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On the asymptotic formula for Goldbach numbers in short intervals / Bazzanella, Danilo; Languasco, A.. - In: STUDIA SCIENTIARUM MATHEMATICARUM HUNGARICA. - ISSN 0081-6906. - STAMPA. - 36:1-2(2000), pp. 185-199.
[10.1556/SscMath.36.2000.1-2.14]

Availability:

This version is available at: 11583/1397857 since:

Publisher:

Akadémiai Kiadó

Published

DOI:10.1556/SscMath.36.2000.1-2.14

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ON THE ASYMPTOTIC FORMULA FOR GOLDBACH NUMBERS IN SHORT INTERVALS

by

D. BAZZANELLA and A. LANGUASCO*

This is the authors' post-print version of an article published on
Stud. Sci. Math. Hung. 36 (2000) n.1-2, 185–199,
DOI:10.1556/SScMath.36.2000.1-2.14.¹

1. INTRODUCTION

Define a Goldbach number (G-number) to be an even number which can be written as a sum of two primes. In the following we denote by N a sufficiently large integer and let $L = \log N$. Let further

$$R(k) = \sum_{N < m \leq 2N} \sum_{\substack{N < l < 2N \\ m+l=k}} \Lambda(l)\Lambda(m)$$

be the weighted counting function of G-numbers,

$$\mathfrak{S}(k) = \begin{cases} 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p|k \\ p>2}} \left(\frac{p-1}{p-2}\right) & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

be the singular series of Goldbach's problem and

$$m(k) = \sum_{N < m \leq 2N} \sum_{\substack{N < l < 2N \\ m+l=k}} 1.$$

We recall that a well-known conjecture states that as $k \rightarrow \infty$

$$R(k) \sim m(k)\mathfrak{S}(k). \tag{1}$$

In this paper we study the asymptotic formula for the average of $R(k)$ over short intervals of type $[n, n + H)$. In the extreme case $H = 1$, Chudakov [1], van der Corput [2] and Estermann [4] proved that, as $N \rightarrow \infty$, (1) holds for all $k \in [1, N]$ but

* Research supported by a postdoctoral grant from the University of Genova.

¹This version does not contain journal formatting and may contain minor changes with respect to the published version. The final publication is available at <http://dx.doi.org/10.1556/SScMath.36.2000.1-2.14>. The present version is accessible on PORTO, the Open Access Repository of Politecnico di Torino (<http://porto.polito.it>), in compliance with the Publisher's copyright policy as reported in the SHERPA-ROMEO website: <http://www.sherpa.ac.uk/romeo/issn/0081-6906/>

$O(NL^{-A})$ exceptions, for every $A > 0$. Moreover, the same techniques prove, for $H \leq L^D$ and $N \rightarrow \infty$, that

$$\sum_{k \in [n, n+H)} R(k) \sim \sum_{k \in [n, n+H)} m(k) \mathfrak{S}(k) \quad (2)$$

holds for all $n \in (\frac{5}{2}N, \frac{7}{2}N]$ but $O(NL^{-A})$ exceptions, for every $A, D > 0$.

We recall that Montgomery-Vaughan [12] improved Chudakov-van der Corput-Estermann's result proving that there exists a (small) constant $\delta > 0$ such that $|E(N)| \ll N^{1-\delta}$, where $E(N) = E \cap [1, N]$ and E is the exceptional set for Goldbach's problem. Montgomery-Vaughan's technique intrinsically does not give any information about the asymptotic formula for $R(k)$.

On the other hand, using the circle method and Ingham-Huxley's zero density estimate, Perelli [14] proved that (2) holds as $n \rightarrow \infty$ uniformly for $H \geq n^{1/6+\varepsilon}$. Our aim here is to show, using the circle method, that the asymptotic formula (2) holds for almost all $n \in (\frac{5}{2}N, \frac{7}{2}N]$, uniformly for $L^D \leq H \leq N^{1/6+\varepsilon}$, for all $D > 0$.

Our result is

Theorem. *Let $D, \varepsilon > 0$ be arbitrary constants and $L^D \leq H \leq N^{1/6+\varepsilon}$. Then, as $N \rightarrow \infty$, (2) holds for all $n \in (\frac{5}{2}N, \frac{7}{2}N]$ but $O(NL^{42+\varepsilon}H^{-2})$ exceptions.*

In fact, following the proof of the Theorem, it is easy to see that we have $O(NL^{f(\theta)}H^{-2})$ exceptions, where

$$H = N^\theta \quad \text{and} \quad f(\theta) = \frac{24 - 18\theta}{1 - 3\theta} + \varepsilon.$$

A direct computation shows that $f(\theta)$ is an increasing function and hence the exponent 42 in the log-factor of the Theorem follows taking $\theta = 1/6 + \varepsilon$.

We observe that our result, for $\theta = 1/6 + \varepsilon$, proves only that the number of exceptions for (2) is $O(N^{2/3-\varepsilon})$ while, from Perelli's [14] result, we know that there are no exceptions.

We recall that Mikawa, see Lemma 4 of [10], proved a slightly weaker, in the log-factor, result without using the circle method. We finally recall that, under the assumption of the Riemann Hypothesis (RH), (2) holds uniformly for $H \geq \infty(\log^2 n)$, where $f = \infty(g)$ means $g = o(f)$, and that, assuming further the Montgomery pair correlation conjecture, (2) holds uniformly for $H \geq \infty(\log n)$.

Acknowledgments. We wish to thank Prof. A.Perelli for some useful discussions.

2. OUTLINE OF THE METHOD

Let

$$Q = \frac{H}{L^\varepsilon}, \quad T = \frac{N}{Q}L^{2+\varepsilon} \quad \text{and} \quad K_H(n) = \sum_{k \in [n, n+H)} e(-k\alpha),$$

where $e(x) = \exp(2\pi ix)$. Let further $\beta + i\gamma$ denote the generic non-trivial zero of $\zeta(s)$,

$$S(\alpha) = \sum_{N < m \leq 2N} \Lambda(m)e(m\alpha), \quad T(\alpha) = \sum_{N < m \leq 2N} e(m\alpha),$$

$$T_\rho(\alpha) = \sum_{N < m \leq 2N} a_\rho(m)e(m\alpha), \quad a_\rho(m) = \int_m^{m+1} t^{\rho-1} dt.$$

Given an interval $I = [a, b] \subset [1/2, 1]$ we define

$$\Sigma_b(\alpha) = \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha), \quad \Sigma_g(\alpha) = \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} T_\rho(\alpha) + \sum_{|\gamma| > T} T_\rho(\alpha) + R(\alpha)$$

where $R(\alpha)$ is defined by difference in the approximation

$$S(\alpha) = T(\alpha) - \Sigma_g(\alpha) - \Sigma_b(\alpha). \quad (3)$$

Subdivide now $(-\frac{1}{2}, \frac{1}{2})$ into $O(\log Q)$ subintervals of the following form

$$A_0 = (-\frac{1}{Q}, \frac{1}{Q}), \quad A_j = (-\frac{1}{2^j}, -\frac{1}{2^{j+1}}] \cup [\frac{1}{2^{j+1}}, \frac{1}{2^j})$$

for $j \in [1, K]$, where $K = [\log Q / \log 2]$. Hence we have

$$\sum_{k \in [n, n+H]} R(k) = \int_{-1/2}^{1/2} S(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/Q}^{1/Q} S(\alpha)^2 K_H(\alpha) d\alpha$$

$$+ \sum_{j=1}^K \int_{A_j} S(\alpha)^2 K_H(\alpha) d\alpha = \Sigma_1 + \Sigma_2, \quad (4)$$

say. We will prove that

$$\Sigma_1 = \sum_{k \in [n, n+H]} m(k) \mathfrak{S}(k) + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha + o(HN), \quad (5)$$

$$\sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} \left| \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha \right|^2 \ll N^3 L^{f(\theta)}, \quad (6)$$

and

$$\Sigma_2 = o(HN). \quad (7)$$

We will need also that

$$\sum_{k \in [n, n+H]} m(k) \mathfrak{S}(k) \gg HN \quad (8)$$

which can be obtained immediately using $\mathfrak{S}(2k) \gg 1$. Since $\varepsilon > 0$ is arbitrarily small, our Theorem follows at once from (4)-(8).

3. PRELIMINARY LEMMAS

In the following we will need two auxiliary lemmas.

Lemma 1. *Let $N(\sigma, T)$ be the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta-function such that $|\gamma| \leq T$ and $\beta \geq \sigma$, and let $I \subset [1/2, 1]$ be an interval. Then*

$$\int_N^{2N} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} x^\rho \frac{(1 + Q/x)^\rho - 1}{\rho} \right|^2 dx \ll Q^2 L^4 \max_{\sigma \in I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}).$$

The proof of Lemma 1 is standard. It can be obtained using, *e.g.*, Saffari-Vaughan's [15] technique and hence we omit it.

Lemma 2. *We have, for $|\gamma| \ll N$ and N sufficiently large, that*

$$T_\rho(\alpha) \ll N^\beta |\gamma|^{-1/2}.$$

Proof. We follow the line of Perelli [13] and hence we give only a brief sketch of the proof. Since

$$a_\rho(m) = \int_m^{m+1} t^{\rho-1} dt = \frac{m^\rho}{\rho} \left(\left(1 + \frac{1}{m}\right)^\rho - 1 \right),$$

and, for P sufficiently large but fixed,

$$\left(1 + \frac{1}{m}\right)^\rho - 1 = \sum_{j=1}^P \frac{\rho(\rho-1)\cdots(\rho-j+1)}{j!} \left(\frac{1}{m}\right)^j + O(N^{-11}),$$

we can write

$$T_\rho(\alpha) = T_{\rho,1}(\alpha) + \sum_{j=2}^P \frac{(\rho-1)(\rho-2)\cdots(\rho-j+1)}{j!} T_{\rho,j}(\alpha) + O(N^{\beta-10}), \quad (9)$$

where

$$T_{\rho,j}(\alpha) = \sum_{N < m \leq 2N} m^{\rho-j} e(m\alpha).$$

From Abel's inequality we have

$$|T_{\rho,j}(\alpha)| \ll N^{\beta-j} \max_{N \leq y \leq 2N} \left| \sum_{N \leq m \leq y} e^{2\pi i f_\rho(\alpha)} \right|,$$

where $f_\rho(\alpha) = \frac{\gamma}{2\pi} \log n + \alpha n$. We can assume that the maximum is attained at $Y = 2N$, and so, using van der Corput's second derivative method, see Theorem 2.2 of Graham-Kolesnik [5], we get

$$T_{\rho,j}(\alpha) \ll N^{\beta-j+1} |\gamma|^{-1/2}. \quad (10)$$

Lemma 2 now follows inserting (10) in (9). \square

4. ESTIMATION OF Σ_2

Letting $S(\alpha) = T(\alpha) + R_1(\alpha)$, where $R_1(\alpha)$ is defined by difference, and using

$$K_H(\alpha) \ll \min(H, \frac{1}{|\alpha|}) \quad \text{for every } \alpha \in [-\frac{1}{2}, \frac{1}{2}], \quad (11)$$

we have

$$\begin{aligned} \Sigma_2 &\ll \sum_{j=1}^K \left(\int_{A_j} |T(\alpha)|^2 |K_H(\alpha)| d\alpha + \int_{A_j} |R_1(\alpha)|^2 |K_H(\alpha)| d\alpha \right) \\ &\ll \sum_{j=1}^K 2^j \left(\int_{A_j} |T(\alpha)|^2 d\alpha + \int_{A_j} |R_1(\alpha)|^2 d\alpha \right) = \Sigma_{2,1} + \Sigma_{2,2}, \end{aligned} \quad (12)$$

say. Using

$$T(\alpha) \ll \min(N, \frac{1}{|\alpha|}) \quad \text{for every } \alpha \in [-\frac{1}{2}, \frac{1}{2}], \quad (13)$$

we obtain

$$\Sigma_{2,1} \ll \sum_{j=1}^K 4^j \ll 4^K \ll Q^2 = o(HN). \quad (14)$$

By Gallagher's lemma, see, *e.g.*, Lemma 1.9 of Montgomery [11], and the Brun-Titchmarsh theorem we get

$$\Sigma_{2,2} \ll \sum_{j=1}^K 2^j \int_{-2^{-j}}^{2^{-j}} \left| \sum_{N < m \leq 2N} (\Lambda(m) - 1) e(m\alpha) \right|^2 d\alpha \ll \sum_{j=1}^K 2^{-j} (J(N, 2^j) + L^2 2^{3j}), \quad (15)$$

where $J(N, h)$ is the Selberg integral. Inserting the estimate $J(N, h) \ll h^2 N + hNL$ for all $h \geq 1$, see the Lemma in Languasco [7], in (15) we have

$$\Sigma_{2,2} \ll \sum_{j=1}^K 2^{-j} (2^{3j} L^2 + 2^{2j} N + 2^j NL) \ll L^2 Q^2 + NQ + NL \log Q = o(HN). \quad (16)$$

Hence, inserting (14) and (16) in (12), we finally have that (7) holds.

 5. ESTIMATION OF Σ_1

Inserting the identity

$$S(\alpha)^2 = (2S(\alpha)T(\alpha) - T(\alpha)^2) - \Sigma_g(\alpha)^2 - 2T(\alpha)\Sigma_g(\alpha) + 2S(\alpha)\Sigma_g(\alpha) + \Sigma_b(\alpha)^2$$

into the definition of Σ_1 , we obtain

$$\Sigma_1 = \Sigma_{1,1} - \Sigma_{1,2} - \Sigma_{1,3} + \Sigma_{1,4} + \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha, \quad (17)$$

where

$$\Sigma_{1,1} = \int_{-1/Q}^{1/Q} (2S(\alpha)T(\alpha) - T(\alpha)^2) K_H(\alpha) d\alpha,$$

$$\begin{aligned}\Sigma_{1,2} &= \int_{-1/Q}^{1/Q} \Sigma_g(\alpha)^2 K_H(\alpha) d\alpha, \\ \Sigma_{1,3} &= \int_{-1/Q}^{1/Q} 2T(\alpha) \Sigma_g(\alpha) K_H(\alpha) d\alpha \\ &\quad \text{and} \\ \Sigma_{1,4} &= \int_{-1/Q}^{1/Q} 2S(\alpha) \Sigma_g(\alpha) K_H(\alpha) d\alpha.\end{aligned}$$

In this section we will prove

$$\Sigma_{1,1} = \sum_{k \in [n, n+H]} m(k) \mathfrak{S}(k) + o(HN) \quad (18)$$

and

$$\Sigma_{1,2} = o(HN), \quad (19)$$

while the estimation of the mean-square of $\int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha$ will be performed in the next section.

Assuming that (19) holds, the contribution of $\Sigma_{1,3}$ and $\Sigma_{1,4}$ can be estimated using the Cauchy-Schwarz inequality and

$$\int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha \ll N, \quad (20)$$

which can be proved using the same argument in the proof of Corollary 3 of Languasco-Perelli [9]. We obtain

$$\Sigma_{1,3} = o(HN) \quad \text{and} \quad \Sigma_{1,4} = o(HN). \quad (21)$$

Hence, by (17)-(19) and (21), we have that (5) holds.

Now we proceed to evaluate $\Sigma_{1,1}$ and $\Sigma_{1,2}$.

Contribution of $\Sigma_{1,1}$

Squaring out we obtain

$$\int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha = \sum_{k \in [n, n+H]} m(k)$$

and hence, using (11) and (13), we get

$$\int_{-1/Q}^{1/Q} T(\alpha)^2 K_H(\alpha) d\alpha = \int_{-1/2}^{1/2} T(\alpha)^2 K_H(\alpha) d\alpha + O(Q^2) = \sum_{k \in [n, n+H]} m(k) + o(HN). \quad (22)$$

Using the Prime Number Theorem, the Cauchy-Schwarz inequality and arguing analogously, we can write

$$\int_{-1/Q}^{1/Q} S(\alpha) T(\alpha) K_H(\alpha) d\alpha = \sum_{k \in [n, n+H]} m'(k) + o(HN), \quad (23)$$

where

$$m'(k) = \sum_{N < m \leq 2N} \Lambda(m) \sum_{\substack{N < h \leq 2N \\ m+h=k}} 1.$$

Again by the Prime Number Theorem, we get

$$\sum_{k \in [n, n+H)} m(k) = \sum_{k \in [n, n+H)} m'(k) + o(HN) \quad (24)$$

and hence, by (22)-(24), we have

$$\Sigma_{1,1} = \sum_{k \in [n, n+H)} m(k) + o(HN). \quad (25)$$

Using the Theorem of Languasco [8] and by partial summation, it is easy to prove

$$\sum_{k \in [n, n+H)} m(k) = \sum_{k \in [n, n+H)} m(k) \mathfrak{S}(k) + o(HN) \quad \text{for } H \geq L^{2/3+\varepsilon}. \quad (26)$$

Now (18) follows from (25) and (26).

Contribution of $\Sigma_{1,2}$

Since

$$\Sigma_g(\alpha)^2 \ll \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} T_\rho(\alpha) \right|^2 + \left| \sum_{|\gamma| > T} T_\rho(\alpha) \right|^2 + |R(\alpha)|^2,$$

we have

$$\Sigma_{1,2} \ll A_1 + A_2 + A_3, \quad (27)$$

where

$$A_1 = \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} T_\rho(\alpha) \right|^2 |K_H(\alpha)| d\alpha,$$

$$A_2 = \int_{-1/Q}^{1/Q} \left| \sum_{|\gamma| > T} T_\rho(\alpha) \right|^2 |K_H(\alpha)| d\alpha$$

and

$$A_3 = \int_{-1/Q}^{1/Q} |R(\alpha)|^2 |K_H(\alpha)| d\alpha.$$

Using (11) and Gallagher's lemma, we obtain

$$\begin{aligned} A_1 &\ll \frac{H}{Q^2} \left(\int_N^{2N} \left| \sum_{x < m < x+Q} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} a_\rho(m) \right|^2 dx + \int_{N-Q}^N \left| \sum_{N < m < x+Q} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} a_\rho(m) \right|^2 dx \right. \\ &\quad \left. + \int_{2N-Q}^{2N} \left| \sum_{x < m \leq 2N} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} a_\rho(m) \right|^2 dx \right) = A_{1,1} + A_{1,2} + A_{1,3}, \end{aligned} \quad (28)$$

say. Interchanging summation and integration in $A_{1,1}$, we get

$$\begin{aligned} A_{1,1} &\ll \frac{H}{Q^2} \int_N^{2N} \left| \int_{[x]+1}^{[x+Q]} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx \\ &\ll \frac{H}{Q^2} \int_N^{2N} \left| \left(\int_x^{x+Q} - \int_x^{[x]+1} - \int_{[x+Q]}^{x+Q} \right) \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx. \end{aligned} \quad (29)$$

To bound the contribution of the integral on $[x, [x] + 1]$ in (29), we argue as follows.

Interchanging summation and integration, we get

$$\int_N^{2N} \left| \int_x^{[x]+1} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx \ll \sum_{N < n \leq 2N} \int_n^{n+1} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} x^\rho \frac{((n+1)/x)^\rho - 1}{\rho} \right|^2 dx$$

and then, using $\frac{((n+1)/x)^\rho - 1}{\rho} \ll \min(\frac{1}{N}, \frac{1}{|\gamma|})$, we have

$$\int_N^{2N} \left| \int_x^{[x]+1} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx \ll L^4 \max_{\sigma \notin I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}). \quad (30)$$

To estimate the integral on $[[x+Q], x+Q]$ in (29) we proceed analogously and hence we get

$$\int_N^{2N} \left| \int_{[x+Q]}^{x+Q} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx \ll L^4 \max_{\sigma \notin I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}). \quad (31)$$

Now we treat the integral on $[x, x+Q]$ in (29). Proceeding as above we obtain

$$\begin{aligned} \int_N^{2N} \left| \int_x^{x+Q} \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} t^{\rho-1} dt \right|^2 dx &\ll \int_N^{2N} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} x^\rho \frac{(1+Q/x)^\rho - 1}{\rho} \right|^2 dx \\ &\ll Q^2 L^4 \max_{\sigma \notin I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}), \end{aligned} \quad (32)$$

where the last inequality follows by Lemma 1.

Choosing, in the definition of the interval I ,

$$a = \frac{1+3\theta}{2} - l \frac{\log L}{L} \quad \text{and} \quad b = \frac{5-3\theta}{6} + k \frac{\log L}{L}, \quad (33)$$

where $l > \frac{27(1-\theta)}{2(1-3\theta)}$ and k is a sufficiently large constant, we have, using Ingham-Huxley's density estimate, see, *e.g.*, Ivić [6], and (29)-(33), that

$$A_{1,1} \ll HL^4 \max_{\sigma \notin I} N^{2\sigma-1} N(\sigma, \frac{N}{Q}) = o(HN). \quad (34)$$

Interchanging summation and integration in $A_{1,2}$, we get

$$A_{1,2} \ll \frac{H}{Q^2} \int_{N-Q}^N \left| \sum_{\substack{|\gamma| \leq T \\ \beta \notin I}} x^\rho c_{\rho,Q} \right|^2 dx,$$

where $c_{\rho,Q} = ((\frac{[x+Q]}{x})^\rho - (\frac{N}{x})^\rho) / \rho$. Splitting the summation according to $|\gamma| \leq N/Q$ and $N/Q \leq |\gamma| \leq T$ and using $c_{\rho,Q} \ll \min(\frac{Q}{N}, \frac{1}{|\gamma|})$, we obtain

$$\begin{aligned} A_{1,2} &\ll \frac{H}{Q^2} \left(\frac{Q^2}{N^2} \int_{N-Q}^N \left| \sum_{\substack{|\gamma| \leq N/Q \\ \beta \notin I}} x^\beta \right|^2 dx + \int_{N-Q}^N \left| \sum_{\substack{N/Q \leq |\gamma| \leq T \\ \beta \notin I}} \frac{x^\beta}{|\gamma|} \right|^2 dx \right) \\ &\ll HQL^4 \max_{\sigma \notin I} N^{2\sigma-2} N(\sigma, \frac{N}{Q})^2. \end{aligned}$$

Using Ingham-Huxley's density estimate, we see that the maximum is attained at $\sigma = 1/2$ and hence we can write

$$A_{1,2} \ll HQL^4 N^{-1} \left(\frac{N}{Q}\right)^2 L^2 = \frac{HNL^6}{Q} = o(HN). \quad (35)$$

$A_{1,3}$ can be bounded following the lines of the estimation of $A_{1,2}$. We have

$$A_{1,3} = o(HN). \quad (36)$$

Inserting (34) and (35)-(36) in (28) we obtain

$$A_1 = o(HN). \quad (37)$$

Now we proceed to estimate A_2 . By (11) we get

$$A_2 \ll H \int_{-1/Q}^{1/Q} \left| \sum_{N < m \leq 2N} \sum_{|\gamma| > T} a_\rho(m) e(m\alpha) \right|^2 d\alpha. \quad (38)$$

Using (38), Gallagher's lemma and the explicit formula for $\psi(x)$, see equations (9)-(10) in ch. 17 of Davenport [3], we have

$$A_2 \ll \frac{H}{Q^2} \int_{N-Q}^{2N} \frac{N^2 L^4}{T^2} dx \ll \frac{HNL^3}{Q^2 T^2} L^4 = o(HN). \quad (39)$$

To bound A_3 we use (11), Gallagher's lemma and the explicit formula for $\psi(x)$, see equation (1) in ch. 17 of Davenport [3]. Hence

$$\begin{aligned} A_3 &\ll \frac{H}{Q^2} \int_{N-Q}^{2N} \left| \sum_{\substack{x < m < x+Q \\ N < m \leq 2N}} (\Lambda(m) - 1 + \sum_{\rho} a_\rho(m)) \right|^2 dx \\ &\ll \frac{H}{Q^2} \int_{N-Q}^{2N} L^4 dx \ll \frac{HNL^4}{Q^2} = o(HN). \end{aligned} \quad (40)$$

Now (19) follows inserting (37) and (39)-(40) in (27).

6. MEAN-SQUARE ESTIMATE OF $\Sigma_b(\alpha)^2$

Squaring out and using the definition of $\Sigma_b(\alpha)$, we get

$$\begin{aligned}
& \sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} \left| \int_{-1/Q}^{1/Q} \Sigma_b(\alpha)^2 K_H(\alpha) d\alpha \right|^2 \\
&= \sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} \int_{-1/Q}^{1/Q} \left(\sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right)^2 K_H(\alpha) d\alpha \int_{-1/Q}^{1/Q} \left(\sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right)^2 \overline{K_H}(\delta) d\delta \\
&\ll \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 \sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} K_H(\alpha) \overline{K_H}(\delta) |d\delta d\alpha = \Sigma_3,
\end{aligned} \tag{41}$$

say. Since $K_H(\alpha) = \frac{\sin \pi H \alpha}{\sin \pi \alpha} e(\frac{1-H}{2}\alpha) e(-n\alpha)$, we have

$$\Sigma_3 \ll H^2 \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 \left(\int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 K_N(\alpha - \delta) d\delta \right) d\alpha, \tag{42}$$

where $K_N(t) = \sum_{\frac{5}{2}N < n \leq \frac{7}{2}N} e(-nt) \ll \min(N, \frac{1}{|t|})$.

Using the latest estimate and (42), we obtain

$$\begin{aligned}
\Sigma_3 &\ll H^2 N \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 \left(\int_{(-\frac{1}{Q}, \frac{1}{Q}) \cap (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 d\delta \right) d\alpha \\
&+ H^2 \int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 \left(\int_{(-\frac{1}{Q}, \frac{1}{Q}) \setminus (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 \frac{1}{|\alpha - \delta|} d\delta \right) d\alpha \tag{43} \\
&= \Sigma_{3,1} + \Sigma_{3,2},
\end{aligned}$$

say. Using (3) and arguing as in section 6, we get

$$\int_{-1/Q}^{1/Q} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right|^2 d\alpha \ll \int_{-1/Q}^{1/Q} |S(\alpha)|^2 d\alpha + O(N) \ll N, \tag{44}$$

where the latest inequality follows from (20).

Now, inserting (44) in $\Sigma_{3,1}$, we have

$$\begin{aligned}
\Sigma_{3,1} &\ll H^2 N^2 \left(\max_{\alpha \in (-1/Q, 1/Q)} \int_{(-\frac{1}{Q}, \frac{1}{Q}) \cap (\alpha - \frac{1}{N}, \alpha + \frac{1}{N})} \left| \sum_{\substack{|\gamma'| \leq T \\ \beta' \in I}} T_{\bar{\rho}'}(\delta) \right|^2 d\delta \right) \\
&\ll H^2 N \left(\max_{\delta \in (-1/Q, 1/Q)} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\delta) \right|^2 \right).
\end{aligned} \tag{45}$$

To bound $\Sigma_{3,2}$, we argue as for $\Sigma_{3,1}$ and we can prove that the bound in (45) holds, with an extra L factor, for $\Sigma_{3,2}$ too. Finally, by (41), (43), (45) and the above remark, we obtain

$$\sum_{\frac{5}{2}N \leq n \leq \frac{7}{2}N} \left| \int_{-1/Q}^{1/Q} \left(\sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\alpha) \right)^2 K_H(\alpha) d\alpha \right|^2 \ll H^2 NL \left(\max_{\delta \in (-1/Q, 1/Q)} \left| \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\delta) \right|^2 \right). \quad (46)$$

Using Lemma 2 and a standard argument to bound sums over zeros of $\zeta(s)$, we have

$$\begin{aligned} \sum_{\substack{|\gamma| \leq T \\ \beta \in I}} T_\rho(\delta) &\ll L^2 \left(\max_{\substack{\sigma \in I \\ \sigma < 7/9}} N^\sigma \max_{|t| \leq T} N(\sigma, t) |t|^{-1/2} + \max_{\substack{\sigma \in I \\ \sigma \geq 7/9}} N^\sigma \max_{|t| \leq T} N(\sigma, t) |t|^{-1/2} \right) \\ &\ll L^2 \left(\max_{\substack{\sigma \in I \\ \sigma < 7/9}} N^\sigma N(\sigma, T) T^{-1/2} + \max_{\substack{\sigma \in I \\ \sigma \geq 7/9}} N^\sigma \right). \end{aligned} \quad (47)$$

By Ingham-Huxley's density estimate, we have that the first maximum is attained at $\sigma = a$ and the second at $\sigma = b$. Hence, by (46) and (47), we see that (6) holds.

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Danilo Bazzanella
Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino, Italy
e-mail : bazzanella@polito.it

Alessandro Languasco
Dipartimento di Matematica Pura e Applicata
Università di Padova
Via Belzoni 7
35131 Padova, Italy
e-mail : languasco@math.unipd.it