Functional estimates for derivatives of the modified Bessel function K0 and related exponential functions

Functional Estimates for Derivatives of the Modified Bessel Function K_0 and related Exponential Functions

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Abstract

Let K_0 denote the modified Bessel function of second kind and zeroth order. In this paper we will studying the function $\tilde{\omega}_n(x) := \frac{(-x)^n K_0^{(n)}(x)}{n!}$ for positive argument. The function $\tilde{\omega}_n$ plays an important role for the formulation of the wave equation in two spatial dimensions as a retarded potential integral equation. We will prove that the growth of the derivatives $\tilde{\omega}_n^{(m)}$ with respect to n can be bounded by $O\left((n+1)^{m/2}\right)$ while for small and large arguments x the growth even becomes independent of n.

These estimates are based on an integral representation of K_0 which involves the function $g_n(t) = \frac{t^n}{n!}\exp(-t)$ and their derivatives. The estimates then rely on a subtle analysis of g_n and its derivatives which we will also present in this paper.

Keywords: Bessel Function K_0 of second kind and order zero, exponential function, functional error estimates.

1. Introduction

The study of Bessel functions is a classical field in mathematics and a vast literature is devoted to its analysis. Classical references include [16, first edition: 1922.], [1], [15, Chap. 9], [14], where the main focus is on the detailed study of the asymptotic behavior in various parameter regimes, functional relations, and recurrences.

Sharp estimates of quotients of modified Bessel functions are proved, e.g., in [3], [6], [7], while Turán-type estimates can be found in [4]. As is common in applied mathematics new questions arise from new applications; in this case, the rapidly increasing interest in *convolution quadrature* methods for solving the wave equation in exterior domains (cf. [10, 11, 13, 12, 9, 2]) leads to the question of functional-type inequalities for derivatives of the Bessel functions which are explicit with respect to the order of the derivatives. This has been invested for the three-dimensional case in [8], where the kernel function is the exponential function.

The modified Bessel function K_0 plays an important role when studying the wave equation in two spatial dimensions as retarded potential integral equations. Their fast numerical solution with panel-clustering and multipole methods requires the approximation of these functions with polynomials. In this paper, we will prove new and sharp functional estimates for the function $\tilde{\omega}_n(x) := \frac{(-x)^n K_0^{(n)}(x)}{n!}$ and its derivatives on the positive real axes. It turns out that the growth of the m-th derivative $\tilde{\omega}_n^{(m)}$ can be bounded, in general, by $O\left((n+1)^{m/2}\right)$. In addition, we will prove that for small and large argument x the derivatives can be bounded independent of n. This is at first glance surprising since a straightforward use of Cauchy's integral theorem leads to an *exponential* growth with respect to n. Our proof is based on an integral representation of K_0, where the function $g_n(t) = \frac{t^n}{n!}\exp(-t)$ is involved. The estimates then rely on a subtle analysis of g_n and its derivatives which we will also present in this paper.

These results imply that fast panel-clustering techniques can efficiently be used to solve retarded potential integral equations in two spatial dimensions.

In [2], cutoff techniques have been introduced to accelerate the computation of two dimensional retarded potential equations. For this purpose, the behavior of the function ω_n for large arguments has been investigated which leads to a speed-up for large times compared to conventional methods.

2. Functional Estimates for Derivatives of the Modified Bessel Function

The kernel function for acoustic retarded potential integral equations – discretized by convolution quadrature with the BDF1 method – is related to the function $\tilde{\omega}_n(x) := \frac{(-x)^n K_0^{(n)}(x)}{n!}$ by

$$\omega_n(d) = \frac{1}{2\pi}\tilde{\omega}_n\left(\frac{d}{\Delta t}\right).$$

The estimate of its local approximability by polynomials requires the investigation of the m-th derivative of $\tilde{\omega}_n$.

Theorem 1. 1. *General estimate. For all $n, m \in \mathbb{N}_0$, and $x > 0$, the estimate*

$$\left|\tilde{\omega}_n^{(m)}(x)\right| \leq m!\left(\frac{\gamma\sqrt{n+1}}{x}\right)^m\left(\frac{\delta_{m,0}}{\sqrt{2x+1}} + \frac{\gamma}{\sqrt{n+1}}\left(1 + \delta_{m+n,0}\log\frac{x+1}{x}\right)\right) \tag{1}$$

holds for some $\gamma \geq 1$ (independent of m, n, and x); $\delta_{m,n}$ denotes the Kronecker delta.

2. *Refined estimates for small and large arguments.*

   (a) *Small argument. For given $C > 1$ independent of m, n, x, there exists some constant $\gamma > 1$ independent of m, n, x such that for all $n \geq 0$ and $0 \leq m \leq \sqrt{n}\left(\frac{C-1}{C}\right)$ with the further restriction on m:*

$$2\left\lfloor\frac{m\log(n+1)}{4\log 2}\right\rfloor - 2 \leq \frac{n-1}{2} \tag{2}$$

   and all

$$0 < x \leq \min\left\{\frac{n}{2C}, \frac{n+1}{4\gamma}\right\} \tag{3}$$

   it holds

$$\left|\tilde{\omega}_n^{(m)}(x)\right| \leq m!\left(\frac{\gamma}{x}\right)^m\left(\frac{\delta_{m,0}}{\sqrt{2x+1}} + \frac{\gamma}{\sqrt{n+1}}\left(1 + \delta_{n+m,0}\log\frac{x+1}{x}\right)\right). \tag{4}$$

   (b) *Large argument. For all $n \geq 0$ and $m \geq 1 + 2\log(n+1)$ it holds*

$$\left|\tilde{\omega}_n^{(m)}(x)\right| \leq m!\left(\frac{\gamma}{x}\right)^m \qquad \forall x > \begin{cases} 0 & n = 0, 1, \\ n + m\left(\sqrt{n}+2\right) & n \geq 2, \end{cases} \tag{5}$$

for some constant $\gamma \geq 1$ (independent of m, n, and x).

3. *Exponential decay. For $m = 0$ and $x \geq \max\{1, n + \sqrt{n}\}$, the function $\tilde{\omega}_n$ is decaying exponentially*

$$|\tilde{\omega}_n(x)| \leq 3 \frac{\exp\left(\sqrt{n} - \frac{x}{1+\sqrt{n}}\right)}{\sqrt{n+1}}. \tag{6}$$

Proof. We have

$$\tilde{\omega}_n(x) := \frac{(-x)^n K_0^{(n)}(x)}{n!} \overset{[1,\ 9.6.23]}{=} \frac{x^n}{n!} \int_1^\infty \frac{t^n \mathrm{e}^{-xt}}{\sqrt{t^2-1}} dt = \frac{1}{n!} \int_0^\infty \frac{(x(s+1))^n \mathrm{e}^{-x(s+1)}}{\sqrt{s(s+2)}} ds.$$

Let $g_n(t) := \frac{t^n \mathrm{e}^{-t}}{n!}$. The m-th derivative of $\tilde{\omega}_n$ is given by

$$\begin{aligned}
\tilde{\omega}_n^{(m)}(x) &= \frac{1}{n!} \int_0^\infty \frac{\left(\frac{d}{dx}\right)^m \left((x(s+1))^n \mathrm{e}^{-x(s+1)}\right)}{\sqrt{s(s+2)}} ds \\
&= \int_0^\infty \frac{(s+1)^m g_n^{(m)}(x(s+1))}{\sqrt{s(s+2)}} ds \\
&= \frac{1}{x^m} \int_x^\infty \frac{t^m g_n^{(m)}(t)}{\sqrt{t^2-x^2}} dt.
\end{aligned}$$

We split this integral into $\tilde{\omega}_n^{(m)} = P_{n,L}^{(m)} + Q_{n,L}^{(m)}$ for some $L \geq x$ with

$$P_{n,L}^{(m)}(x) := \frac{1}{x^m} \int_x^L \frac{t^m g_n^{(m)}(t)}{\sqrt{t^2-x^2}} dt \quad \text{and} \quad Q_{n,L}^{(m)}(x) := \frac{1}{x^m} \int_L^\infty \frac{t^m g_n^{(m)}(t)}{\sqrt{t^2-x^2}} dt$$

and introduce the quantity

$$M_{a,b}^{\ell,m,n} := \sup_{a \leq t \leq b} \left| t^{m+\ell} g_n^{(m)}(t) \right|.$$

Estimate of $P_{n,L}^{(m)}$.

For general $m, n \geq 0$ and $0 < x \leq L$ we get

$$\begin{aligned}
\left| P_{n,L}^{(m)}(x) \right| &\leq \frac{M_{x,L}^{0,m,n}}{x^m} \int_x^L \frac{1}{\sqrt{t^2-x^2}} dt \\
&= \frac{M_{x,L}^{0,m,n}}{x^m} \left(\log \frac{L}{x} + \log\left(1 + \frac{\sqrt{L^2-x^2}}{L}\right)\right) \leq \frac{M_{x,L}^{0,m,n}}{x^m} \log \frac{2L}{x}.
\end{aligned} \tag{7}$$

For our final estimate, we will choose $L \in \{x+1, 2x\}$. For $L = 2x$, we get

$$\left| P_{n,2x}^{(m)}(x) \right| \leq \frac{M_{x,2x}^{0,m,n}}{x^m} \log 4. \tag{8}$$

For $L = x+1$, we have to refine the estimate to see that the logarithmic behavior of $P_{n,L}^{(m)}(x)$ as $x \to 0$ only appears for $n = m = 0$. We obtain

- for $x \geq 1$ directly from (7):

$$\left| P_{n,x+1}^{(m)}(x) \right| \leq \frac{M_{x,x+1}^{0,m,n}}{x^m} \log 4; \tag{9}$$

- for $0 \leq x \leq 1$

  i. for $n = m = 0$

$$\left| P_{0,x+1}^{(0)}(x) \right| \leq \log \frac{2(x+1)}{x}; \tag{10}$$

  ii. for $n = 0 \wedge m \geq 1$

$$\begin{aligned} \left| P_{0,x+1}^{(m)}(x) \right| &\leq \frac{(m-1)!}{x^m} M_{x,x+1}^{0,0,m-1} \int_x^{x+1} \frac{t}{\sqrt{t^2 - x^2}} dt \\ &= \sqrt{3} \frac{(m-1)!}{x^m} M_{x,x+1}^{0,0,m-1}; \end{aligned} \tag{11}$$

  iii. for $n \geq 1 \wedge m = 0$

$$\left| P_{n,x+1}^{(0)}(x) \right| \leq \int_x^{x+1} \frac{t^n \, \mathrm{e}^{-t}}{n! \sqrt{t^2 - x^2}} dt \leq \sqrt{3} \frac{M_{x,x+1}^{0,0,n-1}}{n}; \tag{12}$$

  iv. for $n \geq 1 \wedge m \geq 1$

  By employing the recursion $g_n^{(m)} = g_{n-1}^{(m-1)} - g_n^{(m-1)}$ we get

$$\begin{aligned} \left| P_{n,x+1}^{(m)}(x) \right| &\leq \frac{M_{x,x+1}^{0,m-1,n-1} + M_{x,x+1}^{0,m-1,n}}{x^m} \int_x^{x+1} \frac{t}{\sqrt{t^2 - x^2}} dt \\ &= \sqrt{3} \frac{M_{x,x+1}^{0,m-1,n-1} + M_{x,x+1}^{0,m-1,n}}{x^m}. \end{aligned} \tag{13}$$

For $m = 0$ and $x \geq n + \sqrt{n}$, the estimate can be refined by using (36):

$$\begin{aligned} \left| P_{n,L}^{(0)}(x) \right| &\leq \int_x^L \frac{g_n(t)}{\sqrt{t^2 - x^2}} dt \leq \left(\max_{x \leq t \leq L} g_n(t) \right) \log \frac{2L}{x} \\ &\leq \frac{\log \frac{2L}{x}}{\sqrt{n+1}} \exp\left(\sqrt{n} - \frac{x}{1 + \sqrt{n}} \right). \end{aligned} \tag{14}$$

Estimate of $Q_{n,L}^{(m)}(x)$.

For $m=0$ it is easy to see that

$$Q_{n,L}^{(0)}(x) = \int_L^\infty \frac{g_n(t)}{\sqrt{t^2-x^2}} dt \le \frac{1}{\sqrt{L^2-x^2}} \int_0^\infty g_n(t)\, dt = \frac{1}{\sqrt{L^2-x^2}}. \tag{15}$$

In our application, we will employ the function $Q_{n,L}^{(0)}$ only for $L := x+1$ and obtain from (15)

$$Q_{n,x+1}^{(0)}(x) \le \frac{1}{\sqrt{2x+1}}. \tag{16}$$

Again, for $x \ge n+\sqrt{n}$, this estimate can be refined by using (36):

$$Q_{n,x+1}^{(0)}(x) \le \frac{1}{\sqrt{2x+1}} \frac{1}{\sqrt{n+1}} \int_L^\infty \exp\left(\sqrt{n} - \frac{t}{1+\sqrt{n}}\right) dt \le \sqrt{2} \frac{\exp\left(\sqrt{n} - \frac{x}{1+\sqrt{n}}\right)}{\sqrt{2x+1}}. \tag{17}$$

Next we consider $m \ge 1$ and introduce a splitting $Q_{n,L}^{(m)} = R_{n,L,\mu}^{(m)} + S_{n,L,\mu}^{(m)}$ which we will explain next. First, note that the Taylor expansion of $\left(t^2-x^2\right)^{-1/2}$ around $x=0$ can be written in the form (cf. [5, (5.24.31)])

$$\frac{1}{\sqrt{t^2-x^2}} \approx \frac{T_\mu(x/t)}{t} \quad \text{with} \quad T_\mu(z) := \sum_{\ell=0}^{\mu-1} \binom{2\ell}{\ell} \left(\frac{z}{2}\right)^{2\ell} \qquad \mu \in \mathbb{N}_{\ge 1}. \tag{18}$$

We set

$$\begin{aligned} R_{n,L,\mu}^{(m)}(x) &:= \frac{1}{x^m} \int_L^\infty t^m g_n^{(m)}(t) \left(\frac{1}{\sqrt{t^2-x^2}} - \frac{T_\mu(x/t)}{t}\right) dt, \\ S_{n,L,\mu}^{(m)}(x) &:= \frac{1}{x^m} \int_L^\infty T_\mu(x/t)\, t^{m-1} g_n^{(m)}(t)\, dt. \end{aligned}$$

For the estimate of $R_{n,L,\mu}^{(m)}$, we employ Proposition 5 to obtain

$$R_{n,L,\mu}^{(m)}(x) \le \frac{M_{L,\infty}^{0,m,n}}{x^m} \int_L^\infty \frac{1}{\sqrt{t^2-x^2}} \left(\frac{x}{t}\right)^{2\mu} dt \le \frac{M_{L,\infty}^{0,m,n}}{x^m} \left(\frac{x}{L}\right)^{2\mu}. \tag{19}$$

Estimate of $S_{n,L,\mu}^{(m)}$ for $m \ge 1$ and $n=0$.

By using

$$|T_\mu(z)| \le \sum_{\ell=0}^{\mu-1} \binom{2\ell}{\ell} \left(\frac{1}{2}\right)^{2\ell} = 2^{1-2\mu} \mu \binom{2\mu}{\mu} =: c_\mu \qquad \forall z \in [0,1]$$

the function $S^{(m)}_{n,L,\mu}$ can be estimated for $n=0$ by

$$\left|S^{(m)}_{0,L,\mu}(x)\right| \leq \frac{c_\mu}{x^m}\int_0^\infty t^{m-1}\,\mathrm{e}^{-t}\,dt = c_\mu \frac{(m-1)!}{x^m}. \tag{20}$$

Estimate of $S^{(m)}_{n,L,\mu}$ for $m \geq 1$ and $n \geq 1$

First, observe that for $0 \leq 2r \leq n-1$ it holds (cf. Proposition 6)

$$\int_L^\infty t^{m-1-2r} g_n^{(m)}(t)\,dt = -\frac{(n-1-2r)!}{n!}\left(y^m g_{n-1-2r}^{(m-1-2r)}(y)\right)^{(2r)}\Bigg|_{y=L}.$$

Note that

$$\begin{aligned}\left|\left(y^m g_{n-1-2r}^{(m-1-2r)}(y)\right)^{(2r)}\Bigg|_{y=L}\right| &= \left|\sum_{\ell=0}^{\min(m,2r)}\binom{2r}{\ell}\frac{m!}{(m-\ell)!}L^{m-\ell}g_{n-1-2r}^{(m-1-\ell)}(L)\right| \\ &\leq \sum_{\ell=0}^{\min(m,2r)}\binom{2r}{\ell}\frac{m!}{(m-\ell)!}M_{L,L}^{1,m-1-\ell,n-1-2r} =: K_{n,L}^{m,r}.\end{aligned} \tag{21}$$

Thus,

$$\begin{aligned}\left|S^{(m)}_{n,L,\mu}(x)\right| &\leq \frac{1}{x^m}\sum_{r=0}^{\mu-1}\left(\frac{x}{2}\right)^{2r}\binom{2r}{r}\left|\int_L^\infty t^{m-1-2r}g_n^{(m)}(t)\,dt\right| \\ &\leq \frac{1}{x^m}\sum_{r=0}^{\mu-1}\left(\frac{x}{2}\right)^{2r}\binom{2r}{r}\frac{(n-1-2r)!}{n!}K_{n,L}^{m,r}.\end{aligned} \tag{22}$$

The derived estimates imply the assertion by arguing as follows.

Case 1: Exponential decay.

Estimate (6) follows from (14) and (17) (with $x \geq 1$ and $L := x+1$).

Case 2: General estimate and estimate for large argument.

For $m=0$, estimate (10), (9), (12), (16), (33) imply with $L = x+1$

$$\left|\tilde{\omega}_n(x)\right| \leq \frac{1}{\sqrt{2x+1}} + \frac{1}{\sqrt{n+1}}\begin{cases}\log\frac{4(x+1)}{x} & n=0, \\ \gamma & n \geq 1,\end{cases} \tag{23}$$

for some $\gamma > 1$ so that (1) and (5) follow for $m=0$.

For $m \geq 1$, $n = 0$, the estimates (11), (9), (19), (20) imply with $L = x+1$ and $\mu = 1$ the estimate

$$\left|\tilde{\omega}_0^{(m)}(x)\right| \leq \sqrt{3}\frac{(m-1)!}{x^m}M_{x,x+1}^{0,0,m-1} + \frac{M_{x,x+1}^{0,m,0}}{x^m}\log 4 + \frac{M_{x+1,\infty}^{0,m,0}}{x^m} + \frac{(m-1)!}{x^m}. \tag{24}$$

The combination of (24) with (33) leads to

$$\left|\tilde{\omega}_0^{(m)}(x)\right| \leq m!\left(\frac{\gamma}{x}\right)^m.$$

This implies (1) and (5) for $m \geq 1$ and $n = 0$.

For $m \geq 1$ and $n \geq 1$, the choices $L = x + 1$, and $\mu = 1$ allow to estimate the constant $K_{n,x+1}^{m,0}$ in (21) by $K_{n,x+1}^{m,0} \leq M_{x+1,x+1}^{1,m-1,n-1}$ and, in turn, we have

$$\left|S_{n,x+1,1}^{(m)}(x)\right| \leq \frac{M_{x+1,x+1}^{1,m-1,n-1}}{nx^m}.$$

Thus, the estimates (9), (13), (19) imply

$$\left|\tilde{\omega}_n^{(m)}(x)\right| \leq \frac{M_{x,x+1}^{0,m,n}}{x^m}\log 4+\sqrt{3}\frac{M_{x,x+1}^{0,m-1,n-1}+M_{x,x+1}^{0,m-1,n}}{x^m}+\frac{M_{x+1,\infty}^{0,m,n}}{x^m}+\frac{M_{x+1,x+1}^{1,m-1,n-1}}{nx^m}. \tag{25}$$

Estimate (33) allows to bound the terms $M_{a,b}^{\ell,m,n}$ in (25) which leads to (1) while the combination with (35) gives (5).

Case 3: Estimate for small argument

For $m = 0$, estimate (1) directly implies (4). Note that the condition $0 \leq m \leq \sqrt{n}\left(\frac{C-1}{C}\right)$ implies $m = 0$ for $n = 0, 1$. Hence, for the following we assume $n \geq 2$.

In the following, let $m \geq 1$ and let C and γ be as explained in statement (2a) of the theorem. We assume $1 \leq m \leq \sqrt{n}\left(\frac{C-1}{C}\right)$ and restrict the range of x to

$$0 < 2x =: L \leq \frac{n}{C} \leq n - m\sqrt{n}. \tag{26}$$

Then, (8) and (34) imply

$$\left|P_{n,2x}^{(m)}(x)\right| \leq \frac{M_{x,2x}^{0,m,n}}{x^m}\log 4 \leq \frac{m!}{\sqrt{n+1}}\left(\frac{\gamma}{x}\right)^m, \tag{27}$$

while (19) and (33) yield

$$\left|R_{n,2x,\mu}^{(m)}(x)\right| \leq m!\left(\frac{\gamma}{x}\right)^m (n+1)^{\frac{m-1}{2}}\left(\frac{1}{2}\right)^{2\mu}$$

By choosing $\mu := \left\lfloor \frac{m\log(n+1)}{4\log 2}\right\rfloor$, we get

$$\left|R_{n,2x,\mu}^{(m)}(x)\right| \leq \frac{m!}{\sqrt{n+1}}\left(\frac{\gamma}{x}\right)^m. \tag{28}$$

It remains to estimate the term $S_{n,L,\mu}^{(m)}(x)$. An estimate for the constant $K_{n,2x}^{m,r}$ for $0 \le r \le \mu - 1$ follows from (21), (34), and, by using $2x = L \le \frac{n}{C}$, via

$$K_{n,2x}^{m,r} = L \sum_{\ell=0}^{\min(m,2r)} \binom{2r}{\ell} \frac{m!}{(m-\ell)!} \left| L^{m-\ell-1} g_{n-1-2r}^{(m-1-\ell)}(L) \right| \tag{29}$$

$$\le m! L \sum_{\ell=0}^{\min(m,2r)} (m-\ell+1) \binom{2r}{\ell} 2^{m-\ell} \frac{1}{\sqrt{n-2r}} \left(\frac{n-1-2r}{n-1-2r-L} \right)^{m-1-\ell}. \tag{30}$$

Now, it holds

$$\frac{n-1-2r}{2} \ge \frac{n+1-2\mu}{2} = \frac{1}{2}\left(n - 1 - \left(2 \left\lfloor \frac{m \log(n+1)}{4 \log 2} \right\rfloor - 2 \right) \right) \overset{(2)}{\ge} \frac{n-1}{4}.$$

so that, by choosing $\gamma \ge 6$ in (3), we obtain

$$\frac{n-1-2r}{2} \ge \frac{n-1}{4} \overset{n \ge 2}{\ge} \frac{n+1}{12} \ge 2x = L \quad \text{so that (cf. (30))} \quad \frac{n-1-2r}{n-1-2r-L} \le 2. \tag{31}$$

The combination of (29) and (31) with $n - 2r \ge n + 2 - 2\mu$ and $L \le n/C$ leads to

$$K_{n,2x}^{m,r} \le \frac{(m+1)!n}{2C\sqrt{n+2-2\mu}} \sum_{\ell=0}^{\min(m,2r)} \binom{2r}{\ell} 4^{m-\ell} \le \frac{(m+1)!n}{2C\sqrt{n+2-2\mu}} 4^{m-2r} \sum_{\ell=0}^{2r} \binom{2r}{\ell} 4^{2r-\ell}$$
$$= \frac{(m+1)!n}{2C\sqrt{n+2-2\mu}} 4^{m-2r} 5^{2r} \le \frac{(n+1)\, m!}{\sqrt{n+2-2\mu}} \gamma^{m+2r}$$

with a properly adjusted γ. The combination with (22) and $\left(\frac{1}{2}\right)^{2r} \binom{2r}{r} \le 1$ leads to

$$\left| S_{n,L,\mu}^{(m)}(x) \right| \le m! \left(\frac{\gamma}{x} \right)^m \frac{(n+1)}{(n+2-2\mu)^{3/2}} \sum_{r=0}^{\mu-1} \left(\frac{\gamma x}{n-2r} \right)^{2r}.$$

Next, we use the additional condition (cf. (2)) on m such that

$$2r \le 2\mu - 2 = 2 \left\lfloor \frac{m \log(n+1)}{4 \log 2} \right\rfloor - 2 \le \frac{n-1}{2}$$

holds. This leads to

$$\left| S_{n,L,\mu}^{(m)}(x) \right| \le m! \left(\frac{\gamma}{x} \right)^m \frac{2^{3/2}}{\sqrt{n+1}} \sum_{r=0}^{\mu-1} \left(\frac{2\gamma x}{n+1} \right)^{2r}.$$

By using the assumption $x \leq \frac{n+1}{4\gamma}$ as stated in (2a) of the theorem we end up with the estimate

$$\left| S_{n,L,\mu}^{(m)}(x) \right| \leq m! \left(\frac{\gamma}{x} \right)^m \frac{1}{\sqrt{n+1}}, \tag{32}$$

again with a properly adjusted γ. The combination of (27), (28), and (32) finally leads to the estimate (4) for small argument. ■

3. Functional Estimates for Derivatives of $g_n(t) = \frac{t^n e^{-t}}{n!}$

Proposition 2. 1. *General estimate. For $n \geq 0$, $m \geq 0$ and $\ell = 0, 1$, it holds for all $t \geq 0$*

$$\left| t^{m+\ell} g_n^{(m)}(t) \right| \leq C_m (n+1)^{\frac{m-1}{2}+\ell} \quad \text{with} \quad C_m := (4\,\mathrm{e})^{m+3} (m+2)!. \tag{33}$$

2. *Refined estimates for small and large arguments.*

   (a) *Small argument. For $0 \leq t \leq n - m\sqrt{n}$, the refined estimate holds*[1]

   $$\left| t^m g_n^{(m)}(t) \right| \leq 2^{m+1} \frac{(m+2)!}{\sqrt{n+1}} \left(\frac{n}{n-t} \right)^m. \tag{34}$$

   (b) *Large argument. For $n \geq 0$ and $m \geq 2 \log(n+1)$, the refined estimate*

   $$\left| t^{m+\ell} g_n^{(m)}(t) \right| \leq 4 \left(\frac{3}{\ln \frac{1}{c}} \right)^m (m+2)! (n+1)^\ell \qquad \forall t \geq \begin{cases} 0 & n = 0, 1 \\ n + m(\sqrt{n}+2) & n \geq 2 \end{cases} \tag{35}$$

   holds with $c := 9/10$.

3. *Exponential decay. For $m = 0$ and $t \geq n + \sqrt{n}$, we obtain*

$$g_n(t) \leq \frac{1}{\sqrt{n+1}} \exp\left(\sqrt{n} - \frac{t}{1+\sqrt{n}} \right). \tag{36}$$

Proof. We start to prove some special cases.

Case $n = 0, 1$.

[1] For $n = 0$, the condition $0 \leq t \leq n - m\sqrt{n}$ implies $t = m = 0$ and the factor $\left(\frac{n}{n-t}\right)^m$ is defined as 1 for this case.

For $n=0$, and $\ell=0,1$, it holds

$$\left|t^{m+\ell}g_0^{(m)}(t)\right| = t^{m+\ell}\,\mathrm{e}^{-t} \le \frac{(m+\ell)!}{\sqrt{m+\ell+1}} \qquad \forall t\ge 0, \tag{37}$$

so that (33) holds for all $C_m \ge \frac{(m+\ell)!}{\sqrt{m+\ell+1}}$. This also implies (34) and (36) for $n=0$.

For $n=1$, we get

$$(-t)^{m+\ell}\,g_1^{(m)}(t) = (-t)^{m+\ell}\left(t\,\mathrm{e}^{-t}\right)^{(m)} = (-1)^{\ell}\,(t-m)\,t^{m+\ell}\,\mathrm{e}^{-t}.$$

By estimating $t^{m+\ell+1}\,\mathrm{e}^{-t}$ and $t^{m+\ell}\,\mathrm{e}^{-t}$ as in (37) we get

$$\left|t^{m+\ell}g_1^{(m)}(t)\right| \le 2\frac{(m+\ell+1)!}{\sqrt{m+\ell+1}} \tag{38}$$

so that (33) holds for $n=1$ if $C_m \ge 2^{\frac{3-m}{2}-\ell}\frac{(m+\ell+1)!}{\sqrt{m+\ell+1}}$. Note that this also implies (34) and (36) for $n=1$.

For the rest of the proof we assume $n\ge 2$. Note that (49) directly implies (34) for $n\ge 2$ and it remains to prove the remaining inequalities.

Case $m=0,1$.

For $m=0$ and $\ell=0,1$ the function $t^{\ell}g_n(t)$ has its extremum at $t=n+\ell$, i.e.,

$$\left|t^{\ell}g_n(t)\right| \le \frac{(n+\ell)^{n+\ell}\,\mathrm{e}^{-n-\ell}}{n!} \qquad \forall t\ge 0.$$

Since $n\ge 1$, Stirling's formula gives us

$$\left|t^{\ell}g_n(t)\right| \le \frac{(n+1)^{\ell}}{\sqrt{n+\ell+1}} \le (n+1)^{\ell-1/2}$$

so that (33) holds for this case if $C_0\ge 1$.

For $m=1$ and $\ell=0,1$, the function $t^{1+\ell}g_n^{(1)}(t) = (n-t)\,\frac{t^{n+\ell}\,\mathrm{e}^{-t}}{n!}$ has its extrema at $t_{\pm} = \left(n+\frac{\ell+1}{2}\right)(1\pm\delta_{n,\ell})$ with $\delta_{n,\ell} = \frac{\sqrt{n+\frac{(\ell+1)^2}{4}}}{n+\frac{\ell+1}{2}}$. Hence, with

Stirling's formula we obtain

$$\left|t_{\pm}^{1+\ell} g_n'\left(t_{\pm}\right)\right| = \left(\sqrt{n+\frac{(\ell+1)^2}{4}} \pm \frac{\ell+1}{2}\right) \frac{\left(\left(n+\frac{\ell+1}{2}\right)\left(1\pm\delta_{n,\ell}\right)\right)^{n+\ell} \mathrm{e}^{-\left(n+\frac{\ell+1}{2}\right)\left(1\pm\delta_{n,\ell}\right)}}{n!}$$

$$\leq \frac{1}{\sqrt{2\pi}} \left(\sqrt{1+\frac{(\ell+1)^2}{4n}} \pm \frac{\ell+1}{2\sqrt{n}}\right) \left(n+\frac{\ell+1}{2}\right)^{\ell} \left(1+\frac{\ell+1}{2n}\right)^{n} \times \tag{39}$$

$$\times \left(\left(\left(1\pm\delta_{n,\ell}\right)\right)^{n+\ell} \mathrm{e}^{-\left(\frac{\ell+1}{2} \pm \left(n+\frac{\ell+1}{2}\right)\delta_{n,\ell}\right)}\right). \tag{40}$$

Since $n \geq 1$ and $\ell = 0, 1$, we get

$$\sqrt{1+\frac{(\ell+1)^2}{4n}} \pm \frac{\ell+1}{2\sqrt{n}} \leq \sqrt{2}+1, \quad \left(1+\frac{\ell+1}{2n}\right)^{n} \leq \mathrm{e} \text{ and } \left(n+\frac{\ell+1}{2}\right)^{\ell} = (n+1)^{\ell}.$$

The last factor in (39) is considered first with "+" signs and can be estimated by

$$\begin{aligned}\left(\left(1+\delta_{n,\ell}\right)\right)^{n+\ell} \mathrm{e}^{-\left(\frac{\ell+1}{2}+\left(n+\frac{\ell+1}{2}\right)\delta_{n,\ell}\right)} &= \mathrm{e}^{(n+\ell)\log(1+\delta_n)} \,\mathrm{e}^{-\left(\frac{\ell+1}{2}+\left(n+\frac{\ell+1}{2}\right)\delta_{n,\ell}\right)} \\ &\leq \mathrm{e}^{(n+\ell)\delta_{n,\ell}-\left(\frac{\ell+1}{2}+\left(n+\frac{\ell+1}{2}\right)\delta_{n,\ell}\right)} = \mathrm{e}^{-c_{n,\ell}}\end{aligned}$$

with

$$c_{n,\ell} = \frac{1-\ell}{2}\delta_{n,\ell} + \frac{(\ell+1)}{2} \geq 0$$

so that, in this case, the last factor in (39), (40) is bounded by 1. For the "−" signs, we get

$$(n+\ell)\log\left(1-\delta_{n,\ell}\right) = -(n+\ell)\log\left(1+\frac{\delta_{n,\ell}}{1-\delta_{n,\ell}}\right) \leq -(n+\ell)\,\delta_{n,\ell}$$

so that

$$\begin{aligned}\left(\left(1-\delta_{n,\ell}\right)\right)^{n+\ell} \mathrm{e}^{-\left(\frac{\ell+1}{2}-\left(n+\frac{\ell+1}{2}\right)\delta_{n,\ell}\right)} &\leq \exp\left(-(n+\ell)\,\delta_{n,\ell} - \left(\frac{\ell+1}{2} - \left(n+\frac{\ell+1}{2}\right)\delta_{n,\ell}\right)\right) \\ &= \exp\left(\frac{1-\ell}{2}\delta_{n,\ell} - \frac{\ell+1}{2}\right)\end{aligned}$$

Since $\delta_{n,\ell} \leq \frac{1}{\sqrt{2}}$ we arrive at the estimate

$$\left|t^{1+\ell} g_n'(t)\right| \leq \frac{3\,\mathrm{e}}{\sqrt{2\pi}}\,(n+1)^{\ell}$$

and this proves (33) for $m = 1$.

Case $0 \le t \le n - \sqrt{n}$.

Proposition 3 implies for $\ell = 0, 1$ and $t \le n$

$$\left| t^{m+\ell} g_n^{(m)}(t) \right| \le (m+2)!\, \mathrm{e}^m\, n^{\frac{m-1}{2}+\ell}. \tag{41}$$

Case $n - \sqrt{n} \le t \le n + \sqrt{n}$.

We start with the simple recursions

$$g_n' = g_{n-1} - g_n \quad \text{and} \quad g_n = \frac{t}{n} g_{n-1}$$

from which we conclude $g_n' = \left(1 - \frac{t}{n}\right) g_{n-1}$. By differentiating this relation m times we get via Leibniz' rule

$$g_n^{(m+1)} = \left(1 - \frac{t}{n}\right) g_{n-1}^{(m)} - \frac{m}{n} g_{n-1}^{(m-1)} \quad \text{with (cf. (37))} \quad \left| g_0^{(m)} \right| \le 1, \quad \left| g_1^{(m)} \right| \le m.$$

For $n = 0, 1$ we get

$$\left| g_0^{(m)}(t) \right| = \mathrm{e}^{-t} \le 1 \quad \text{and} \quad g_1^{(m)}(t) = \left| (t-m) \exp(-t) \right| \le m.$$

It is easy to verify that the coefficients $A_n^{(m)}$ in the recursion

$$A_n^{(m)} := \begin{cases} 1 & n = 0, m \in \mathbb{N}_0, \\ m & n = 1, m \in \mathbb{N}_0, \\ \dfrac{1}{\sqrt{n+1}} & m = 0, n \in \mathbb{N}_0, \\ \frac{\sqrt{2}}{n+1} & m = 1, n \in \mathbb{N}_{\ge 2}, \\ A_n^{(m)} = \frac{1}{\sqrt{n}} A_{n-1}^{(m-1)} + \frac{m-1}{n} A_{n-1}^{(m-2)} & n \ge 2, m \ge 2 \end{cases} \tag{42}$$

majorate $\left| g_n^{(m)} \right|$. For the estimate of $A_n^{(m)}$ we distinguish between two cases. Recall that we may restrict to the cases $m \ge 2$ and $n \ge 2$.

a) Let $n \ge m/2$. Then it holds

$$A_n^{(m)} \le \frac{m! a^m}{n^{(m+1)/2}} \quad \text{for any} \quad a \ge 1 + \sqrt{3}. \tag{43}$$

This is proved by induction: It is easy to see that the right-hand side in (43) majorates $A_n^{(m)}$ for the first four cases in (42). Then, by induction we get

$$\begin{aligned}\frac{1}{\sqrt{n}}A_{n-1}^{(m-1)}+\frac{m-1}{n}A_{n-1}^{(m-2)} &\leq \frac{1}{\sqrt{n}}\frac{(m-1)!a^{m-1}}{(n-1)^{m/2}}+\frac{m-1}{n}\frac{(m-2)!a^{m-2}}{(n-1)^{(m-1)/2}}\\ &\leq \frac{m!a^m}{n^{(m+1)/2}}\left(\frac{1}{ma^2}\left(a\left(\frac{n}{n-1}\right)^{m/2}+\left(\frac{n}{n-1}\right)^{(m-1)/2}\right)\right).\end{aligned} \tag{44}$$

For $n \geq m/2$, we have

$$\left(\frac{n}{n-1}\right)^{(m-1)/2} \leq \left(\frac{n}{n-1}\right)^{m/2} \leq \left(\frac{n}{n-1}\right)^{n} \leq 4$$

so that the factor $(\ldots)$ in the right-hand side of (44) can be estimated from above by $2\,(a+1)\,/a^2$, which is ≤ 1 for $a \geq 1+\sqrt{3}$. Thus, estimate (43) is proved .

b) For $n < m/2$, it holds

$$A_n^{(m)} \leq G_n^{(m)} := \frac{2^n\,(m-1)!!}{n!\,(m-1-2n)!!}. \tag{45}$$

This can be seen by first observing that this holds for $n=0$. Next we prove the auxiliary statement**:** For $1 \leq n < m/2$, it holds

$$\frac{1}{\sqrt{n}}G_{n-1}^{(m)} \leq \frac{m}{n}G_{n-1}^{(m-1)}. \tag{46}$$

This is equivalent to

$$Q_n^{(m)} := \frac{\sqrt{n}}{m}\frac{(m-1)!!}{(m-2)!!}\frac{(m-2n)!!}{(m+1-2n)!!} \leq 1.$$

It is easy to see that the quotient $\frac{(m-2n)!!}{(m+1-2n)!!}$ increases with increasing $1 \leq n < m/2$ so that $Q_n^{(m)}$ can be bounded from above by setting $n = \frac{m-1}{2}$:

$$Q_n^{(m)} \leq \frac{\sqrt{m-1}}{2\sqrt{2}m}\frac{(m-1)!!}{(m-2)!!} \overset{\text{Lem. 7}}{\leq} \frac{m-1}{\sqrt{2}m} \leq 1$$

and the auxiliary statement is proved.

Hence, by induction it holds

$$\begin{aligned}A_n^{(m+1)} &= \frac{1}{\sqrt{n}}A_{n-1}^{(m)}+\frac{m}{n}A_{n-1}^{(m-1)} \leq \frac{1}{\sqrt{n}}G_{n-1}^{(m)}+\frac{m}{n}G_{n-1}^{(m-1)} = 2\frac{m}{n}G_{n-1}^{(m-1)}\\ &= \frac{2^n m!!}{n!\,(m-2n)!!} = G_n^{(m+1)}\end{aligned}$$

and (45) is proved.

c) For $1 \le n < m/2$, we will show that

$$G_n^{(m)} \le \frac{m! a^m}{n^{(m+1)/2}} \tag{47}$$

holds. For the smallest values of m, i.e., $m = 2n+1$, this follows from Stirling's formula with $a \ge 1+\sqrt{3}$

$$\begin{aligned} G_n^{(2n+1)} &= 4^n \le \left(\frac{2a}{\mathrm{e}}\right)^{2n} \le 2\sqrt{\pi}\left(\frac{2a}{\mathrm{e}}\right)^{2n+1}\left(n+\frac{1}{2}\right)^{n+1/2}\left(1+\frac{1}{2n}\right)^{n+1} \\ &= a^{2n+1}\sqrt{2\pi}\frac{(2n+1)^{2n+3/2}\,\mathrm{e}^{-(2n+1)}}{n^{n+1}} \le \frac{a^m m!}{n^{(m+1)/2}}. \end{aligned}$$

Hence, for $m > 2n+1$ we get by induction

$$G_n^{(m)} \overset{(46)}{\le} \frac{m}{\sqrt{n+1}} G_n^{(m-1)} \overset{\text{induction}}{\le} \frac{m}{\sqrt{n+1}} \frac{(m-1)! a^{m-1}}{n^{m/2}} \le \frac{m! a^m}{n^{(m+1)/2}}$$

and (47) is proved.

Since $n \ge 2$, it follows

$$\left|g_n^{(m)}(t)\right| \le \frac{\delta_m}{(n+1)^{(m+1)/2}} \quad \text{with} \quad \delta_m = m!\left(1+\sqrt{3}\right)^m \left(\frac{3}{2}\right)^{(m+1)/2}.$$

From $t^{m+\ell} \le (n+\sqrt{n})^{m+\ell} \le 2^{m+\ell}(n+1)^{m+\ell}$, the assertion follows:

$$\left|t^{m+\ell} g_n^{(m)}(t)\right| \le 2^{m+1}\delta_m (n+1)^{\frac{m-1}{2}+\ell}.$$

Case $t \ge n+\sqrt{n}$.

Estimate (33) in this case follows from Proposition 4.

The estimate (36) follows trivially, for $n = 0$, from the definition of g_n and, for $n \ge 2$, directly from Proposition 4. For $n = 1$, the estimate follows by observing that (65) also holds for $n = 1$. The refined estimate (35) follows for $n \ge 2$ from (63) and the case $n = 0, 1$ have been treated already at the beginning of the proof. ∎

Proposition 3. *Let $n \ge 2$ and $m \ge 0$. For $0 \le t \le n - \sqrt{n}$, it holds*

$$\left|t^m g_n^{(m)}(t)\right| \le \mathrm{e}^m (m+2)! n^{\frac{m-1}{2}}. \tag{48}$$

For $0 \le t \le n - m\sqrt{n}$, the refined estimate holds

$$\left| t^m g_n^{(m)}(t) \right| \le 2^{m+1} \frac{(m+2)!}{\sqrt{n+1}} \left(\frac{n}{n-t} \right)^m . \tag{49}$$

Proof. Note that

$$t^m g_n^{(m)}(t) = g_n(t)\, s_{n,m}(t) \tag{50}$$

with

$$s_{n,m}(t) = \sum_{\ell=0}^{\min(m,n)} \tbinom{m}{\ell} (-1)^{m-\ell} \frac{n! t^{m-\ell}}{(n-\ell)!}. \tag{51}$$

Estimate of g_n.

We set $t = n/c$ for some $c = 1 + \frac{\delta}{n-\delta}$ and $0 \le \delta < n$. Stirling's formula gives us

$$g_n\left(\frac{n}{c}\right) \le \frac{w^n(c)}{\sqrt{n+1}},$$

where

$$\begin{aligned} w^n(c) &:= \left(\frac{\exp\left(1 - \frac{1}{c}\right)}{c} \right)^n = \left(1 - \frac{\delta}{n}\right)^n \exp(\delta) \\ &= \exp\left(n \log\left(1 - \frac{\delta}{n}\right) + \delta\right) = \exp\left(-n \sum_{k=2}^{\infty} \left(\frac{\delta}{n}\right)^k / k\right) \le \exp\left(-\frac{\delta^2}{2n}\right). \end{aligned}$$

Note that the range $t \in [0, n - \sqrt{n}]$ corresponds to the range of $\delta \in [\sqrt{n}, n]$. Thus

$$g_n(n-\delta) \le \frac{\exp\left(-\delta^2/(2n)\right)}{\sqrt{n+1}} \quad \forall \delta \in \left[\sqrt{n}, n\right]. \tag{52}$$

This proves the assertion (48) for $m = 0$ so that, for the following, we assume $m \ge 1$.

Estimate of $s_{n,m}$ **for** $m \ge 1$.

The definition of $s_{n,m}$ directly leads to the estimate

$$|s_{n,m}(t)| \le \sum_{\ell=0}^{m} \tbinom{m}{\ell} n^\ell t^{m-\ell} \le (t+n)^m . \tag{53}$$

Next this estimate will be refined for $0 \le t \le n - \sqrt{n}$. We set $\delta := n - t$ and introduce the function $\tilde{s}_{n,m}(\delta) = (-1)^m s_{n,m}(n-\delta)$ so that an estimate of $|\tilde{s}_{n,m}|$ at δ implies the same estimate of $|s_{n,m}|$ at $n - \delta$. For later use, we will estimate $\tilde{s}_{n,m}(\delta)$ not only for $\delta \in [\sqrt{n}, n]$ but for all $\delta \in \mathbb{R}$ with $|\delta| \ge \sqrt{n}$.

From (51) one concludes that $\tilde{s}_{n,m}$ satisfies the recursion

$$\tilde{s}_{n,m+1}(\delta) = (m-\delta)\,\tilde{s}_{n,m}(t) + (n-\delta)\,\tilde{s}'_{n,m}(\delta) \quad \text{with} \quad \tilde{s}_{0,0} := 1. \tag{54}$$

By inspection of (51) we conclude that

$$\tilde{s}_{n,m}(\delta) = \sum_{\ell=0}^{m} n^{\ell} p_{\ell,m}(\delta), \tag{55}$$

where $p_{\ell,m} \in \mathbb{P}_{m-\ell}$. From (54) we obtain the recursion

$$p_{\ell,m+1}(\delta) = (m-\delta)\,p_{\ell,m}(\delta) - \delta p'_{\ell,m}(\delta) + p'_{\ell-1,m}(\delta) \quad \text{with} \quad p_{0,0} := 1, \tag{56}$$

where we formally set $p_{-1,m} = 0$ and $p_{\ell,m} = 0$ for $\ell > m$. It is easy to prove by induction that

$$p_{0,m}(\delta) = (-1)^m \delta^m$$

and $p_{\ell,m} \in \mathbb{P}_{m-2\ell}$, where $\mathbb{P}_\ell := \{0\}$ for $\ell < 0$. Hence,

$$\tilde{s}_{n,m}(\delta) = \sum_{\ell=0}^{\lfloor m/2 \rfloor} n^{\ell} p_{\ell,m}(\delta). \tag{57}$$

Next, we will estimate the polynomial $p_{\ell,m}$. We write

$$p_{\ell,m}(\delta) = \sum_{k=0}^{m-2\ell} a_{\ell,m,k}\delta^k \quad \text{with} \quad a_{\ell,m,k} = c_{\ell,m-k}(-1)^{k+\ell}\frac{m!}{k!}.$$

Plugging this ansatz into (56) gives

$$a_{\ell,m+1,k} + a_{\ell,m,k-1} = (m-k)\,a_{\ell,m,k} + (k+1)\,a_{\ell-1,m,k+1}$$

and, in turn, the recursion[2]

$$c_{\ell,k+1} = \frac{k}{k+1}c_{\ell,k} + \frac{c_{\ell-1,k-1}}{k+1} \quad \text{with} \quad c_{\ell,2\ell} = \frac{1}{2^\ell \ell!}.$$

[2]Particular results are

$$\begin{aligned} c_{0,k} &= \delta_{0,k} \\ c_{1,k} &= \frac{1}{k} \\ c_{2,k} &= \frac{H_{k-2}-1}{k} \\ c_{3,k} &= -\frac{-1+2H_{k-3}-H_{k-2}^2+H_{k-2,2}}{2k} \end{aligned}$$

with the harmonic numbers $H_{n,r} = \sum_{\ell=1}^{n} 1/\ell^r$.

By induction it is easy to prove that

$$c_{\ell,k} \leq 1 \quad \forall k \geq 2\ell$$

so that

$$|p_{\ell,m}(\delta)| \leq m! \sum_{k=0}^{m-2\ell} \frac{|\delta|^k}{k!}.$$

Note that $t^{k-1}/(k-1)! \leq t^r/r!$ for all $t \geq r \geq k \geq 1$ so that

$$|p_{\ell,m}(\delta)| \leq \begin{cases} \dfrac{(m+1)!}{(m-2\ell)!} |\delta|^{m-2\ell} & |\delta| \geq m-2\ell, \\ m!\, \mathrm{e}^{|\delta|} & \delta \in \mathbb{R}. \end{cases} \tag{58}$$

Hence, it holds for $|\delta| \geq \sqrt{n}$

$$\begin{aligned} |s_{n,m}(n-\delta)| = |\tilde{s}_{n,m}(\delta)| &\leq \begin{cases} n^{m/2} (m+2)! \max\limits_{0 \leq \ell \leq m/2} \dfrac{\left(|\delta|/\sqrt{n}\right)^{m-2\ell}}{(m-2\ell)!} & |\delta| \geq m, \\ n^{m/2} (m+1)!\, \mathrm{e}^{|\delta|} & \delta \in \mathbb{R}, \end{cases} \\ &\leq \begin{cases} (m+2)^2 \delta^m & m\sqrt{n} \leq \delta, \\ n^{m/2} (m+2)!\, \mathrm{e}^{|\delta|/\sqrt{n}} & m \leq \delta \leq m\sqrt{n}, \\ n^{m/2} (m+1)!\, \mathrm{e}^{|\delta|} & \delta \in \mathbb{R}. \end{cases} \end{aligned} \tag{59}$$

Estimate of $t^m g_n^{(m)}(t)$.

Let $t = n - \delta$ for some $\delta \in [\sqrt{n}, n]$.

Case $\sqrt{n} \leq \delta \leq m$. The combination of the third case in (59) with (52) yields

$$\left| t^m g_n^{(m)}(t) \right| \leq n^{m/2} (m+1)! \frac{\exp\left(\delta - \frac{\delta^2}{2n}\right)}{\sqrt{n+1}}. \tag{60}$$

From $\delta \leq m$ we conclude that

$$\left| t^m g_n^{(m)}(t) \right| \leq n^{\frac{m-1}{2}} (m+1)!\, \mathrm{e}^m.$$

Case $\max(m, \sqrt{n}) \leq \delta \leq \sqrt{n} \min(m, \sqrt{n})$. Here we get from the second case in (59) and (52) the estimate

$$\left| t^m g_n^{(m)}(t) \right| \leq n^{m/2} (m+2)! \frac{\exp\left(\frac{\delta}{\sqrt{n}} - \frac{\delta^2}{2n}\right)}{\sqrt{n+1}}.$$

The exponent is monotonously decreasing for $\delta \geq \sqrt{n}$ so that

$$\left| t^m g_n^{(m)}(t) \right| \leq \sqrt{\mathrm{e}} n^{\frac{m-1}{2}} (m+2)!.$$

Case $m\sqrt{n} \leq \delta \leq n$. In this case, we obtain, by using $\mathrm{e}^t \geq t^k/k!$ for $t \geq 0$, from the first case in (59) and (52)

$$\left| t^m g_n^{(m)}(t) \right| \leq \frac{(m+2)^2}{\sqrt{n+1}} \frac{\delta^m}{\exp\left(\delta^2/(2n)\right)} \leq 2^m \frac{(m+2)^2}{\sqrt{n+1}} m! \left(\frac{n}{\delta}\right)^m.$$

■

Proposition 4. *Let $n \geq 2$.*

1. *For $t \geq n + \sqrt{n}$, it holds*

$$g_n(t) \leq \frac{1}{\sqrt{n+1}} \exp\left(\sqrt{n}\left(1 - \frac{t}{n+\sqrt{n}}\right)\right). \tag{61}$$

2. *For $m \geq 0$ and $\ell = 0, 1$, we get the estimate*

$$\left| t^{m+\ell} g_n^{(m)}(t) \right| \leq (4\,\mathrm{e})^{m+3} (m+2)! n^{\frac{m-1}{2}+\ell} \quad \forall t \geq n + \sqrt{n}. \tag{62}$$

3. *For $t \geq n + m(\sqrt{n}+2)$ and $m \geq 2\log n$, the refined estimate*

$$\left| t^{m+\ell} g_n^{(m)}(t) \right| \leq \left(\frac{3}{\ln\frac{1}{c}}\right)^m (m+2)! (4n)^\ell. \tag{63}$$

   holds with $c := 9/10$.

Proof.

For any $0 < c \leq 1$, we write

$$g_n(t) = \frac{t^n \exp(-ct)}{n!} \exp(-(1-c)t).$$

The first fraction has its maximum at $t = n/c$ so that Stirling's formula leads to

$$g_n(t) \leq \frac{1}{\sqrt{n+1}} \left(\frac{1}{c}\right)^n \exp(-(1-c)t). \tag{64}$$

We choose $c = \frac{n}{n+\sqrt{n}}$ and obtain

$$g_n(t) \leq \frac{1}{\sqrt{n+1}} \left(1 + \frac{1}{\sqrt{n}}\right)^n \exp\left(-\frac{\sqrt{n}}{n+\sqrt{n}} t\right) \leq \frac{1}{\sqrt{n+1}} \exp\left(\sqrt{n}\left(1 - \frac{t}{n+\sqrt{n}}\right)\right) \tag{65}$$

which shows the exponential decay for $t \geq n + \sqrt{n}$ (cf. (61)). This also implies (62) for $m = \ell = 0$. For $m = 0$ and $\ell = 1$, we obtain

$$|tg_n(t)| \leq \frac{t^{n+1}\,\mathrm{e}^{-t}}{n!} \leq \frac{(n+1)^{n+1}\,\mathrm{e}^{-n-1}}{n!} \overset{\text{Stirling}}{\leq} \sqrt{n+1}$$

so that (62) is satisfied for $m = 0$, $\ell = 0, 1$.

For the rest of the proof, we assume $m \geq 1$ and employ the representation (50).

Estimate of $t^{m+\ell} g_n^{(m)}(t)$**.**

The combination of (50), (65), and (53) leads to

$$\left|t^{m+\ell} g_n^{(m)}(t)\right| \leq \frac{\left(n+\sqrt{n}\right)^{m+\ell}}{\sqrt{n+1}} (x+1)^{m+\ell} \exp\left(\sqrt{n}\,(1-x)\right) \quad \text{with} \quad x = \frac{t}{n+\sqrt{n}} \in [1, \infty[\,. \tag{66}$$

The right-hand side in (66) is maximal for $x = \frac{m+\ell}{\sqrt{n}} - 1$ if $m + \ell \geq 2\sqrt{n}$ and for $x = 1$ otherwise.

Case 1) $m + \ell \geq 2\sqrt{n}$.

For $m + \ell \geq 2\sqrt{n}$, the right-hand side in (66) is maximal for $x = \frac{m+\ell}{\sqrt{n}} - 1$ and we get

$$\left|t^{m+\ell} g_n^{(m)}(t)\right| \leq n^{\frac{m+\ell}{2}} \frac{2^{m+\ell}}{\sqrt{n+1}} (m+\ell)^{m+\ell} \exp\left(2\sqrt{n} - m - \ell\right) \leq (2\,\mathrm{e})^{m+\ell} \frac{(m+1)!}{\sqrt{m+1}} n^{\frac{m+\ell-1}{2}}.$$

Case 2) $\sqrt{n} \leq m + \ell \leq 2\sqrt{n}$.

In this case, the right-hand side in (66) is maximal for $x = 1$ and we arrive, by Stirling's formula and by using $\sqrt{n} \leq m + \ell$, at the estimate

$$\begin{aligned} \left|t^{m+\ell} g_n^{(m)}(t)\right| &\leq 2^{m+\ell} \frac{\left(n+\sqrt{n}\right)^{m+\ell}}{\sqrt{n+1}} \leq \frac{4^{m+\ell} n^{m/2+\ell}}{\sqrt{n+1}} n^{m/2} \\ &\leq \frac{4^{m+\ell} n^{m/2+\ell}}{\sqrt{n+1}} (m+\ell)^m \leq \frac{(4\,\mathrm{e})^{m+\ell}}{(m+1)^{1/2}} m! n^{\frac{m-1}{2}+\ell}. \end{aligned}$$

Case 3) $m + \ell \leq \sqrt{n}$.

Note that

$$|s_{n,m}(n+\delta)| = |\tilde{s}_{n,m}(-\delta)| = \left|\sum_{\ell=0}^{m} n^\ell p_{\ell,m}(-\delta)\right|$$

with $p_{\ell,m}$ as in (57). Since $\delta \geq \sqrt{n} \geq m+\ell$, we get (cf. (58)

$$|p_{k,m}(-\delta)| \leq \frac{(m+1)!}{(m-2k)!}\delta^{m-2k} \qquad \forall 0 \leq k \leq m/2$$

and, in turn, (cf. (55))

$$\begin{aligned} n^k |p_{k,m}(-\delta)| &\leq \frac{(m+1)!}{(m-2k)!}\delta^{m-2k} n^k = (m+1)! n^{m/2} \frac{\left(\frac{\delta}{\sqrt{n}}\right)^{m-2k}}{(m-2k)!} \\ &\overset{(59)}{\leq} \begin{cases} (m+1)\,\delta^m & \delta \geq m\sqrt{n}, \\ (m+1)! n^{m/2}\,\mathrm{e}^{\delta/\sqrt{n}} & \delta \geq \sqrt{n}. \end{cases} \end{aligned}$$

This implies for $s_{n,m}$ the estimate

$$|s_{n,m}(n+\delta)| = |\tilde{s}_{n,m}(-\delta)| \leq \begin{cases} (m+1)^2\,\delta^m & \delta \geq m\sqrt{n}, \\ (m+2)! n^{m/2}\,\mathrm{e}^{\delta/\sqrt{n}} & \delta \geq \sqrt{n}. \end{cases}$$

The combination with (50) yields with $t = n+\delta$

$$\left|t^{m+\ell} g_n^{(m)}(t)\right| \leq \begin{cases} \dfrac{(n+\delta)^{n+\ell}\,\mathrm{e}^{-n-\delta}}{n!}(m+1)^2\,\delta^m & \delta \geq m\sqrt{n}, \\ \dfrac{(n+\delta)^{n+\ell}\,\mathrm{e}^{-n-\delta}}{n!}(m+2)! n^{m/2}\,\mathrm{e}^{\delta/\sqrt{n}} & \delta \geq \sqrt{n}. \end{cases} \tag{67}$$

Case 3a) $\sqrt{n} \leq \delta \leq m(\sqrt{n}+2)$.

We recall that in this case $m \leq \sqrt{n}$ so that the (generous) estimate $\delta \leq n + 2\sqrt{n} \leq 3n$ can be applied.

We employ the second case in the right-hand side of (67). First, we estimate one factor $(n+\delta)^\ell$ by $4n^\ell$ and then observe that the maximum of the remaining expression is taken at $\delta = \frac{n}{\sqrt{n}-1}$. Thus, we have

$$\begin{aligned} \left|t^{m+\ell} g_n^{(m)}(t)\right| &\leq 4\frac{\left(\frac{n^{\frac{3}{2}}}{\sqrt{n}-1}\right)^n \exp\left(-\left(n+\sqrt{n}\right)\right)}{n!} n^{m/2+\ell}(m+2)! \\ &\overset{\text{Stirling}}{\leq} \frac{4}{\sqrt{n+1}}\left(\frac{\sqrt{n}}{\sqrt{n}-1}\right)^n \exp\left(-\sqrt{n}\right) n^{m/2+\ell}(m+2)!. \end{aligned}$$

Since the function $\left(\frac{\sqrt{n}}{\sqrt{n}-1}\right)^n \exp\left(-\sqrt{n}\right)$ is monotonously decreasing, we get, for $n \geq 2$, the estimate

$$\left|t^{m+\ell} g_n^{(m)}(t)\right| \leq 12 n^{\frac{m-1}{2}+\ell}(m+2)!.$$

Case 3b) $m(\sqrt{n}+2) \leq \delta$. We employ the first case in the right-hand side of (67). Since $m \geq 1$, the right-hand side in (67) (first case) is maximal for $\delta_0(m) = \frac{m+\ell}{2}\left(1+\sqrt{1+\frac{4nm}{(m+\ell)^2}}\right)$. Note that the quantity $\frac{\delta_0(m)}{m}$ is monotonously decreasing with respect to m so that, for $\ell = 0, 1$, it holds

$$\delta_0(m) \leq m\left(1+\sqrt{1+n}\right) \leq m\left(\sqrt{n}+2\right).$$

Hence, the right-hand side in (67) (first case) has its maximum at $\delta_\star = m(\sqrt{n}+2)$ which is given by (recall $m(\sqrt{n}+2) \leq 3n$)

$$\text{rhs} := \frac{(n+\delta_\star)^{n+\ell}\,\mathrm{e}^{-n-\delta_\star}}{n!}(m+1)^2\,\delta_\star^m \tag{68}$$

$$\overset{\text{Stirling}}{\leq} (4n)^\ell \left(\frac{1+\frac{\delta_\star}{n}}{\mathrm{e}^{\delta_\star/n}}\right)^n (m+1)^2\,\delta_\star^m. \tag{69}$$

Case 3b$_1$) General range: $m \in [1, \sqrt{n}]$. We consider the numerator in the right-hand side of (68) as a function of a free positive variable $\delta_\star$ with maximum at $\delta_\star = \ell$. This leads to

$$\text{rhs} \leq (n+1)^{\ell-1/2}(m+1)^2\, m^m 3^m n^{m/2} \overset{\text{Stirling}}{\leq} \sqrt{2}\,(3\,\mathrm{e})^m\,(m+2)!n^{\frac{m-1}{2}+\ell}.$$

Case 3b$_2$) Restricted range: $m \in [2\log n, \sqrt{n}]$

We will estimate (69) from above. Note that the function $(1+x)\,\mathrm{e}^{-x}$ is monotonously decreasing for $x \geq 0$. Since $\frac{\delta_\star}{n} \geq \frac{m}{\sqrt{n}}$we get

$$\text{rhs} \leq (4n)^\ell \left(\frac{1+\frac{m}{\sqrt{n}}}{\mathrm{e}^{m/\sqrt{n}}}\right)^n (m+1)^2\,\delta_\star^m. \tag{70}$$

Case 3b$_{2\,\mathrm{I}}$) $m \leq \frac{3\sqrt{n}}{4}$.

A Taylor argument for the logarithm implies

$$1+x \leq \exp\left(x - \frac{1}{4}x^2\right) \qquad \forall 0 \leq x \leq 3/4$$

and, in turn,

$$\text{rhs} \leq \frac{(3\,\mathrm{e})^m\,(m+2)!}{\sqrt{m+1}}(4n)^\ell \left(n\,\mathrm{e}^{-m/2}\right)^{m/2}.$$

For $m \geq 2\log n$ the last bracket is bounded by 1 and we have proved

$$\text{rhs} \leq (3\,\mathrm{e})^m\,(m+2)!\,(4n)^\ell.$$

Case $\mathbf{b}_{2\,\mathrm{II}}$) $\frac{3\sqrt{n}}{4} \leq m \leq \sqrt{n}$.

Note that the bracket in the right-hand side of (70) is monotonously decreasing as a function of $m/\sqrt{n}$ and hence, by choosing $m = 3\sqrt{n}/4$ and using $\delta_\star \leq 3n$, we end up with

$$\text{rhs} \leq 3^m (4n)^\ell (m+1)^2 (c^n n^m) \quad \text{with} \quad c = \frac{9}{10}.$$

The last bracket is maximal for $n = \frac{m}{\log \frac{1}{c}}$ so that

$$\text{rhs} \leq \left(\frac{3}{\log \frac{1}{c}} \right)^m (4n)^\ell (m+1)^2 \, \mathrm{e}^{-m} m^m \leq \left(\frac{3}{\log \frac{1}{c}} \right)^m (m+2)! \, (4n)^\ell .$$

■

Appendix A. Some Auxiliary Estimates

In this section we first provide an estimate for the approximation of the function $f(t,x) := \frac{1}{\sqrt{t^2 - x^2}}$ by its Taylor polynomial $t^{-1} T_\mu (x/t)$ with respect to x around $x_0 = 0$, where T_μ is as in (18).

Proposition 5. *Let* $0 \leq x < t$. *Then*

$$\left| f(t,x) - t^{-1} T_\mu (x/t) \right| \leq \left(\frac{x}{t} \right)^{2\mu} \frac{1}{\sqrt{t^2 - x^2}}. \tag{A.1}$$

Proof. We will prove

$$\left| 1 - \sqrt{1 - x^2} T_\mu (x) \right| \leq x^{2\mu} \qquad \forall 0 \leq x < 1$$

from which (A.1) follows for $t = 1$ and, for general $t > 0$, by a simple scaling argument. By using [5, (5.24.31)] we obtain

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \binom{2k}{k} \left(\frac{x}{2} \right)^{2k} \qquad |x| < 1.$$

Using $\left(\sqrt{1-x^2} \right)' = -\frac{x}{\sqrt{1-x^2}}$ we conclude that

$$\sqrt{1-x^2} = -\sum_{k=0}^{\infty} \frac{1}{(2k-1)} \binom{2k}{k} \left(\frac{x}{2} \right)^{2k} \qquad |x| < 1.$$

Hence,

$$1-\sqrt{1-x^2}T_\mu(x) = 1+\left(\sum_{k=0}^{\infty}\frac{1}{(2k-1)}\binom{2k}{k}\left(\frac{x}{2}\right)^{2k}\right)\left(\sum_{\ell=0}^{\mu-1}\binom{2\ell}{\ell}\left(\frac{x}{2}\right)^{2\ell}\right) = 1+\sum_{r=0}^{\infty}c_{r,\mu}\left(\frac{x}{2}\right)^{2r}$$

with $c_{r,\mu} := \sum_{\ell=0}^{\min\{r,\mu-1\}}\frac{1}{2(r-\ell)-1}\binom{2(r-\ell)}{r-\ell}\binom{2\ell}{\ell}$. For all $\mu\in\mathbb{N}_{\geq 1}$ and $r\in\mathbb{N}_0$ it follows by induction

$$c_{r,\mu} = \begin{cases} -1 & r=0,\\ 0 & 1\leq r<\mu,\\ \frac{\mu}{r}\binom{2\mu}{\mu}\binom{2(r-\mu)}{r-\mu} & \mu\leq r \end{cases}$$

so that

$$\begin{aligned} 1-\sqrt{1-x^2}T_\mu(x) &= \mu\binom{2\mu}{\mu}\left(\frac{x}{2}\right)^{2\mu}\sum_{r=0}^{\infty}\frac{1}{r+\mu}\binom{2r}{r}\left(\frac{x}{2}\right)^{2r}\\ &\leq \mu\binom{2\mu}{\mu}\left(\frac{x}{2}\right)^{2\mu}\sum_{r=0}^{\infty}\frac{1}{r+\mu}\binom{2r}{r}\left(\frac{1}{2}\right)^{2r}\\ &= x^{2\mu}. \end{aligned}$$

■

Proposition 6. *For any $m,n\in\mathbb{N}_0$ and $0\leq 2r\leq n-1$, it holds*

$$\int_x^\infty t^{m-1-2r}g_n^{(m)}(t)\,dt = -\frac{(n-1-2r)!}{n!}\left(\frac{d}{dx}\right)^{2r}\left(x^m g_{n-1-2r}^{(m-1-2r)}(x)\right)\quad \forall x>0.$$

Proof.

Since the limit $x\to\infty$ of both sides converges to zero due to the exponential decay of $g_n^{(k)}(x)$ it sufficient to prove that the derivatives of both sides coincide. We set

$$\begin{aligned} L_r^{m,n}(x) &:= -x^{m-1-2r}g_n^{(m)}(x)\\ R_r^{m,n}(x) &:= -\frac{(n-1-2r)!}{n!}\left(\frac{d}{dx}\right)^{2r+1}\left(x^m g_{n-1-2r}^{(m-1-2r)}(x)\right) \end{aligned}$$

and prove $L_r^{m,n}(x) = R_r^{m,n}(x)$ by induction over r.

Start of induction: $r=0$. Then,

$$L_0^{m,n}(x) = -x^{m-1}g_n^{(m)}(x).$$

By $m-$times differentiating the relation $g_n(x) = \frac{x}{n} g_{n-1}(x)$ we obtain

$$\begin{aligned} L_0^{m,n}(x) &= -x^{m-1} g_n^{(m)}(x) = \frac{-x^{m-1}}{n} \left(x g_{n-1}^{(m)}(x) + m g_{n-1}^{(m-1)}(x) \right) \\ &= -\frac{1}{n} \frac{d}{dx} \left(x^m g_{n-1}^{(m-1)}(x) \right) = R_0^{m,n}(x). \end{aligned}$$

Induction step: We assume that $L_k^{m,n} = R_k^{m,n}$ for $0 \le k \le r$. Then

$$\begin{aligned} R_{r+1}^{m,n}(x) &:= -\frac{(n-3-2r)!}{n!} \left(\frac{d}{dx} \right)^{2r+1} \left(\frac{d}{dx} \right)^2 \left(x^m g_{n-3-2r}^{(m-3-2r)}(x) \right) \\ &= \frac{m(m-1) R_r^{m-2,n-2}(x) + 2m R_r^{m-1,n-2}(x) + R_r^{m,n-2}(x)}{n(n-1)}. \end{aligned}$$

By using the induction assumption and several times the recurrence relation

$$g_n(x) = \frac{x}{n} g_{n-1}(x) \quad \text{which implies} \quad g_n^{(k)}(x) = \frac{x}{n} g_{n-1}^{(k)}(x) + \frac{k}{n} g_{n-1}^{(k-1)}(x)$$

we obtain

$$\begin{aligned} R_{r+1}^{m,n}(x) &= \frac{m(m-1) L_r^{m-2,n-2}(x) + 2m L_r^{m-1,n-2}(x) + L_r^{m,n-2}(x)}{n(n-1)} \\ &= -\frac{m(m-1) x^{m-3-2r} g_{n-2}^{(m-2)}(x) + 2m x^{m-2-2r} g_{n-2}^{(m-1)}(x) + x^{m-1-2r} g_{n-2}^{(m)}(x)}{n(n-1)} \\ &= -\frac{m x^{m-3-2r}}{n} g_{n-1}^{(m-1)} - \frac{x^{m-2-2r}}{n} g_{n-1}^{(m)} \\ &= -x^{m-3-2r} g_n^{(m)} = L_{r+1}^{m,n}(x). \end{aligned}$$

■

Lemma 7. *For $m \ge 0$ and $0 \le k \le m$, it holds*

$$\left(\frac{3}{5} \right)^k \sqrt{\frac{m!}{(m-k)!}} \le \frac{m!!}{(m-k)!!} \le 2^k \sqrt{\frac{m!}{(m-k)!}}$$

Proof.

The case $m = 0$ and the case $k = 0$ are trivial. Let $k = 1$. The formula is easy to check for $m = 1, 2$ and we restrict in the following to $m \ge 3$.

We use Stirling's formula in the form

$$n! = C_n n^{n+\frac{1}{2}} \exp(-n) \quad \text{with} \quad C_n = \sqrt{2\pi} \exp(\theta/(12n)) \text{ for } n \in \mathbb{N}_{\ge 1} \text{ and } \theta \in]0,1[$$

so that $\frac{C_n}{\sqrt{2\pi}} \in \left]1, \mathrm{e}^{1/12}\right[$.

For $m = 2r+1$, we get with $r \geq 1$

$$\frac{(2r+1)!!}{(2r)!!} = \frac{(2r+1)!}{4^r\,(r!)^2} = \frac{C_m}{C_r^2}\frac{(2r+1)^{2r+\frac{3}{2}}\exp(-2r-1)}{4^r r^{2r+1}\exp(-2r)}$$
$$= \frac{C_m}{C_r^2\,\mathrm{e}}2\left(1+\frac{1}{2r}\right)^{2r+1}\sqrt{2r+1}\begin{cases} \leq \frac{\mathrm{e}^{1/12-1}}{\sqrt{2\pi}}2\left(\frac{3}{2}\right)^3\sqrt{m} & \leq \frac{11}{10}\sqrt{m}, \\ \geq \frac{2}{\sqrt{2\pi}\,\mathrm{e}^{1/6}}\sqrt{m} & \geq \frac{3}{5}\sqrt{m}. \end{cases}$$

For $m = 2r$ and $r \geq 2$, we get

$$\frac{(2r)!!}{(2r-1)!!} = \frac{(2r)!!\,(2r-2)!!}{(2r-1)!} = \frac{2^{2r-1}r!\,(r-1)!}{(2r-1)!} = \frac{C_r C_{r-1} 2^{2r-1} r^{r+1/2}\,(r-1)^{r-1/2}\,\mathrm{e}^{1-2r}}{C_{m-1}\,(2r-1)^{2r-1/2}\,\mathrm{e}^{1-2r}}$$
$$= \sqrt{m-1}\frac{C_r C_{r-1}}{2C_{m-1}}\left(\frac{r}{r-1/2}\right)^{r+1/2}\left(\frac{r-1}{r-1/2}\right)^{r-1/2}$$
$$\begin{cases} \leq \sqrt{\frac{\pi(m-1)}{2}}\,\mathrm{e}^{-1/3}\left(\frac{4}{3}\right)^{5/2} \leq 2\sqrt{m-1} \leq 2\sqrt{m}, \\ \geq \sqrt{\frac{\pi(m-1)}{2}}\,\mathrm{e}^{\frac{5}{12}}\left(\frac{2}{3}\right)^{3/2} \geq \frac{11}{10}\sqrt{m-1} \geq \frac{4}{5}\sqrt{m}. \end{cases}$$

In summary, we have proved

$$\frac{3}{5}\sqrt{m} \leq \frac{m!!}{(m-1)!!} \leq 2\sqrt{m}.$$

From this the assertion follows by induction. ■

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