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Linking over cones and nontrivial solutions for p-Laplace equations with p-superlinear nonlinearity ☆

Enlacement sur cones et solutions non triviales pour équations avec p-laplacien et nonlinéarité p-surlinéaire

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Abstract

We prove that the quasilinear equation $-\Delta_p u = \lambda V|u|^{p-2}u + g(x,u)$, with g subcritical and p-superlinear at 0 and at infinity, admits a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ for any $\lambda \in \mathbb{R}$. A minimax approach, allowing also an estimate of the corresponding critical level, is used. New linking structures, associated to certain variational eigenvalues of $-\Delta_p u = \lambda V|u|^{p-2}u$, are recognized, even in absence of any direct sum decomposition of $W_0^{1,p}(\Omega)$ related to the eigenvalue itself.
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Résumé

On démontre que l'équation quasilinéaire $-\Delta_p u = \lambda V|u|^{p-2}u + g(x,u)$, avec g souscritique et p-surlinéaire en 0 et à l'infini, admet une solution faible non triviale. Une approche de minimax est utilisée, qui permet aussi une estimation du niveau critique correspondant. De nouvelles structures d'enlacement, associées à certaines valeurs propres variationnelles de $-\Delta_p u = \lambda V|u|^{p-2}u$, sont reconnues, même en l'absence d'une décomposition directe de $W_0^{1,p}(\Omega)$ liée à la valeur propre.
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MSC: 58E05; 35J65

Keywords: Linking theorem; Cohomological index; p-Laplace equations; Nontrivial solutions

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$ and let $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denote the p-Laplace operator. We are interested in the solutions u of the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda V|u|^{p-2}u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Assume that $V \in L^\infty(\Omega)$ and that $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following conditions:

(g1) we have

$$\big|g(x,s)\big| \leqslant C\big(1 + |s|^{q-1}\big), \quad \text{with } C > 0 \text{ and } p < q < p^* := \frac{np}{n-p}, \quad \text{if } p < n,$$

$$\big|g(x,s)\big| \leqslant C\big(1 + |s|^{q-1}\big), \quad \text{with } C > 0 \text{ and } q > p, \quad \text{if } p = n,$$

while no condition is required if $p > n$;

(g2) we have $\lim_{s\to 0} g(x,s)/|s|^{p-1} = 0$ uniformly for $x \in \overline{\Omega}$;

(g3) there exist $\mu > p$ and $R > 0$ such that

$$|s| \geqslant R \quad \Longrightarrow \quad 0 < \mu G(x,s) \leqslant s g(x,s),$$

where $G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t$;

(g4) we have $sg(x,s) \geqslant 0$.

Of course, by (g2) we have $g(x,0) = 0$ for every $x \in \overline{\Omega}$. Therefore, (1.1) admits the trivial solution $u = 0$. We prove the following

Theorem 1.1. *Let us suppose that assumptions* (g1)–(g4) *hold and let $V \in L^\infty(\Omega)$. Then, for every $\lambda \in \mathbb{R}$, the quasilinear elliptic problem* (1.1) *admits a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$.*

In the case $p = 2$, the result is a classical application of the Linking Theorem (see e.g. [28, Theorem 5.16]). More precisely, let us assume, without loss of generality, that $\lambda \geqslant 0$. If the set

$$M := \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega V|u|^p\,\mathrm{d}x = 1 \right\} \tag{1.2}$$

is empty or if $M \neq \emptyset$ and

$$\lambda < \lambda_1 := \min\left\{ \int_\Omega |\nabla u|^p\,\mathrm{d}x : u \in M \right\},$$

then Theorem 1.1 can be proved by the Mountain Pass Theorem for any $p > 1$ (see [1] for the case $p = 2$ and [12] for the case $p \neq 2$). On the contrary, if $M \neq \emptyset$ and $\lambda \geqslant \lambda_1$, the classical proof is based on the fact that each eigenvalue λ_m of $-\Delta_2$ induces a suitable direct sum decomposition of $W_0^{1,2}(\Omega)$. In the case $p \neq 2$, even with $V \equiv 1$, the properties of the set $\sigma(-\Delta_p)$ of the eigenvalues of $-\Delta_p$, i.e. the set of real numbers η for which the equation $-\Delta_p u = \eta|u|^{p-2}u$ admits a nontrivial solution $u \in W_0^{1,p}(\Omega)$, are not yet well understood. It is known that there exist a first eigenvalue $\lambda_1 = \min\sigma(-\Delta_p) > 0$ and a second eigenvalue $\lambda_2 > \lambda_1$, both possessing several equivalent variational characterizations (see [2,20,21,3,13,9]). It is also possible to define, in at least three different variational ways, a diverging sequence (λ_m) in $\sigma(-\Delta_p)$ (see [18,13,26,7,27]), but it is not known if these definitions are equivalent for $m \geqslant 3$, if the whole set $\sigma(-\Delta_p)$ is covered and if there exists an induced direct sum decomposition. Actually, nobody has so far excluded the possibility that $\sigma(-\Delta_p) = \{\lambda_1\} \cup [\lambda_2, +\infty[$. Only in the case $n = 1$, it is known that $\sigma(-\Delta_p)$ is just the image of a positively divergent sequence (see [11]) and only for λ_1, and any dimension n, a linking structure, suitable for problem (1.1), has been so far recognized (see [16, Proposition 2.2 and Remark 2.2]). Let us also mention that, for different variational definitions of (λ_m), it has been shown in [13,7,27] that each λ_m induces generalized saddle

structures which are useful when $g(x,s)/|s|^{p-1} \to 0$ as $|s| \to \infty$. However, these geometries seem to be of no help when (g3) holds.

For these reasons, few papers have treated so far the case $\lambda \geqslant \lambda_1$. In [16, Theorem 1.2] the case $\lambda < \lambda_2$ is covered by the linking argument we have mentioned. Since it was hard to recognize a linking structure, other authors [22,25] have used Morse theory, which has however different features with respect to the minimax approach. In particular, Morse theory does not allow an easy estimate of the associated critical level, an information which plays a crucial role when $q \to p^*$. In [22] the case $\lambda < \lambda_2$ is treated using Morse theory, while [25] deals with the general case, provided that the further condition $\lambda \notin \sigma(-\Delta_p)$ is satisfied.

In this paper we show that, if we set as in [7]

$$\lambda_m = \inf\left\{ \sup_{u\in A} \int_\Omega |\nabla u|^p \,\mathrm{d}x \colon\ A \subseteq M,\ A \text{ is symmetric and } \operatorname{Index}(A) \geqslant m \right\}, \tag{1.3}$$

where Index is the $\mathbb{Z}_2$-cohomological index of [14,15], then each λ_m with $\lambda_m < \lambda_{m+1}$ induces a generalized linking structure associated with the cones

$$C_- = \left\{ u \in W_0^{1,p}(\Omega) \colon \int_\Omega |\nabla u|^p \,\mathrm{d}x \leqslant \lambda_m \int_\Omega V|u|^p \,\mathrm{d}x \right\}, \tag{1.4}$$

$$C_+ = \left\{ u \in W_0^{1,p}(\Omega) \colon \int_\Omega |\nabla u|^p \,\mathrm{d}x \geqslant \lambda_{m+1} \int_\Omega V|u|^p \,\mathrm{d}x \right\}. \tag{1.5}$$

This is the key tool to prove Theorem 1.1 by a minimax technique, without any restriction on λ.

In Section 2 we develop some ideas from [7] to recognize such a generalized linking. In the main results (Theorem 2.8 and Corollary 2.9) we also describe geometries of the type "Splitting spheres" and "Links and bounds", in the language of [23,24], which are of independent interest. In Section 3 we recall some basic properties of the eigenvalues of $-\Delta_p$. Finally, in Section 4 we prove the existence of a nontrivial solution for (1.1) under more general conditions than those stated above, and in the last section we derive Theorem 1.1 as a particular case.

2. Link and cohomological link

Throughout this section, X will denote a metric space and H^* Alexander–Spanier cohomology [30]. The next definition is a variant of [17, Definition 2.3].

Definition 2.1. Let D, S, A, B be four subsets of X with $S \subseteq D$ and $B \subseteq A$. We say that (D,S) *links* (A,B), if $S \cap A = B \cap D = \emptyset$ and, for every deformation $\eta : D \times [0,1] \to X \setminus B$ with $\eta(S \times [0,1]) \cap A = \emptyset$, we have that $\eta(D \times \{1\}) \cap A \neq \emptyset$.

It is readily seen that, if (D,S) links (A,B), then $D \cap A \neq \emptyset$. As usual, such geometries are designed for minimax theorems. The next one is a particular case of [17, Theorem 3.1].

Theorem 2.2. *Let X be a complete Finsler manifold of class C^1 and let $f : X \to \mathbb{R}$ be a function of class C^1. Let D, S, A, B be four subsets of X, with $S \subseteq D$ and $B \subseteq A$, such that (D,S) links (A,B) and such that*

$$\sup_S f < \inf_A f, \qquad \sup_D f < \inf_B f$$

(we agree that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$). Define

$$c = \inf_{\eta \in \mathcal{N}} \sup f\big(\eta\big(D \times \{1\}\big)\big),$$

where $\mathcal{N}$ is the set of deformations $\eta : D \times [0,1] \to X \setminus B$ with $\eta(S \times [0,1]) \cap A = \emptyset$. Then we have

$$\inf_A f \leqslant c \leqslant \sup_D f.$$

Moreover, if f satisfies $(PS)_c$, then c is a critical value of f.

The previous Definition 2.1 has a cohomological counterpart, which is much more suited for Morse theory, but also useful in general as sufficient condition. We recall it, in an equivalent form, from [10, Definition 5.1]. It is an adaptation of the well-known homological link [6, Definition II.1.2] (see also [7, Definition 3.1]).

Definition 2.3. Let D, S, A, B be four subsets of X with $S \subseteq D$ and $B \subseteq A$, let m be a nonnegative integer and let $\mathbb{K}$ be a field. We say that (D, S) *links* (A, B) *cohomologically in dimension* m *over* $\mathbb{K}$, if $S \cap A = B \cap D = \emptyset$ and the restriction homomorphism $H^m(X \setminus B, X \setminus A; \mathbb{K}) \to H^m(D, S; \mathbb{K})$ is not identically zero.

In the setting of Definitions 2.1 and 2.3, if $B = \emptyset$ (resp. $S = \emptyset$), we simply write A instead of $(A, \emptyset)$ (resp., D instead of $(D, \emptyset)$).

Proposition 2.4. *If* (D, S) *links* (A, B) *cohomologically, then* (D, S) *links* (A, B).

Proof. Let η be as in Definition 2.1. If $i : (D, S) \to (X \setminus B, X \setminus A)$ is the inclusion map, we have

$$H^*\big(\eta(\cdot, 1)\big) = H^*(i) : H^*(X \setminus B, X \setminus A; \mathbb{K}) \to H^*(D, S; \mathbb{K}).$$

If, by contradiction, $\eta(D \times \{1\}) \cap A = \emptyset$, then $H^*(\eta(\cdot, 1))$ can be factorized through $H^*(X \setminus A, X \setminus A; \mathbb{K})$ and is therefore identically zero. □

The aim of the section is to prove that particular classes of subsets cohomologically link. For this purpose, a powerful tool is constituted by the cohomological index of [14,15], we now recall in a particular case.

Definition 2.5. Let X be a real normed space. A subset A of X is said to be *symmetric*, if $-u \in A$ whenever $u \in A$. A subset A of X is said to be a *cone*, if $tu \in A$ whenever $u \in A$ and $t > 0$.

Definition 2.6. If X is a real normed space and A a symmetric subset of $X \setminus \{0\}$, we denote by $\mathrm{Index}(A)$ the $\mathbb{Z}_2$-index of A, as defined in [14,15].

Let us recall that, if $2 \leqslant \dim X \leqslant \infty$, the index can be defined as follows. Let $\sim$ be the equivalence relation in $X \setminus \{0\}$ which identifies u with $-u$. It is well known that $H^1((X \setminus \{0\})/\sim; \mathbb{Z}_2) \approx \mathbb{Z}_2$. Let α be the generator of $H^1((X \setminus \{0\})/\sim; \mathbb{Z}_2)$. If $A \subseteq X \setminus \{0\}$ is symmetric, then $\mathrm{Index}(A)$ is the least integer k such that $\alpha^k|_{A/\sim} = 0$. If no such integer exists, then $\mathrm{Index}(A) = \infty$.

Let us also recall that $\mathrm{Index}(Y \setminus \{0\}) = \dim Y$, whenever Y is a linear subspace of X. Moreover, we have $\gamma^+(A) \leqslant \mathrm{Index}(A) \leqslant \gamma^-(A)$, where, according to [6],

$$\gamma^+(A) = \sup\big\{m \in \mathbb{N}\text{: there exists an odd continuous map } \psi : \mathbb{R}^m \setminus \{0\} \to A\big\},$$
$$\gamma^-(A) = \inf\big\{m \in \mathbb{N}\text{: there exists an odd continuous map } \psi : A \to \mathbb{R}^m \setminus \{0\}\big\}.$$

The next result is a cohomological analogue of [7, Theorem 3.6]. It establishes a key connection between the equivariant notion of index and the nonequivariant notion of cohomological link.

Theorem 2.7. *Let* X *be a real normed space and let* S, A *be two symmetric subsets of* X *such that* $S \cap A = \emptyset$, $0 \in A$ *and* $\mathrm{Index}(S) = \mathrm{Index}(X \setminus A) < \infty$. *Then* (X, S) *links* A *cohomologically in dimension* $\mathrm{Index}(S)$ *over* $\mathbb{Z}_2$.

Proof. Let $m = \mathrm{Index}(S)$ and consider the exact sequence

$$H^m(X, X \setminus A; \mathbb{Z}_2) \to H^m(X, S; \mathbb{Z}_2) \to H^m(X \setminus A, S; \mathbb{Z}_2)$$

associated with the triple $(X, X \setminus A, S)$. The assertion we have to prove is equivalent to say that the restriction homomorphism $H^m(X, S; \mathbb{Z}_2) \to H^m(X \setminus A, S; \mathbb{Z}_2)$ is not injective. For this fact, we refer the reader to the second part of the proof of [7, Theorem 3.6]. □

We can now prove the main results of this section.

Theorem 2.8. *Let X be a real normed space and let C_-, C_+ be two cones in X such that C_+ is closed in X, $C_- \cap C_+ = \{0\}$ and such that $(X, C_- \setminus \{0\})$ links C_+ cohomologically in dimension m over $\mathbb{K}$. Let $r_-, r_+ > 0$ and let*

$$D_- = \{u \in C_-\colon \|u\| \leqslant r_-\}, \qquad S_- = \{u \in C_-\colon \|u\| = r_-\},$$
$$D_+ = \{u \in C_+\colon \|u\| \leqslant r_+\}, \qquad S_+ = \{u \in C_+\colon \|u\| = r_+\}.$$

Then the following facts hold:

(a) (D_-, S_-) *links* C_+ *cohomologically in dimension* m *over* $\mathbb{K}$;
(b) (D_-, S_-) *links* (D_+, S_+) *cohomologically in dimension* m *over* $\mathbb{K}$.

Moreover, let $e \in X$ with $-e \notin C_-$, let

$$Q = \{u + te\colon u \in C_-,\ t \geqslant 0,\ \|u + te\| \leqslant r_-\},$$
$$H = \{u + te\colon u \in C_-,\ t \geqslant 0,\ \|u + te\| = r_-\},$$

and assume that $r_- > r_+$. Then the following facts hold:

(c) $(Q, D_- \cup H)$ *links* S_+ *cohomologically in dimension* $m + 1$ *over* $\mathbb{K}$;
(d) $D_- \cup H$ *links* (D_+, S_+) *cohomologically in dimension* m *over* $\mathbb{K}$.

In particular, in each case (a)–(d) *there is a geometry of the type described in Definition* 2.1.

Proof. For the sake of simplicity, the coefficient field $\mathbb{K}$ will not be displayed.

(a) Since the inclusion maps $S_- \to C_- \setminus \{0\}$ and $D_- \to X$ are homotopy equivalences, from the Five Lemma (see [30, Lemma 4.5.11]) we deduce that the restriction homomorphism $H^m(X, C_- \setminus \{0\}) \to H^m(D_-, S_-)$ is an isomorphism. Then the assertion readily follows.

(b) Let

$$E_+ = \{u \in C_+\colon \|u\| \geqslant r_+\}.$$

Since X and $X \setminus E_+$ are star-shaped with respect to the origin, we have that $H^*(X, X \setminus E_+)$ is trivial. From the exact sequence of triple $(X, X \setminus E_+, X \setminus C_+)$ we deduce that the restriction homomorphism

$$H^m(X, X \setminus C_+) \longrightarrow H^m(X \setminus E_+, X \setminus C_+)$$

is an isomorphism. Therefore, assertion (a) is equivalent to the fact that the restriction homomorphism

$$H^m(X \setminus E_+, X \setminus C_+) \longrightarrow H^m(D_-, S_-)$$

is not identically zero. On the other hand, $E_+ \cap (X \setminus S_+)$ is a closed subset of $X \setminus S_+$ contained in the open set $X \setminus D_+$. Therefore, we also have the excision isomorphism

$$H^m(X \setminus S_+, X \setminus D_+) \longrightarrow H^m(X \setminus E_+, X \setminus C_+)$$

and assertion (b) follows.

(c) Consider the diagram

$$\begin{array}{ccccc}
H^m(X, X \setminus D_+) & \longrightarrow & H^m(X \setminus S_+, X \setminus D_+) & \xrightarrow{\delta^*} & H^{m+1}(X, X \setminus S_+) \\
\downarrow & & \downarrow & & \downarrow \\
H^m(Q, H) & \longrightarrow & H^m(D_- \cup H, H) & \xrightarrow{\delta^*} & H^{m+1}(Q, D_- \cup H) \\
 & & \downarrow & & \\
 & & H^m(D_-, S_-) & &
\end{array}$$

where vertical rows are restriction homomorphisms and horizontal rows come from the exact sequences of the triples $(X, X \setminus S_+, X \setminus D_+)$ and $(Q, D_- \cup H, H)$.

Since the restriction homomorphism $H^m(X \setminus S_+, X \setminus D_+) \to H^m(D_-, S_-)$ is not identically zero by assertion (b), a fortiori the restriction homomorphism $H^m(X \setminus S_+, X \setminus D_+) \to H^m(D_- \cup H, H)$ does the same. On the other hand, since $-e \notin C_-$, we can define a contraction $\mathcal{K} : H \times [0,1] \to H$ of H in itself by

$$\mathcal{K}(v,s) = \frac{r_-}{\|(1-s)v + se\|}\big((1-s)v + se\big).$$

It follows that $H^m(Q, H)$ is trivial, as Q also is clearly contractible in itself, hence that $\delta^* : H^m(D_- \cup H, H) \to H^{m+1}(Q, D_- \cup H)$ is injective, by the exactness of the second row. Therefore, also the restriction homomorphism $H^{m+1}(X, X \setminus S_+) \to H^{m+1}(Q, D_- \cup H)$ is not identically zero, by the commutativity of the right square.

(d) Consider the commutative square

$$\begin{array}{ccc} H^m(X \setminus S_+, X \setminus D_+) & \xrightarrow{\delta^*} & H^{m+1}(X, X \setminus S_+) \\ \downarrow & & \downarrow \\ H^m(D_- \cup H) & \xrightarrow{\delta^*} & H^{m+1}(Q, D_- \cup H) \end{array}$$

where vertical rows are restriction homomorphisms and horizontal rows come from the exact sequences of the triples $(X, X \setminus S_+, X \setminus D_+)$ and $(Q, D_- \cup H, \emptyset)$.

In the previous point, we have found an element in $H^{m+1}(X, X \setminus S_+)$ coming from $H^m(X \setminus S_+, X \setminus D_+)$ through δ^* and with nonzero restriction in $H^{m+1}(Q, D_- \cup H)$. From the commutativity of the square, it follows that the restriction homomorphism $H^m(X \setminus S_+, X \setminus D_+) \to H^m(D_- \cup H)$ is not identically zero. □

Corollary 2.9. *Let X be a real normed space and let C_-, C_+ be two symmetric cones in X such that C_+ is closed in X, $C_- \cap C_+ = \{0\}$ and such that*

$$\operatorname{Index}\big(C_- \setminus \{0\}\big) = \operatorname{Index}(X \setminus C_+) < \infty.$$

Then the assertions (a)–(d) *of Theorem* 2.8 *hold for $m = \operatorname{Index}(C_- \setminus \{0\})$ and $\mathbb{K} = \mathbb{Z}_2$.*

Proof. It is enough to combine Theorems 2.7 and 2.8. □

Remark 2.10. Assertion (a) of Corollary 2.9 is essentially contained in [7], while assertions (b)–(d) are new. In particular, assertion (c) will be used in the proof of Theorem 1.1.

Remark 2.11. If C_-, C_+ are closed linear subspaces of a Banach space X with $X = C_- \oplus C_+$ and $\dim C_- < \infty$, then the assertions of Corollary 2.9 are either well known or essentially known (see e.g. [6,10]). As for the geometry involved, assertions (a) and (c) correspond to the well known Saddle Theorem and Linking Theorem (see e.g. [28]). Assertion (b) corresponds to the "Splitting Spheres Theorem" of [23,24], while assertions (c) and (d) correspond to the "Links and Bounds Theorem" of [23,24] (see, in particular, [24, Theorems 8.1 and 8.2]).

Proposition 2.12. *Let X be a real normed space, C a closed cone in X and $e \in X$ with $-e \notin C$.*

Then there exists $\beta \geqslant 1$ such that

$$\|u\| + \|e\| \leqslant \beta \|u + e\| \quad \forall u \in C. \tag{2.1}$$

In particular, $C + \mathbb{R}^+ e$ is closed in X.

Proof. If we examine the cases $\|u\| \leqslant 2\|e\|$ and $\|u\| > 2\|e\|$, we easily find β satisfying (2.1). Now, if $(u_k + t_k e)$ is a convergent sequence in X with $u_k \in C$ and $t_k \geqslant 0$, from (2.1) we deduce that

$$\|u_k\| + t_k\|e\| \leqslant \beta \|u_k + t_k e\|.$$

Therefore (t_k) is bounded, hence convergent, up to a subsequence, to some $t \geqslant 0$. It follows that (u_k) is convergent to some $u \in C$. Therefore, $C + \mathbb{R}^+ e$ is closed in X. □

3. Eigenvalues of the p-Laplace operator

Let Ω be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$ and let

$$\mathcal{V}(\Omega) := \begin{cases} L^{n/p}(\Omega) & \text{if } 1 < p < n, \\ \bigcup_{r>1} L^r(\Omega) & \text{if } p = n, \\ L^1(\Omega) & \text{if } p > n. \end{cases}$$

Finally, let $V \in \mathcal{V}(\Omega)$.

We define an even functional $\mathcal{E} : W_0^{1,p}(\Omega) \to \mathbb{R}$ of class C^1 by

$$\mathcal{E}(u) = \int_\Omega |\nabla u|^p \, dx$$

and M according to (1.2).

Proposition 3.1. *If $M \neq \emptyset$, then M is a closed, symmetric submanifold in $W_0^{1,p}(\Omega)$ of class C^1 with $0 \notin M$ and* $\mathrm{Index}(M) = \infty$. *Moreover, for every symmetric, open subset A of M, we have*

$$\mathrm{Index}(A) = \sup\big\{\mathrm{Index}(K):\ K \text{ is compact and symmetric with } K \subseteq A\big\}.$$

Finally, the map $\{u \mapsto V|u|^{p-2}u\}$ is weak-to-strong sequentially continuous from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, while the map $\{u \mapsto V|u|^p\}$ is weak-to-strong sequentially continuous from $W_0^{1,p}(\Omega)$ to $L^1(\Omega)$.

Proof. It is easily seen that M is a closed, symmetric submanifold in $W_0^{1,p}(\Omega)$ of class C^1. If $M \neq \emptyset$, it is proved in [32, p. 199] that $\gamma^+(M) = \infty$, whence also $\mathrm{Index}(M) = \infty$.

Let now A be a symmetric, open subset of M, let α be the generator of $H^1((W_0^{1,p}(\Omega) \setminus \{0\})/\sim; \mathbb{Z}_2) \approx \mathbb{Z}_2$ and let $0 \leqslant m < \mathrm{Index}(A)$. It follows that $\alpha^m|_{A/\sim} \neq 0$. Since $A/\sim$ is an open subset of the manifold $M/\sim$, Alexander–Spanier and singular cohomology are naturally isomorphic. Then there exists $z \in H_m(A/\sim; \mathbb{Z}_2)$ such that $\langle \alpha^m|_{A/\sim}, z\rangle \neq 0$, as $\mathbb{Z}_2$ is a field. Since singular homology is a theory with compact supports, there exist a symmetric, compact subset K of A and $z' \in H_m(K/\sim; \mathbb{Z}_2)$ such that $H_*(i)z' = z$, where $i : K/\sim \to A/\sim$ denotes the inclusion map. In particular, for every symmetric, open neighborhood U of K, we have

$$\big\langle \alpha^m|_{U/\sim}, H_*(j)z' \big\rangle = \big\langle \alpha^m|_{A/\sim}, z \big\rangle \neq 0,$$

where $j\colon K/\sim \to U/\sim$ denotes the inclusion map. Observe that the natural isomorphism between Alexander–Spanier and singular cohomology is still valid for $U/\sim$, which is open in $M/\sim$. It follows that $\alpha^m|_{U/\sim} \neq 0$, hence that $\mathrm{Index}(U) > m$ for every symmetric, open neighborhood U of K. From the continuity of the index we deduce that $\mathrm{Index}(K) > m$ and the assertion concerning $\mathrm{Index}(A)$ follows by the arbitrariness of m.

Finally, in [32, Lemma 4.2] it is proved that the map $\{u \mapsto V|u|^{p-2}u\}$ is weak-to-strong sequentially continuous from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. The proof that the map $\{u \mapsto V|u|^p\}$ is weak-to-strong sequentially continuous from $W_0^{1,p}(\Omega)$ to $L^1(\Omega)$ is similar. □

If $M \neq \emptyset$, for every integer $m \geqslant 1$ we define λ_m according to (1.3). Let us remark that, by the previous proposition,

$$\lambda_m = \inf\Big\{\max_{u\in K} \mathcal{E}(u):\ K \subseteq M,\ K \text{ is compact and symmetric with } \mathrm{Index}(K) \geqslant m\Big\},$$

which is essentially the definition of λ_m given in [27]. Indeed, for every $\varepsilon > 0$, there exists a symmetric subset A of M with $\mathrm{Index}(A) \geqslant m$ and $A \subseteq \{u \in M:\ \mathcal{E}(u) < \lambda_m + \varepsilon\}$, whence $\mathrm{Index}(\{u \in M:\ \mathcal{E}(u) < \lambda_m + \varepsilon\}) \geqslant m$. By the previous proposition, there exists a symmetric, compact subset K of $\{u \in M:\ \mathcal{E}(u) < \lambda_m + \varepsilon\}$ with $\mathrm{Index}(K) \geqslant m$ and the assertion follows from the arbitrariness of ε.

We also refer the reader to [7] for a comparison with other variational definitions of λ_m given in the literature.

Theorem 3.2. *Let $M \neq \emptyset$. Then the functional $\mathcal{E}|_M$ satisfies $(PS)_c$ for every $c \in \mathbb{R}$ and we have*

$$\lambda_1 = \min_{u \in M} \mathcal{E}(u) > 0, \qquad \lim_{m \to \infty} \lambda_m = +\infty.$$

Moreover, if $m \geqslant 1$ is such that $\lambda_m < \lambda_{m+1}$, then we have

$$\operatorname{Index}\left(\left\{u \in W_0^{1,p}(\Omega) \setminus \{0\}\colon\ \mathcal{E}(u) \leqslant \lambda_m \int_\Omega V|u|^p \,\mathrm{d}x\right\}\right)$$
$$= \operatorname{Index}\left(\left\{u \in W_0^{1,p}(\Omega)\colon\ \mathcal{E}(u) < \lambda_{m+1} \int_\Omega V|u|^p \,\mathrm{d}x\right\}\right) = m.$$

Proof. It is clear that λ_1 is just the infimum of $\mathcal{E}|_M$. Moreover, in [32, Theorem 4.4] it is proved that $\mathcal{E}|_M$ satisfies $(PS)_c$ for every $c \in \mathbb{R}$. On the other hand, the deformation theorem has been extended, also in the equivariant case, to Finsler manifolds of class C^1 (see e.g., at various levels of generality, [31,4,8,5,29,19]). Therefore, the infimum of $\mathcal{E}|_M$ is achieved and the sequence (λ_m) is divergent.

Let now $m \geqslant 1$ be such that $\lambda_m < \lambda_{m+1}$. If we set

$$C = \left\{u \in M\colon\ \mathcal{E}(u) \leqslant \lambda_m\right\}, \qquad U = \left\{u \in M\colon\ \mathcal{E}(u) < \lambda_{m+1}\right\},$$

we clearly have $\operatorname{Index}(C) \leqslant m \leqslant \operatorname{Index}(U)$. Assume, for a contradiction, that $\operatorname{Index}(C) \leqslant m-1$. By the continuity of the index, there exists a symmetric neighborhood W of C with $\operatorname{Index}(W) = \operatorname{Index}(C)$. Such a W is also a neighborhood of the critical set of $\mathcal{E}|_M$ at level λ_m. By the equivariant deformation theorem, there exist $\varepsilon > 0$ and an odd continuous map

$$\psi : \left\{u \in M\colon\ \mathcal{E}(u) \leqslant \lambda_m + \varepsilon\right\} \to \left\{u \in M\colon\ \mathcal{E}(u) \leqslant \lambda_m - \varepsilon\right\} \cup W = W.$$

It follows that $\operatorname{Index}(\{u \in M\colon\ \mathcal{E}(u) \leqslant \lambda_m + \varepsilon\}) \leqslant m-1$, which contradicts the definition of λ_m. We also refer the reader to [10, Theorem 2.3], where this kind of argument is described in more detail. Since the index is invariant by odd deformation retractions, it follows that

$$\operatorname{Index}\left(\left\{u \in W_0^{1,p}(\Omega) \setminus \{0\}\colon\ \mathcal{E}(u) \leqslant \lambda_m \int_\Omega V|u|^p \,\mathrm{d}x\right\}\right) = \operatorname{Index}(C) = m.$$

Assume now, for a contradiction, that $\operatorname{Index}(U) \geqslant m+1$. By Proposition 3.1 there exists a symmetric, compact subset K of U with $\operatorname{Index}(K) \geqslant m+1$. Since $\max\{\mathcal{E}(u)\colon\ u \in K\} < \lambda_{m+1}$, we contradict the definition of λ_{m+1}. Again, since the index is invariant by odd deformation retractions, it follows that

$$\operatorname{Index}\left(\left\{u \in W_0^{1,p}(\Omega)\colon\ \mathcal{E}(u) < \lambda_{m+1} \int_\Omega V|u|^p \,\mathrm{d}x\right\}\right) = \operatorname{Index}(U) = m$$

and the proof is complete. □

4. Existence of a nontrivial solution

Let Ω be a bounded open subset of $\mathbb{R}^n$, let $1 < p < \infty$ and let $V \in \mathcal{V}(\Omega)$. Let also $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following assumptions:

($g1'$) we have that

for every $\varepsilon > 0$ there exists $a_\varepsilon \in \mathcal{V}(\Omega)$ such that

$$\left|g(x,s)\right| \leqslant a_\varepsilon(x)|s|^{p-1} + \varepsilon|s|^{p^*-1}, \quad \text{if } p < n;$$

there exist $a \in \mathcal{V}(\Omega)$, $C > 0$ and $q > p$ such that

$$\left|g(x,s)\right| \leqslant a(x)|s|^{p-1} + C|s|^{q-1}, \quad \text{if } p = n;$$

for every $S > 0$ there exists $a_S \in \mathcal{V}(\Omega)$ such that

$$\left|g(x,s)\right| \leqslant a_S(x)|s|^{p-1} \quad \text{whenever } |s| \leqslant S, \quad \text{if } p > n;$$

(g2′) for a.e. $x \in \Omega$, we have

$$\lim_{s\to 0} \frac{G(x,s)}{|s|^p} = 0 \quad \text{and} \quad \lim_{|s|\to\infty} \frac{G(x,s)}{|s|^p} = +\infty,$$

where $G(x,s) = \int_0^s g(x,t)\,\mathrm{d}t$;

(g3′) there exist $\mu > p$, $\gamma_0 \in L^1(\Omega)$ and $\gamma_1 \in \mathcal{V}(\Omega)$ such that

$$\mu G(x,s) \leqslant sg(x,s) + \gamma_0(x) + \gamma_1(x)|s|^p$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$;

(g4′) we have $G(x,s) \geqslant 0$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

The main result of the section is the following

Theorem 4.1. *Let us suppose that assumptions* (g1′)–(g4′) *hold and let $V \in \mathcal{V}(\Omega)$. Then, for every $\lambda \in \mathbb{R}$, the quasilinear elliptic problem*

$$\begin{cases} -\Delta_p u = \lambda V|u|^{p-2}u + g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

admits a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$.

The proof will be given at the end of the section. First of all, let us define a functional $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ of class C^1 by

$$f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \frac{\lambda}{p}\int_\Omega V|u|^p\,\mathrm{d}x - \int_\Omega G(x,u)\,\mathrm{d}x$$

and set $\|u\| = (\int_\Omega |\nabla u|^p\,\mathrm{d}x)^{1/p}$ for every $u \in W_0^{1,p}(\Omega)$.

Lemma 4.2. *Let E be a measurable subset of $\mathbb{R}^n$, let $1 \leqslant \alpha < \infty$, $1 \leqslant \beta < \infty$ and let $h : E \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Assume that, for every $\varepsilon > 0$, there exists $a_\varepsilon \in L^\beta(E)$ such that*

$$\big|h(x,s)\big| \leqslant a_\varepsilon(x) + \varepsilon|s|^{\alpha/\beta}$$

for a.e. $x \in E$ and every $s \in \mathbb{R}$.

Then, if (u_k) is a sequence bounded in $L^\alpha(E)$ and convergent to u a.e. in E, we have that $(h(x,u_k))$ is convergent to $h(x,u)$ strongly in $L^\beta(E)$.

Proof. From Fatou's Lemma, it easily follows that $u \in L^\alpha(E)$. Moreover, there exists a constant $c_\beta > 0$ such that

$$\big|h(x,s_1) - h(x,s_2)\big|^\beta \leqslant c_\beta\big(a_\varepsilon(x)\big)^\beta + c_\beta\varepsilon^\beta|s_1|^\alpha + c_\beta\varepsilon^\beta|s_2|^\alpha.$$

Therefore, we can apply Fatou's Lemma also to the sequence of nonnegative functions

$$c_\beta a_\varepsilon^\beta + c_\beta\varepsilon^\beta|u_k|^\alpha + c_\beta\varepsilon^\beta|u|^\alpha - \big|h(x,u_k) - h(x,u)\big|^\beta,$$

obtaining

$$\begin{aligned} c_\beta\int_E \big(a_\varepsilon^\beta + 2\varepsilon^\beta|u|^\alpha\big)\,\mathrm{d}x &\leqslant \liminf_{k\to\infty}\int_E \big[c_\beta a_\varepsilon^\beta + c_\beta\varepsilon^\beta|u_k|^\alpha + c_\beta\varepsilon^\beta|u|^\alpha - \big|h(x,u_k) - h(x,u)\big|^\beta\big]\,\mathrm{d}x \\ &\leqslant c_\beta\int_E \big(a_\varepsilon^\beta + \varepsilon^\beta|u|^\alpha\big)\,\mathrm{d}x + c_\beta\varepsilon^\beta\sup_{k\in\mathbb{N}}\int_E |u_k|^\alpha\,\mathrm{d}x - \limsup_{k\to\infty}\int_E \big|h(x,u_k) - h(x,u)\big|^\beta\,\mathrm{d}x, \end{aligned}$$

hence

$$\limsup_{k\to\infty} \int_E \big|h(x,u_k)-h(x,u)\big|^\beta \,\mathrm{d}x \leqslant c_\beta \varepsilon^\beta \left(\sup_{k\in\mathbb{N}} \int_E |u_k|^\alpha \,\mathrm{d}x - \int_E |u|^\alpha \,\mathrm{d}x \right).$$

By the arbitrariness of ε, the assertion follows. □

Proposition 4.3. *The following facts hold*:

(a) *we have*

$$\frac{\int_\Omega G(x,u)\,\mathrm{d}x}{\|u\|^p} \to 0 \quad \text{as } \|u\| \to 0;$$

(b) *if $b>0$ and (u_k) is a sequence in $W_0^{1,p}(\Omega)$ with $\|u_k\|\to\infty$ and*

$$\int_\Omega |\nabla u_k|^p \,\mathrm{d}x \leqslant b \int_\Omega V|u_k|^p \,\mathrm{d}x,$$

then we have

$$\frac{\int_\Omega G(x,u_k)\,\mathrm{d}x}{\|u_k\|^p} \to +\infty;$$

(c) *for every $\lambda\in\mathbb{R}$, the map $\{u\mapsto \lambda V|u|^{p-2}u + g(x,u)\}$ is weak-to-strong sequentially continuous from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$*;

(d) *for every $\lambda\in\mathbb{R}$ and $c\in\mathbb{R}$, the functional f satisfies $(PS)_c$.*

Proof. (a) Consider the case in which $p<n$. If we set

$$G_0(x,s) = \begin{cases} \dfrac{G(x,s)}{|s|^p} & \text{if } s\neq 0, \\ 0 & \text{if } s=0, \end{cases}$$

by (g1′), with $\varepsilon=1$, and (g2′) we have that G_0 is a Carathéodory function such that

$$\big|G_0(x,s)\big| \leqslant \frac{1}{p}a_1(x) + \frac{1}{p^*}|s|^{p^*-p}.$$

By the continuous embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, it follows that $G_0(x,u)$ goes to 0 in $L^{n/p}(\Omega)$ as $\|u\|\to 0$. On the other hand, by Hölder inequality we have

$$\begin{aligned} \int_\Omega \big|G(x,u)\big|\,\mathrm{d}x &= \int_\Omega \big|G_0(x,u)\big||u|^p \,\mathrm{d}x \\ &\leqslant \left(\int_\Omega \big|G_0(x,u)\big|^{n/p}\,\mathrm{d}x \right)^{p/n} \left(\int_\Omega |u|^{p^*}\,\mathrm{d}x \right)^{p/p^*}. \end{aligned}$$

Again, by the continuous embedding of $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, the assertion follows in the case $p<n$. The case $p\geqslant n$ is similar.

(b) Let $w_k = u_k/\|u_k\|$. Up to a subsequence, (w_k) is convergent to some w weakly in $W_0^{1,p}(\Omega)$ and a.e. in Ω. By Proposition 3.1, it follows that

$$b\int_\Omega V|w|^p \,\mathrm{d}x \geqslant 1,$$

in particular we can exclude that $w=0$ a.e. in Ω. Then by (g2′) we have

$$\lim_{k\to\infty} \frac{G(x,u_k(x))}{\|u_k\|^p} = \lim_{k\to\infty} \frac{G(x,\|u_k\|w_k(x))}{\|u_k\|^p|w_k(x)|^p}\big|w_k(x)\big|^p = +\infty$$

on a set of positive measure. On the other hand, by (g4′) it is possible to apply Fatou's Lemma to the sequence $(G(x,u_k)/\|u_k\|^p)$ and the assertion follows.

(c) Consider the case in which $p<n$ and set $\beta=p^*/(p^*-1)$. Let (u_k) be a sequence weakly convergent to u in $W_0^{1,p}(\Omega)$. Then (u_k) is bounded in $L^{p^*}(\Omega)$ and, up to a subsequence, convergent to u a.e. in Ω. On the other hand, by (g1′) and Young's inequality we have

$$\begin{aligned}\big|g(x,s)\big| &\leqslant a_\varepsilon(x)|s|^{p-1}+\varepsilon|s|^{p^*-1}\\ &\leqslant \frac{\beta p}{n}\left(\frac{a_\varepsilon(x)}{\varepsilon}\right)^{n/\beta p}+\frac{p-1}{p^*-1}\varepsilon^{p^*-1/p-1}|s|^{p^*-1}+\varepsilon|s|^{p^*-1}.\end{aligned}$$

By Lemma 4.2, $(g(x,u_k))$ is convergent to $g(x,u)$ strongly in $L^\beta(\Omega)$, hence strongly in $W^{-1,p'}(\Omega)$. Combining this fact with Proposition 3.1, the assertion follows in the case $p<n$. If $p\geqslant n$, the proof is similar and even simpler.

(d) Let $\lambda,c\in\mathbb{R}$ and let (u_k) be a sequence in $W_0^{1,p}(\Omega)$ with $f'(u_k)\to 0$ in $W^{-1,p'}(\Omega)$ and $f(u_k)\to c$. First of all, we claim that (u_k) is bounded in $W_0^{1,p}(\Omega)$. Arguing by contradiction, we may assume that $\|u_k\|\to\infty$. By (g3′) we have

$$\begin{aligned}\mu f(u_k)-\big\langle f'(u_k),u_k\big\rangle &= \left(\frac{\mu}{p}-1\right)\int_\Omega\big(|\nabla u_k|^p-\lambda V|u_k|^p\big)\,\mathrm{d}x+\int_\Omega\big(g(x,u_k)u_k-\mu G(x,u_k)\big)\,\mathrm{d}x\\ &\geqslant \left(\frac{\mu}{p}-1\right)\int_\Omega\big(|\nabla u_k|^p-\lambda V|u_k|^p\big)\,\mathrm{d}x-\int_\Omega\big(\gamma_0+\gamma_1|u_k|^p\big)\,\mathrm{d}x.\end{aligned}$$

Since

$$\mu f(u_k)-\big\langle f'(u_k),u_k\big\rangle+\int_\Omega\gamma_0\,\mathrm{d}x\leqslant\frac{1}{2}\left(\frac{\mu}{p}-1\right)\int_\Omega|\nabla u_k|^p\,\mathrm{d}x$$

eventually as $k\to\infty$, there exists $b>0$ such that

$$\int_\Omega|\nabla u_k|^p\,\mathrm{d}x\leqslant\int_\Omega(2\lambda V+b\gamma_1)|u_k|^p\,\mathrm{d}x$$

eventually as $k\to\infty$. Since $(2\lambda V+b\gamma_1)\in\mathcal{V}(\Omega)$, from assertion (b) we deduce that

$$\lim_{k\to\infty}\frac{\int_\Omega G(x,u_k)\,\mathrm{d}x}{\|u_k\|^p}=+\infty.$$

Therefore, we have

$$0=\lim_{k\to\infty}\frac{f(u_k)}{\|u_k\|^p}=\frac{1}{p}-\lim_{k\to\infty}\left(\frac{\lambda\int_\Omega V|u_k|^p\,\mathrm{d}x}{p\|u_k\|^p}+\frac{\int_\Omega G(x,u_k)\,\mathrm{d}x}{\|u_k\|^p}\right)=-\infty$$

and a contradiction follows. Then (u_k) is weakly convergent, up to a sequence, to some u in $W_0^{1,p}(\Omega)$. From assertion (c) it follows that $(\Delta_p u_k)$ is strongly convergent in $W^{-1,p'}(\Omega)$, hence that (u_k) is strongly convergent in $W_0^{1,p}(\Omega)$. □

Proof of Theorem 4.1. By exchanging (λ,V) with $(-\lambda,-V)$, we may suppose that $\lambda\geqslant 0$.

Let us first consider the case in which the manifold M defined in (1.2) is not empty. Let (λ_m) be the sequence defined in (1.3) and assume that $\lambda\geqslant\lambda_1$. Since the sequence (λ_m) is divergent, there exists $m\geqslant 1$ such that $\lambda_m\leqslant\lambda<\lambda_{m+1}$. If we define C_-,C_+ according to (1.4), (1.5), we have that C_-,C_+ are two symmetric closed cones in $W_0^{1,p}(\Omega)$ with $C_-\cap C_+=\{0\}$. By Theorem 3.2 we also have that

$$\mathrm{Index}\big(C_-\setminus\{0\}\big)=\mathrm{Index}\big(W_0^{1,p}(\Omega)\setminus C_+\big)=m.$$

Since $\lambda<\lambda_{m+1}$, by (a) of Proposition 4.3 there exist $r_+>0$ and $\alpha>0$ such that $f(u)\geqslant\alpha$ whenever $u\in C_+$ and $\|u\|=r_+$. On the other hand, since $\lambda\geqslant\lambda_m$, by (g4′) we have $f(u)\leqslant 0$ for every $u\in C_-$.

Now let $e \in W_0^{1,p}(\Omega) \setminus C_-$. Consider, on $W_0^{1,p}(\Omega)$, also the norm defined by

$$\|u\|_V := \left(\int_\Omega \big(|V|+1\big)|u|^p \,\mathrm{d}x \right)^{1/p}.$$

If $u \in C_-$ and $t > 0$, we have

$$\begin{aligned} \|u+te\| &= t\left\|\frac{u}{t}+e\right\| \leqslant t\left(\left\|\frac{u}{t}\right\|+\|e\|\right) \\ &\leqslant t\left(\lambda_m^{1/p}\left\|\frac{u}{t}\right\|_V + \frac{\|e\|}{\|e\|_V}\|e\|_V\right) \leqslant \max\left\{\lambda_m^{1/p}, \frac{\|e\|}{\|e\|_V}\right\} t\left(\left\|\frac{u}{t}\right\|_V + \|e\|_V\right). \end{aligned}$$

Since C_- is closed in $W_0^{1,p}(\Omega)$ also with respect to the norm $\|\ \|_V$, by Proposition 2.12 there exists $\beta \geqslant 1$ such that

$$\left\|\frac{u}{t}\right\|_V + \|e\|_V \leqslant \beta\left\|\frac{u}{t}+e\right\|_V.$$

Therefore, there exists $b > 0$ such that

$$\|u+te\| \leqslant b\|u+te\|_V \quad \text{for every } u \in C_- \text{ and every } t \geqslant 0.$$

Since $|V|+1 \in \mathcal{V}(\Omega)$, by (b) of Proposition 4.3 it follows that

$$\frac{\int_\Omega G(x,u_k)\,\mathrm{d}x}{\|u_k\|^p} \to +\infty, \quad \text{whenever } \|u_k\| \to \infty \text{ with } u_k \in C_- + \mathbb{R}^+ e.$$

In particular, there exists $r_- > r_+$ such that $f(u) \leqslant 0$ whenever $u \in C_- + \mathbb{R}^+ e$ and $\|u\| \geqslant r_-$. If we define D_-, S_+, Q and H as in Theorem 2.8, by Corollary 2.9 we have that $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $m+1$ over $\mathbb{Z}_2$. In particular, $(Q, D_- \cup H)$ links S_+. Moreover, f is bounded on Q and we have $f(u) \leqslant 0$ for every $u \in D_- \cup H$ and $f(u) \geqslant \alpha > 0$ for every $u \in S_+$. Finally, $(PS)_c$ also holds by (d) of Proposition 4.3.

By Theorem 2.2, f admits a critical value $c \geqslant \alpha$, hence a critical point u with $f(u) > 0$. Then u is a nontrivial weak solution of (4.1).

If we have $M \neq \emptyset$, $0 \leqslant \lambda < \lambda_1$ or $M = \emptyset$, $\lambda \geqslant 0$, we set $C_- = \{0\}$, $C_+ = W_0^{1,p}(\Omega)$ and the argument is similar and even simpler. □

5. Proof of the main result

Proof of Theorem 1.1. Of course, if $V \in L^\infty(\Omega)$, we have $V \in \mathcal{V}(\Omega)$. Moreover, if $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying (g1)–(g4), it is well known that (g2′) and (g4′) are satisfied. From (g1) and (g3), condition (g3′) also follows. Finally, if $p \leqslant n$, from (g1) and (g2) we deduce that there exists $C > 0$ such that

$$\big|g(x,s)\big| \leqslant C\big(|s|^{p-1} + |s|^{q-1}\big).$$

If $p = n$, it is clear that (g1′) holds. If $p < n$ then, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\big|g(x,s)\big| \leqslant C_\varepsilon |s|^{p-1} + \varepsilon |s|^{p^*-1}.$$

Therefore (g1′) holds also in this case. If $p > n$, (g2) and the continuity of g directly imply (g1′).

By Theorem 4.1, the assertion follows. □

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