c$. We have that

$$\int_\Omega u_j(v-u_j)\,dx \le 0 \qquad \forall v \in \mathbf{K}_\infty$$

and, as $j \to +\infty$, we obtain

$$\int_\Omega u(v-u)\,dx \le 0 \qquad \forall v \in \mathbf{K}_\infty \ :$$

a contradiction, because $\mathbf{K}_\infty$ and S_ρ are not tangent at u.

Because of (vii), there exists $R > 0$ such that $\tilde{f}_\infty^{c+\tilde{\varepsilon}} \subseteq \mathrm{B}_R(0)$. Let $\tilde{h} \in \mathbf{N}$, $R_1, R_2 > 0$ with $R < R_1 < R_2$ be such that

$$\left[\left(\bigcup_{h \ge \tilde{h}} \tilde{f}_h^{c+\tilde{\varepsilon}}\right) \cap \mathrm{B}_{R_2}(0)\right] \subseteq \mathrm{B}_{R_1}(0)\ .$$

Now, by contradiction, let us assume there exist $\varepsilon > 0$ and $h_k \to +\infty$ such that $\tilde{f}_{h_k}$ has no essential value in $]c-\varepsilon, c+\varepsilon[$. Without loss of generality, let us assume that $\varepsilon < \tilde{\varepsilon}$.

Let $a, b \in]c-\varepsilon, c+\varepsilon[$ with $a < b$. Let us prove that the pair $\left(\tilde{f}_\infty^b, \tilde{f}_\infty^a\right)$ is trivial. Let $[\alpha', \alpha'']$ be a neighbourhood of a and $[\beta', \beta'']$ be a neighbourhood of b with $\beta'' < c+\tilde{\varepsilon}$. Since $\tilde{f}_{h_k}$ has no essential value in $]a, b[$, the pair $\left(\tilde{f}_{h_k}^b, \tilde{f}_{h_k}^a\right)$ is trivial by Theorem (2.6). Let $a', a'', b', b'' \in \mathbf{R}$ be such that $\alpha' < a' < a < a'' < \alpha''$ and $\beta' < b' < b < b'' < \beta''$. For every $k \in \mathbf{N}$ there exists a continuous function $\mathcal{K}_k : \tilde{f}_{h_k}^{b'} \times [0,1] \to \tilde{f}_{h_k}^{b''}$ such that

$$\mathcal{K}_k(u,0) = u\,,$$

$$\mathcal{K}_k\left(\tilde{f}_{h_k}^{b'} \times \{1\}\right) \subseteq \tilde{f}_{h_k}^{a''}\,,$$

$$\mathcal{K}_k\left(\tilde{f}_{h_k}^{a'} \times [0,1]\right) \subseteq \tilde{f}_{h_k}^{a''}\,.$$

Let $\hat{\varepsilon} \in]0, \tilde{\varepsilon}[$ be such that $\alpha' + \hat{\varepsilon} \le a'$, $a'' + \hat{\varepsilon} \le \alpha''$, $\beta' + \hat{\varepsilon} \le b'$, $b'' + \hat{\varepsilon} \le \beta''$ and such that $R_1 + \hat{\varepsilon} \le R_2$.

Now let $\overline{h}$, P_h and Q_h be related to R_1, $(b'' - \hat{\varepsilon})$ and $\hat{\varepsilon}$ as in Lemma (4.10) and let $k \in \mathbf{N}$ be such that $h_k \ge \max\{\tilde{h}, \overline{h}\}$. Let us define $\mathcal{H} : \tilde{f}_\infty^{\beta'} \times [0,1] \to \tilde{f}_\infty^{\beta''}$ by

$$\mathcal{H}(u,t) \,=\, Q_{h_k}\left(\mathcal{K}_k(P_{h_k}(u), t)\right)\,.$$

Of course $\mathcal{H}\left(\tilde{f}_\infty^{\beta'} \times \{1\}\right) \subseteq \tilde{f}_\infty^{\alpha''}$ and $\mathcal{H}\left(\tilde{f}_\infty^{\alpha'} \times [0,1]\right) \subseteq \tilde{f}_\infty^{\alpha''}$. By Lemma (4.10) $\mathcal{H}(\cdot, 0) : \left(\tilde{f}_\infty^{\beta'}, \tilde{f}_\infty^{\alpha'}\right) \to \left(\tilde{f}_\infty^{\beta''}, \tilde{f}_\infty^{\alpha''}\right)$ is homotopic to the inclusion map. Therefore the pair $\left(\tilde{f}_\infty^b, \tilde{f}_\infty^a\right)$ is trivial.

We conclude that c is not an essential value of $\tilde{f}_\infty$: a contradiction. ∎

5 A more specific case

Throughout this section, Ω will denote a bounded open subset of $\mathbf{R}^n$ with $n \geq 3$. Let $(\mathbf{K}_h)$, $h \in \overline{\mathbf{N}}$, be a family of closed convex subsets of $H^1_0(\Omega)$ with $0 \in \mathbf{K}_h$. We assume that $(\mathbf{K}_h)$ is convergent to $\mathbf{K}_\infty$ in the sense of Mosco.

Let $P_h : \Omega \times \mathbf{R} \to \mathbf{R}$, $h \in \overline{\mathbf{N}}$, be Carathéodory functions such that

$(H1)$ for every $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that

$$|P_h(x,s)| \leq a_\varepsilon(x) + \varepsilon |s|^{\frac{2n}{n-2}}$$

for a.e. $x \in \Omega$ and all $s \in \mathbf{R}$ and $h \in \overline{\mathbf{N}}$;

$(H2)$ for a.e. $x \in \Omega$ we have

$$P_\infty(x,s) = \lim_h P_h(x,s)$$

uniformly on compact subsets of $\mathbf{R}$;

$(H3)$ we have $P_\infty(x,s) \geq 0$ for a.e. $x \in \Omega$ and all $s \in \mathbf{R}$.

Finally, let (μ_h) be a sequence strongly convergent to $\mu_\infty = 0$ in $H^{-1}(\Omega)$.

Now define $f_h : H^1_0(\Omega) \to \mathbf{R} \cup \{+\infty\}$, $h \in \overline{\mathbf{N}}$, by

$$f_h(u) = \begin{cases} \frac{1}{2}\int_\Omega |Du|^2\,dx + \int_\Omega P_h(x,u)\,dx - \langle \mu_h, u\rangle & \forall u \in \mathbf{K}_h \\ +\infty & \text{elsewhere} \end{cases}.$$

Lemma 5.1 *The sequence (f_h) satisfies all the conditions $(i), \ldots, (vii)$ of the previous section.*

Proof. Let (u_h) be a sequence weakly convergent to u in $H^1_0(\Omega)$. Up to a subsequence, (u_h) is convergent to u a.e. in Ω. For every $\varepsilon > 0$, we have

$$P_h(x,u_h) + \varepsilon |u_h|^{\frac{2n}{n-2}} \geq -a_\varepsilon(x).$$

From $(H2)$ and Fatou's Lemma it follows that

$$\int_\Omega P_\infty(x,u)\,dx + \varepsilon \int_\Omega |u|^{\frac{2n}{n-2}}\,dx \leq \liminf_h \int_\Omega P_h(x,u_h)\,dx + \varepsilon \limsup_h \int_\Omega |u_h|^{\frac{2n}{n-2}}\,dx,$$

hence

$$\int_\Omega P_\infty(x,u)\,dx \leq \liminf_h \int_\Omega P_h(x,u_h)\,dx + \varepsilon \sup_h \|u_h\|^{\frac{2n}{n-2}}_{\frac{2n}{n-2}}.$$

By the arbitrariness of ε, we have

$$\int_\Omega P_\infty(x,u)\,dx \leq \liminf_h \int_\Omega P_h(x,u_h)\,dx.$$

In a similar way, we can prove that

$$\int_\Omega P_\infty(x,u)\,dx \geq \limsup_h \int_\Omega P_h(x,u_h)\,dx\,,$$

so that

$$\int_\Omega P_\infty(x,u)\,dx = \lim_h \int_\Omega P_h(x,u_h)\,dx\,.$$

Therefore we have

$$\lim_h \left[\int_\Omega [P_h(x,u) - P_\infty(x,u)]\,dx - \langle \mu_h, u\rangle\right] = 0$$

uniformly on bounded subsets of $H_0^1(\Omega)$.

Let us consider

$$\hat{f}_h(u) = \begin{cases} \frac{1}{2}\int_\Omega |Du|^2\,dx + \int_\Omega P_\infty(x,u)\,dx & \forall u \in \mathbf{K}_h \\ +\infty & \text{elsewhere} \end{cases}.$$

It is easy to see that $(\hat{f}_h)$ satisfies $(i),\dots,(vii)$. From Proposition (4.3) we conclude that (f_h) satisfies $(i),\dots,(vii)$. ■

As in the previous section, let us set

$$\text{for } \rho > 0: \qquad S_\rho = \{u \in H_0^1(\Omega) : \textstyle\int_\Omega u^2\,dx = \rho^2\}\,,$$

$$\text{for any } h \in \overline{\mathbf{N}}: \qquad \tilde{f}_h := f_{h|\mathbf{K}_h \cap S_\rho}\,.$$

Theorem 5.2 *Let $c \in \mathbf{R}$ be an essential value of $\tilde{f}_\infty$. Let us assume that $\mathbf{K}_\infty$ and S_ρ are not tangent at any point of $\tilde{f}_\infty^c$.*

Then for every $\varepsilon > 0$ there exists $\overline{h} \in \mathbf{N}$ such that for every $h \geq \overline{h}$ the functional $\tilde{f}_h$ has an essential value in $]c-\varepsilon, c+\varepsilon[$.

Proof. The assertion follows from Lemma (5.1) and Theorem (4.5). ■

Finally, let us mention that, in more particular situations, it is possible to give sufficient conditions for nontangency and for the existence of essential values. Moreover, it is possible to show that for any essential value c of $\tilde{f}_h$ there exists $(\lambda, u) \in \mathbf{R} \times H_0^1(\Omega)$ such that

$$\begin{cases} u \in \mathbf{K}_h \cap S_\rho \\ \int_\Omega [DuD(v-u) + p_h(x,u)(v-u)]\,dx \geq \lambda \int_\Omega u(v-u)\,dx \quad \forall v \in \mathbf{K}_h \\ f_h(u) = c \end{cases},$$

where $p_h(x,s) = \frac{\partial P_h}{\partial s}(x,s)$. For all this aspects, we refer the reader to [6, 10].

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