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PERTURBATIONS OF EVEN NONSMOOTH FUNCTIONALS

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Abstract. An eigenvalue problem for elliptic variational inequalities is considered. The existence of multiple solutions is proved, when the operator is a small (non-odd) perturbation of an odd operator. To this aim, techniques of nonsmooth critical point theory are employed.

1. INTRODUCTION

Let us consider a nonlinear eigenvalue problem of the form

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K} \\ \int_\Omega \left[DuD(v-u) + p(x,u)(v-u)\right] dx \geq \lambda \int_\Omega u(v-u)\, dx \quad \forall v \in \mathbb{K} \\ \int_\Omega u^2\, dx = \rho^2 \end{cases} , \qquad (1.1)$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\mathbb{K}$ is a convex subset of $H_0^1(\Omega)$ of the form

$$\mathbb{K} = \left\{ u \in H_0^1(\Omega) : \psi_1 \leq u \leq \psi_2 \right\} ,$$

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p is a given nonlinearity and $\rho > 0$.

If $p(x, \cdot)$ is odd, $\psi_1 = -\psi_2$ and suitable qualitative conditions are satisfied, it has been shown that (1.1) admits a sequence (λ_h, u_h) of solutions with $\lambda_h \to +\infty$ (see [5, 7, 18]).

It is then natural to ask what happens, if (1.1) is perturbed by a "small" non-symmetric term. For instance, we can consider a perturbed problem of the form

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K} \\ \int_\Omega \left[DuD(v-u) \,+\, (p(x,u) + q(x,u))\,(v-u)\right] dx + \\ + < \mu, v-u > \geq \lambda \int_\Omega u(v-u)\,dx \quad \forall v \in \mathbb{K} \\ \int_\Omega u^2\,dx \,=\, \rho^2 \end{cases}, \tag{1.2}$$

where $\mu \in H^{-1}(\Omega)$ and q is another nonlinearity. Of course we do not assume that $q(x, \cdot)$ is odd, while we could impose some smallness condition on μ and q.

Roughly speaking, our main result (Theorem (3.12)) asserts that the number of solutions of (1.2) goes to infinity, as μ and q become smaller and smaller.

In the case of equations, results of this kind are well-known in the literature (see [1, 3, 17]). Actually, in that context, also perturbative results have been proved, in which the perturbed problem has still infinitely many solutions (see [2, 3, 4, 19, 21, 23]).

Here the presence of the constraint $\mathbb{K}$ causes some new difficulties which must be overcome.

First of all, problems (1.1) and (1.2) have a variational structure, but the associated functionals are not smooth in a classical sense. In the last years, several authors have treated variational problems with lack of regularity, providing useful tools to handle such situations. Here we follow the approach of [8, 10, 12].

Another difficulty is topological in nature. In the study of perturbed problems, a key role is played by the well-known fact that the manifold

$$S_\rho \,=\, \left\{ u : \int_\Omega u^2\,dx \,=\, \rho^2 \right\}$$

is contractible in itself. In our context, we prove (Theorem (3.8)) the contractibility of $\mathbb{K} \cap S_\rho$.

In the next section we introduce the auxiliary notion of "essential value" and we prove some related results. In a slightly different form, this notion appears also in [11]. Roughly speaking, essential values are candidate critical values, which are stable under small perturbations. However, the notion of essential value is purely topological.

In the third section we treat the concrete problem and we prove the main result.

2. Essential values of continuous functionals

In the following X will denote a metric space endowed with the metric d and $f : X \to \mathbb{R}$ a continuous function. If $b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, let us set

$$f^b = \{u \in X : f(u) \le b\} .$$

For the topological notions mentioned in the paper, the reader is referred to [20] and [22, Chapter 1, Sections 4 and 8].

(2.1) Definition. Let $a, b \in \overline{\mathbb{R}}$ with $a \le b$. The pair $\left(f^b, f^a\right)$ is said to be *trivial*, if for every neighbourhood $[\alpha', \alpha'']$ of a and $[\beta', \beta'']$ of b ($\alpha', \alpha'', \beta', \beta'' \in \overline{\mathbb{R}}$) there exist two closed subsets A and B of X such that $f^{\alpha'} \subseteq A \subseteq f^{\alpha''}$, $f^{\beta'} \subseteq B \subseteq f^{\beta''}$ and such that A is a strong deformation retract of B.

(2.2) Theorem. *Let $a, c, d, b \in \overline{\mathbb{R}}$ with $a < c < d < b$. Let us assume that the pairs $\left(f^b, f^c\right)$ and $\left(f^d, f^a\right)$ are trivial.*

Then the pair $\left(f^b, f^a\right)$ is trivial.

Proof. Let $[\alpha', \alpha'']$ be a neighbourhood of a and $[\beta', \beta'']$ a neighbourhood of b. Without loss of generality, we can assume $\alpha'' < c$ and $\beta' > d$. Moreover, let $c < \gamma < d$. There exist two closed subsets A_1 and B of X such that $f^{\alpha''} \subseteq A_1 \subseteq f^{\gamma}$, $f^{\beta'} \subseteq B \subseteq f^{\beta''}$ and there exists a strong deformation retraction $\mathcal{H}_1 : B \times [0,1] \to B$ of B to A_1. Moreover there exist two closed subsets A and B_2 of X such that $f^{\alpha'} \subseteq A \subseteq f^{\alpha''}$, $f^{\gamma} \subseteq B_2 \subseteq f^{\beta'}$ and there exists a strong deformation retraction $\mathcal{H}_2 : B_2 \times [0,1] \to B_2$ of B_2 to A. If we define $\mathcal{H} : B \times [0,1] \to B$ by

$$\mathcal{H}(u,t) = \begin{cases} \mathcal{H}_1(u, 2t) & 0 \le t \le \frac{1}{2} \\ \mathcal{H}_2\left(\mathcal{H}_1(u,1), 2t-1\right) & \frac{1}{2} \le t \le 1 \end{cases} ,$$

it turns out that $\mathcal{H}$ is a strong deformation retraction of B to A, and the thesis follows. ■

(2.3) DEFINITION. A real number c is said to be *an essential value* of f, if for every $\varepsilon > 0$ there exist $a, b \in]c-\varepsilon, c+\varepsilon[$ with $a < b$ such that the pair $\left(f^b, f^a\right)$ is not trivial.

(2.4) *Remark.* The set of essential values of f is closed in $\mathbb{R}$.

(2.5) THEOREM. *Let $a, b \in \overline{\mathbb{R}}$ with $a < b$. Let us assume that f has no essential value in $]a, b[$.*

Then the pair $\left(f^b, f^a\right)$ is trivial.

Proof. Let $[\alpha', \alpha'']$ be a neighbourhood of a, $[\beta', \beta'']$ a neighbourhood of b and let $a' \in]a, \alpha''[$, $b' \in]\beta', b[$. There exist $a' \leq c_1 < \cdots < c_k \leq b'$ and $\varepsilon_i > 0$ for $i = 1, \cdots, k$, such that

$$[a', b'] \subseteq \bigcup_{i=1}^{k}]c_i - \varepsilon_i, c_i + \varepsilon_i[$$

and such that for every $\overline{a}, \overline{b} \in]c_i - \varepsilon_i, c_i + \varepsilon_i[$ with $\overline{a} < \overline{b}$ the pair $\left(f^{\overline{b}}, f^{\overline{a}}\right)$ is trivial. Arguing by induction on k and taking into account Theorem (2.2), it is easy to see that the pair $\left(f^{b'}, f^{a'}\right)$ is trivial. Then there exist two closed subsets A and B of X such that $f^{\alpha'} \subseteq A \subseteq f^{\alpha''}$, $f^{\beta'} \subseteq B \subseteq f^{\beta''}$ and such that A is a strong deformation retract of B. It follows that the pair $\left(f^b, f^a\right)$ is trivial. ■

Now let us show the two main properties of essential values.

(2.6) THEOREM. *Let $c \in \mathbb{R}$ be an essential value of f. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every continuous function $g : X \to \mathbb{R}$ with*

$$\sup \{|g(u) - f(u)| : u \in X\} < \delta$$

admits an essential value in $]c - \varepsilon, c + \varepsilon[$.

Proof. By contradiction, assume there exist $\varepsilon > 0$ and a sequence of continuous functions $g_k : X \to \mathbb{R}$ such that

$$\sup \{|g_k(u) - f(u)| : u \in X\} < \frac{1}{k}$$

and such that g_k has no essential value in $]c - \varepsilon, c + \varepsilon[$.

Let $a, b \in]c-\varepsilon, c+\varepsilon[$ with $a < b$. Let us show that the pair $\left(f^b, f^a\right)$ is trivial. Let $\alpha' < a < \alpha''$ and $\beta' < b < \beta''$. Since the function g_k has no essential value in $]a, b[$, the pair $\left(g_k^b, g_k^a\right)$ is trivial by Theorem (2.5). Moreover, if k is sufficiently large, we have

$\alpha' + 1/k < a < \alpha'' - 1/k$ and $\beta' + 1/k < b < \beta'' - 1/k$. Then there exist two closed subsets A_k and B_k of X such that $g_k^{\alpha'+1/k} \subseteq A_k \subseteq g_k^{\alpha''-1/k}$, $g_k^{\beta'+1/k} \subseteq B_k \subseteq g_k^{\beta''-1/k}$ and such that A_k is a strong deformation retract of B_k. It follows $f^{\alpha'} \subseteq A_k \subseteq f^{\alpha''}$ and $f^{\beta'} \subseteq B_k \subseteq f^{\beta''}$, so that the pair $\left(f^b, f^a\right)$ is trivial.

Therefore c is not an essential value of f: a contradiction. ■

In order to prove the next result, let us recall some notions from [8, 12].

(2.7) Definition. For every $u \in X$ let us denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(u) \times [0, \delta] \to X$ with

$$d(\mathcal{H}(v,t), v) \leq t \,,$$

$$f(\mathcal{H}(v,t)) \leq f(v) - \sigma t \,.$$

The extended real number $|df|(u)$ is called *the weak slope* of f at u.

If X is a Finsler manifold of class C^1 and f a function of class C^1, it turns out that $|df|(u) = \|df(u)\|$ for every $u \in X$.

Let us point out that the above notion has been independently introduced also in [16], while a similar notion can be found in [15].

(2.8) Definition. An element $u \in X$ is said to be *a critical point* of f, if $|df|(u) = 0$. A real number c is said to be *a critical value* of f, if there exists a critical point $u \in X$ of f such that $f(u) = c$. Otherwise c is said to be *a regular value* of f.

(2.9) Definition. Let c be a real number. The function f is said to satisfy *the Palais - Smale condition at level* c ($(PS)_c$ in short), if every sequence (u_h) in X with $|df|(u_h) \to 0$ and $f(u_h) \to c$ admits a subsequence (u_{h_k}) converging in X.

Now let us prove the second basic property of essential values.

(2.10) Theorem. *Let $c \in \mathbb{R}$ be an essential value of f. Let us assume that X is complete and that $(PS)_c$ holds.*

Then c is a critical value of f.

Proof. By contradiction, let us assume that c is not a critical value of f. Since the function $|df|$ is lower semicontinuous (see [12, Proposition 2.6]) and $(PS)_c$ holds, there exists $\varepsilon > 0$ such that

$$\inf \left\{|df|(u) : \, u \in X, \, c - \varepsilon < f(u) < c + \varepsilon\right\} \, > \, 0 \,.$$

In particular, f has no critical value in $]c-\varepsilon, c+\varepsilon[$ and $(PS)_d$ holds whenever $c-\varepsilon < d < c+\varepsilon$. Let $a, b \in]c-\varepsilon, c+\varepsilon[$ with $a < b$. By [8, Theorem (2.15)] there exists $\eta : X \times [0,1] \to X$ continuous such that

$$\eta(u,0) = u\,,$$

$$f(\eta(u,t)) \leq f(u)\,,$$

$$f(u) \leq b \Longrightarrow f(\eta(u,1)) \leq a\,,$$

$$f(u) \leq a \Longrightarrow \eta(u,t) = u\,.$$

In particular, f^a is a strong deformation retract of f^b, so that the pair $\left(f^b, f^a\right)$ is trivial.

Therefore c is not an essential value of f: a contradiction. ■

Let us conclude the section by proving two results concerning the existence of essential values. First of all, let us show that the values, arising from usual min – max procedures, are all essential.

(2.11) THEOREM. *Let Γ be a non-empty family of closed non-empty subsets of X and let $d \in \mathbb{R} \cup \{-\infty\}$. Let us assume that for every $C \in \Gamma$ and for every deformation $\eta : X \times [0,1] \to X$ with $\eta(u,t) = u$ on $f^d \times [0,1]$, we have $\overline{\eta(C \times \{1\})} \in \Gamma$. Let us set*

$$c = \inf_{C \in \Gamma} \sup_{u \in C} f(u)$$

and let us suppose that $d < c < +\infty$.

Then c is an essential value of f.

Proof. By contradiction, let us assume that c is not an essential value of f. Let $d < a < c$ and $b > c$ be such that $\left(f^b, f^a\right)$ is trivial. Let $a < \alpha < c < \gamma < \beta < b$. There exist two closed subsets A and B of X such that $f^d \subseteq A \subseteq f^\alpha$, $f^\beta \subseteq B$ and there exists a strong deformation retraction $\mathcal{H} : B \times [0,1] \to B$ of B to A. Let $\vartheta : X \to [0,1]$ be a continuous function such that $\vartheta(u) = 1$ for $f(u) \leq \gamma$ and $\vartheta(u) = 0$ for $f(u) \geq \beta$. Let us define $\eta : X \times [0,1] \to X$ by

$$\eta(u,t) = \begin{cases} \mathcal{H}(u, \vartheta(u)t) & \text{if } f(u) \leq \beta \\ u & \text{if } f(u) \geq \beta \end{cases}.$$

Let $C \in \Gamma$ be such that $C \subseteq f^\gamma$. Then we have $\overline{\eta(C \times \{1\})} \in \Gamma$ and $\overline{\eta(C \times \{1\})} \subseteq f^\alpha$. This is absurd, as $\alpha < c$. ■

Finally, let us prove the main abstract result, in view of our applications. Let us point out that the argument is similar to that of [3, 17, 23].

(2.12) THEOREM. *Let E be a normed space, D a symmetric subset of E with respect to the origin with $0 \notin D$ and $f : D \to \mathbb{R}$ an even continuous function. Let us assume that D is non-empty and $k-$connected for every $k \geq 0$. For every $h \geq 1$ let us set*

$$c_h = \inf_{C \in \Gamma_h} \sup_{u \in C} f(u),$$

where Γ_h is the family of compact subsets of D of the form $\varphi(S^{h-1})$ with $\varphi : S^{h-1} \to D$ continuous and odd.

Then $\Gamma_h \neq \emptyset$ for every $h \geq 1$ and we have

$$\sup_h c_h \leq \sup \{c \in \mathbb{R} : c \text{ is an essential value of } f\}$$

with the convention $\sup \emptyset = -\infty$.

Proof. Since $D \neq \emptyset$, we have $\Gamma_1 \neq \emptyset$. On the other hand, if $\varphi : S^{h-1} \to D$ is continuous and odd, φ is homotopic to a constant map. By [17, Lemma VI.4.5] there exists an odd continuous map $\psi : S^h \to D$. Therefore $\Gamma_h \neq \emptyset$ for every $h \geq 1$.

Let us set

$$\gamma = \sup \{c \in \mathbb{R} : c \text{ is an essential value of } f\}.$$

It is readily seen that $c_1 = \inf_D f$ is an essential value of f or $c_1 = -\infty$. Therefore $c_1 \leq \gamma$. By contradiction let us assume that $\sup_h c_h > \gamma$. Hence there exists $h \geq 1$ such that $c_h \leq \gamma < c_{h+1}$. Let $a, \alpha', \alpha'' \in \mathbb{R}$ be such that $\gamma < \alpha' < a < \alpha'' < c_{h+1}$. There exists $\varphi : S^{h-1} \to D$ continuous and odd with $\varphi(S^{h-1}) \subseteq f^{\alpha'}$ and there exists a homotopy $\mathcal{H} : S^{h-1} \times [0, 1] \to D$ between φ and a constant map. Since $a > \gamma$, f has no essential value in $]a, +\infty[$. By Theorem (2.5) the pair (D, f^a) is trivial. Let

$$\beta = \max \left\{ f(\mathcal{H}(x, t)) : x \in S^{h-1}, \, t \in [0, 1] \right\}.$$

Then there exist two closed subsets A and B of D such that $f^{\alpha'} \subseteq A \subseteq f^{\alpha''}$, $f^\beta \subseteq B$ and there exists a strong deformation retraction $\eta : B \times [0, 1] \to B$ of B to A. Let us define $\mathcal{K} : S^{h-1} \times [0, 1] \to f^{\alpha''}$ by $\mathcal{K}(x, t) = \eta(\mathcal{H}(x, t), 1)$. Then $\mathcal{K}$ is a homotopy between $\varphi : S^{h-1} \to f^{\alpha''}$ and a constant map. By [17, Lemma VI.4.5] there exists $\psi : S^h \to f^{\alpha''}$ continuous and odd. This is absurd, as $\alpha'' < c_{h+1}$. ■

3. Perturbations of variational inequalities with symmetry

In this section we want to show an application of the previous results to a perturbation problem for variational inequalities.

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $n \geq 3$, let $p : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$p(x, -s) \;=\; -p(x, s)\,,$$

$$s\, p(x, s) \;\geq\; 0\,,$$

$$|p(x, s)| \;\leq\; a_1(x) + b|s|^r$$

with $a_1 \in L^{\frac{2n}{n+2}}(\Omega)$, $b \in \mathbb{R}$ and $0 < r < \frac{n+2}{n-2}$, let $\psi : \Omega \to [0, +\infty]$ be a quasi-lower semicontinuous function and let $\rho > 0$ with

$$\rho^2 \;<\; \int_\Omega \psi^2\, dx\,.$$

In the following $\|\cdot\|_p$ will denote the norm in $L^p(\Omega)$, $\|\cdot\|_{1,p}$ the norm in $W^{1,p}(\Omega)$ and $< \cdot, \cdot >$ the pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$. As usual, $L^1_{loc}(\Omega)$ will be identified with a subspace of $\mathcal{D}'(\Omega)$. For notions and results related to capacities, the reader is referred to [9].

We start from the nonlinear eigenvalue problem

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K} \\ \int_\Omega \left[DuD(v-u) \;+\; p(x,u)(v-u)\right] dx \;\geq\; \lambda \int_\Omega u(v-u)\, dx \quad \forall v \in \mathbb{K} \\ \int_\Omega u^2\, dx \;=\; \rho^2 \end{cases} \tag{3.1}$$

where

$$\mathbb{K} = \left\{ u \in H^1_0(\Omega) :\; -\psi(x) \leq \tilde{u}(x) \leq \psi(x) \text{ cap. q.e. in } \Omega \right\}$$

and $\tilde{u}$ is a quasi-continuous representative of u. It is readily seen that (3.1) possesses a symmetry. In fact, if (λ, u) is a solution of (3.1), also $(\lambda, -u)$ is a solution of (3.1).

We want to study a perturbation of (3.1) of the form

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K} \\ \int_\Omega [DuD(v-u) \,+\, (p(x,u)+q(x,u))(v-u)]\, dx + \\ + < \mu, v-u > \geq \, \lambda \int_\Omega u(v-u)\, dx \quad \forall v \in \mathbb{K} \\ \int_\Omega u^2\, dx \,=\, \rho^2 \end{cases} \tag{3.2}$$

where $q : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\mu \in H^{-1}(\Omega)$. We assume that, for some $\delta > 0$, we have $\|\mu\|_{H^{-1}(\Omega)} \,\leq\, \delta$ and

$$|q(x,s)| \,\leq\, a_2(x) + \delta |s|^r$$

with $a_2 \in L^{\frac{2n}{n+2}}(\Omega)$ and $\|a_2\|_{\frac{2n}{n+2}} \leq \delta$.

We want to show that, as $\delta \to 0$, the number of solutions of (3.2) becomes greater and greater.

Problems (3.1) and (3.2) have a variational structure. Let us set

$$B_\rho \,=\, \left\{ u \in L^2(\Omega) : \int_\Omega u^2\, dx \,<\, \rho^2 \right\},$$

$$S_\rho \,=\, \left\{ u \in L^2(\Omega) : \int_\Omega u^2\, dx \,=\, \rho^2 \right\}$$

and let us define $f : \mathbb{K} \cap S_\rho \to \mathbb{R}$ by

$$f(u) \,=\, \frac{1}{2} \int_\Omega |Du|^2\, dx \,+\, \int_\Omega P(x,u)\, dx\,,$$

where $P(x,s) \,=\, \int_0^s p(x,t)\, dt$. In the following, the set $\mathbb{K} \cap S_\rho$ will be endowed with the H^1_0-metric.

First of all, we want to apply Theorem (2.12) to the functional f. To this aim, let us recall a definition from [5, 6].

(3.3) Definition. Let C be a convex subset of a Banach space X, let M be a hypersurface of class C^1 in X, let $u \in C \cap M$ and let $\nu(u) \in X'$ be a unit normal vector to M at u. The sets C and M are said to be *tangent at* u, if we have either

$$< \nu(u), v-u > \leq\, 0 \qquad \forall v \in C$$

or

$$< \nu(u), v-u > \geq\, 0 \qquad \forall v \in C\,,$$

where $< \cdot, \cdot >$ is the pairing between X' and X.

The sets C and M are said to be *tangent*, if they are tangent at some point of $C \cap M$.

(3.4) LEMMA. *Let $\psi_1 : \Omega \to \overline{\mathbb{R}}$ be a quasi-upper semicontinuous function, $\psi_2 : \Omega \to \overline{\mathbb{R}}$ a quasi-lower semicontinuous function,*

$$C = \left\{ u \in H_0^1(\Omega) : \psi_1(x) \leq \tilde{u}(x) \leq \psi_2(x) \text{ cap. q.e. in } \Omega \right\} ,$$

$u \in C$ and $\alpha \in L^1_{loc}(\Omega) \cap H^{-1}(\Omega)$.

Then the following facts are equivalent:

a) we have

$$< \alpha, v - u > \leq 0 \qquad \forall v \in C ;$$

b) we have

$$\begin{cases} \alpha(x) \geq 0 & \text{a.e. in } \{x \in \Omega : u(x) > \psi_1(x)\} \\ \alpha(x) \leq 0 & \text{a.e. in } \{x \in \Omega : u(x) < \psi_2(x)\} \end{cases} .$$

Proof. It is easy to see that $b)$ implies $a)$. The converse is also a rather standard fact. Let us prove it for reader's convenience. Let (w_h) be a sequence in $H_0^1(\Omega) \cap L^\infty(\Omega)$ of non-negative functions such that

$$\psi_2(x) - \tilde{u}(x) = \sum_{h=0}^{\infty} \tilde{w_h}(x) \qquad \text{cap. q.e. in } \Omega .$$

If we set

$$w = \sum_{h=0}^{\infty} 2^{-h} \left(1 + \|w_h\|_{1,2} + \|w_h\|_\infty\right)^{-1} w_h ,$$

it is readily seen that $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and cap. q.e. in Ω we have

$$0 \leq \tilde{w}(x) \leq \psi_2(x) - \tilde{u}(x) ,$$

$$\tilde{w}(x) = 0 \implies \tilde{u}(x) = \psi_2(x) .$$

Then, for every $\vartheta \in C_c^\infty(\Omega)$ with $0 \leq \vartheta(x) \leq 1$, we have $(u + w\vartheta) \in C$, which yields

$$\int_\Omega \alpha w \vartheta \, dx = < \alpha, w\vartheta > \leq 0 .$$

It follows $\alpha(x)w(x) \leq 0$ a.e. in Ω, hence $\alpha(x) \leq 0$ a.e. in

$$\{x \in \Omega : u(x) < \psi_2(x)\} .$$

In a similar way, it is possible to show that $\alpha(x) \geq 0$ a.e. in

$$\{x \in \Omega : u(x) > \psi_1(x)\} . \blacksquare$$

Now we can characterize the tangency condition between $\mathbb{K}$ and S_ρ. If $\psi \in H^1(\Omega)$, similar characterizations have been proved in [6, 7].

(3.5) THEOREM. *The following facts hold:*
a) given $u \in \mathbb{K} \cap S_\rho$, the sets $\mathbb{K}$ and S_ρ are tangent at u, if and only if

$$\tilde{u}(x) \neq 0 \Longrightarrow |\tilde{u}(x)| = \psi(x) \qquad \text{cap. q.e. in } \Omega ;$$

b) the sets $\mathbb{K}$ and S_ρ are tangent, if and only if there exists a measurable subset E of Ω such that the function $\psi\chi_E$ is quasi-continuous and belongs to $H^1_0(\Omega) \cap S_\rho$.

Proof.

a) If $u \in \mathbb{K} \cap S_\rho$ is a function with the above property, it is readily seen that $\mathbb{K}$ and S_ρ are tangent at u.

Conversely, let us assume that $\mathbb{K}$ and S_ρ are tangent at $u \in \mathbb{K} \cap S_\rho$. Since $0 \in \mathbb{K}$, we have

$$\int_\Omega u(v-u)\,dx = < u, v-u > \leq 0 \qquad \forall v \in \mathbb{K} .$$

If we set $E = \{x \in \Omega : \tilde{u}(x) \neq 0\}$, we deduce from the previous lemma that $|u(x)| = \psi(x)\chi_E(x)$ a.e. in Ω. Since $\psi\chi_E$ is quasi-lower semicontinuous, this implies $|\tilde{u}(x)| \geq \psi(x)\chi_E(x)$ cap. q.e. in Ω. On the other hand, the opposite inequality is also true, as $u \in \mathbb{K}$. We deduce that $|\tilde{u}(x)| = \psi(x)\chi_E(x)$ cap. q.e. in Ω, and the thesis follows.

b) It follows from *a)*. $\blacksquare$

(3.6) THEOREM. *The space $\mathbb{K} \setminus \{0\}$ is contractible in itself.*

Proof. Let $V = \bigcup_{t>0} (t\mathbb{K})$. Since $-\mathbb{K} = \mathbb{K}$ and $\mathbb{K} \neq \{0\}$, V is an infinite dimensional linear subspace of $H^1_0(\Omega)$. Therefore $V \setminus \{0\}$ is contractible in itself by Dugundji's theorem [13].

Let $w \in H_0^1(\Omega)$ be such that $0 \leq \tilde{w}(x) \leq \psi(x)$ and $\tilde{w}(x) = 0 \Longrightarrow \psi(x) = 0$ cap. q.e. in Ω. Let us define $\mathcal{H} : (V \setminus \{0\}) \times [0,1] \to V \setminus \{0\}$ by

$$\mathcal{H}(u,t) \;=\; (1-t)u + t[(u \vee (-w)) \wedge w]\,.$$

Then $\mathcal{H}$ is a weak deformation retraction of $V \setminus \{0\}$ to $\mathbb{K} \setminus \{0\}$ and the thesis follows. ■

(3.7) LEMMA. *Let us assume that $\mathbb{K}$ and S_ρ are not tangent. Then for every compact subset C of $\mathbb{K} \setminus \{0\}$ there exist two continuous maps $r : C \to \mathbb{K} \setminus B_\rho$ and $\mathcal{H} : (C \setminus B_\rho) \times [0,1] \to \mathbb{K} \setminus B_\rho$ such that $\mathcal{H}(u,0) = u$ and $\mathcal{H}(u,1) = r(u)$ for every $u \in C \setminus B_\rho$.*

Proof. Let C be a compact subset of $\mathbb{K} \setminus \{0\}$. For every $u \in \mathbb{K}$ and $\varphi \in H_0^1(\Omega)$, let us set $\mathbb{K}(\varphi, u) = \left\{v \in H_0^1(\Omega) : \; |v| \leq \varphi \vee |u| \text{ a.e. in } \Omega\right\}$. Let us show that there exists a non-negative function φ in $\mathbb{K}$ such that $\int_\Omega \varphi^2\,dx > \rho^2$ and such that for every $u \in C \cap S_\rho$ the sets $\mathbb{K}(\varphi, u)$ and S_ρ are not tangent at u. By contradiction, let (φ_h) be an increasing sequence in $\mathbb{K} \setminus B_\rho$ with $(\tilde{\varphi}_h)$ convergent to ψ cap. q.e. in Ω and for every $h \in \mathbb{N}$ let $u_h \in C \cap S_\rho$ be such that

$$\int_\Omega u_h(v - u_h)\,dx \;\leq\; 0 \quad \forall v \in \mathbb{K}(\varphi_h, u_h)\,.$$

Since $\mathbb{K}(\varphi_h, u_h)$ contains $\mathbb{K}_{\varphi_h} := \left\{u \in H_0^1(\Omega) : \, -\varphi_h \leq u \leq \varphi_h \text{ a.e. in } \Omega\right\}$, we have

$$\int_\Omega u_h(v - u_h)\,dx \;\leq\; 0 \quad \forall v \in \mathbb{K}_{\varphi_h}\,.$$

Under a subsequence, (u_h) is convergent to $u \in C \cap S_\rho$, so that

$$\int_\Omega u(v - u)\,dx \;\leq\; 0 \quad \forall v \in \bigcup_{h=0}^\infty \mathbb{K}_{\varphi_h}\,.$$

By [9, Lemma 1.6] we have $\mathbb{K} = \overline{\bigcup_{h=0}^\infty \mathbb{K}_{\varphi_h}}$. It follows that $\mathbb{K}$ and S_ρ are tangent at u, which is absurd.

Let us define a continuous map $\Phi : \mathbb{K} \times H^1(\mathbb{R}^n) \to \mathbb{K}$ by

$$\Phi(u,v) \;=\; [v \vee (-(\varphi \vee |u|))] \wedge (\varphi \vee |u|)\,.$$

Let $\vartheta \in C^\infty(\mathbb{R})$ be such that $0 \leq \vartheta \leq 1$, $\vartheta(t) = 0$ if $t \geq \rho$ and $\vartheta(t) > 0$ if $t < \rho$. For every $u \in C$ let us set

$$T(u) \;=\; \{\sigma \in [0, +\infty[: \; \|\Phi(u, (1+\sigma)\mathcal{U}(u, \vartheta(\|u\|_2)))\|_2 > \rho\}\;,$$

where $\mathcal{U} : H^1(\mathbb{R}^n) \times [0, +\infty[\to H^1(\mathbb{R}^n)$ is the semiflow associated with the parabolic problem

$$\begin{cases} \frac{\partial \mathcal{U}}{\partial t}(u,t) = \Delta \mathcal{U}(u,t) \\ \mathcal{U}(u,0) = u \end{cases} .$$

For every $t > 0$ the function $\mathcal{U}(u,t)$ is real analytic on $\mathbb{R}^n$ and not identically zero. Since the gradient of a smooth function vanishes a.e. where the function is zero (see e.g. [14]), we have $\mathcal{U}(u,t)(x) \neq 0$ a.e. in $\mathbb{R}^n$. Then it is readily seen that $T(u) \neq \emptyset$ whenever $\|u\|_2 < \rho$. If $\|u\|_2 = \rho$, the sets $\mathbb{K}(\varphi, u)$ and S_ρ are not tangent at u, so that $T(u) \neq \emptyset$ by Theorem (3.5). Finally, it is obvious that $T(u) \neq \emptyset$ for $\|u\|_2 > \rho$. Therefore, for every $u \in C$ the set $T(u)$ is not empty. Moreover, for every $u \in C$ there exist a neighbourhood U of u and $\sigma \geq 0$ such that $[\sigma, +\infty[\subseteq \bigcap\limits_{v \in U} T(v)$. Because of the compactness of C, there exists $\tau \in \bigcap\limits_{u \in C} T(u)$. Let us define $\mathcal{K} : C \times [0,1] \to \mathbb{K} \setminus \{0\}$ by

$$\mathcal{K}(u,s) \;=\; \Phi(u, (1 + s\tau)\mathcal{U}(u, \vartheta(\|u\|_2)))\,.$$

For every $u \in C$ we have $\mathcal{K}(u,1) \in \mathbb{K} \setminus B_\rho$. Moreover, we have $\mathcal{K}(u,0) = u$ for every $u \in C \setminus B_\rho$ and $\mathcal{K}((C \setminus B_\rho) \times [0,1]) \subseteq \mathbb{K} \setminus B_\rho$. The thesis follows by setting $r = \mathcal{K}(\cdot, 1)$ and $\mathcal{H} = \mathcal{K}_{|(C \setminus B_\rho) \times [0,1]}$. ∎

Now we can show the main property of $\mathbb{K} \cap S_\rho$, in view of the result we want to prove.

(3.8) Theorem. *Let us assume that $\mathbb{K}$ and S_ρ are not tangent. Then $\mathbb{K} \cap S_\rho$ is an absolute retract. In particular, $\mathbb{K} \cap S_\rho$ is contractible in itself.*

Proof. First of all, let us show that every compact subset of $\mathbb{K} \cap S_\rho$ is contractible in $\mathbb{K} \cap S_\rho$. Since $\mathbb{K} \cap S_\rho$ is a strong deformation retract of $\mathbb{K} \setminus B_\rho$, it is sufficient to show that every compact subset of $\mathbb{K} \setminus B_\rho$ is contractible in $\mathbb{K} \setminus B_\rho$. Let C be a compact subset of $\mathbb{K} \setminus B_\rho$. By Theorem (3.6) there exists a contraction $\mathcal{K} : C \times [0,1] \to \mathbb{K} \setminus \{0\}$ of C in $\mathbb{K} \setminus \{0\}$. Moreover, by Lemma (3.7) there exist two continuous maps $r : \mathcal{K}(C \times [0,1]) \to \mathbb{K} \setminus B_\rho$ and $\mathcal{H} : [\mathcal{K}(C \times [0,1]) \setminus B_\rho] \times [0,1] \to \mathbb{K} \setminus B_\rho$ such that $\mathcal{H}(u,0) = u$ and $\mathcal{H}(u,1) = r(u)$. Let us define $\eta : C \times [0,1] \to \mathbb{K} \setminus B_\rho$ by

$$\eta(u) \;=\; \begin{cases} \mathcal{H}(u, 2t) & 0 \leq t \leq \frac{1}{2} \\ r(\mathcal{K}(u, 2t-1)) & \frac{1}{2} \leq t \leq 1 \end{cases} .$$

Then η is a contraction of C in $\mathbb{K} \setminus B_\rho$.

It follows that $\mathbb{K} \cap S_\rho$ is $k-$connected for every $k \geq 0$. On the other hand, $\mathbb{K} \cap S_\rho$ is an absolute neighbourhood retract by [5, Theorem 2.5]. The thesis follows by [20, Corollary to Theorem 15]. ■

(3.9) Theorem. *Let us assume that* $\mathbb{K}$ *and* S_ρ *are not tangent. Then the functional* $f : \mathbb{K} \cap S_\rho \to \mathbb{R}$ *admits a sequence* (d_h) *of essential values with* $d_h \to +\infty$.

Proof. Let us consider $E = H_0^1(\Omega)$ and $D = \mathbb{K} \cap S_\rho$. The set D is non-empty, contractible in itself and symmetric with respect to the origin with $0 \notin D$. The function $f : D \to \mathbb{R}$ is continuous, even and bounded from below. By Theorem (2.12), to conclude the proof, it is sufficient to show that $c_h \to +\infty$, where c_h is defined as in Theorem (2.12). By [5, Proposition 3.3] the functional f verifies $(PS)_c$ for every $c \in \mathbb{R}$. Moreover, $\mathbb{K} \cap S_\rho$ is an absolute neighbourhood retract. Arguing as in [8, Theorem 3.5] and taking into account [8, Theorems 2.16 and 2.17], it is easy to see that f^b has finite genus for every $b \in \mathbb{R}$. It follows $c_h \to +\infty$. ■

Now let us consider problem (3.2). Let $\vartheta_R \in C_c^\infty(\mathbb{R})$ with $0 \leq \vartheta_R \leq 1$, $\operatorname{supt} \vartheta_R \subseteq [-R-1, R+1]$ and $\vartheta_R(x) = 1$ on $[-R, R]$ and let $g : \mathbb{K} \cap S_\rho \to \mathbb{R}$ be the functional defined by

$$\begin{aligned} g(u) \;=\; & \frac{1}{2}\int_\Omega |Du|^2\,dx \;+\; \int_\Omega P(x,u)\,dx\,+ \\ & +\; \vartheta_R\left(\frac{1}{2}\int_\Omega |Du|^2\,dx\right)\left(\int_\Omega Q(x,u)\,dx + \;<\mu, u>\right), \end{aligned}$$

where $Q(x,s) = \int_0^s q(x,t)\,dt$.

In order to apply Theorem (2.6), we have to consider a “uniformly small” perturbation of f. This is the reason because the cut-off function ϑ_R has been introduced.

(3.10) Theorem. *Let us assume that* $\mathbb{K}$ *and* S_ρ *are not tangent. Then for every* $R > 0$ *and for every* $\sigma > 0$ *there exists* $\delta > 0$ *for which the following facts hold:*

a) *the functional* g *is continuous and*

$$\sup\{|g(u) - f(u)| : u \in \mathbb{K} \cap S_\rho\} \;<\; \sigma\,;$$

b) *for every* $u \in \mathbb{K} \cap S_\rho$ *there exist* $\lambda \in \mathbb{R}$ *and* $\eta \in H^{-1}(\Omega)$ *such that* $\|\eta\| = |dg|(u)$ *and*

$$\int_\Omega [DuD(v-u) + p(x,u)(v-u)]\,dx +$$

$$+\vartheta_R'\left(\frac{1}{2}\int_\Omega |Du|^2\,dx\right)\left(\int_\Omega Q(x,u)\,dx+ <\mu,u>\right)\int_\Omega DuD(v-u)\,dx+$$

$$+\vartheta_R\left(\frac{1}{2}\int_\Omega |Du|^2\,dx\right)\left(\int_\Omega q(x,u)(v-u)\,dx+ <\mu,v-u>\right)\geq$$

$$\geq \lambda\int_\Omega u(v-u)\,dx+ <\eta,v-u> \qquad \forall v\in \mathbb{K}\,;$$

c) the function g verifies $(PS)_c$ for every $c\in\mathbb{R}$.

Proof. Of course g is continuous. Moreover, we have $\|\mu\|_{H^{-1}}\leq\delta$ and

$$|Q(x,s)|\;\leq\; a_2(x)|s|\;+\;\frac{\delta}{r+1}\,|s|^{r+1}$$

with $\|a_2\|_{\frac{2n}{n+2}}\leq\delta$. Therefore, if δ is sufficiently small, we have

$$\sup_{u\in H^1_0}\,\vartheta_R\left(\frac{1}{2}\int_\Omega |Du|^2\,dx\right)\left|\int_\Omega Q(x,u)\,dx+\;<\mu,u>\right|\;<\;\sigma\,,$$

$$\sup_{u\in H^1_0}\left|\vartheta_R'\left(\frac{1}{2}\int_\Omega |Du|^2\,dx\right)\right|\left|\int_\Omega Q(x,u)\,dx+\;<\mu,u>\right|\;\leq\;\frac{1}{2}\,.$$

In particular, property a) holds.

Property b) follows from the Lagrange's multiplier theorem proved in [5, Theorem 2.5].

Let us prove property c). Let (u_h) be a sequence in $\mathbb{K}\cap S_\rho$ with $|dg|(u_h)\to 0$ and $g(u_h)\to c\in\mathbb{R}$. It is readily seen that (u_h) is bounded in $H^1_0(\Omega)$. Up to a subsequence, (u_h) is weakly convergent in $H^1_0(\Omega)$ to some $u\in\mathbb{K}\cap S_\rho$. Again up to a subsequence, we have that $(p(x,u_h))$, $(q(x,u_h))$ and (u_h) are strongly convergent in $H^{-1}(\Omega)$. According to the previous point, we have

$$\left[1+\vartheta_R'\left(\frac{1}{2}\int_\Omega |Du_h|^2\,dx\right)\left(\int_\Omega Q(x,u_h)\,dx+ <\mu,u_h>\right)\right]\int_\Omega Du_hD(v-u_h)\,dx+$$

$$+\int_\Omega p(x,u_h)(v-u_h)\,dx+$$

$$+\vartheta_R\left(\frac{1}{2}\int_\Omega |Du_h|^2\,dx\right)\left(\int_\Omega q(x,u_h)(v-u_h)\,dx+ <\mu,v-u_h>\right)\geq$$

$$\geq\lambda_h\int_\Omega u_h(v-u_h)\,dx+\;<\eta_h,v-u_h> \quad \forall v\in\mathbb{K} \tag{3.11}$$

with $\lambda_h\in\mathbb{R}$ and $\eta_h\to 0$ in $H^{-1}(\Omega)$.

Since $\mathbb{K}$ and S_ρ are not tangent at u, there exists $u^+ \in \mathbb{K}$ with

$$\int_\Omega u(u^+ - u)\, dx \, > \, 0\,.$$

If we put $v = u^+$ and $v = 0$ in (3.11), we deduce that (λ_h) is bounded in $\mathbb{R}$. Up to a subsequence, it follows that

$$\int_\Omega Du_h D(v - u_h)\, dx \, \geq < \beta_h, v - u_h > \qquad \forall v \in \mathbb{K}\,,$$

with (β_h) strongly convergent in $H^{-1}(\Omega)$. Therefore (u_h) is strongly convergent to u in $H_0^1(\Omega)$. ■

Now we can prove our main result.

(3.12) Theorem. *Let us assume that $\mathbb{K}$ and S_ρ are not tangent. Then for every $m \in \mathbb{N}$ there exists $\delta > 0$ for which problem* (3.2) *has at least m solutions $(\lambda_1, u_1), \cdots, (\lambda_m, u_m)$ with $u_1, \cdots, u_m$ all distinct.*

Proof. By Theorem (3.9) we can find m distinct essential values $d_1 < \cdots < d_m$ of f. Let $\varepsilon > 0$ be such that $2\varepsilon < d_i - d_{i-1}$ for every i and let $\delta_i > 0$ be obtained by applying Theorem (2.6) to the essential value d_i. Let $R > 0$ be such that

$$\forall u \in \mathbb{K} \cap S_\rho : \, f(u) \, < \, d_m + 2\varepsilon \implies \frac{1}{2}\int_\Omega |Du|^2\, dx \, < \, R\,.$$

Now let us apply Theorem (3.10) with $\sigma = \min\{\varepsilon, \delta_1, \cdots, \delta_m\}$ and let $\delta > 0$ be the value we obtain. Let us show that δ satisfies our claim.

By Theorem (2.6) g has an essential value in every $]d_i - \varepsilon, d_i + \varepsilon[$, hence it has at least m distinct essential values in $]-\infty, d_m + \varepsilon[$. By Theorem (2.10) each essential value of g is a critical value of g. Let $u_1, \cdots, u_m$ be distinct critical points of g with $g(u_i) < d_m + \varepsilon$. Since $f(u_i) < d_m + 2\varepsilon$, we have $\frac{1}{2}\int_\Omega |Du|^2\, dx \, < \, R$. By Theorem (3.10) it follows that for every u_i there exists $\lambda_i \in \mathbb{R}$ such that (λ_i, u_i) is a solution of (3.2). ■

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